Scalar Product of Two Vectors

Definition 11. The scalar product (also known as the dot product) of two vectors $\vec{a} = \langle a_x, a_y, a_z \rangle$ and $\vec{b} = \langle b_x, b_y, b_z \rangle$ is defined as the product of two matrices (if the second vector is written as a row matrix):

$$\boxed{\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z}$$

Note that the **scalar product is a number**, not a vector.

Properties of Scalar Product

1. The scalar product equals the **product of the two magnitudes and the cosine of the angle between the vectors**: $\boxed{\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\alpha}$ where $\alpha = \measuredangle(\vec{a}, \vec{b})$.

Note that to determine the angle correctly, both vectors must have the same initial point.

2. Commutative property: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.

3. Associative property: $k\left(\vec{a} \cdot \vec{b}\right) = \left(k\vec{a}\right) \cdot \vec{b} = \vec{a} \cdot \left(k\vec{b}\right)$

4. Distributive property: $\left(\vec{a} + \vec{b}\right) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

5. The **scalar product of two nonzero vectors equals 0 if and only if the vectors are perpendicular**:

For $\vec{a} \neq \vec{0}, \vec{b} \neq \vec{0}$, $\boxed{\vec{a} \cdot \vec{b} = 0 \Leftrightarrow \vec{a} \perp \vec{b}}$

Applications of Scalar Product

1. To find the **angle between two vectors**: $\cos\alpha = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{a}| \cdot |\vec{b}|}$.

2. To find the **magnitude of a vector**: $\vec{a} \cdot \vec{a} = a_x^{\ 2} + a_y^{\ 2} + a_z^{\ 2} = |\vec{a}|^2$ therefore $\vec{a}^{\,2} = |\vec{a}|^2$ and $|\vec{a}| = \sqrt{\vec{a}^{\,2}}$.

3. To find the projection of a vector: $proj_{\vec{b}}\,\vec{a} = \dfrac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \vec{a} \cdot \overrightarrow{b^0}$.

4. To find mechanical work. The **mechanical work** done by the force $\vec{F}$ while moving a mass particle along a straight line by the vector $\vec{s}$ equals $A = \vec{F} \cdot \vec{s}$.

Example. Given A(-1, 3, 0), B(2, 2, 1), C(-4, 0, *k*), use the scalar product to find the value of *k* for which $\measuredangle A$ is right. Verify by the Pythagorean Theorem. For this value of *k*, use the scalar product to find $\measuredangle B$ to a degree.

The vectors $\overrightarrow{AB} = \begin{pmatrix} 2-(-1) \\ 2-3 \\ 1-0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$; $\overrightarrow{AC} = \begin{pmatrix} -4-(-1) \\ 0-3 \\ k-0 \end{pmatrix} \begin{pmatrix} -3 \\ -3 \\ k \end{pmatrix}$; ;

$$\overrightarrow{AB} \perp \overrightarrow{AC} \Rightarrow \overrightarrow{AB} \cdot \overrightarrow{AC} = 3(-3) - 1(-3) + 1 \cdot k = 0 \Rightarrow k = 6;\ \overrightarrow{BC} = \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix}.$$

To verify: $\left|\overrightarrow{AB}\right|^2 = 3^2 + 1^2 + 1^2 = 11; \left|\overrightarrow{AC}\right|^2 = 3^2 + 3^2 + 6^2 = 54; \left|\overrightarrow{BC}\right|^2 = 6^2 + 2^2 + 5^2 = 65; \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = \left|\overrightarrow{BC}\right|^2$.

$$\cos\measuredangle B=\frac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\cdot\left|\overrightarrow{BC}\right|}=\frac{-3\cdot(-6)+1\cdot 2-1\cdot 5}{\sqrt{11}\cdot\sqrt{65}}=\frac{15}{\sqrt{11}\cdot\sqrt{65}}\approx 0.56097;\ \measuredangle B\approx 56^\circ.$$

Exercises

1. Use the scalar product to show that in each of the following the three given points are the vertices of a right–angled triangle:
   a) $(5;-1);(-2;4);(3;11)$ b) $(1;2;-3);(-3;4;-2);(2;-1;7)$
2. Find the angles of the triangle whose vertices are:
   (a) (3, 1, 1), (4, –1, 3), (5, 3, 2) ;
   (b) (2, –1, 3), (3, 2, 4), (6, –4, –1) ;
   (c) (1, 2, –2), (3, 3, –1), (–1, 1, –1) ;
   (d) (2, 4, 1), (0, 2, 2), (3, 5, 1).
3. Determine whether the vectors $\vec{a}=6\vec{i}+2\vec{j}+9\vec{k}$ and $\vec{b}=7\vec{i}+6\vec{j}-6\vec{k}$ can be the sides of a square.
4. Find the shape of the quadrilateral whose vertices are:
   (a) (3, 4, 1), (5, 0, –1), (1, –1, 6), (–1, 3, 8) ;
   (b) (2, 3, –1), (12, 14, 1), (2, 4, –4), (–8, –7, –6) ;
   (c) (–1, 2, –2), (3, –2, 1), (5, 3, 5), (1, 7, 2) ;
   (d) (2, –1, –1), (4, 1, 0), (3, 3, –2), (1, 1, –3) ;
   (e) (5, 1, 2), (3, –1, 5), (9, 0, 4), (7, –2, 7).
5. Find the mechanical work done by force $\vec{F}=2\vec{i}-4\vec{j}+5\vec{k}$ while moving a particle in staright line from the point A(3, -1, 2) to the point B(5, -2, 3).
6. Given the points A(-1, 2, 1); B(2, 6, -3); C(0, 2, 2); D(4, -2, 0), find the projection of $\overrightarrow{AB}$ on $\overrightarrow{CD}$.
7. Given that $|\vec{a}|=4$, $|\vec{b}|=2$, and $\measuredangle(\vec{a},\vec{b})=120^\circ$, find

   (a) $(\vec{a}-3\vec{b})\cdot(2\vec{a}+\vec{b})$ (b) $|2\vec{a}+\vec{b}|$ (c) $\cos\alpha$ where $\alpha=\measuredangle(\vec{a},2\vec{a}+\vec{b})$

ANSWERS

1. a) $A(5;-1); B(-2;4); C(3;11);$ $\overrightarrow{AB}=\begin{pmatrix}-7\\5\end{pmatrix}; \overrightarrow{BC}=\begin{pmatrix}5\\7\end{pmatrix}; \overrightarrow{AB}\cdot\overrightarrow{BC}=0\Rightarrow \overrightarrow{AB}\perp\overrightarrow{BC}.$

b) $A(1;2;-3); B(-3;4;-2); C(2;-1;7).$ $\overrightarrow{AB}=\begin{pmatrix}-4\\2\\1\end{pmatrix}; \overrightarrow{AC}=\begin{pmatrix}1\\-3\\10\end{pmatrix}; \overrightarrow{AB}\cdot\overrightarrow{AC}=0\Rightarrow \overrightarrow{AB}\perp\overrightarrow{AC}.$

2. (a) 90°; 45°;45°. (b) $\approx 115°;\approx 44°;\approx 21°$. (c) $\approx 132°;\approx 24°;\approx 24°$.(d) $\approx 161°;\approx 6°;\approx 13°$.

3. $\vec{a}\cdot\vec{b}=0\Rightarrow \vec{a}\perp\vec{b}$; $|\vec{a}|=|\vec{b}|=11$ units; therefore, yes.

4. (a) parallelogram; (b) rhombus; (c) rectangle; (d) square; (e) rectangle.

5. $\vec{s}=\overrightarrow{AB}=2\vec{i}-\vec{j}+\vec{k}$; $A=\vec{F}\cdot\vec{s}=13$ force units.

6. $\overrightarrow{AB}=\langle 3,4,-4\rangle; \overrightarrow{CD}=\langle 4,-4,-2\rangle; proj_{\overrightarrow{CD}}\overrightarrow{AB}=\dfrac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{CD}|}=\dfrac{4}{6}=\dfrac{2}{3}.$

7. (a) $(\vec{a}-3\vec{b})\cdot(2\vec{a}+\vec{b})=2\vec{a}^2-6\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}-3\vec{b}^2=2\cdot16-5\cdot(-4)-3\cdot4=40$

(b) $|2\vec{a}+\vec{b}|^2=(2\vec{a}+\vec{b})^2=4\vec{a}^2+4\vec{a}\cdot\vec{b}+\vec{b}^2=4\cdot16+4\cdot(-4)+4=52; |2\vec{a}+\vec{b}|=\sqrt{52}=2\sqrt{13}.$

(c) Let $2\vec{a}+\vec{b}=\vec{c}$; $\vec{a}\cdot\vec{c}=\vec{a}\cdot(2\vec{a}+\vec{b})=(2\vec{a}^2+\vec{a}\cdot\vec{b})=32-4=28; \cos\alpha=\dfrac{\vec{a}\cdot\vec{c}}{|\vec{a}|\cdot|\vec{c}|}=\dfrac{28}{4\cdot2\sqrt{13}}=\dfrac{7\sqrt{13}}{26}$