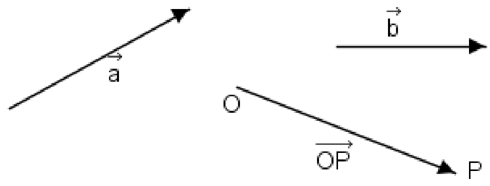
Vector Algebra: Basic Definitions and Linear Operations

Definition 1. ***A vector*** is a ***displacement*** *in the plane or in 3-D space;* geometrically vectors are denoted and defined as ***directed segments.*** A vector has both *length* and *direction* (as opposed to a *scalar* – a number - which has only a magnitude).

Definition 2. The length of a vector is called its ***modulus*** or ***magnitude***.



The starting point of the vector is called the ***initial*** point, and the endpoint is called the ***terminal*** point.

Vectors are denoted by small letters of the Latin alphabet or by the initial (first) and terminal (second) points; when in print, **bold** font may be used to distinguish a vector from a number or an unoriented segment: **a, r, AB**; or an arrow above the vector name: $\overrightarrow{AB}$; $\vec{a}, \vec{v}, \vec{i}, \vec{j}, \vec{k}$; when writing by hand, the arrow is used.

A vector can be **free** (a vector that can be moved to any position in the space), **sliding** (can only be moved along the line that contains it) or **fixed** (cannot be moved at all) – the type of the vector depends on its application, e.g, in physics. In this course only free vectors will be considered.

Definition 3. Two vectors are called ***collinear*** if the lines containing them are parallel or coincident. If the vectors have the same direction they are called ***parallel*** ($\vec{a} \uparrow\uparrow \vec{b}$); if their directions are opposite they are called ***antiparallel*** ($\vec{a} \uparrow\downarrow \vec{b}$).

Definition 4. Two vectors are called ***equal*** if they have ***the same magnitude and are parallel***, and ***opposite*** if they have ***the same magnitude and are antiparallel***.

Definition 5. Three vectors are called ***coplanar*** if they lie in the same plane or in parallel planes.

Definition 6. A vector whose length is 1 is called a ***unit vector***.

Definition 7. A vector whose starting point and endpoint is the same is called the ***zero vector***; it corresponds to ***zero displacement*** and is denoted $\vec{0}$.

Vector Algebra: The Linear Operations

1. Scalar multiples: The product of a vector $\vec{a}$ and a scalar k is a vector $k \cdot \vec{a} = \vec{b}$ such that

(1) its magnitude is $|\vec{b}| = |k| \cdot |\vec{a}|$

(2) $\vec{a} \uparrow\uparrow \vec{b}$ if $k > 0$, and $\vec{a} \uparrow\downarrow \vec{b}$ if $k < 0$.

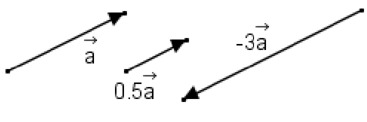
If $k = 0$ then the product is the zero vector: $0 \cdot \vec{a} = \vec{0}$

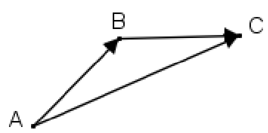
Theorem. Two vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if there exists a scalar $k \neq 0$ such that $k \cdot \vec{a} = \vec{b}$.

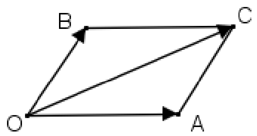
2. The sum of two vectors

A. The ***Triangle Rule***: If the initial point of the second vector coincides with the terminal point of the first vector, then the sum of the two vectors is a vector whose initial point is the same as the first vector's and the terminal point the same as the second vector's: $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$.

B. The ***Parallelogram Rule*** is applied if the initial points of both vectors coincide. Complete the parallelogram whose sides are the given vectors; the sum of the two vectors is the diagonal vector with the same initial point: $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ (see below).





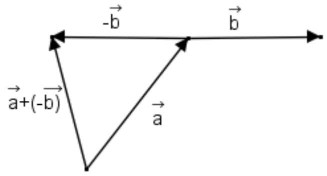


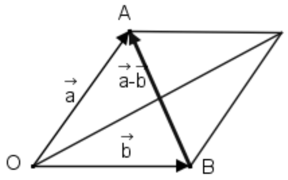
3. The difference of two vectors:

The difference of two vectors $\vec{a}$ and $\vec{b}$ is defined as such a vector $\vec{d}$ that must be added to $\vec{b}$ to obtain $\vec{a}$: if $\vec{b}+\vec{d}=\vec{a}$ then $\vec{a}-\vec{b}=\vec{d}$.

A. Applying the Triangle Rule, add vectors $\vec{a}$ and $-\vec{b}$: $\vec{a}-\vec{b}=\vec{a}+(-\vec{b})$

B. In general, if $\overrightarrow{OB}+\overrightarrow{BA}=\overrightarrow{OA}$,then $\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OC}$. Therefore, the difference is the vector of the second diagonal of the parallelogram, starting from the terminal point of the **second** vector and ending at the initial point of the **first** vector.



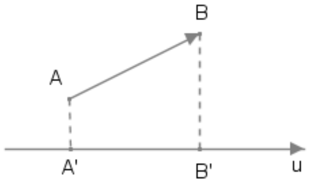


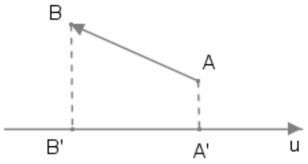
4. The projection of a vector *on the axis* (or vector).

Let the projections of the points A and B on the axis *u* be the points A' and B' respectively.

Then the projection of the vector $\overrightarrow{AB}$ on the axis *u* is the length of the segment $A'B'$ used with the (+) sign: $proj_u \overrightarrow{AB} = |A'B'|$ if $\overrightarrow{A'B'} \uparrow\uparrow \vec{u}$ and with the (–) sign $proj_u \overrightarrow{AB} = -|A'B'|$ if $\overrightarrow{A'B'} \uparrow\downarrow \vec{u}$;

geometrically this means $proj_u \overrightarrow{AB} = |\overrightarrow{AB}| \cdot \cos\alpha$ where α is the angle between $\overrightarrow{AB}$ and the axis.



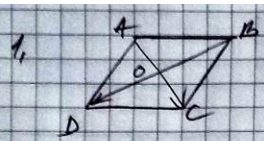


Exercise 1.

1. Given a parallelogram ABCD, express the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$ in terms of $\vec{a}=\overrightarrow{AC}$ and $\vec{b}=\overrightarrow{BD}$.
2. $|\vec{a}|=6, |\vec{b}|=8$ and $\vec{a}\perp\vec{b}$. Find $|\vec{a}+\vec{b}|$ and $|\vec{a}-\vec{b}|$.
3. The medians of the triangle ABC are AD, BE and CF. Prove that $\overrightarrow{AD}+\overrightarrow{BE}+\overrightarrow{CF}=\vec{0}$.
4. The mass centre of the triangle ABC is O. Express the vector $\overrightarrow{BO}$ using the vectors $\vec{a}=\overrightarrow{AB}$ and $\vec{b}=\overrightarrow{AC}$.
5. The point M lies on the segment AB so that the ratio (a) $\frac{AM}{AB}=\frac{2}{5}$ (b) $\frac{AM}{MB}=\frac{a}{b}$. The point P is outside the line AB. Express the vector $\overrightarrow{PM}$ using the vectors $\vec{p}=\overrightarrow{PA}$ and $\vec{q}=\overrightarrow{PB}$.

ANSWERS

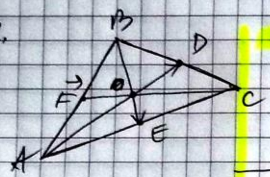
1.



$$\vec{AB} = \vec{AO} + \vec{OB} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$$

$$\vec{AD} = \vec{AO} + \vec{OD} = \frac{1}{2}\vec{a} + \frac{1}{2}\vec{b}$$

2.



$$\vec{AD} = \frac{1}{2}\vec{AB} + \frac{1}{2}\vec{AC}$$

$$\vec{BE} = \frac{1}{2}\vec{BC} + \frac{1}{2}\vec{BA}$$

$$\vec{CF} = \frac{1}{2}\vec{CA} + \frac{1}{2}\vec{CB}$$

ADD: $\vec{AB} + \vec{BE} + \vec{CF} =$

$$= \frac{1}{2}\left(\vec{AB} + \vec{BA} + \vec{AC} + \vec{CA} + \vec{BC} + \vec{CB}\right) =$$

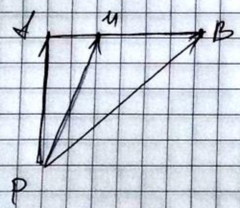
$$= \vec{O} \quad (\text{as } \vec{AB} = -\vec{BA} \text{ etc.})$$

3. $\vec{BO} = \vec{BA} + \vec{AO} = -\vec{a} + \frac{2}{3}\vec{AD} =$

$$= -\vec{a} + \frac{2}{3}\cdot\frac{1}{2}\left(\vec{AB} + \vec{AC}\right) =$$

$$= -\vec{a} + \frac{1}{3}\vec{a} + \frac{1}{3}\vec{b} = \frac{1}{3}\vec{b} - \frac{2}{3}\vec{a}.$$

4.



(a) $\vec{PM} = \vec{PA} + \vec{AM} = \vec{PA} + \frac{2}{5}\vec{AB} =$

$$= \vec{PA} + \frac{2}{5}\left(\vec{PB} - \vec{PA}\right) =$$

$$= \frac{3}{5}\vec{p} + \frac{2}{5}\vec{q}$$

(b) $\vec{PM} = \vec{PA} + \vec{AM} = \vec{PA} + \frac{a}{a+b}\vec{AB} =$

$$= \vec{p} + \frac{a}{a+b}\left(\vec{q} - \vec{p}\right) =$$

$$= \frac{b}{a+b}\vec{p} + \frac{a}{a+b}\vec{q}$$