The Determinant of a Matrix

For every square matrix a number called **determinant** can be calculated.

If the matrix $\mathbf{A} = \begin{pmatrix} a_{11}\ a_{12}\ \ldots a_{1n} \\ a_{21}\ a_{22}\ \ldots a_{2n} \\ \ldots\ldots \\ a_{n1}\ a_{n2}\ \ldots a_{nn} \end{pmatrix}$ then its determinant is written $\det \mathbf{A} = \begin{vmatrix} a_{11}\ a_{12}\ \ldots a_{1n} \\ a_{21}\ a_{22}\ \ldots a_{2n} \\ \ldots\ldots \\ a_{n1}\ a_{n2}\ \ldots a_{nn} \end{vmatrix}$ or |A|.

First, Second and Third Order Determinants

Definition 1. The **determinant of a 1×1 matrix** is the value of its only element.

Definition 2. The **determinant of a 2×2 matrix** $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det \mathbf{A} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

that is, the difference of the products on the main and secondary diagonal.

For example, $\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 3\cdot 4 - 2\cdot 5 = 12 - 10 = 2$; $\begin{vmatrix} 8 & 1 \\ -5 & 0 \end{vmatrix} = 0 - (-5) = 5$.

Exercise 2.A

1. Find det**A** given that the matrix **A** equals (a) $\begin{pmatrix} 3 & 7 \\ 2 & 4 \end{pmatrix}$ (b) $\begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$ (c) $O_{2\times 2}$ (d) I_2

(e) $\begin{pmatrix} 3 & -2 \\ 7 & 4 \end{pmatrix}$ (f) $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ (g) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (h) $\begin{pmatrix} a & -a \\ 1 & a \end{pmatrix}$

2. Find (a) $\begin{vmatrix} 51 & 47 \\ 52 & 48 \end{vmatrix}$ (b) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix}$ (c) $\begin{vmatrix} 1-\sqrt{2} & 2+\sqrt{5} \\ 2-\sqrt{5} & 1+\sqrt{2} \end{vmatrix}$

3. Solve the equations (a) $\begin{vmatrix} x-1 & x \\ x-1 & 5 \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1-x & -1 \\ 1 & 3-x \end{vmatrix} = 0$ (c) $\begin{vmatrix} x^2 & 3x \\ 3 & x \end{vmatrix} = 0$

4. Given that $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$, find (a) |**A**| (b) |2**A**| (c) |-**A**| (d) |-3**B**| (e) |**AB**|

5. Compare the determinants $\begin{vmatrix} 3 & 1 \\ -4 & 7 \end{vmatrix}$, $\begin{vmatrix} 3x & 1 \\ -4x & 7 \end{vmatrix}$, $\begin{vmatrix} 3 & 1 \\ -4x & 7x \end{vmatrix}$ and $\begin{vmatrix} 3x & x \\ -4x & 7x \end{vmatrix}$.

6. Let **A** = $\begin{pmatrix} 3 & 2 \\ k & 4 \end{pmatrix}$ and **B** = $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Find, in terms of *k*, (a) 2**A**− **B** (b) det(2**A**− **B**)

Definition 3. The **determinant of a 3×3 square matrix** is defined as follows:

$$\det \mathbf{A} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \cdot \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \cdot \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_3b_1c_2 - a_2b_1c_3 + a_2b_3c_1 - a_3b_2c_1.$$

Definition 4. A ***minor*** M_{ij} is a determinant of order (*n–1*) which is obtained by deleting the *i*-th row and *j*-th column of the *n*th order determinant or matrix.

For example, to find the minor M_{23}, delete Row 2 and Column 3: $M_{23} = \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$

The above determinant then can be written as $\det \mathbf{A} = a_1M_{11} - b_1M_{12} + c_1M_{13}$; this expression is called the ***expansion* of determinant *by the 1st row***.

Method 1.

Notice that the second term $-b_1M_{12}$ has the ***opposite*** sign. The determinant 3x3 can be found by expanding by **any row** or **any column** in a similar way, using the elements of that row or columns multiplied by the corresponding minors; the sign diagram for each product is $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$.

For example, the expansion by 2nd row

$$\det \mathbf{A} = -a_2 \cdot M_{21} + b_2 \cdot M_{22} - c_2 \cdot M_{23} = -a_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \cdot \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \cdot \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} =$$

$$= -a_2b_1c_3 + a_2b_3c_1 + a_1b_2c_3 - a_3b_2c_1 - a_1b_3c_2 + a_3b_1c_2 \ ,$$

or by 3rd column: $\det \mathbf{A} = c_1 \cdot M_{13} - c_2 \cdot M_{23} + c_3 \cdot M_{33}$, giving the same result again.

Method 2.

1. Copy the 1st and 2nd column again to the right of the determinant.

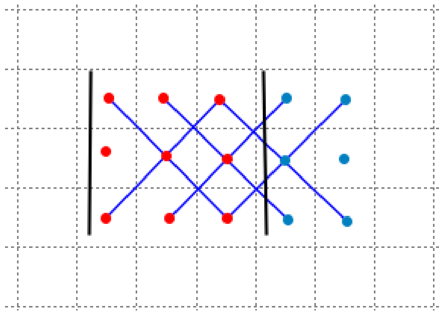
2.Find the three <u>products</u> on the diagonal lines 'top left – bottom right' (red) and **add** them .

3. Then find the three <u>products</u> on the diagonal lines 'bottom left –top right' (green) and ***subtract*** them from the first sum.

Example.

$$\begin{vmatrix} 2 & 4 & 1 \\ -4 & 0 & -2 \\ 3 & 5 & -1 \end{vmatrix} \begin{matrix} 2 & 4 \\ -4 & 0 \\ 3 & 5 \end{matrix} = \underline{2\cdot 0\cdot(-1) + 4\cdot(-2)\cdot 3 + 1\cdot(-4)\cdot 5} -$$

$$\underline{-3\cdot 0\cdot 1 - 5\cdot(-2)\cdot 3 - (-1)\cdot(-4)\cdot 4} = -40$$



Exercise 2.B

1. Given the determinant $\begin{vmatrix} 2 & -4 & 1 \\ 1 & -3 & 3 \\ -1 & 2 & 0 \end{vmatrix}$, expand and compute by (a) 1st row, (b) 2nd column (c) 3rd column.

2. Find the determinants:

17. $\begin{vmatrix} 2 & -3 & 1 \\ 2 & 0 & 2 \\ 3 & -2 & 4 \end{vmatrix}$ **18.** $\begin{vmatrix} 3 & 1 & -2 \\ 2 & -5 & 4 \\ 3 & 2 & 1 \end{vmatrix}$ **19.** $\begin{vmatrix} -2 & 3 & 2 \\ 1 & 2 & -3 \\ -4 & -2 & 1 \end{vmatrix}$ **20.** $\begin{vmatrix} 3 & -2 & 0 \\ 2 & -3 & 2 \\ 8 & -2 & 5 \end{vmatrix}$

21. $\begin{vmatrix} 2 & -3 & 10 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{vmatrix}$ **22.** $\begin{vmatrix} 6 & 0 & 0 \\ 2 & -3 & 0 \\ 7 & -8 & 2 \end{vmatrix}$ **23.** $\begin{vmatrix} 0 & -2 & 4 \\ 1 & 0 & -7 \\ 5 & -6 & 0 \end{vmatrix}$ **24.** $\begin{vmatrix} 5 & -8 & 0 \\ 2 & 0 & -7 \\ 0 & -2 & -1 \end{vmatrix}$

27. $\begin{vmatrix} 1 & 5 & 0 \\ -2 & 4 & 0 \\ 3 & 2 & 5 \end{vmatrix}$ 28. $\begin{vmatrix} 2 & 1 & -1 \\ 6 & 0 & 3 \\ -2 & -1 & 2 \end{vmatrix}$ 29. $\begin{vmatrix} 3 & 2 & 4 \\ -4 & 7 & 5 \\ -2 & 3 & -3 \end{vmatrix}$ 30. $\begin{vmatrix} 12 & 2 & 3 \\ 2 & -2 & -1 \\ -5 & -1 & 1 \end{vmatrix}$

3. For what value of m is det**C** = 0, given that $\mathbf{C} = \begin{pmatrix} 2 & 1 & 0 \\ m & 3 & 4 \\ m-2 & 2 & -2 \end{pmatrix}$?

4. Let $\mathbf{A} = \begin{pmatrix} 2 & 1 & 4 \\ -1 & 3 & -1 \\ 2 & 0 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 2 & -1 \\ 3 & 0 & -2 \end{pmatrix}$. Verify that (a) det (2**B**)=8 det**B** (b) det(**A**·**B**)=det**A**·det**B**