Complex Analysis Reference Book

Stuyvesant Class of 2022

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Complex Calculus, Mr. Stern

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1 THE COMPLEX NUMBER SYSTEM

1.1 The Algebra of Complex Numbers

Definition 1.1 (Complex Numbers):
The *complex numbers* are the set

$$\mathbb{C} := \{ [(a,b)] \mid a, b \in \mathbb{R} \}$$

where

$$[(a,b)] := \begin{cases} a & b = 0 \\ (a,b) & b \neq 0 \end{cases}$$

Definition 1.2 (Real and Imaginary Parts of an element of $\mathbb{C}$):
The real part of $z \in \mathbb{C}$ where $z = [(a,b)]$ is defined as:

$$\mathrm{Re}(()z) := a.$$

Similarly, the imaginary part of $z \in \mathbb{C}$ is defined by $\mathrm{Im}(()z) := b$.

Definition 1.3 (Addition and Multiplication in $\mathbb{C}$):
For $[(a,b)], [(c,d)] \in \mathbb{C}$, $[(a,b)] + [(c,d)] := [(a+c, b+d)]$. Additionally, we will define multiplication as follows:

$$[(a,b)] \cdot [(c,d)] := [(ac - bd, ad + bc)].$$

This may seem to be an arbitrary defintion of multiplication, but as we explore the properties of this definition its motivation will become apparant.

Theorem 1.1 (Properties of Addition and Multiplication $\mathbb{C}$):
We can now verify a number of properties that will allow this rigorous bracketed point construction of $\mathbb{C}$ to be recgonized as equivalent to the familiar (non-rigorous) introduction of $\mathbb{C}$. Starting with properties of addition in $\mathbb{C}$, for $[(a,b)],[(c,d)] \in \mathbb{C}$ we have commutativity, associativity, and an identity.

Starting with commutativity, we know $[(a,b)]+[(c,d)] = [(a+c,b+d)] = [(c+a,d+b)]$ by the commutativity of $\mathbb{R}$. $[(c,d)]+[(a,b)] = [(c+a,d+b)]$, and thus we have $[(a,b)]+[(c,d)] = [(c,d)]+[(a,b)]$.

Next, associativity for $[(a,b)],[(c,d)],[(e,f)] \in \mathbb{C}$ is proven easily as well: We know $([(a,b)]+[(c,d)])+[(e,f)] = [((a+c)+e,(b+d)+f)]$. By the associativity of $\mathbb{R}$, we have $[(a+(c+e),b+(d+f))] = [(a,b)]+([(c,d)]+[(e,f)])$, and thus addition in $\mathbb{C}$ is associative.

Finally we have an identity element, $[(0,0)]$. Note that $[(0,0)]$ is equal to the real number 0. We quickly verify that it is indeed an indentity: For any $[(a,b)] \in \mathbb{C}$, $[(a,b)]+[(0,0)] = [(a+0,b+0)] = [(a,b)]$.

Next, we verify the same properties for multiplication. (In these more or less straightforward proofs, it will become clear how notationally clumsy our bracketed point notation is, and we will abadon it soon after these proofs. Terrence Tao calls this type of idea mathematical scaffolding in his book Analysis I: necessary to sturdy and rigorous construction, but discarded after use.)

Once again, starting with commutativity, we know $[(a,b)]\cdot[(c,d)] = [(ac-bd,ad+bc)]$ by definition. Similarly, $[(c,d)]\cdot[(a,b)] = [(ca-db,cb+da)]$ and that is equal to $[(ac-bd,ad+bc)]$ by commutativity of multiplication and addtion in $\mathbb{R}$.

Next we prove associativity. We know $([(a,b)]\cdot[(c,d)])\cdot[(e,f)] = ([(ac-bd,ad+bc)])\cdot[(e,f)] = [((ac-bd)e-(ad+bc)f,(ac-bd)f+(ad+bc)e]$. This is equal to $[(a,b)]\cdot([(c,d)]\cdot[(e,f)])$ by the standard properties of multiplication and addition in $\mathbb{R}$.

As an intermediate step, we can also use this oppurtunity to prove distributivity of multiplication over addition. For $[(a,b)],[(c,d)],[(e,f)] \in \mathbb{C}$ we know

$$[(a,b)]\cdot([(c,d)]+[(e,f)]) = [(a,b)]\cdot[(c+e,d+f)] = [(ac+ae-bd-bf,ad+af+bc+be)]$$

Then, by the properties of addition and multiplication in $\mathbb{R}$ we knnow

$$[(ac-bd,ad+bc)]+[(ae-bf,af+be)] = [(a,b)]\cdot[(c,d)]+[(a,b)]\cdot[(e,f)]$$

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3.1 Neighborhoods, Open and Closed Sets

Definition 3.1 (Open Disk):
The *open disk* of radius r around the point p is defined as follows:

$$\mathrm{D_p}(r) = \{z \in \mathbb{C} \mid |z-p| < r\}$$

Definition 3.2 (Neighborhood of a Point):
A set $S \subseteq \mathbb{C}$ is called a *neighborhood* of a point p if there exists some $r > 0$ such that $\mathrm{D_r}(p) \subseteq S$. We write this as $S \in \mathcal{N}(p)$, where $\mathcal{N}(p)$ denotes the *neighborhood-system* of p.

Definition 3.3 (Punctured Disk):
The *punctured disk* of radius r around the point p is defined as follows:

$$\mathrm{D^*_p}(r) = \{z \in \mathbb{C} \mid 0 < |z-p| < r\}$$

Definition 3.4 (Punctured Neighborhood of a Point):
A set $S \subseteq \mathbb{C}$ is called a *punctured neighborhood* of a point p if there exists some $r > 0$ such that $\mathrm{D^*_r}(p) \subseteq S$ and $p \notin S$. We write this as $S \in \mathcal{N}^*(p)$, where $\mathcal{N}^*(p)$ denotes the *punctured neighborhood-system* of p.

Definition 3.5 (Open Set):
A set S is called an *open set* iff $\forall z \in S, S \in \mathcal{N}(z)$.

Lemma 3.1 (Open Disks are Open Sets):
Any open disk $\mathrm{D_r}(p)$ is an open set.

Proof: We need to show that $\mathrm{D_r}(p)$ is a neighborhood of everyone one of its points. Let $q \in \mathrm{D_r}(p)$. We can show that $\mathrm{D_{r\text{-}|p\text{-}q|}}(q) \subseteq \mathrm{D_r}(p)$. To do this, pick any point z in the new disk. To show it's in our original disk, we need $|z-p| < r$. Since z is in the second disk, we can write $|z-q| < r - |p-q|$. We can then use the triangle inequality to show that $|z-p| = |z-q+q-p| < |z-q|+|p-q| < r-|p-q|+|p-q| < r$. Therefore, our original open disk contains a smaller open disk around every point, making it a neighborhood of every one of its points, i.e. an open set.

Lemma 3.2 (Supersets of Neighborhoods):
If $N \in \mathcal{N}(p)$ and $N \subseteq M$, $M \in \mathcal{N}(p)$.

Proof: If $N \in \mathcal{N}(p)$, then there exists some $\mathrm{D_r}(p) \subseteq N$. Since $N \subseteq M$, $\mathrm{D_r}(p) \subseteq M$. Therefore, $M \in \mathcal{N}(p)$.

Lemma 3.3 (Properties of Open Sets):
The union of any collection of open sets is open, and the intersection of any *finite* collection of open sets is open.

Proof: The union of any collection of sets is the set containing all of the points in all of the sets. Denote the union as U, and let p be an element of one of the sets, S. Since S is an open set, it is a neighborhood of p, and therefore there exists a $\mathrm{D_r}(p) \in S$. Since S is a subset of U, so is the disk, and since this is true of any point in the union, the union must be open. If a point p is in the intersection of a collection of sets C, it is in everyone one of the elements of the collection. Since each set in the collection is open, they are all neighborhoods of p. For every set S in the collection, there is therefore some radius r_S such that $\mathrm{D_{r_S}}(p) \in S$. Note that we can choose any radius less than or equal to r_S and still get an open disk around p fully contained in S Since there are only finitely many sets in the collection, we can let $\varepsilon = \min r_S$, which will be greater than 0. But since $\varepsilon \leq r_S$ for any S, $\mathrm{D}_\varepsilon(p)$ will be a subset of every S, and therefore will be in the intersection. The intersection is therefore a neighborhood of all of its points, making it an open set.

Definition 3.6 (Closed Set):
A set $S \subseteq \mathbb{C}$ is called *closed* iff its complement S^C is open.

Lemma 3.4 (Properties of Closed Sets):
The intersection of any collection of closed sets is closed, and the union of any *finite* collection of closed sets is closed.

Proof: This proof follows directly from De Morgan's Laws for sets and the union and intersection properties of open sets.

Definition 3.7 (Topology on a Set):
Given any set X, a *topology* on X τ_X is a collection of subsets called open sets, which satisfy the following properties:

- $\emptyset \in \tau_X, X \in \tau_X$
- The union of a collection of elements in τ_X is in τ_X
- The intersection of a *finite* collection of elements in τ_X is in τ_X

A neighborhood of a point p in any general topology is a set which contains an open set containing p.

Definition 3.8 (Relative Topology):
Given any type of set in the topology of $\mathbb{C}$ (open set, closed set, open disk, punctured disk, etc.) and some fixed subset $X \subseteq \mathbb{C}$, we can define a type of set *relative* to X as the intersection of that type of set with X. For instance, if $N \in \mathcal{N}(p)$ and $p \in X$, then the intersection $N \cap X$ is called a *relative neighborhood* of p, denoted as $N \cap X \in \mathcal{N}_\mathrm{X}(p)$. The collection of all open sets relative to X forms a topology, called the *relative topology* of X.

3.2 Accumulation Points and the Closure of a Set

Definition 3.9 (Accumulation Point):
A point p is called an *accumulation point* or *limit point* of a set S iff there does not exist a neigborhood $N \in \mathcal{N}(p)$ such that $N \cap S = \emptyset$.

Lemma 3.5 (Accumulation Points of a Closed Set):
$K \subseteq \mathbb{C}$ is closed iff it contains all of its accumulation points.

Proof: First, we will prove that a closed set contains all of its limit points. Suppose for the sake of contradiction that there exists a p which is an accumulation point of K and is in K^C. Since K is closed, its complement is open, and therefore is a neighborhood of every one of its points. $p \in K^C$, so $K^C \in \mathcal{N}(p)$. Therefore, there exists some disk centered at p which lies entirely in K^C. But this disk is a neighborhood of p which does not contain any points in K, which means p cannot be an accumulation point of S. Now for the converse. Suppose K contains all of its limit points. This means that there isn't a limit point of K in K^C. Thus, for any point $p \in K^C$, there exists a neighborhood of p which is entirely in K^C. But since any superset of a neighborhood of p is still a neighborhood of p, $K^C \in \mathcal{N}(p)$. K^C is therefore a neighborhood of all of its points, meaning it is open, and K is closed.

Definition 3.10 (Closure of a Set):
Given a set S, its *closure* $\overline{S}$ is the union of S and all of its accumulation points.

Lemma 3.6 (Closure of a Closed Set):
K is a closed set iff $\overline{K} = K$.

Proof: Since the closure of a set is the set together with all of its accumulation points, a set equalling its closure means that the set contains all of its limit points. By a previous lemma, this is equivalent to the set being closed.

Lemma 3.7 (Closure of a Set is Closed):
The closure of any set S is a closed set.

Proof: Let $p \in \left(\overline{S}\right)^C$. Since p is not in S nor is it a limit point, there exists a disk around p which is entirely contained in $\left(\overline{S}\right)^C$. The entire complement of the closure is therefore the union of such open disks. But since an open disk is an open set and the union of open sets is open, the complement of the closure is open, meaning the closure is closed.

3.3 Interior, Exterior, and Boundary

Definition 3.11 (Interior of a Set):
Given any set S, its *interior* $\text{int}(S)$ is defined as the complement of the closure of the complement of S.

Lemma 3.8 (Properties of the Interior):
If S is some set, and $\text{int}(S)$ is its interior, the following will hold:

1. $\text{int}(S)$ is open.
2. $$\text{int}(S) = \{p \in S \mid S \in \mathcal{N}(p)\}$$

Proof: The interior is an open set because it is the complement of the closure of some other set. Since the closure of a set is closed, this means that the interior is open. To prove the second part, take any point p in the set where the set is a neighborhood of p. Clearly, p is not in the complement of S. Since there is a neighborhood of p which does not contain any points in the complement of S, namely S itself, it is not a limit point of the complement of S. Therefore, p is not in the complement of the closure, and therefore is in the complement of the closure of the complement, also called the interior.

Definition 3.12 (Exterior of a Set):
Given any set S, its *exterior* $\mathrm{ext}(S)$ is defined as the interior of its complement.

Definition 3.13 (Boundary of a Set):
Given any set S, its *boundary* $\mathrm{bd}(S)$ consists of all points not in the interior of exterior of S.

3.4 Sequences in the Complex Plane, Limits of Sequences

Definition 3.14 (Sequence):
Given a set S, a *sequence* in S is a function $a : \mathbb{N} \longrightarrow S$. The sequence can be written as $(a_n)_{n=0}^{\infty}$, and the nth member of the sequence is written a_n.

Definition 3.15 (Topological Limit of a Sequence):
If $(a_n)_{n=0}^{\infty} \in S \subseteq X$ and X has a topology, we say the sequence has a *limit* of $a \in \overline{S}$ iff for every $U \in \mathcal{N}_{\overline{S}}(a)$ there exists some natural number N such that $n > N \implies a_n \in U$.

Definition 3.16 (Metric Limit of a Sequence):
If $(a_n)_{n=0}^{\infty} \in S \subseteq \mathbb{C}$ we say the sequence has a *limit* of $a \in \overline{S}$ iff for every $\varepsilon > 0$ there exists some natural number N such that $n > N \implies |a_n - a| < \varepsilon$.

Theorem 3.1 (Equivalence of Limit Types for Sequences):
If $(a_n)_{n=0}^{\infty} \in S \subseteq \mathbb{C}$ then the sequence has a topological limit of $a \in \overline{S}$ iff it has a metric limit of $a \in \overline{S}$.

Proof: First we will show that a topological limit implies a metric limit. We are given some $\varepsilon > 0$, and we need to find the corresponding N such that all values in the sequence past N are within ε of a. This is equivalent to saying all of the values in the sequence past N are in $\mathrm{D}_\varepsilon(a)$. We also know that all elements of the sequence are in S, and they are therefore in $\overline{S}$. Those past N are therefore in $\mathrm{D}_\varepsilon(a) \cap \overline{S}$. This set is a neighborhood of a relative to the set S. Since we know the sequence converges topologically, we can find such an N.
Now we will show the converse: a metric limit implies a topological limit. We are given some $U \in \mathcal{N}_{\overline{S}}(a)$, and we need to find the corresponding N. We can write U as the intersection of some larger neighborhood of a and the closure of the output set: $U = \tilde{U} \cap \overline{S}$. Since $\tilde{U}$ is a neighborhood of a, there exists some $\mathrm{D_r}(a) \subseteq \tilde{U}$. Since we know there is a metric limit, we can find an N such that all sequence elements past N are within r of a, i.e. in the disk. Since all of these points are in the disk, they are in $\tilde{U}$, and since this is a sequence in S, they are in $\overline{S}$. Therefore, all points in the sequence past N are in U, showing that the sequence converges to the same point topologically.

Definition 3.17 (Convergent Sequence):
A sequence is *convergent* if it has a limit.

Definition 3.18 (Bounded Set):
A set $S \subseteq \mathbb{C}$ is *bounded* if it is a subset of some disk centered around 0.

Lemma 3.9 (Convergent Sequences are Bounded):
If $(a_n)_{n=0}^{\infty}$ is a convergent sequence then its range is a bounded set.

Proof: Since $(a_n)_{n=0}^{\infty}$ is convergent, we know it has a limit, which we'll call a. We can pick $\varepsilon = 1$ and find some N such that all sequence members past N are within 1 unit of a. By the triangle inequality, this means they are in $D0|a| + 1$. This disk includes all of the points after N, but not necessarily those before N. However, there are only finitely many sequence members before or at N, so we can take the maximum modulus of all of these points, which we'll call R. Therefore, all points are contained within $\mathrm{D}_{\mathrm{|a|+R+2}}(0)$, so the sequence is bounded.

Theorem 3.2 (Uniqueness of Limits):
If $(a_n)_{n=0}^{\infty} \in S \subseteq \mathbb{C}$ has a limit of a, then it does not have any other limit.

Proof: Since $a_n \to a$, we know that for every $\varepsilon > 0$, there exists some cutoff point after which all elements of the sequence are within ε of a. Now assume for contradiction that there is another point $\tilde{a}$ with the same property. Since these points are not equal, the distance between them is a positive number. Let $\varepsilon_0 = \frac{1}{2}|a - \tilde{a}|$. Since the sequence converges to both points, there is a corresponding cuttoff N_a for a and a cutoff $N_{\tilde{a}}$ for $\tilde{a}$. Now consider the following point:

$$p = a_{N_a + N_{\tilde{a}} + 1}$$

Since p is beyond the cutoff for a, $|p - a| < \frac{1}{2}|\tilde{a} - a|$. But since p is beyond the cutoff for $\tilde{a}$, $|p - \tilde{a}| < \frac{1}{2}|\tilde{a} - a|$. Adding these inequalities together, we get $|a - p| + |p - \tilde{a}| < |a - \tilde{a}|$. This violates the triangle inequality, so $\tilde{a}$ cannot be different from a.

Theorem 3.3 (Limit Laws for Sequences):
Suppose $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are sequences in the complex plane with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then we can conclude the following:

1. $$\lim_{n\to\infty}(a_n + b_n) = a + b$$
2. $$\lim_{n\to\infty}(a_n b_n) = ab$$
3. $$\lim_{n\to\infty}\frac{1}{a_n} = \frac{1}{a}$$
   provided $a \neq 0$.
4. $$\lim_{n\to\infty}\mathrm{Re}(a_n) = \mathrm{Re}(a)$$
5. $$\lim_{n\to\infty}|a_n| = |a|$$

Proof:

1. We are given an $\varepsilon > 0$ and need to find the corresponding N. Since both sequences converge, let N_a and N_b be the cutoffs for each sequence corresponding to a distance of $\varepsilon/2$. Let $N = N_a + N_b + 1$. Now we can bound the distance from $a_N + b_N$ to $a + b$:

$$\begin{aligned} |(a+b) - (a_N + b_N)| &= |(a - a_N) + (b - b_N)| \\ &\leq |a - a_N| + |b - b_N| \\ &\leq \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

Therefore, we can use this N to force the sum of the sequence elements to be within ε of the sum of the limits.

2. Since both a_n and b_n converge to a and b respectively, we can find an N such that all sequence members past N in both sequences are within $\delta > 0$ of their limits. Knowing this, we'll want to estimate the distance between any two terms in the product sequence and the product of the limits whenever $n > N$:

$$\begin{aligned} |ab - a_n b_n| &= |ab - a_n b + a_n b - a_n b_n| \\ &= |b(a - a_n) + a_n(b - b_n)| \\ &\leq |b|\,|a - a_n| + |a_n|\,|b - b_n| \\ &\leq |b|\,\delta + |a_n|\,\delta \end{aligned}$$

Since convergent sequences are bounded, we can bound $|a_n|$ by some M:

$$\leq \delta\,(|b| + M)$$

Therefore, given any $\varepsilon > 0$, since we can force both sequences to be within $\frac{\varepsilon}{|b|+M}$ of their limits via some N, we can get the product $a_n b_n$ to be within ε of ab.

3. Since a_n converges to a, we can find an N such that all sequence members past N are within $\delta > 0$ of a. In addition, since convergent sequences are bounded, we can also find an M such that $|a_n| \leq M$. We can now bound the difference between the reciprocal of the limit and the reciprocal of any sequence element past N:

$$\begin{aligned} \left|\frac{1}{a} - \frac{1}{a_n}\right| &= \left|\frac{a_n - a}{a a_n}\right| \\ &= \frac{|a - a_n|}{|a|\,|a_n|} \\ &\leq \frac{\delta}{M\,|a|} \end{aligned}$$

Therefore, given any $\varepsilon > 0$, finding an N that forces a_n to be within $M\,|a|\,\varepsilon$ of a will force the reciprocal of a_n to be within ε of the reciprocal of a.

4. If $|a - a_n| < \varepsilon$, then $|\text{Re}(a) - \text{Re}(a_n)| = |\text{Re}(a - a_n)| \leq |a - a_n| < \varepsilon$.

5. If $|a - a_n| < \varepsilon$, then $||a| - |a_n|| \leq |a - a_n| < \varepsilon$.

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