The Inverse of a Square Matrix

Definition 8. Two matrices are called mutually ***inverse*** if their product is the identity matrix. The inverse of the matrix A is denoted A^{-1}. Therefore, $A\,A^{-1} = A\,A^{-1} = I$.
If det**A** = 0 then the inverse does not exist and the matrix **A** is called ***singular***.

Finding the Inverse Matrix

The Cofactor Method

1. Find the **determinant** of the given matrix **A**.
2. Find **all cofactors** of the given matrix.
3. Form the **cofactor matrix C** (replace all elements of the given matrix by their cofactors).
4. **Transpose** the cofactor matrix. The matrix $\mathbf{C}^T$ is called the ***adjoint matrix*** (often denoted adj**A**) of the given matrix.
5. The inverse of **A** is $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}} \cdot \mathbf{C}^T$ or $\dfrac{1}{\det \mathbf{A}} \cdot \mathrm{adj}\mathbf{A}$.

The Inverse of a 2×2 Matrix:

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; then the cofactors are $C_{11} = d, C_{12} = -c, C_{21} = -b, C_{22} = a$ and the cofactor matrix is

$\mathbf{C} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$ and the adjoint matrix $\mathbf{C}^T = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Therefore $\mathbf{A}^{-1} = \dfrac{1}{\det \mathbf{A}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Example 1. Find the inverse matrix $\mathbf{A}^{-1}$ if the matrix **A**= $\begin{pmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ -2 & -1 & 2 \end{pmatrix}$

1. $\det \mathbf{A} = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 0 & 5 \\ -2 & -1 & 2 \end{vmatrix} = -40$

2. Cofactors (it is advised to place the cofactors on your page in an order corresponding to their indices):

$C_{11} = \begin{vmatrix} 0 & 5 \\ -1 & 2 \end{vmatrix} = 5$	$C_{12} = -\begin{vmatrix} 4 & 5 \\ -2 & 2 \end{vmatrix} = -18$	$C_{13} = \begin{vmatrix} 4 & 0 \\ -2 & -1 \end{vmatrix} = -4$
$C_{21} = -\begin{vmatrix} 3 & -1 \\ -1 & 2 \end{vmatrix} = -5$	$C_{22} = \begin{vmatrix} 2 & -1 \\ -2 & 2 \end{vmatrix} = 2$	$C_{23} = -\begin{vmatrix} 2 & 3 \\ -2 & -1 \end{vmatrix} = -4$
$C_{31} = \begin{vmatrix} 3 & -1 \\ 0 & 5 \end{vmatrix} = 15$	$C_{32} = -\begin{vmatrix} 2 & -1 \\ 4 & 5 \end{vmatrix} = -14$	$C_{33} = \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = -12$

3. Cofactor matrix $\mathbf{C} = \begin{pmatrix} 5 & -18 & -4 \\ -5 & 2 & -4 \\ 15 & -14 & -12 \end{pmatrix}$; 4. Adjoint matrix: $\mathrm{adj}\mathbf{A} = \mathbf{C}^T = \begin{pmatrix} 5 & -5 & 15 \\ -18 & 2 & -14 \\ -4 & -4 & -12 \end{pmatrix}$

5. Inverse matrix $\mathbf{A}^{-1} = -\dfrac{1}{40}\begin{pmatrix} 5 & -5 & 15 \\ -18 & 2 & -14 \\ -4 & -4 & -12 \end{pmatrix} = \begin{pmatrix} -0.125 & 0.125 & -0.375 \\ 0.45 & -0.05 & 0.35 \\ 0.1 & 0.1 & 0.3 \end{pmatrix}$

Example 2. To find the inverse of $\mathbf{B} = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$, det**B**= -2 and $\mathbf{B}^{-1} = \dfrac{1}{-2}\begin{pmatrix} -3 & 5 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 1.5 & -2.5 \\ 1 & -2 \end{pmatrix}$

The Row Reduction Method of Finding Inverse Matrix

We will work with a 'double matrix' (**A | I**). It can be shown that the same row operations that transform the matrix **A** into the identity matrix **I**, will transform **I** into $\mathbf{A}^{-1}$: $(\mathbf{A} \mid \mathbf{I}) \rightarrow (\mathbf{I} \mid \mathbf{A}^{-1})$.

Operations allowed:
1. Exchanging two rows
2. Dividing/multiplying a row by a nonzero number
3. Adding a multiple of one row to another

Notice that the operations are the same we could perform when changing the determinants into diagonal form. However,

- In the *determinant* the first two operations changed its value; for the inverse matrix we do *not* consider the changes in the determinant.
- In the determinant, similar operations are allowed for columns; here it is strictly ***rows only.***

We **aim** to obtain the identity matrix **I** on the left-hand side of the double matrix; don't forget to perform the same operations with the right-hand side.

Example 3. Let $\mathbf{M}=\begin{pmatrix}3&2&2\\1&-2&1\\-1&1&-1\end{pmatrix}$. Then $(\mathbf{M}|\mathbf{I})=\left(\begin{array}{ccc|ccc}3&2&2&1&0&0\\1&-2&1&0&1&0\\-1&1&-1&0&0&1\end{array}\right)$

$$\left(\begin{array}{ccc|ccc}3&2&2&1&0&0\\1&-2&1&0&1&0\\-1&1&-1&0&0&1\end{array}\right)\sim\begin{pmatrix}R_1-2R_2\\R_2\\R_3\end{pmatrix}\sim\left(\begin{array}{ccc|ccc}1&6&0&1&-2&0\\1&-2&1&0&1&0\\-1&1&-1&0&0&1\end{array}\right)\sim\begin{pmatrix}R_1\\R_2-R_1\\R_3+R_1\end{pmatrix}\sim\left(\begin{array}{ccc|ccc}1&6&0&1&-2&0\\0&-8&1&-1&3&0\\0&7&-1&1&-2&1\end{array}\right)\sim$$

$$\begin{pmatrix}R_1\\R_2+R_3\\R_3\end{pmatrix}\sim\left(\begin{array}{ccc|ccc}1&6&0&1&-2&0\\0&-1&0&0&1&1\\0&7&-1&1&-2&1\end{array}\right)\sim\begin{pmatrix}R_1\\-R_2\\R_3\end{pmatrix}\sim\left(\begin{array}{ccc|ccc}1&6&0&1&-2&0\\0&1&0&0&-1&-1\\0&7&-1&1&-2&1\end{array}\right)\sim\begin{pmatrix}R_1\\R_2\\R_3-7R_2\end{pmatrix}\sim\left(\begin{array}{ccc|ccc}1&6&0&1&-2&0\\0&1&0&0&-1&-1\\0&0&-1&1&5&8\end{array}\right)\sim$$

$$\begin{pmatrix}R_1\\R_2\\-R_3\end{pmatrix}\sim\left(\begin{array}{ccc|ccc}1&6&0&1&-2&0\\0&1&0&0&-1&-1\\0&0&1&-1&-5&-8\end{array}\right)\sim\begin{pmatrix}R_1-6R_2\\R_2\\R_3\end{pmatrix}\sim\left(\begin{array}{ccc|ccc}1&0&0&1&4&6\\0&1&0&0&-1&-1\\0&0&1&-1&-5&-8\end{array}\right);\qquad \mathbf{M}^{-1}=\begin{pmatrix}1&4&6\\0&-1&-1\\-1&-5&-8\end{pmatrix}$$

Exercise 4.

1. Find the inverse matrices of (a) $\begin{pmatrix}1&-3\\-2&5\end{pmatrix}$ (b) $\begin{pmatrix}-2&3\\6&-8\end{pmatrix}$ (c) $\begin{pmatrix}4&2\\-3&1\end{pmatrix}$

2. Find the inverse matrices:

(a) $\begin{pmatrix}1&2&-1\\2&5&1\\3&6&-2\end{pmatrix}$ (b) $\begin{pmatrix}1&3&-2\\-1&-5&6\\2&6&-3\end{pmatrix}$ (c) $\begin{pmatrix}1&2&-1\\2&6&1\\3&6&-4\end{pmatrix}$ (d) $\begin{pmatrix}2&1&-1\\6&4&-1\\4&2&-3\end{pmatrix}$ (e) $\begin{pmatrix}2&4&-4\\1&3&-4\\2&4&-3\end{pmatrix}$ (f) $\begin{pmatrix}1&-2&2\\2&-3&1\\3&-6&6\end{pmatrix}$

(g) $\begin{pmatrix}2&-1&0\\1&-3&4\\3&-2&1\end{pmatrix}$ (h) $\begin{pmatrix}2&3&-4\\3&2&-4\\3&3&-5\end{pmatrix}$ (i) $\begin{pmatrix}1&-1&1\\-3&1&2\\1&2&-6\end{pmatrix}$ (j) $\begin{pmatrix}3&-2&-4\\2&0&-1\\-4&1&4\end{pmatrix}$ (k) $\begin{pmatrix}-2&0&2\\1&2&2\\-2&1&4\end{pmatrix}$

3. Show that $\mathbf{N}=\mathbf{M}^2-8\mathbf{I}$ is the inverse of the matrix $\mathbf{M}=\begin{pmatrix}3&2&2\\1&-2&1\\-1&1&-1\end{pmatrix}$.

4. Given $\mathbf{C}=\begin{pmatrix}2&3&-4\\3&2&-4\\3&3&-5\end{pmatrix}$, evaluate $\mathbf{C}^2$ and find $\mathbf{C}^{-1}$.

5. Consider the matrices $\mathbf{A}=\begin{pmatrix}7&8\\6&7\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3&-2\\4&-3\end{pmatrix}$.

Find (a) $\mathbf{A}^{-1}$ (b) $\mathbf{B}^{-1}$ (c) $\mathbf{A}^{-1}\mathbf{B}^{-1}$ (d) $\mathbf{B}^{-1}\mathbf{A}^{-1}$

(e) $\mathbf{AB}$ and $(\mathbf{AB})^{-1}$ (f) $\mathbf{BA}$ and $(\mathbf{BA})^{-1}$.

ANSWERS

1. (a) $\frac{1}{-1}\begin{pmatrix}5 & 3\\ 2 & 1\end{pmatrix}=\begin{pmatrix}-5 & -3\\ -2 & -1\end{pmatrix}$ (b) $\begin{pmatrix}4 & 1.5\\ 3 & 1\end{pmatrix}$ (c) $\begin{pmatrix}0.1 & -0.2\\ 0.3 & 0.4\end{pmatrix}$

2. (a) $\det\mathbf{A}=1;\quad \mathbf{A}^{-1}=\begin{pmatrix}-16 & -2 & 7\\ 7 & 1 & -3\\ -3 & 0 & 1\end{pmatrix}$ (b) $\det\mathbf{A}=-2;\quad \mathbf{A}^{-1}=\begin{pmatrix}10.5 & 1.5 & -4\\ -4.5 & -0.5 & 2\\ -2 & 0 & 1\end{pmatrix}$ (c) $\det\mathbf{A}=-2;\quad \mathbf{A}^{-1}=\begin{pmatrix}15 & -1 & -4\\ -5.5 & 0.5 & 1.5\\ 3 & 0 & -1\end{pmatrix}$

(d) $\det\mathbf{A}=-2;\quad \mathbf{A}^{-1}=\begin{pmatrix}5 & -0.5 & -1.5\\ -7 & 1 & 2\\ 2 & 0 & -1\end{pmatrix}$ (e) $\det\mathbf{A}=2;\quad \mathbf{A}^{-1}=\begin{pmatrix}3.5 & -2 & -2\\ -2.5 & 1 & 2\\ -1 & 0 & 1\end{pmatrix}$ (f) det**A**=0; singular matrix, no inverse

(g) $\mathbf{A}^{-1}=\begin{pmatrix}-5 & -1 & 4\\ -11 & -2 & 8\\ -7 & -1 & 5\end{pmatrix}$ (h) $\mathbf{C}^{-1}=\begin{pmatrix}2 & 3 & -4\\ 3 & 2 & -4\\ 3 & 3 & -5\end{pmatrix}=\mathbf{C}$ (i) $\mathbf{A}^{-1}=\begin{pmatrix}10 & 4 & 3\\ 16 & 7 & 5\\ 7 & 3 & 2\end{pmatrix}$

(j) $\det\mathbf{A}=3;\quad \mathbf{A}^{-1}=\begin{pmatrix}\frac{1}{3} & \frac{4}{3} & \frac{2}{3}\\ -\frac{4}{3} & -\frac{4}{3} & -\frac{5}{3}\\ \frac{2}{3} & \frac{5}{3} & \frac{4}{3}\end{pmatrix}$ (k) $\det\mathbf{A}=-2;\quad \mathbf{A}^{-1}=\begin{pmatrix}-3 & -1 & 2\\ 4 & 2 & -3\\ -2.5 & -1 & 2\end{pmatrix}$

3. $\mathbf{M}^2=\begin{pmatrix}9 & 4 & 6\\ 0 & 7 & -1\\ -1 & -5 & 0\end{pmatrix};\quad \mathbf{N}=\begin{pmatrix}9 & 4 & 6\\ 0 & 7 & -1\\ -1 & -5 & 0\end{pmatrix}-\begin{pmatrix}8 & 0 & 0\\ 0 & 8 & 0\\ 0 & 0 & 8\end{pmatrix}=\begin{pmatrix}1 & 4 & 6\\ 0 & -1 & -1\\ -1 & -5 & -8\end{pmatrix};$

N=$\mathbf{M}^{-1}$ if **MN=I**: check that $\begin{pmatrix}3 & 2 & 2\\ 1 & -2 & 1\\ -1 & 1 & -1\end{pmatrix}\cdot\begin{pmatrix}1 & 4 & 6\\ 0 & -1 & -1\\ -1 & -5 & -8\end{pmatrix}=\mathbf{I}$

4. $\mathbf{C}^2=\mathbf{I}$, therefore $\mathbf{C}=\mathbf{C}^{-1}$.

5. (a) $\mathbf{A}^{-1}=\begin{pmatrix}7 & -8\\ -6 & 7\end{pmatrix}$ (b) $\mathbf{B}^{-1}=\begin{pmatrix}3 & -2\\ 4 & -3\end{pmatrix}$ (c) $\mathbf{A}^{-1}\mathbf{B}^{-1}=\begin{pmatrix}-11 & 10\\ 10 & -9\end{pmatrix}$ (d) $\mathbf{B}^{-1}\mathbf{A}^{-1}=\begin{pmatrix}33 & -38\\ 46 & -53\end{pmatrix}$

(e) $\mathbf{AB}=\begin{pmatrix}53 & -38\\ 46 & -33\end{pmatrix}$ (f) $(\mathbf{AB})^{-1}=\begin{pmatrix}33 & -38\\ 46 & -53\end{pmatrix}$ (f) $\mathbf{BA}=\begin{pmatrix}9 & 10\\ 10 & 11\end{pmatrix}$, $(\mathbf{BA})^{-1}=\begin{pmatrix}-11 & 10\\ 10 & -9\end{pmatrix}$