Arithemetic

算术

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To Begin With

QR Mathematical Convention 1

Any number in QR is a real number.
Imaginary Numbers are out of scope of QR.

Integers

Presentation Overview for Integers

1. Integers
   - Even v.s. Odd
   - Divisor and the Greatest Common Divisor
   - Prime V.s. Composite
   - Multiple and Least Common Multiple
   - Quotient and Remainder
2. Fractions

Even v.s. Odd

Can negative numbers be odd or even?
For example, is -2 Even?

Can negative numbers be odd or even?

For example, is -2 Even?

YES!

Even v.s. Odd

奇偶运算: 看剩下的尾巴

定义

x is an odd number if $x = 2k + 1$, where $k = ... - 2, -1, 0, 1, 2, ...$

x is an even number if $x = 2k$, where $k = ... - 2, -1, 0, 1, 2, ...$

Facts about Odd and Even Numbers

- *odd* $\pm$ *even* = *odd* $\quad (2k_1 + 1) \pm 2k_2 = 2(k_1 \pm k_2) + \mathbf{1}$
- *odd* $\pm$ *odd* = *even* $\quad (2k_1 + 1) \pm (2k_2 + 1) = 2(k_1 \pm k_2)$
- *even* $\pm$ *even* = *even* $\quad 2k_1 \pm 2k_2 = 2(k_1 \pm k_2)$

- *odd* $\times$ *even* = *even* $\quad (2k_1 + 1) \times 2k_2 = 2(2k_1k_2 + k_2)$
- *odd* $\times$ *odd* = *odd* $\quad (2k_1 + 1) \times (2k_2 + 1) = 2(2k_1k_2 + k_1 + k_2) + \mathbf{1}$
- *even* $\times$ *even* = *even* $\quad 2k_1 \times 2k_2 = 4k_1$

First Try!

用奇偶运算算如下题目 (看尾巴)；注意读题（“must be”)；

If a and b are both positive integers, and a −b and a/b are even, which of following must be an odd integer?

A. $\frac{a}{2}$
B. $\frac{b}{2}$
C. $\frac{(a+b)}{2}$
D. $\frac{(a+2)}{2}$
E. $\frac{(b+2)}{2}$

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C. $\frac{(a+b)}{2}$
D. $\frac{(a+2)}{2}$
E. $\frac{(b+2)}{2}$

$\because$ a - b is even
$\therefore$ a is even and b is even;
Or, a is odd and b is odd;

$\because$ a/b is even
$\therefore$ a = 2kb;
$\therefore$ a/2 is even
b/2 is even or odd

First Try!

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C. $\frac{(a+b)}{2}$
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Or, a is odd and b is odd;

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$\therefore$ a $=$ 2kb;
$\therefore$ a/2 is even
b/2 is even or odd

A. $\frac{a}{2}$ even
B. $\frac{b}{2}$ even or odd
C. $\frac{(a+b)}{2}$ even or odd
D. $\frac{(a+2)}{2}$ even $+ 1 =$ odd
E. $\frac{(b+2)}{2}$ even or odd $+ 1 =$ even or odd

Answer **D**

A Real QR Problem!

用奇偶运算算如下题目 (看尾巴)；注意读题（“must be”)；

If p is an even integer, which of the following must be an odd integer?

- $\dfrac{3p}{2}$
- $\dfrac{3p}{2}+1$
- $\dfrac{3p^2}{2}$
- $\dfrac{3p^2}{2}+1$
- p^3

图: 2-Sec1-9

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- ○ p^3

图: 2-Sec1-9

$\because p = 2k$

$\therefore \frac{p}{2} = k$, which is even or odd.

$\therefore \frac{p^2}{2} = 2k^2$, which is even.

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Answer **D** 请把 5 个选项大小排序

Divisor and the Greatest Common Divisor

Definitions & Examples For A Divisor(Factor)

约数（因数）

定义

When integers are multiplied, each of the multiplied integers is called a factor or divisor of the resulting product

例

- $(2)(3)(10) = 60$, so 2,3, and 10 are factors of 60.
- The integers 4, 15, 5, and 12 are also factors of 60.
- $(-2)(-30) = 60$. The negatives of the positive factors are also factors of 60,

推论

Every positive integers a is divisible by the trivial divisors 1 and a.

定义

The greatest common divisor (or greatest common factor) of two nonzero integers c and d is the greatest positive integer that is a divisor of both c and d.

例

The greatest common divisor of 30 and 75 is 15

- The positive divisors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30.
- The positive divisors of 75 are 1, 3, 5, 15, 25, and 75.
- The common positive divisors of 30 and 75 are 1, 3, 5, and 15.
- The greatest of these is 15.

Is there a better way to find the gcd of two nonzero integers c and d?

Prime V.s. Composite

定义

A prime number is an integer greater than 1 that has only two positive divisors: 1 and itself.

定义

An integer greater than 1 that is not a prime number is called a composite number.

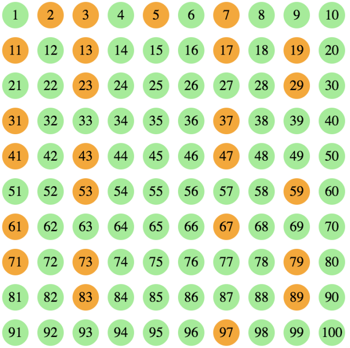

图: There are 25 prime numbers which are less than 100.

Divisible Rules

定理 (Divisible by 2)

The last digit is even (0, 2, 4, 6, or 8). Thus, Any even number who is greater than 2 is not a prime.

定理 (Divisible by 3)

The Sum of digits if divisible by 3. For example, 12, 36, 93, 102.

定理 (Divisible by 5)

The last digit is 0 or 5.

定理 (Prime Factorization)

Every integer greater than 1 either is a prime number or can be uniquely expressed as a product of factors that are prime numbers, or prime divisors

例

- $12 = 2^2 \cdot 3$
- $81 = 3^3$
- $3398 =$

Prime Factorization
质数分解

定理 (Prime Factorization)

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例

- $12 = 2^2 \cdot 3$
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- $3398 = 2 \cdot 13^2$
- $1155 = 3 \cdot 5 \cdot 7 \cdot 11$

First Try!

If y is the smallest positive integer such that 3150 multiplied by y is the square of an integer, then y must be?

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If y is the smallest positive integer such that 3150 multiplied by y is the square of an integer, then y must be?

$3150 = 2 \cdot 3^2 \cdot 5^2 \cdot 7$

Prime Factorization
质数分解

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例

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- $81 = 3^3$
- $3398 = 2 \cdot 13^2$
- $1155 = 3 \cdot 5 \cdot 7 \cdot 11$

First Try!

If y is the smallest positive integer such that 3150 multiplied by y is the square of an integer, then y must be?

$3150 = 2 \cdot 3^2 \cdot 5^2 \cdot 7$

$y \cdot 2 \cdot 3^2 \cdot 5^2 \cdot 7 = x^2$, in which x and y are positive integers.

The smallest y $= 2 \cdot 7 =$ **14**

A Real QR Problem!

What is the greatest prime factor of $3^{100} - 3^{97}$?

- 3
- 5
- 7
- 11
- 13

图: 6-Sec3-20

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$3^{100} - 3^{97} = 3^{97} \cdot (3^3 - 1) = 3^{97} \cdot 26 = 3^{97} \cdot 2 \cdot 13,$
in which 3 , 2 and 11 are the prime factors.

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$3^{100} - 3^{97} = 3^{97} \cdot (3^3 - 1) = 3^{97} \cdot 26 = 3^{97} \cdot 2 \cdot 13,$
in which 3 , 2 and 11 are the prime factors.

Answer **E**

Find GCD with Prime Factorization
用质数分解找公因数

What is the gcd of 168 and 96?

1. Prime Factorization of 168:
   $168 = 2^3 \cdot 3 \cdot 7$
2. Prime Factorization of 96:
   $96 = 2^5 \cdot 3$
3. Gcd equals the products of the common factors with smaller exponent.
   取公因子的最小指数
   $gcd(168, 96) = 2^3 \cdot 3 = \mathbf{24}$

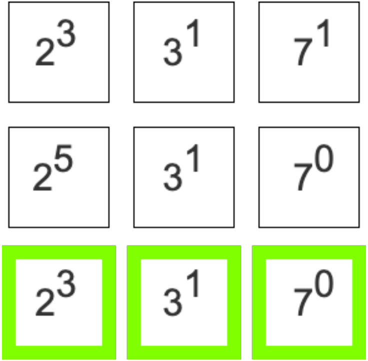

图: The Factors with the smaller exponents

Have a try!

先 Prime Factorization，然后找 Common Factors

What is the gcd of 42 and 56?

Have a try!

先 Prime Factorization，然后找 Common Factors

What is the gcd of 42 and 56?

1. Prime Factorization of 42: $42 = 3 \cdot 2 \cdot 7$

Have a try!

先 Prime Factorization，然后找 Common Factors

What is the gcd of 42 and 56?

1. Prime Factorization of 42: $42 = 3 \cdot 2 \cdot 7$
2. Prime Factorization of 56: $56 = 2^3 \cdot 7$

Have a try!

先 Prime Factorization，然后找 Common Factors

What is the gcd of 42 and 56?

1. Prime Factorization of 42: $42 = 3 \cdot 2 \cdot 7$
2. Prime Factorization of 56: $56 = 2^3 \cdot 7$
3. $gcd(42, 56) = 2 \cdot 7 = \mathbf{14}$

A Real QR Problem!

n and q are different positive integers.

Quantity A	Quantity B
The greatest common factor of n and q	The greatest common factor of $201n+2q$ and $100n+q$

- ◯ Quantity A is greater.
- ◯ Quantity B is greater.
- ◯ The two quantities are equal.
- ◯ The relationship cannot be determined from the information given.

图: 7-Sec2-7

Answer

$A = gcd(n, q)$
$\therefore n = k_1 \cdot gcd(n, q)$ and $q = k_2 \cdot gcd(n, q)$

$B = gcd(201n + 2q, 100n + q)$
$201n + 2q = 67 \cdot 3n + 2q = gcd(n, q)(67 \cdot 3k_1 + 2k_2)$
$100n + q = 2^2 \cdot 5^2 n + q = gcd(n, q)(2^2 \cdot 5^2 k_1 + k_2)$
$B = gcd(201n + 2q, 100n + q) = gcd(n, q) \cdot gcd(67 \cdot 3k_1 + 2k_2, 2^2 \cdot 5^2 k_1 + k_2)$

$\because gcd(67 \cdot 3k_1 + 2k_2, 2^2 \cdot 5^2 k_1 + k_2) \geq 1$
$\therefore A \leq B$

Answer

$A = gcd(n, q)$
$\therefore n = k_1 \cdot gcd(n, q)$ and $q = k_2 \cdot gcd(n, q)$

$B = gcd(201n + 2q, 100n + q)$
$201n + 2q = 67 \cdot 3n + 2q = gcd(n, q)(67 \cdot 3k_1 + 2k_2)$
$100n + q = 2^2 \cdot 5^2 n + q = gcd(n, q)(2^2 \cdot 5^2 k_1 + k_2)$
$B = gcd(201n + 2q, 100n + q) = gcd(n, q) \cdot gcd(67 \cdot 3k_1 + 2k_2, 2^2 \cdot 5^2 k_1 + k_2)$

$\because gcd(67 \cdot 3k_1 + 2k_2, 2^2 \cdot 5^2 k_1 + k_2) \geq 1$
$\therefore A \leq B$

Answer **D:** The relationship cannot be determined from the information given.

Have Another Try!

$$A = gcd(n, q)$$

$$B = gcd(201n + 3q, 96n + 81q)$$

Have Another Try!

$A = gcd(n, q)$

$B = gcd(201n + 3q, 96n + 81q)$

$B \geq 3 \cdot gcd(n, q) > A$
Answer **B:** Quantity B is greater

Multiple and Least Common Multiple

Definitions & Examples For A Divisor(Factor)

约数（因数）

定义

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推论

Every positive integers a is divisible by the trivial divisors 1 and a.

Definitions & Examples For The Greatest Common Divisors

gcd（最大公因数）

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The greatest common divisor (or greatest common factor) of two nonzero integers c and d is the greatest positive integer that is a divisor of both c and d.

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The Big Question

Is there a better way to find the gcd of two nonzero integers c and d?

Quotient and Remainder

Fractions

Presentation Overview for Fractions

1. Integers

2. Fractions

1 Min Break

Questions? Comments?