# Complex Analysis Reference Book

Stuyvesant Class of 2022

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Complex Calculus, Mr. Stern

# Contents

# 1 THE COMPLEX NUMBER SYSTEM

## 1.1 The Algebra of Complex Numbers

**Definition 1.1** (Complex Numbers):
The *complex numbers* are the set

$$\mathbb{C} := \{ [(a,b)] \mid a,b \in \mathbb{R} \}$$

where

$$[(a,b)] := \begin{cases} a & b = 0 \\ (a,b) & b \neq 0 \end{cases}$$

**Definition 1.2** (Real and Imaginary Parts of an element of $\mathbb{C}$):
The real part of $z \in \mathbb{C}$ where $z = [(a,b)]$ is defined as:

$$Re(z) := a.$$

Similarly, the imaginary part of $z \in \mathbb{C}$ is defined by $Im(z) := b$.

**Definition 1.3** (Addition and Multiplication in $\mathbb{C}$):
For $[(a,b)], [(c,d)] \in \mathbb{C}$, $[(a,b)] + [(c,d)] := [(a+c, b+d)]$. Additionally, we will define multiplication as follows:

$$[(a,b)] \cdot [(c,d)] := [(ac - bd, ad + bc)].$$

This may seem to be an arbitrary defintion of multiplication, but as we explore the properties of this definition its motivation will become apparant.

**Theorem 1.1** (Properties of Addition and Multiplication $\mathbb{C}$):
We can now verify a number of properties that will allow this rigorous bracketed point construction of $\mathbb{C}$ to be recgonized as equivalent to the familiar (non-rigorous) introduction of $\mathbb{C}$. Starting with properties of addition in $\mathbb{C}$, for $[(a,b)],[(c,d)] \in \mathbb{C}$ we have commutativity, associativity, and an identity.

Starting with commutativity, we know $[(a,b)]+[(c,d)] = [(a+c,b+d)] = [(c+a,d+b)]$ by the commutativity of $\mathbb{R}$. $[(c,d)]+[(a,b)] = [(c+a,d+b)]$, and thus we have $[(a,b)]+[(c,d)] = [(c,d)]+[(a,b)]$.

Next, associativity for $[(a,b)],[(c,d)],[(e,f)] \in \mathbb{C}$ is proven easily as well: We know $([(a,b)]+[(c,d)])+[(e,f)] = [((a+c)+e,(b+d)+f)]$. By the associativity of $\mathbb{R}$, we have $[(a+(c+e),b+(d+f))] = [(a,b)]+([(c,d)]+[(e,f)])$, and thus addition in $\mathbb{C}$ is associative.

Finally we have an identity element, $[(0,0)]$. Note that $[(0,0)]$ is equal to the real number 0. We quickly verify that it is indeed an indentity: For any $[(a,b)] \in \mathbb{C}$, $[(a,b)]+[(0,0)] = [(a+0,b+0)] = [(a,b)]$.

Next, we verify the same properties for multiplication. (In these more or less straightforward proofs, it will become clear how notationally clumsy our bracketed point notation is, and we will abadon it soon after these proofs. Terrence Tao calls this type of idea mathematical scaffolding in his book Analysis I: necessary to sturdy and rigorous construction, but discarded after use.)

Once again, starting with commutativity, we know $[(a,b)]\cdot[(c,d)] = [(ac-bd,ad+bc)]$ by definition. Similarly, $[(c,d)]\cdot[(a,b)] = [(ca-db,cb+da)]$ and that is equal to $[(ac-bd,ad+bc)]$ by commutativity of multiplication and addtion in $\mathbb{R}$.

Next we prove associativity. We know $([(a,b)]\cdot[(c,d)])\cdot[(e,f)] = ([(ac-bd,ad+bc)])\cdot[(e,f)] = [((ac-bd)e-(ad+bc)f,(ac-bd)f+(ad+bc)e]$. This is equal to $[(a,b)]\cdot([(c,d)]\cdot[(e,f)])$ by the standard properties of multiplication and addition in $\mathbb{R}$.

As an intermediate step, we can also use this oppurtunity to prove distributivity of multiplication over addition. For $[(a,b)],[(c,d)],[(e,f)] \in \mathbb{C}$ we know

$$[(a,b)]\cdot([(c,d)]+[(e,f)]) = [(a,b)]\cdot[(c+e,d+f)] = [(ac+ae-bd-bf,ad+af+bc+be)]$$

Then, by the properties of addition and multiplication in $\mathbb{R}$ we knnow

$$[(ac-bd,ad+bc)]+[(ae-bf,af+be)] = [(a,b)]\cdot[(c,d)]+[(a,b)]\cdot[(e,f)]$$

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## 3.1 Neighborhoods, Open and Closed Sets

**Definition 3.1** (Open Disk):
The *open disk* of radius $r$ around the point $p$ is defined as follows:

$$\mathrm{D_p}(r) = \{z \in \mathbb{C} \mid |z - p| < r\}$$

**Definition 3.2** (Neighborhood of a Point):
A set $S \subseteq \mathbb{C}$ is called a *neighborhood* of a point $p$ if there exists some $r > 0$ such that $\mathrm{D_r}(p) \subseteq S$. We write this as $S \in \mathcal{N}(p)$, where $\mathcal{N}(p)$ denotes the *neighborhood-system* of $p$.

**Definition 3.3** (Punctured Disk):
The *punctured disk* of radius $r$ around the point $p$ is defined as follows:

$$\mathrm{D^*_p}(r) = \{z \in \mathbb{C} \mid 0 < |z - p| < r\}$$

**Definition 3.4** (Punctured Neighborhood of a Point):
A set $S \subseteq \mathbb{C}$ is called a *punctured neighborhood* of a point $p$ if there exists some $r > 0$ such that $\mathrm{D^*_r}(p) \subseteq S$ and $p \notin S$. We write this as $S \in \mathcal{N}^*(p)$, where $\mathcal{N}^*(p)$ denotes the *punctured neighborhood-system* of $p$.

**Definition 3.5** (Open Set):
A set $S$ is called an *open set* iff $\forall z \in S, S \in \mathcal{N}(z)$.

**Lemma 3.1** (Properties of Open Sets):
The union of any collection of open sets is open, and the intersection of any *finite* collection of open sets is open.

*Proof:* The union of any collection of sets is the set containing all of the points in all of the sets. Denote the union as $U$, and let $p$ be an element of one of the sets, $S$. Since $S$ is an open set, it is a neighborhood of $p$, and therefore there exists a $\mathrm{D_r}(p) \in S$. Since $S$ is a subset of $U$, so is the disk, and since this is true of any point in the union, the union must be open. If a point $p$ is in the intersection of a collection of sets $C$, it is in everyone one of the elements of the collection. Since each set in the collection is open, they are all neighborhoods of $p$. For every set $S$ in the collection, there is therefore some radius $r_S$ such that $\mathrm{D}_{\mathrm{r_S}}(p) \in S$. Note that we can choose any radius less than or equal to $r_S$ and still get an open disk around $p$ fully contained in $S$ Since there are only finitely many sets in the collection, we can let $\epsilon = \min r_S$, which will be greater than 0. But since $\epsilon \leq r_S$ for any $S$, $\mathrm{D}_\epsilon(p)$ will be a subset of every $S$, and therefore will be in the intersection. The intersection is therefore a neighborhood of all of its points, making it an open set.

**Definition 3.6** (Closed Set):
A set $S \subseteq \mathbb{C}$ is called *closed* iff its complement $S^C$ is open.

**Lemma 3.2** (Properties of Closed Sets):
The intersection of any collection of closed sets is closed, and the union of any *finite* collection of closed sets is closed.

*Proof:* This proof follows directly from De Morgan's Laws for sets and the union and intersection properties of open sets.

**Definition 3.7** (Topology on a Set):
Given any set $X$, a *topology* on $X$ $\tau_X$ is a collection of subsets called open sets, which satisfy the following properties:

- $\emptyset \in \tau_X, X \in \tau_X$
- The union of a collection of elements in $\tau_X$ is in $\tau_X$
- The intersection of a *finite* collection of elements in $\tau_X$ is in $\tau_X$

**Definition 3.8** (Relative Topology):
Given any type of set in the topology of $\mathbb{C}$ (open set, closed set, open disk, punctured disk, etc.) and some fixed subset $X \subseteq \mathbb{C}$, we can define a type of set *relative* to $X$ as the intersection of that type of set with $X$. For instance, if $N \in \mathcal{N}(p)$ and $p \in X$, then the intersection $N \cap X$ is called a *relative neighborhood* of $p$, denoted as $N \cap X \in \mathcal{N}_{\mathrm{X}}(p)$. The collection of all open sets relative to $X$ forms a topology, called the *relative topology* of $X$.

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