Geometry
几何

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GRE 冲分班数学
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To Begin With

QR Mathematical Convention 3

All figures are assumed to lie in a plane unless otherwise indicated.

QR Mathematical Convention 4

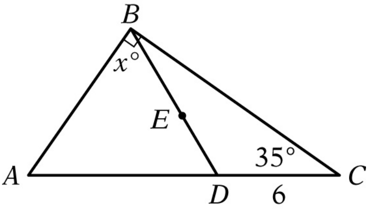
Geometric figures are not necessarily drawn to scale.

例

- Can not assume that quantities such as lengths and are as they appear in a figure
- Can not assume that angle measures such as lengths and are as they appear in a figure
- Can assume all geometric objects are in the relative positions shown.

Rely on Your Geometric Reasoning, not Estimating or Comparing Quantities By Eyesight
用几何推理做题!

Which of the following statements Must Be right?



1. Points A, D, and C are distinct. Point D lies between points A and C, and the line containing them is straight.
2. The length of line segment AD is less than the length of line segment AC.
3. ABC, ABD, and DBC are triangles.
4. Point E lies on line segment BD.
5. Angle ABC is a right angle, as indicated by the small square symbol at point B.
6. The length of line segment DC is 6, and the measure of angle C is 35 degrees.
7. The measure of angle ABD is x degrees,

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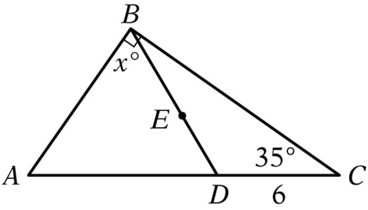

Answer: **They all must be right!**

1. Points A, D, and C are distinct. Point D lies between points A and C, and the line containing them is straight.
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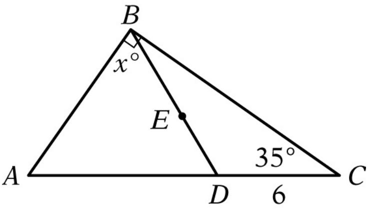


1. The length of line segment AD is greater than the length of line segment DC.
2. The measures of angles BAD and BDA are equal.
3. The measure of angle is less than x degrees.
4. The area of triangle ABD is greater than the area of triangle DBC.

Answer: **They are all not necessarily right!**

Lines and Angles

Presentation Overview for Lines and Angles

Lines

Congruent line segments

用几何推理做题!

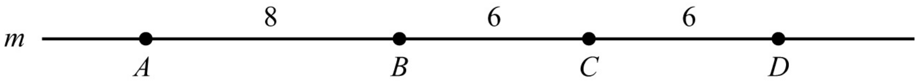


图: BC and CD are congruent line segments.

定义

Line segments that have equal lengths are called congruent line segments .

congruent

/kən rooənt, käNG rooənt/

(of figures) identical in form; coinciding exactly when superimposed.
全等：相同，叠加的时候完全重合

Angles

Opposite Angles
对角相等

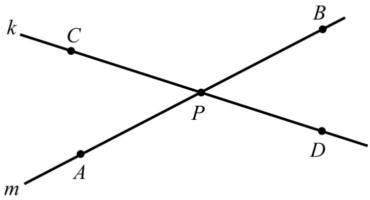


图: $\angle APC$ and $\angle BPD$ are opposite angles; So are $\angle CPB$ and $\angle DPA$.

定义

Opposite angles have equal measure, and angles that have equal measure are called congruent angles. Hence, opposite angles are congruent.

Acute,Right, Obtuse Angles
锐角 直角 钝角

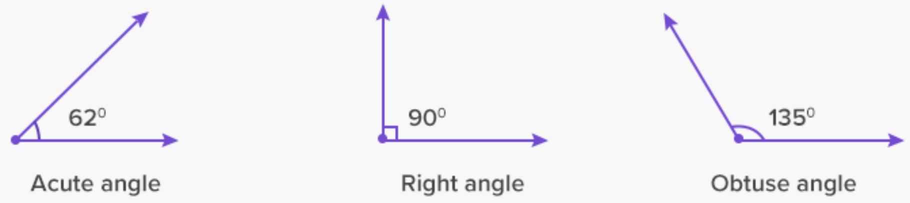


图: BC and CD are congruent line segments.

定义

- An angle with measure less than 90° is called an acute angle.
- An angle with a measure of 90° is called a right angle.
- an angle with measure between 90° and 180° is called an obtuse angle.

Parallel Lines

Parallel Lines

平行线同位角相等，内错角之和为 180 度

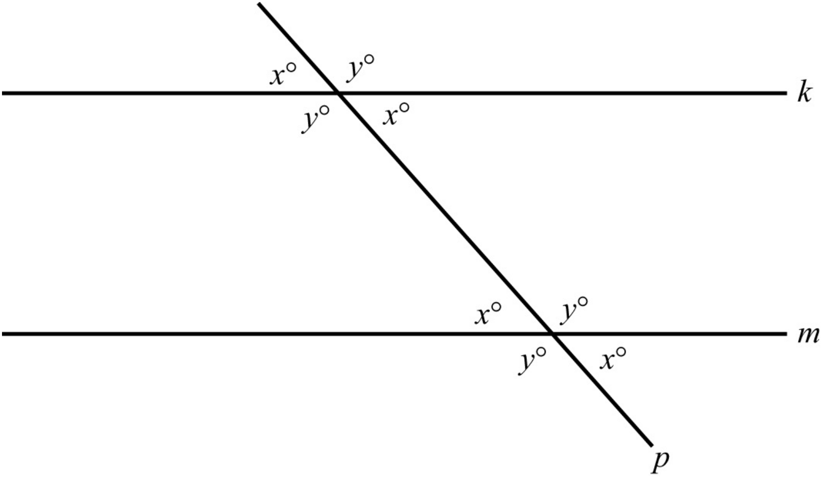


图: $k \parallel m$

Triangles

Presentation Overview for Triangles

Equilateral Triangles

Equilateral

/ ēkwə ladərəl, ekwə ladərəl/

(of figures) having all its sides of the same length..
等边：所有边长相等

Equilateral Triangles

等边三角形：内角均为 60 度

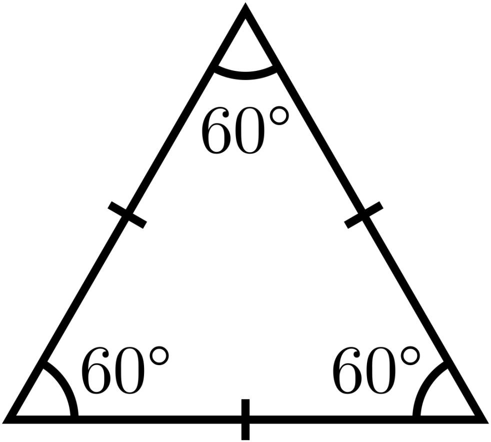


图: The measures of the three interior angles of a equilateral triangle are equal, and each measure is 60°.

A Real QR Problem!

自己画图!

In the xy-plane, the vertices of an equilateral triangle are $(0, 1)$, $(4, 3)$, and (a, b).

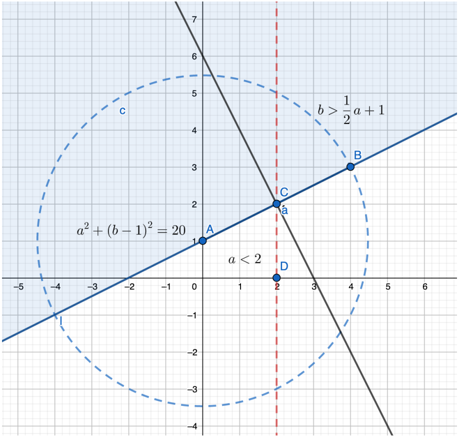
Which of the following statements individually provide(s) sufficient additional information to determine the vertex (a, b) ?

Indicate all such statements.

- ☐ $2b - a > 2$
- ☐ $a < 2$
- ☐ $a^2 + (b - 1)^2 = 20$

图: 2-Sec2-13

Answer
自己画图!



Answer **AB** $b = \frac{1}{2}a + 1$ $a < 2$

Isosceles Triangles

Isosceles

/ī säsə lēz/

(of a triangle) having two sides of equal length.
等腰三角形：两边长相等

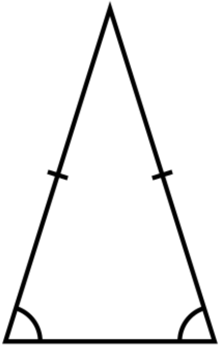


图: Congruent sides suggest congruent angles.

定理 (两角相等互推两边相等)

If a triangle has two congruent sides, then the angles opposite the two congruent sides are congruent. The converse is also true.

定理 (Law Of Sines 正弦定理)

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

An isosceles triangle lies on the rectangular coordinate plane, the coordinates of point A are (0, 0), and the coordinates of point B are (3, 1), point C could lie at one of 6 positions such that (1, 3), (-1, 3), (-3, 1), (-1, -3), (1, -3), (3, -1). How many lengths of side BC are possible?

Answer **5**

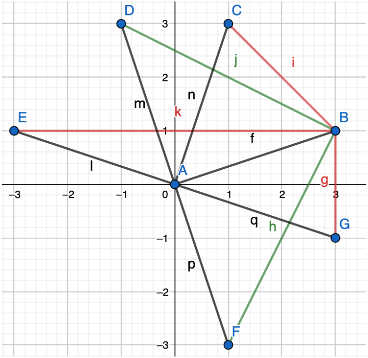
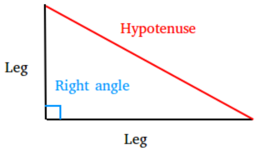


图: *BD* and *BF* have the same length

Right Triangles

hypotenuse

/hī pätn (y)oos/



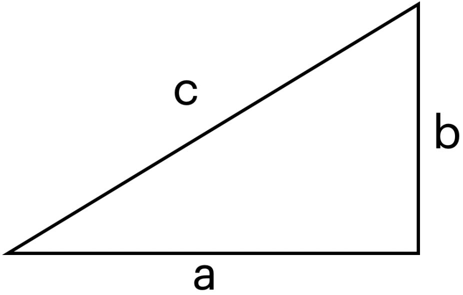
the longest side of a right triangle, opposite the right angle.

斜边：直角三角形直角对边

leg

直角边

The Pythagorean Theorem
勾股定理



$$c^2 = a^2 + b^2$$

Pythagorean

/ī säsə lēz/

relating to or characteristic of the Greek philosopher Pythagoras or his ideas.
毕达哥拉斯

Have a try!

A ladder 25 feet long is leaning against a wall that is perpendicular to level ground. The bottom of the ladder is 7 feet from the base of the wall. If the top of the ladder slips down 4 feet, how many feet will the bottom of the ladder slip?

$$\sqrt{(25^2-(7-4)^2)-(25^2-7^2)} = 2\sqrt{10} \approx= 6.32\textit{feet}$$

Answer **6.32 Feet**

QR 只能一个空之能填一个数，没有根号输入

A Real QR Problem!
注意“could be”

A right triangle has sides of length 2, 5, and x. A second right triangle has sides of length 4, 7, and y. A third right triangle has sides of length x, y, and n. Which of the following could be the value of n?

- 3
- 6
- 8
- 9
- 10

图: 8-Sec3-18

Answer

Answer **B** 6

case $1:$ *x is hypotenuse*

$x^2 = 2^2 + 5^2 = 29$

case $2:$ 5 *is hypotenuse*

$x^2 = 5^2 - 2^2 = 21$

case $3:$ *y is hypotenuse*

$y^2 = 4^2 + 7^2 = 65$

case $4:$ 7 *is hypotenuse*

$y^2 = 7^2 - 4^2 = 33$

case $5:$ *n is hypotenuse*

$n^2 = x^2 + y^2$

$= 29 + 65 = 94$ *discarded!*

$= 29 + 33 = 62$ *discarded!*

$= 21 + 65 = 86$ *discarded!*

$= 21 + 33 = 54$ *discarded!*

case $6:$ *y is hypotenuse*

$n^2 = y^2 - x^2$

$= 65 - 29 = 36$ *B!*

$= 33 - 29 = 4$ *Not shown!*

$= 65 - 21 = 44$ *discarded!*

$= 33 - 21 = 12$ *discarded!*

45°-45°-90° Triangle

边长比为 $1:1:\sqrt{2}$

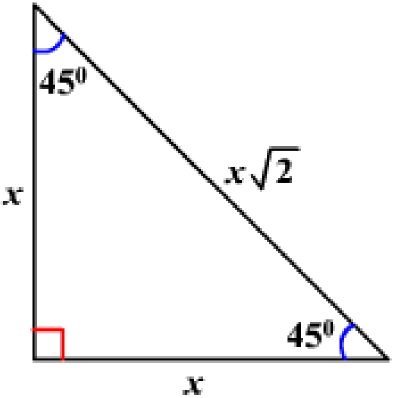
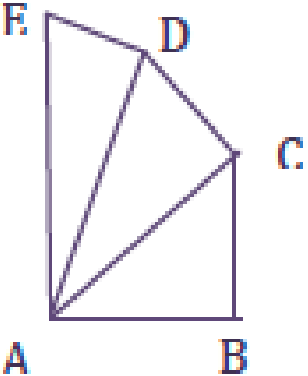


图: Isosceles Right Triangle

Have a try!

In the figure above, AB = BC = CD = DE, all triangles are right triangles. If AE = 10, what is the length of AB?

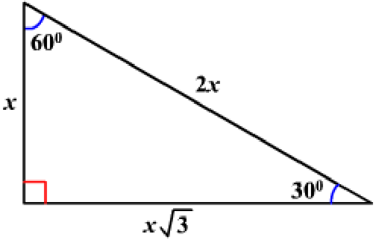


$$AE^2 = ED^2 + AD^2$$
$$= ED^2 + (CD^2 + AC^2)$$
$$= ED^2 + (CD^2 + AB^2 + BC^2)$$
$$= 4AB^2$$
$$= 100$$
$$\therefore AB = \sqrt{25} = 5$$

Answer **5**

30°-60°-90° Triangle

边长比为 1:$\sqrt{3}$:2



A Real QR Problem!

How many noncongruent triangles are there such that the length of each side of each triangle is an integer and the perimeter of each triangle is 15 ?

- Five
- Six
- Seven
- Eight
- Nine

图: 8-Sec2-12

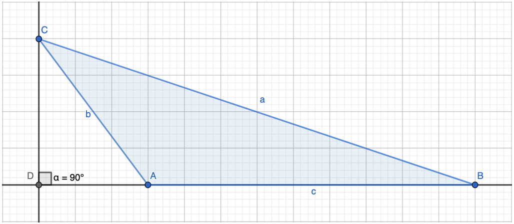
The Pythagorean Inequality For The Obtuse Triangles

钝角三角形长边平方大于两短边平方和

定理

In an obtuse triangle, the square of the longest side will be greater than adding the two squares of the shorter sides.

$$BC^2 = DC^2 + DB^2$$
$$= (AC^2 - AD^2) +$$
$$AD^2 + AB^2 + 2AD \cdot AB$$
$$> AC^2 + AB^2 \;\; Q.E.D.$$



Have a try!

In an obtuse triangle, if two sides are 9 and 40, what is range of the possible length of the unknown one?

$40 - 9 < x < 40 + 9$

$31 < x < 49$

case 1 : 40 *is longest*

$31 < x \leq 40$

$x^2 + 9^2 < 40^2$

$31 < x < 38.97$

case 2 : x *is longest*

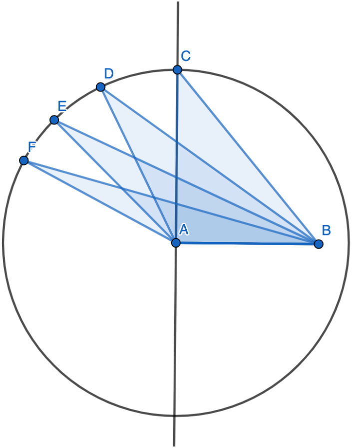
$40 < x < 49$

$40^2 + 9^2 < x^2$

$41 < x$

$41 < x < 49$

Answer $31 < x < 38.97$ *or* $41 < x < 49$



定理

In an acute triangle, the sum of the square of two sides will be greater than the square of the the other side.

$\because AE^2 + BE^2 = AB^2$

$\because BC^2 - CE^2 = BE^2$

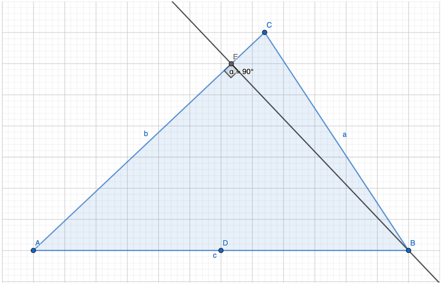
$\therefore (AE^2 - CE^2) + BC^2 = AB^2$

$\therefore AC^2 + BC^2$

$= (AE^2 + 2AE * CE + CE^2) + BC^2$

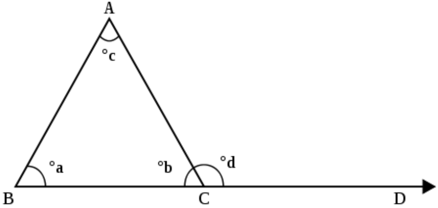
$> (AE^2 - CE^2) + BC^2$

$= AB^2$ *Q.E.D.*



Exterior Angle of Triangles

外角等于相对应内对角之和

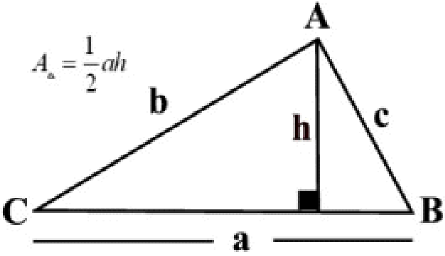


定理

$$d = a + c$$

The Area of a Triangle

The Area of a Triangle
底乘高除以二



Congruent Triangles

SSS, SAS, ASA Congruence

边边边 边角边 角边角 全等

定理 (Side-Side-Side Congruence)

If the three sides of one triangle are congruent to the three sides of another triangle, then the triangles are congruent.

定理 (Side-Angle-Side Congruence)

If two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle, then the triangles are congruent.

定理 (Angle-Side-Angle Congruence)

If two angles and the included side of one triangle are congruent to two angles and the included side of another triangle, then the triangles are congruent.

What about AAS?**Yes!**

A Real QR Problem!

一个字！试！

How many noncongruent triangles are there such that the length of each side of each triangle is an integer and the perimeter of each triangle is 15 ?

- ◯ Five
- ◯ Six
- ◯ Seven
- ◯ Eight
- ◯ Nine

图: 8-Sec2-12

Answer

$a + b + c = 15$
$\because c < a + b = 15 - c$
$\therefore c < 7.5$. By symmetry, $a < 7.5$ and $b < 7.5$
The non-congruent triangles suggest that one triangle have different sides from other triangles.

c=7 **a+b=7**	**c=6** **a+b=9**	**c=5** **a+b=10**	**c=4** **a+b=11**
1 7	~~1 8~~	~~1 9~~	~~1 10~~
2 6	~~2 7~~	~~2 8~~	~~2 9~~
3 5	3 6	~~3 7~~	~~3 8~~
4 4	4 5	~~4 6~~	~~4 7~~
...	...	5 4	~~5 6~~

Answer **C Seven**

Similar Triangles

Scale Factor Of Similarity
相似比例

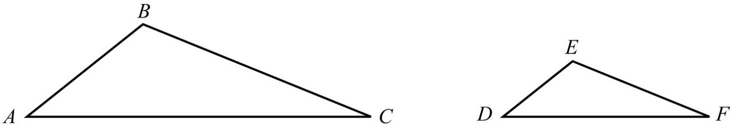


图: Two similar triangles

定义

More precisely, two triangles are similar if their vertices can be matched up so that the corresponding angles are congruent or, equivalently, the lengths of the corresponding sides have the same ratio, called the scale factor of similarity.

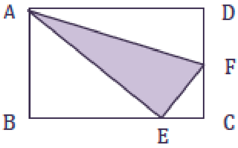
How to prove similarity? **AA!**

vertices

The plural noun of vertex
顶点点的复数

Have a try!

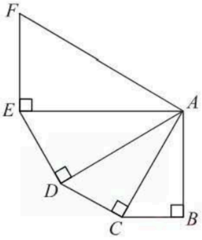
In the figure shown below, fold the rectangle alone AF, and point D launch at E which separate BC into two part. $BE = 6$, $EC = 2$. What is the value of $AE : EF$?



Answer $\frac{AE}{EF} \approx= 2.65$

$$\because \angle FEC = \angle BAE$$
$$\angle BEA = \angle EFC$$
$$\therefore \triangle ABE \sim \triangle ECF$$
$$\therefore \frac{AE}{EF} = \frac{AB}{EC} = \frac{BE}{CF}$$
$$\because AE = AD = BC = BE + EC = 6 + 2 = 8$$
$$\therefore AB = \sqrt{AE^2 - BE^2} = \sqrt{8^2 - 6^2} = 2\sqrt{7}$$
$$\therefore \frac{AE}{EF} = \frac{AB}{EC} = \frac{2\sqrt{7}}{2} = \sqrt{7} \approx= 2.65$$

A Real QR Problem!

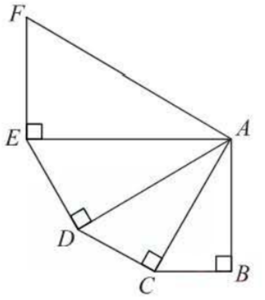


In the figure shown, the measure of angle BAC is 30 degrees. Triangles ABC, ACD, ADE, and AEF are similar. The area of triangle AEF is how many times the area of triangle ABC ?

- $\frac{16}{9}$
- $\frac{64}{27}$
- $\frac{27}{8}$
- $2\sqrt{3}$
- $3\sqrt{3}$

图: 2-Sec2-18

Answer



$$BC = x$$

$$DC = \frac{1}{\sqrt{3}}AC = \frac{2}{\sqrt{3}}x$$

$$DE = \frac{1}{\sqrt{3}}AD = \frac{1}{\sqrt{3}}\frac{4}{\sqrt{3}}x = \frac{4}{3}x$$

$$EF = \frac{1}{\sqrt{3}}AE = \frac{1}{\sqrt{3}}\frac{8}{3}x = \frac{8}{3\sqrt{3}}x$$

$$\frac{S_{\triangle AFE}}{S_{\triangle ABC}} = \frac{\frac{\sqrt{3}}{2}EF^2}{\frac{\sqrt{3}}{2}BC^2} = (\frac{EF}{BC})^2 = (\frac{\frac{8}{3\sqrt{3}}x}{x})^2 = \frac{64}{27}$$

Answer **B** $\frac{64}{27}$

Quadrilaterals

Presentation Overview for Quadrilaterals

quadrilateral

/ kwädrə ladərəl, kwädrə latrəl/

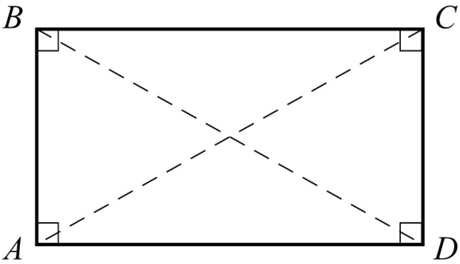
a four-sided figure.
四边形

Rectangle

Rectangle
矩形

定义

A quadrilateral with four right angles is called a rectangle. Opposite sides of a rectangle are parallel and congruent, and the two diagonals are also congruent.
A rectangle with four congruent sides is called a square.



$$\textit{Area}: \quad A = \textit{base} \cdot \textit{height}$$

Parallelogram

Math Vocab!
专业名词记忆时间!

parallelogram

/ perə lelə ram/

a four-sided plane rectilinear figure with opposite sides parallel.

平行四边形

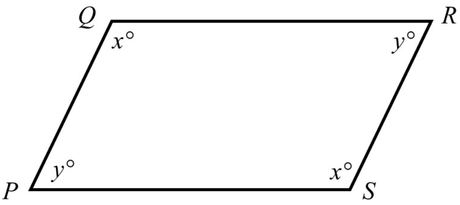
Parallelogram

平行四边形

定义

A quadrilateral in which both pairs of opposite sides are parallel is called a parallelogram. In a parallelogram, opposite sides are congruent and opposite angles are congruent.

Note that all rectangles are parallelograms.



$$Area: \quad A = base \cdot height$$

Trapezoid

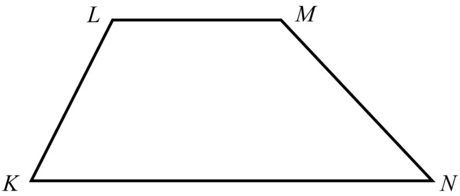
trapezoid

/ trap z d,trə pi z d/

a four-sided plane rectilinear figure with opposite sides parallel.
梯形

定义

A quadrilateral in which at least one pair of opposite sides is parallel is called a trapezoid.
Two opposite, parallel sides of the trapezoid are called bases of the trapezoid.



$$\mathit{Area}:\quad A = \frac{\mathit{base}_1 + \mathit{base}_2}{2} \cdot \mathit{height}$$

Polygons

Presentation Overview for Polygons

polygons

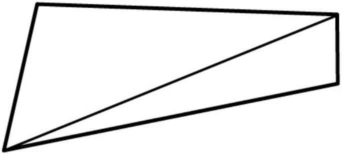
/ pälē än/ a plane figure with at least three

straight sides and angles, and typically five or more.

多边形

定理

If a polygon has n sides, it can be divided into $n-2$ triangles. Since the sum of the measures of the interior angles of a triangle is 180°, it follows that the sum of the measures of the interior angles of an n-sided polygon is $(n-2)(180^\circ)$.



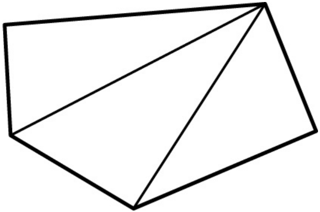


图: $(4-2)(180^\circ) = 360^\circ$(Left); $(5-2)(180^\circ) = 540^\circ$(Right)

perimeter

/pə rimidər/

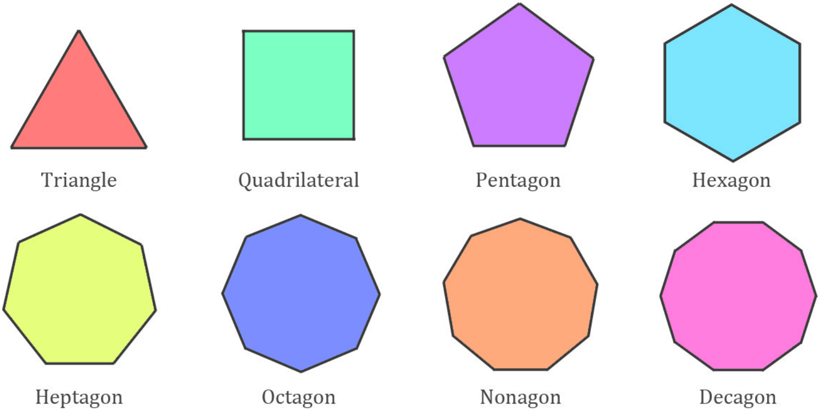
the continuous line forming the boundary of a closed geometric figure.

周长

Regular Polygon
正多边形

定义

A polygon in which all sides are congruent and all interior angles are congruent is called a regular polygon.



Math Vocab!

专业名词记忆时间!

pentagon

/ pen(t)ə än/

五边形

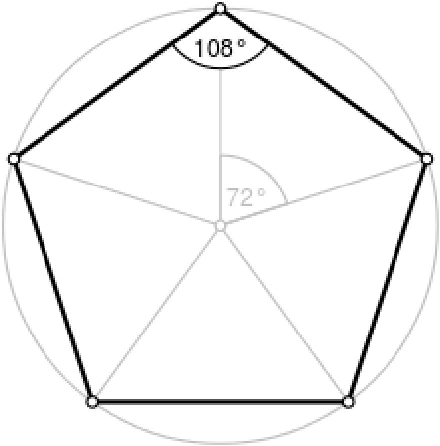


图: The regular pentagon

Math Vocab!
专业名词记忆时间!

hexagon

/ heksə än/

六边形

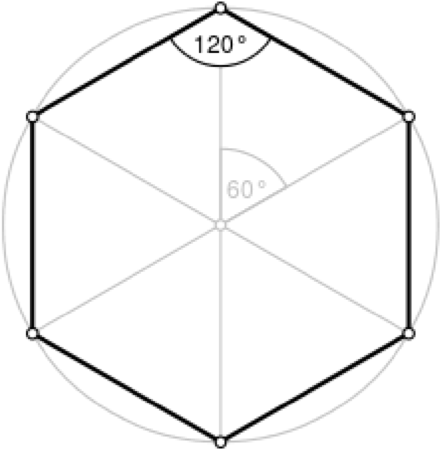


图: The regular hexagon

heptagon

/ heptə   än/

七边形

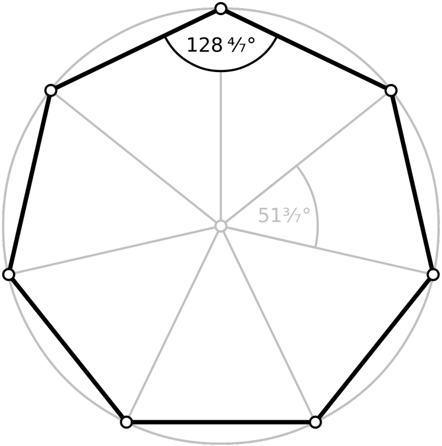


图: The regular heptagon

octagon

/ äktə än, äktə ən/

八边形

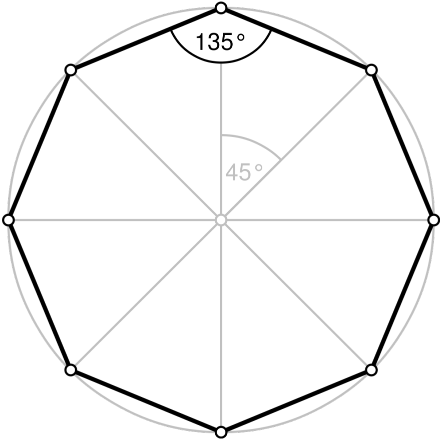


图: The regular octagon

Math Vocab!
专业名词记忆时间!

nonagon

/ nänə än, nōnə än/

九边形

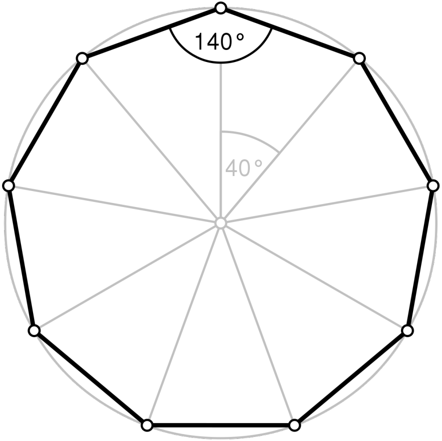


图: The regular nonagon

decagon

/ dekə än/

十边形

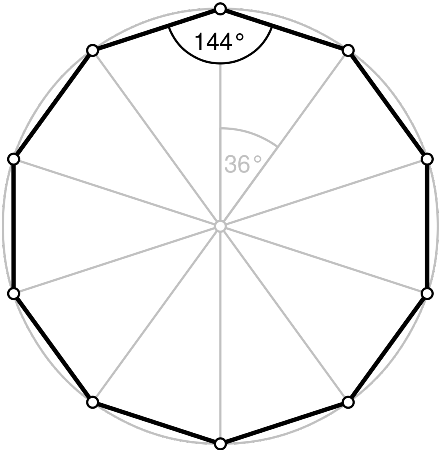


图: The regular decagon

Mnemonic

助记方法：结尾都是 gon，开头是数字前缀

The Number Prefix	Derived Words	The Interior Angle $\frac{(n-2)360°}{n}$
penta-	美国国防部五角大楼	$108°$
hexa-	hexacode 六位数字表达颜色，蓝色 #0000FF	$120°$
hepta-	semtemper 罗马日历的第 7 个月	$128\frac{4}{7}°$
octa-	octopus 八爪鱼	$135°$
nona-	November 罗马日历的第 9 个月	$140°$
deca-	decade 十年	$144°$

In-class Quiz
写不出来的，可以写一个大概

1 Min

State the name of polygons with 5 to 10 sides.

Circles

Presentation Overview for Circles

1. Lines and Angles
2. Triangles
3. Quadrilaterals
4. Polygons
5. Circles
   - Radius, Diameter, And Chord
   - Circumference,Area, Central Angle and Arc
   - Tangent
   - Inscribe v.s. Circumscribe
   - Concentric Circles

Radius, Diameter, And Chord

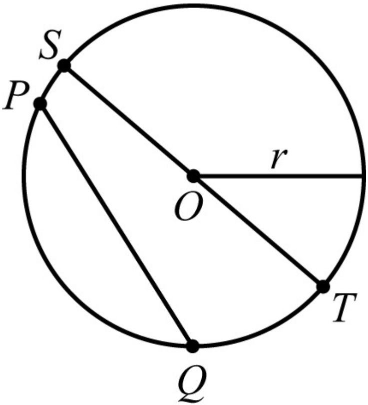


图: r is the radius; PQ is the chord; ST is the diameter as well as the chord.

定义

The point O is called the center of the circle and the distance r is called the radius of the circle.
The diameter of the circle is twice the radius.
Two circles with equal radii are called congruent circles.

radii

The plural noun of radius
半径的复数

Circumference,Area, Central Angle and Arc

$$c = 2\pi r$$
$$A = \pi r^2$$

π 取 3.14 或者 $\frac{22}{7}$

Central Angle and Arc

圆心角和弧

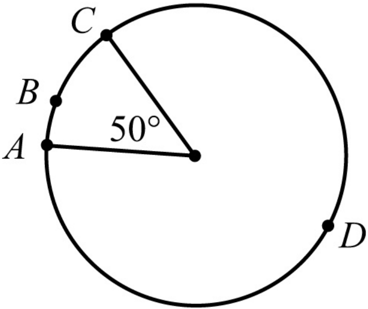


图: The measure of an arc is the measure of its central angle, which is the angle formed by two radii that connect the center of the circle to the two endpoints of the arc.

$$\textit{length of the arc} = 2\pi r \frac{360}{\textit{central angle}}$$

$$\textit{sector area} = \pi r^2 \frac{360}{\textit{central angle}}$$

Central Angle Property
圆心角是圆周角的两倍

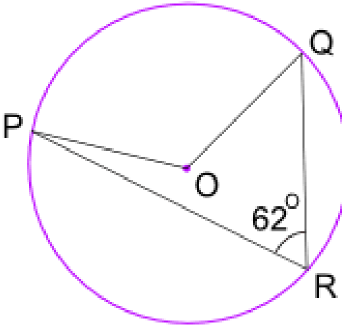


图: Angle POQ is the central angle of arc PQ; angle PRQ is the inscribed angle of the arc PQ;$\angle POQ = 2 \cdot \angle PRQ$

定理

An inscribed angle is half the measure of a central angle subtended by the same arc.

By the property of exterior angles of triangles,

$\angle CAF = \angle ACE + \angle CEA$

$\angle FAD = \angle AED + \angle ADE$

$\because AC = AE = r$

$\therefore \angle ACE = \angle CEA$

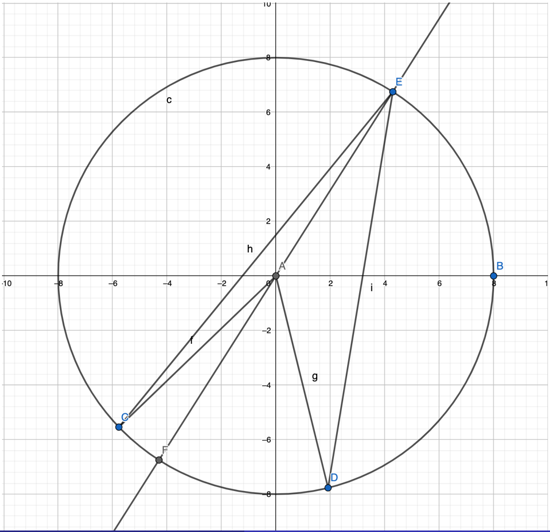
$\because AD = AE = r$

$\therefore \angle AED = \angle ADE$

$\therefore \angle CAD = \angle CAF + \angle FAD$

$= 2(\angle CEA + \angle AED)$

$= 2\angle CED$ *Q.E.D.*

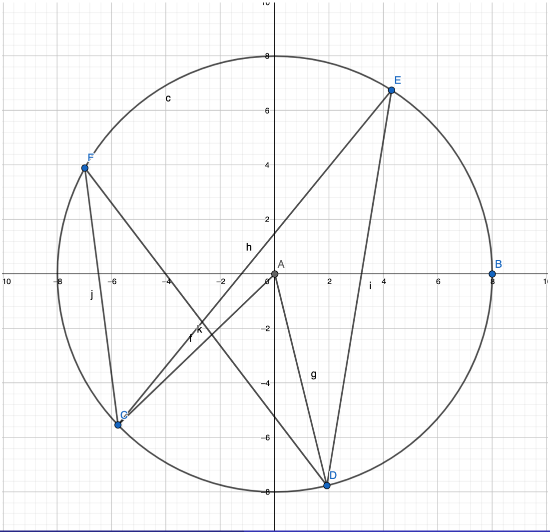


The Proof of Inscribed Angle Property

相同弧圆周角相同

By the property of central angles of triangles,

$$\angle CFD = \frac{1}{2}\angle CAD = \angle CED$$



Tangent

Tangent
切线

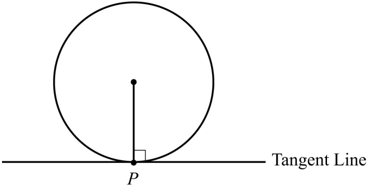


图: $\angle p = 90°$

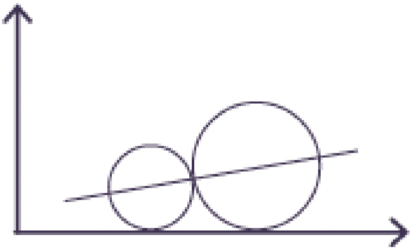
定义

A tangent to a circle is a line that lies in the same plane as the circle and intersects the circle at exactly one point, called the point of tangency

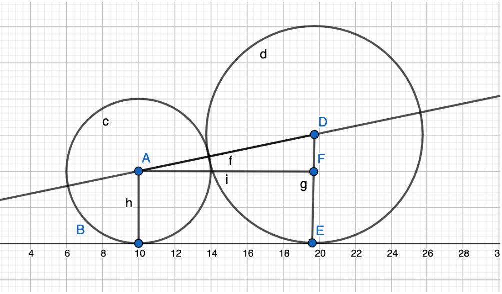
定理 (切线和交点半径垂直)

that is, if a radius and a line intersect at a point on the circle and the line is perpendicular to the radius, then the line is a tangent to the circle at the point of intersection.

In the rectangular coordinate system below, both of two tangent circles are tangent to the x-axis. If the radii of the two circles are 4 and 6, respectively, what is the slope of the line on which two centers lie?



Have a try!
自己用手画图!



$$
\begin{aligned}
\mathit{slope} &= \frac{DF}{AF} \\
&= \frac{DE - AB}{\sqrt{AD^2 - DF^2}} \\
&= \frac{6-4}{\sqrt{10^2 - 2^2}} \\
&= \frac{2}{\sqrt{96}} \\
&= 0.20
\end{aligned}
$$

Answer **0.20**

Chord Tangent Angle Property

弦切角圆周角相等

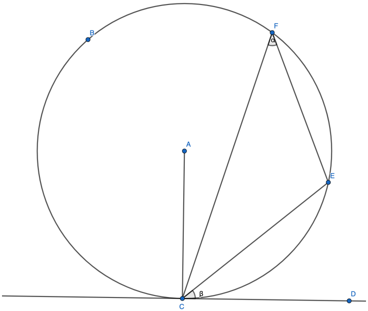


图: Angle ECD is the chord tangent angle for chord *EC* and tangent *CD*; angle PRQ is the inscribed angle of the arc CE;$\angle CDE = \angle CDF$

定理 (弦切角定理)

The angle formed between a chord and a tangent line to a circle is equal to the inscribed angle on the other side of the chord.

The Proof of Chord Tangent Angle Property
弦切角定理证明

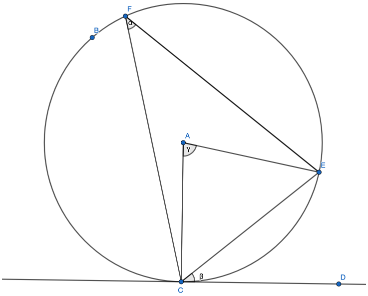


图: $\angle$ $CDE = \angle CDF$

$\because AC = AE$

$\therefore \angle ACE = \angle AEC$

$\because \angle CAE = 2\angle CFE$

$\therefore \angle CAE + \angle ACE + \angle AEC \quad = 2(\angle CFE + \angle A$

$= 360°$

$\therefore \angle CFE + \angle ACE = 90°$

$\because \angle ECD + \angle ACE = 90°$

$\therefore \angle CFE = \angle ECD$ *Q.E.D.*

Inscribe v.s. Circumscribe

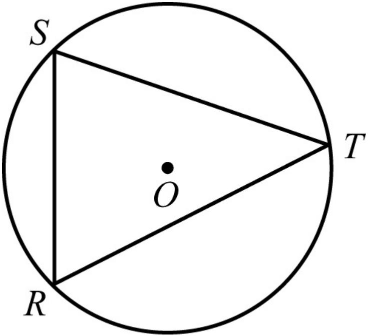


图: The vertices of triangle STR are located on the circle O.

定义

A polygon is inscribed in a circle if all its vertices lie on the circle.

"the circle is circumscribed about the polygon." 谁在外面谁在里面? **还是外接圆**

inscribe

/in skrīb/

draw (a figure) within another so that their boundaries touch but do not intersect. A is inscribed in B: A 被 B 外接, 在里面画

circumscribe

/ sərkəm skrīb/

draw (a figure) around another, touching it at points but not cutting it. A is circumscribed about B: A 外接 B, 在外面画

Thales's theorem

如果三角形边长为外接圆直径，直径对角为直角

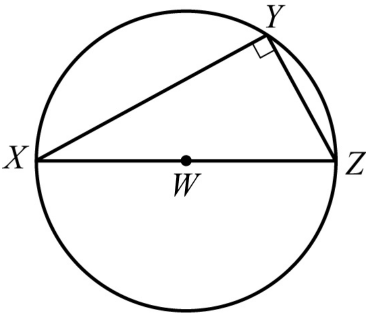


图: XZ is the diameter of the circle W; $\angle XYZ = 90°$

定义

if X, Z, and Y are distinct points on a circle where the line XZ is a diameter, the angle ABC is a right angle.

A Real QR Problem!

自己画图!

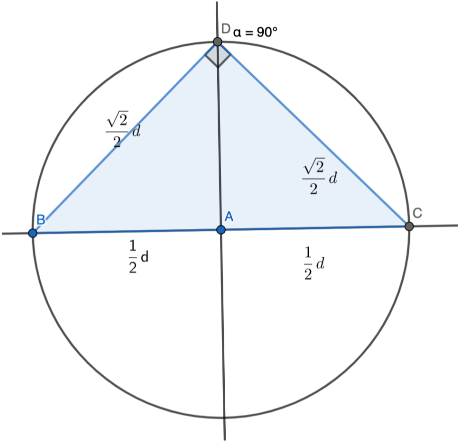
Right isosceles triangle T is inscribed in circle C with diameter d.

Quantity A	Quantity B
The perimeter of triangle T	$\frac{5}{2}d$

- ◯ Quantity A is greater.
- ◯ Quantity B is greater.
- ◯ The two quantities are equal.
- ◯ The relationship cannot be determined from the information given.

图: 7-Sec2-5

Answer



$$perimeter = \frac{\sqrt{2}}{2}d + \frac{\sqrt{2}}{2}d + d$$
$$= \sqrt{2}d + d$$
$$> \frac{5}{2}d$$

Answer **B** Quantity B is greater

Circumscribed Polygon in a Circle
内切圆

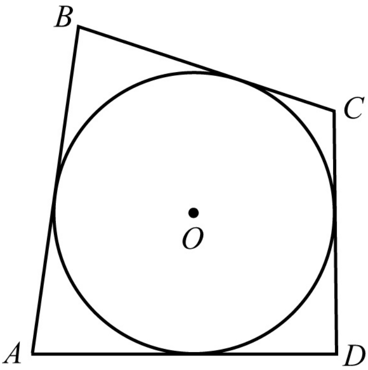


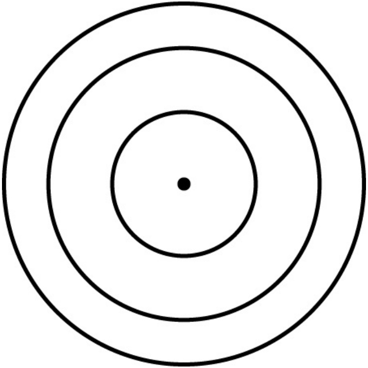
图: quadrilateral $ABCD$ circumscribed about a circle with center O.

定义

A polygon is circumscribed about a circle if each side of the polygon is tangent to the circle, or equivalently, the circle is inscribed in the polygon.

Concentric Circles

Concentric Circles
同心圆



定义

Two or more circles with the same center are called concentric circles.

You must have known eccentric!

Three-Dimensional Figures

Presentation Overview for Three-Dimensional Figures

Rectangular Solid(Right Rectangular Prism)

Rectangular Solid(Right Rectangular Prism)
立方体（正四棱柱）

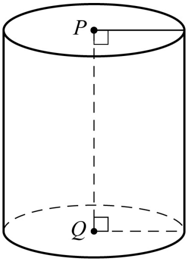
$$\begin{aligned} &\textit{Volume}: \quad V = lwh \\ &\textit{Surface Area}: \quad A = 2(lw + lh + wh) \end{aligned}$$

Circular Cylinder And Right Circular Cylinder

Right Circular Cylinder
正圆柱

$$Volume: \quad V = \pi r^2 h$$

$$Surface\ Area: \quad A = 2\pi r^2 + 2\pi rh$$



1 Min Break

Questions? Comments?