The Vector Product Of Two Vectors. The Triple Product

Definition 11. The ***vector product*** of two vectors $\vec{a}$ and $\vec{b}$, (also known as ***cross product***) is a ***vector***, according to the formula

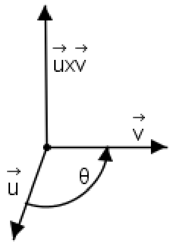
$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix}=(a_yb_z-a_zb_y)\vec{i}+(a_xb_z-a_zb_x)\vec{j}+(a_xb_y-a_yb_x)\vec{k}.$$

For example, if $\vec{a}=\langle -1,2,3\rangle$ and $\vec{b}=\langle 4,0,-5\rangle$ then

$$\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ -1 & 2 & 3\\ 4 & 0 & -5\end{vmatrix}=\vec{i}\cdot\begin{vmatrix}2 & 3\\ 0 & -5\end{vmatrix}-\vec{j}\cdot\begin{vmatrix}-1 & 3\\ 4 & -5\end{vmatrix}+\vec{k}\cdot\begin{vmatrix}-1 & 2\\ 4 & 0\end{vmatrix}=-10\vec{i}+7\vec{j}-8\vec{k}=\langle -10,7,-8\rangle.$$

Properties Of Vector Product

1. Geometrically, the vector product of $\vec{a}$ and $\vec{b}$ will be ***perpendicular*** to both the vectors $\vec{a}$ and $\vec{b}$, its ***direction*** according to the right-handed system (that is, a system where sequence of vectors between them is listed anti-clockwise according to the smallest angles).
2. The ***magnitude*** of the vector product is $|\vec{a}\times\vec{b}|=|\vec{a}|\cdot|\vec{b}|\cdot\sin\theta$ where θ is the positive angle between vectors.
3. **Anticommutative property**: Switching the order of the factors changes the the vector product to the opposite: $\vec{a}\times\vec{b}=-(\vec{b}\times\vec{a})$.



4. **Associative property**: $k(\vec{a}\times\vec{b})=(k\vec{a})\times\vec{b}=\vec{a}\times(k\vec{b})$.
5. **Distributive property**: $(\vec{a}+\vec{b})\times\vec{c}=\vec{a}\times\vec{c}+\vec{b}\times\vec{c}$.
6. $\vec{a}\times\vec{b}=\vec{0}$ if and only if $\vec{a}\parallel\vec{b}$; in particular, $\vec{a}\times\vec{a}=\vec{0}$.
7. $\vec{i}\times\vec{j}=\vec{k}$; $\vec{j}\times\vec{k}=\vec{i}$; $\vec{k}\times\vec{i}=\vec{j}$.

Applications Of The Vector Cross Product

1. The area of a ***parallelogram*** $A_{pg}=|\vec{a}\times\vec{b}|$ where $\vec{a}$ and $\vec{b}$ are vectors defining its non-parallel sides (can be seen from (**2.**)).

2. Subsequently, the area of a ***triangle*** $A_{\Delta}=\frac{1}{2}|\vec{a}\times\vec{b}|$ where $\vec{a}$ and $\vec{b}$ are vectors defining two of its sides.

3. If the force $\vec{F}$ is applied at the point A then the ***moment of force*** (***torque***) $\overrightarrow{M}$ equals the vector product of $\vec{F}$ and the position vector (radius vector) $\vec{r}=\overrightarrow{OA}$: $\overrightarrow{M}=\vec{F}\times\vec{r}$ **.**

The Triple Product (Mixed Product, Box Product) of Three vectors

The ***triple product*** of three vectors $\vec{a}$**,** $\vec{b}$ and $\vec{c}$ and $\vec{b}$ is defined as $\vec{a}\vec{b}\vec{c}=(\vec{a}\times\vec{b})\cdot\vec{c}$.

The triple product is a *scalar* (as the cross product is calculated first, producing a vector, and the scalar product as second).

Properties Of Triple Product

1. The triple product can be computed as the determinant $\vec{a}\vec{b}\vec{c}=\begin{vmatrix}a_x & a_y & a_z\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}$.

2. The sign of the triple product indicates the orientation of the vector system $\vec{a}$, $\vec{b}$, $\vec{c}$: if $\vec{a}\vec{b}\vec{c}>0$ then the system is **right-handed**; if $\vec{a}\vec{b}\vec{c}<0$ then the system is **left-handed**.

3. $\vec{a}\vec{b}\vec{c}=0$ if and only if the vectors $\vec{a}$**,** $\vec{b}$ and $\vec{c}$ are **coplanar.**

4. **Associative** property: $\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$

5. Change of order: if any **two** vectors are exchanged, the sign of the triple product changes to **opposite**; if the vector order is changed in a **cyclic** manner then the triple product is **unchanged**:

$$\vec{a}\,\vec{b}\,\vec{c}=\vec{b}\,\vec{c}\,\vec{a}=\vec{c}\,\vec{a}\,\vec{b}=-\vec{a}\,\vec{c}\,\vec{b}=-\vec{b}\,\vec{a}\,\vec{c}=-\vec{c}\,\vec{b}\,\vec{a}$$

Applications Of The Triple Product

1. Its absolute value gives the ***volume of the parallelepiped*** constructed on the three vectors: $V_{pp}=\left|\vec{a}\,\vec{b}\,\vec{c}\right|$.

2. The volume of a triangular prism: $V_{prism}=\frac{1}{2}\left|\vec{a}\,\vec{b}\,\vec{c}\right|$

3. The volume of the ***tetrahedron*** (triangular pyramid) defined by the three vectors is $V_{th}=\frac{1}{6}\left|\vec{a}\,\vec{b}\,\vec{c}\right|$.

Exercises.

1. Given that $\vec{a}=2\vec{i}+2\vec{j}-\vec{k}$ and $\vec{b}=\vec{i}+2\vec{j}+2\vec{k}$, find $\vec{a}\times\vec{b}$ and show that it is perpendicular to both $\vec{a}$ and $\vec{b}$.

2. Given that $\vec{p}=2\vec{i}+\vec{k}$; $\vec{q}=\vec{i}-2\vec{j}+\vec{k}$ and $\vec{r}=-2\vec{i}+4\vec{j}-3\vec{k}$, find:

a) $\vec{p}\times\vec{q}$ b) $\vec{q}\times\vec{p}$ c) $\left|\vec{p}\times\vec{r}\right|$ d) $\left|\vec{r}\times\vec{q}\right|$ e) $2\vec{p}\times4\vec{r}$

f) $\left(\vec{p}+\vec{r}\right)\times\vec{r}$ g) $\vec{p}\times\left(\vec{q}\times\vec{r}\right)$ h) $\left(\vec{p}+2\vec{q}\right)\times\vec{r}$

3. Find ***all*** vectors that are perpendicular to both

(a) $\vec{\boldsymbol{a}}=\langle 0,-1,3\rangle$ and $\vec{\boldsymbol{b}}=\langle -2,1,2\rangle$; (b) $\vec{\boldsymbol{a}}=\langle -1,3,4\rangle$ and $\vec{\boldsymbol{b}}=\langle 5,0,2\rangle$

4. ABCD is a parallelogram where A is (-1, 3, 2), B(2, 0, 4) and C(-1, -2, 5). Find the
(a) coordinates of D; (b) area of ABCD.

5. The triangle ABC is given by $A(2,0,-1), B(1,3,-2), C(5,5,2)$. Calculate the area and height from A.

6. Find the area of each of the following triangles whose vertices are given:
(a) A(2, 1, 1), B(4, 3, 0) and C(1, 3, -2) (b) A(0, 0, 0), B(-1, 2, 3) and C(1, 2, 6)

7. Forces $\overrightarrow{f_1}=\langle -1,3,4\rangle$, $\overrightarrow{f_2}=\langle 2,-2,1\rangle$, $\overrightarrow{f_3}=\langle 3,1,-3\rangle$ are applied at the point A(1,0,-2). Find the torque (moment of force) with respect to the point B(2,1,-4).

8. Given the vectors $\vec{\boldsymbol{a}}=\langle 1,-1,2\rangle$, $\vec{\boldsymbol{b}}=\langle 2,0,3\rangle$, $\vec{\boldsymbol{c}}=\langle 4,1,-1\rangle$, find:

(a) $\mathbf{a}\times\mathbf{b}$ (b) $\mathbf{b}\times\mathbf{c}$ (c) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$

9. Use triple product to decide if

(a) vectors $\vec{\boldsymbol{a}}=\langle 4,-2,3\rangle$, $\vec{\boldsymbol{b}}=\langle 1,-1,5\rangle$, $\vec{\boldsymbol{c}}=\langle 2,0,-7\rangle$

(b) points $A(1,-1,2)$; $B(3,4,0)$; $C(2,-3,1)$; $D(4,2,5)$ are **coplanar**.

10. In the parallelepiped ABCDEFGH, AE, BF, CG and DH are parallel edges. Given that A, B, D and E have coordinates $(1;2;3)$, $(3;1;1)$, $(2;4;0)$ and $(-1;4;4)$ respectively, find volume of the parallelepiped.

11. The points $A(1;2;3)$, $B(2;4;1)$, $C(-2;3;-1)$ and $D(0;-2;4)$ are the vertices of a tetrahedron. Calculate the volume of this tetrahedron.

12. Given the points K(3, 2, 0), L(1, -2, 1), M(-1, 0, 2), find (a) the volume, (b) the total surface area of the tetrahedron OKLM where O is the origin of coordinates (0; 0; 0).

ANSWERS

1. $\vec{a}\times\vec{b}=\begin{pmatrix}6\\-5\\2\end{pmatrix}=\vec{c}$; as $\vec{c}\cdot\vec{a}=\vec{c}\cdot\vec{b}=0$, $\vec{c}\perp\vec{a}$, $\vec{c}\perp\vec{b}$.

2. a) $\begin{pmatrix}2\\-1\\-4\end{pmatrix}$ b) $\begin{pmatrix}-2\\1\\4\end{pmatrix}$ c) $\begin{pmatrix}-4\\4\\8\end{pmatrix}$; $4\sqrt{6}$ d) $\begin{pmatrix}-2\\-1\\0\end{pmatrix}$; $\sqrt{5}$ e) $\begin{pmatrix}-32\\32\\64\end{pmatrix}$ f) $\begin{pmatrix}-4\\4\\8\end{pmatrix}$ g) $\begin{pmatrix}-1\\2\\2\end{pmatrix}$ h) $\begin{pmatrix}0\\6\\8\end{pmatrix}$

3. (a) $\begin{pmatrix}5k\\6k\\2k\end{pmatrix}, k\in\mathbb{R}$ (b) $\begin{pmatrix}6k\\22k\\-15k\end{pmatrix}, k\in\mathbb{R}$

4. D(-4, 1, 3); Area= $\sqrt{307}$

5. $\overrightarrow{AB}=\langle -1,3,-1\rangle$; $\overrightarrow{AC}=\langle 3,5,3\rangle$; Area A = $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\frac{14\sqrt{2}}{2}=7\sqrt{2}$ square units;

$\overrightarrow{BC}=\langle 4,2,4\rangle$; $\left|\overrightarrow{BC}\right|=6$ units; as $A=\frac{1}{2}\left|\overrightarrow{BC}\right|\cdot h_{BC}$ then $h_{BC}=\frac{7\sqrt{2}}{3}$.

6.

(a) $\overrightarrow{AB}=\langle 2,2,-1\rangle$
$\overrightarrow{AC}=\langle -1,2,-3\rangle$

$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}-4\\7\\6\end{pmatrix}$; $\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|=\sqrt{16+49+36}=\sqrt{101}$;

$A_{(ABC)}=\frac{1}{2}\sqrt{101}\approx 5.02$ (sq. units)

(b) $\begin{pmatrix}-1\\2\\3\end{pmatrix}\times\begin{pmatrix}1\\2\\6\end{pmatrix}=\begin{pmatrix}6\\9\\-4\end{pmatrix}$; $A_{(ABC)}=\frac{1}{2}\sqrt{36+81+16}=\frac{1}{2}\sqrt{133}\approx 5.77$ (sq. u.)

7. $\vec{F}=\vec{\mathbf{f}}_1+\vec{\mathbf{f}}_2+\vec{\mathbf{f}}_3=\langle 4,2,2\rangle$; $\vec{r}=\overrightarrow{BA}=\langle -1,-1,2\rangle$; $\vec{M}=\vec{F}\times\vec{r}=\langle 6,-10,-2\rangle$.

8. (a) $\begin{pmatrix}-3\\1\\2\end{pmatrix}$ (b) $\begin{pmatrix}-3\\14\\2\end{pmatrix}$ (c) -13

9. (a) $\vec{a}\vec{b}\vec{c}=\begin{vmatrix}4&-2&3\\1&-1&5\\2&0&-7\end{vmatrix}=0$ - the vectors are coplanar

(b) $\overrightarrow{AB}=\langle 2,5,-2\rangle$, $\overrightarrow{AC}=\langle 1,-2,-1\rangle$, $\overrightarrow{AD}=\langle 3,3,3\rangle$; $\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}=\begin{vmatrix}2&5&-2\\1&-2&-1\\3&3&3\end{vmatrix}=-54\neq 0$

The points are not coplanar.

10. $\overrightarrow{AB} = \langle 2,-1,-2 \rangle$, $\overrightarrow{AD} = \langle 1,2,-3 \rangle$, $\overrightarrow{AE} = \langle -2,2,1 \rangle$; V = $\left|\left(\overrightarrow{AB}\,\overrightarrow{AD}\,\overrightarrow{AE}\right)\right|$ = 1 (cubic unit)

11. $\overrightarrow{AB} = \langle 1,2,-2 \rangle$, $\overrightarrow{AC} = \langle -3,1,-4 \rangle$, $\overrightarrow{AD} = \langle -1,-4,1 \rangle$

$$V = \frac{1}{6}\left|\left(\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\right)\right| = 4.5\, units^3$$

12. Volume = 3 cub.un.

$$A_{(OKL)} = \frac{1}{2}\left|\overrightarrow{OK} \times \overrightarrow{OL}\right| = \frac{1}{2}\left|\begin{pmatrix}3\\2\\0\end{pmatrix} \times \begin{pmatrix}1\\-2\\1\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}2\\-3\\-8\end{pmatrix}\right| \approx 4.39 \text{ sq.u.}$$

$$A(OLM) = \frac{1}{2}\left|\overrightarrow{OL} \times \overrightarrow{OM}\right| = \frac{1}{2}\left|\begin{pmatrix}1\\-2\\1\end{pmatrix} \times \begin{pmatrix}-1\\0\\2\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}-4\\-3\\-2\end{pmatrix}\right| \approx 2.69 \text{ sq. u.}$$

$$A(OKM) = \frac{1}{2}\left|\overrightarrow{OK} \times \overrightarrow{OM}\right| = \frac{1}{2}\left|\begin{pmatrix}3\\2\\0\end{pmatrix} \times \begin{pmatrix}-1\\0\\2\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}4\\-6\\+2\end{pmatrix}\right| \approx 3.74$$

$$A(KLM) = \frac{1}{2}\left|\overrightarrow{KL} \times \overrightarrow{KM}\right| = \frac{1}{2}\left|\begin{pmatrix}-2\\-4\\+1\end{pmatrix} \times \begin{pmatrix}-4\\-2\\2\end{pmatrix}\right| = \frac{1}{2}\left|\begin{pmatrix}-6\\0\\-12\end{pmatrix}\right| \approx 9.48$$

$$A_{surface} \approx 20.3 \text{ sq.u}$$