

Complex Analysis Reference Book

Stuyvesant Class of 2022

Nathaniel J. Strout, Joshua L. Yagupsky, Francis Zweifler

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Complex Calculus, Mr. Stern

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1 THE COMPLEX NUMBER SYSTEM

1.1 The Algebra of Complex Numbers

Definition 1.1 (Complex Numbers):

The *complex numbers* are the set

$$\mathbb{C} := \{ [(a, b)] \mid a, b \in \mathbb{R} \}$$

where

$$[(a, b)] := \begin{cases} a & b = 0 \\ (a, b) & b \neq 0 \end{cases}$$

Definition 1.2 (Real and Imaginary Parts of an element of \mathbb{C}):

The real part of $z \in \mathbb{C}$ where $z = [(a, b)]$ is defined as:

$$Re(z) := a.$$

Similarly, the imaginary part of $z \in \mathbb{C}$ is defined by $Im(z) := b$.

Definition 1.3 (Addition and Multiplication in \mathbb{C}):

For $[(a, b)], [(c, d)] \in \mathbb{C}$, $[(a, b)] + [(c, d)] := [(a + c, b + d)]$. Additionally, we will define multiplication as follows:

$$[(a, b)] \cdot [(c, d)] := [(ac - bd, ad + bc)].$$

This may seem to be an arbitrary definition of multiplication, but as we explore the properties of this definition its motivation will become apparent.

Theorem 1.1 (Properties of Addition and Multiplication \mathbb{C}):

We can now verify a number of properties that will allow this rigorous bracketed point construction of \mathbb{C} to be recognized as equivalent to the familiar (non-rigorous) introduction of \mathbb{C} . Starting with properties of addition in \mathbb{C} , for $[(a, b)], [(c, d)] \in \mathbb{C}$ we have commutativity, associativity, and an identity.

Starting with commutativity, we know $[(a, b)] + [(c, d)] = [(a + c, b + d)] = [(c + a, d + b)]$ by the commutativity of \mathbb{R} . $[(c, d)] + [(a, b)] = [(c + a, d + b)]$, and thus we have $[(a, b)] + [(c, d)] = [(c, d)] + [(a, b)]$.

Next, associativity for $[(a, b)], [(c, d)], [(e, f)] \in \mathbb{C}$ is proven easily as well: We know $(([(a, b)] + [(c, d)]) + [(e, f)]) = [((a + c) + e, (b + d) + f)]$. By the associativity of \mathbb{R} , we have $[(a + (c + e), b + (d + f))] = [(a, b)] + (([(c, d)] + [(e, f)])$, and thus addition in \mathbb{C} is associative.

Finally we have an identity element, $[(0, 0)]$. Note that $[(0, 0)]$ is equal to the real number 0. We quickly verify that it is indeed an identity: For any $[(a, b)] \in \mathbb{C}$, $[(a, b)] + [(0, 0)] = [(a + 0, b + 0)] = [(a, b)]$.

Next, we verify the same properties for multiplication. (In these more or less straightforward proofs, it will become clear how notationally clumsy our bracketed point notation is, and we will abandon it soon after these proofs. Terrence Tao calls this type of idea mathematical scaffolding in his book *Analysis I*: necessary to sturdy and rigorous construction, but discarded after use.)

Once again, starting with commutativity, we know $[(a, b)] \cdot [(c, d)] = [(ac - bd, ad + bc)]$ by definition. Similarly, $[(c, d)] \cdot [(a, b)] = [(ca - db, cb + da)]$ and that is equal to $[(ac - bd, ad + bc)]$ by commutativity of multiplication and addition in \mathbb{R} .

Next we prove associativity. We know $(([(a, b)] \cdot [(c, d)]) \cdot [(e, f)]) = (((ac - bd, ad + bc)) \cdot [(e, f)]) = [((ac - bd)e - (ad + bc)f, (ac - bd)f + (ad + bc)e)]$. This is equal to $[(a, b)] \cdot (([(c, d)] \cdot [(e, f)])$ by the standard properties of multiplication and addition in \mathbb{R} .

As an intermediate step, we can also use this opportunity to prove distributivity of multiplication over addition. For $[(a, b)], [(c, d)], [(e, f)] \in \mathbb{C}$ we know

$$[(a, b)] \cdot (([(c, d)] + [(e, f)])) = [(a, b)] \cdot [(c + e, d + f)] = [(ac + ae - bd - bf, ad + af + bc + be)]$$

Then, by the properties of addition and multiplication in \mathbb{R} we know

$$[(ac - bd, ad + bc)] + [(ae - bf, af + be)] = [(a, b)] \cdot [(c, d)] + [(a, b)] \cdot [(e, f)]$$

1.2 The Geometry of Complex Numbers

1.2.1 Möbius Transformations and the Riemann Sphere

2 COMPLEX FUNCTIONS

2.1 The Complex Exponential

2.2 Complex Trigonometry

2.3 The Argument Functions and Complex Logarithm

3 TOPOLOGY OF THE COMPLEX PLANE

3.1 Neighborhoods, Open and Closed Sets

Definition 3.1 (Open Disk):
The *open disk* of radius r around the point p is defined as follows:

$$D_p(r) = \{z \in \mathbb{C} \mid |z - p| < r\}$$

Definition 3.2 (Neighborhood of a Point):
A set $S \subseteq \mathbb{C}$ is called a *neighborhood* of a point p if there exists some $r > 0$ such that $D_r(p) \subseteq S$. We write this as $S \in \mathcal{N}(p)$, where $\mathcal{N}(p)$ denotes the *neighborhood-system* of p .

Definition 3.3 (Punctured Disk):
The *punctured disk* of radius r around the point p is defined as follows:

$$D_p^*(r) = \{z \in \mathbb{C} \mid 0 < |z - p| < r\}$$

Definition 3.4 (Punctured Neighborhood of a Point):
A set $S \subseteq \mathbb{C}$ is called a *punctured neighborhood* of a point p if there exists some $r > 0$ such that $D_r^*(p) \subseteq S$ and $p \notin S$. We write this as $S \in \mathcal{N}^*(p)$, where $\mathcal{N}^*(p)$ denotes the *punctured neighborhood-system* of p .

Definition 3.5 (Open Set):
A set S is called an *open set* iff $\forall z \in S, S \in \mathcal{N}(z)$.

Theorem 3.1 (Properties of Open Sets):
The union of any collection of open sets is open, and the intersection of any *finite* collection of open sets is open.

Proof: The union of any collection of sets is the set containing all of the points in all of the sets. Denote the union as U , and let p be an element of one of the sets, S . Since S is an open set, it is a neighborhood of p , and therefore there exists a $D_r(p) \subseteq S$. Since S is a subset of U , so is the disk, and since this is true of any point in the union, the union must be open.

3.1.1 Accumulation Points and the Closure of a Set

3.2 Connectedness and Compactness

3.3 Sequences in the Complex Plane, Limits of Sequences

3.4 Limits of Functions, Continuity

3.4.1 Continuous Images of Connected and Compact Sets

4 COMPLEX DIFFERENTIATION

4.1 Differentiability and Analyticity

4.1.1 The Cauchy-Riemann Equations

4.2 Rules for Derivatives

4.3 Conformal Maps

5 COMPLEX INTEGRATION

5.1 Real Integrals of Complex Functions

5.2 Contours and Contour Integration

5.2.1 Contour Parameterization

5.3 Introduction to Cauchy's Theorem

5.3.1 Cauchy's Theorem on Tiles

5.3.2 Tile-Centered Regions and the Not-Too-Bad Condition

5.4 Cauchy's Integral Formula

5.4.1 Winding Numbers and Continuous Angle Functions

6 TAYLOR SERIES AND SINGULARITIES

6.1 Taylor Series

6.2 Meromorphic Functions, Classification of Singularities

6.2.1 Properties of Essential Singularities

7 HOMOTOPY AND HOMOLOGY