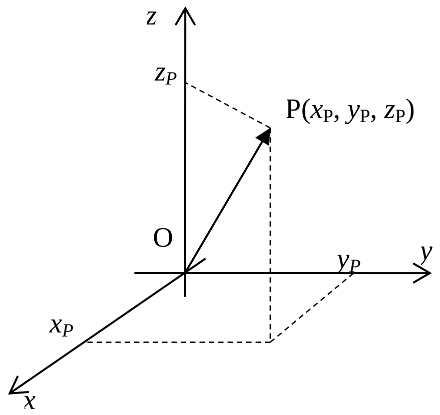
Vector Coordinates in Space



A vector can be expressed in terms of ***coordinates*** *of the vector* – horizontal, vertical and *z*-axis displacements - with respect to the standard unit vectors (representing unit displacements on the axis). The coordinates of a vector are, in fact, the **coordinates of its terminal point (endpoint) if the initial (starting) point is the origin.**

Vector coordinates are often represented as a ***column matrix*** $\vec{a}=\begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$,

or as a row matrix $\vec{a}=\langle a_x, a_y, a_z\rangle$.

The coordinates of the z***ero vector*** are $\langle 0,0,0\rangle$.

If the initial point A(x_1, y_1, z_1) and terminal point B(x_2, y_2, z_2) of the vector are given then to obtain the coordinates of the vector the coordinates of the initial point are subtracted from the corresponding coordinates of the terminal point: $\overrightarrow{AB}=\langle x_2-x_1, y_2-y_1, z_2-z_1\rangle$

If the unit vectors on the *Ox*, *Oy* and *Oz* axes are denoted $\vec{i}$, $\vec{j}$ and $\vec{k}$ respectively, we have $\vec{i}=\langle 1,0,0\rangle$, $\vec{j}=\langle 0,1,0\rangle$ and $\vec{k}=\langle 0,0,1\rangle$. Then the vector can be written using its components as $\vec{a}=a_x\vec{i}+a_y\vec{j}+a_z\vec{k}$.

Magnitude (length) of vector is found as $|\vec{a}|=\sqrt{a_x^{\;2}+a_y^{\;2}+a_z^{\;2}}$.

Note also that the coordinates of the vector $\vec{a}$ are projections of this vector on the corresponding axes. Therefore, if α, β, γ are the angles between the vector and the positive directions of the axes *Ox*, *Oy* and *Oz*, the projections of $\vec{a}$ on the axes (as seen previously) are $a_x=|\vec{a}|\cos\alpha$, $a_y=|\vec{a}|\cos\beta$ and $a_z=|\vec{a}|\cos\gamma$.

The cosines $\cos\alpha=\dfrac{a_x}{|\vec{a}|}$, $\cos\beta=\dfrac{a_y}{|\vec{a}|}$, $\cos\gamma=\dfrac{a_z}{|\vec{a}|}$ are called the ***direction cosines*** of the vector $\vec{a}$. They determine the direction of the vector and are the coordinates of the unit vector $\vec{a^0}$ in the direction of $\vec{a}$:
$\vec{a^0}=\langle\cos\alpha, \cos\beta, \cos\gamma\rangle$. Note that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$ (as the magnitude $|\vec{a^0}|=1$).

Position vectors can be used ***to define points*** by representing their displacement from the origin: $\overrightarrow{OA}=\vec{a}$, where O is the origin of coordinates and A is the given point.

Vector Operations Using Coordinates

1. Scalar multiples: To multiply a vector by a scalar, all the components are multiplied by that scalar:

$k\vec{a}=\begin{pmatrix} ka_x \\ ka_y \\ ka_z \end{pmatrix}$; the opposite vector of $\vec{a}$ is $-\vec{a}=\begin{pmatrix} -a_x \\ -a_y \\ -a_z \end{pmatrix}$.

Two vectors are collinear if their corresponding coordinates are proportional: $\vec{a}\,||\,\vec{b} \Leftrightarrow \dfrac{a_x}{b_x}=\dfrac{a_y}{b_y}=\dfrac{a_z}{b_z}$.

2. The sum and difference of two vectors: To add or subtract two vectors, their corresponding components are added/subtracted: $\vec{a}+\vec{b}=\begin{pmatrix} a_x+b_x \\ a_y+b_y \\ a_z+b_z \end{pmatrix}$ and $\vec{a}-\vec{b}=\begin{pmatrix} a_x-b_x \\ a_y-b_y \\ a_z-b_z \end{pmatrix}$.

Linear Dependence of Vectors

Definition 8. The sum $c_1\overrightarrow{a_1}+c_2\overrightarrow{a_2}+...+c_n\overrightarrow{a_n}$ is called a ***linear combination*** of the vector system $\overrightarrow{a_1},\overrightarrow{a_2},...,\overrightarrow{a_n}$.

Definition 9. The system $\overrightarrow{a_1},\overrightarrow{a_2},...,\overrightarrow{a_n}$ is called ***linearly dependent*** if $c_1\overrightarrow{a_1}+c_2\overrightarrow{a_2}+...+c_n\overrightarrow{a_n}=\vec{0}$ for some ***nonzero coefficients*** $c_1,c_2,...,c_n$ (that is, at least one coefficient is not 0).

Definition 10. The system $\overrightarrow{a_1},\overrightarrow{a_2},...,\overrightarrow{a_n}$ is called ***linearly independent*** if $c_1\overrightarrow{a_1}+c_2\overrightarrow{a_2}+...+c_n\overrightarrow{a_n}=\vec{0}$ only when all coefficients $c_1=c_2=...=c_n=0$.

Two non-collinear vectors are linearly independent. Such vectors form a **base** in the plane, that is, **any vector in the same plane can be expressed as a linear combination of the base vectors.**

Three vectors are linearly dependent if and only if they are **coplanar**. Three non-coplanar vectors form a base in the 3D space.

Three vectors $\vec{a},\vec{b},\vec{c}$ are linearly dependent if and only if the determinant $\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=0$.

Exercises

1. Given the points $A(4,-2,5)$ and $B(1,4,3)$, find the coordinates, magnitude and the unit vector in the direction of the vector of the vector $\overrightarrow{AB}$.
2. Given $\vec{a}=2\vec{i}+3\vec{j}-6\vec{k}$ and $\vec{b}=4\vec{j}+3\vec{k}$, find
   (a) $\vec{a}+\vec{b}$; (b) $\vec{a}-\vec{b}$; (c) $2\vec{a}-3\vec{b}$; (d) $|\vec{a}|$; (e) $|\vec{b}|$; (f) $|\vec{a}+\vec{b}|$.
3. Find the vector $\vec{a}$ given that it is antiparallel to the vector $\vec{b}=3\vec{i}-6\vec{j}-6\vec{k}$ and its magnitude is 3.
4. The vector $\vec{a}$ has angle 60° with the *Ox* axis and 120° with the *Oy* axis, and its magnitude $|\vec{a}|=4$. Find the coordinates of $\vec{a}$.
5. For what values of *a* and *b* are the vectors $\vec{c}=\langle 6,a,-9\rangle$ and $\vec{d}=\langle 4,10,b\rangle$ collinear?
6. Prove that the points A(1, 1, 0), B(5, -5, 2), C(-1, 4, -1) and D(13, -17, 6) are collinear (that is, they lie on the same line).
7. *ABCD* is a parallelogram. Find the coordinates of the vertex *D* and the intersection point of the diagonals *M* given that $A(2,-3,4)$, $B(3,1,-1)$ and $C(2,-1,2)$.
8. Decide whether the vectors $\vec{a},\vec{b},\vec{c}$ are linearly dependent or independent. If they are dependent, express $\vec{c}$ as a linear combination of $\vec{a}$ and $\vec{b}$.
   (a) $\vec{a}=5\vec{i}+2\vec{j}+\vec{k},\quad \vec{b}=\vec{i}-\vec{j}+2\vec{k},\quad \vec{c}=\vec{j}-3\vec{k}$
   (b) $\vec{a}=3\vec{i}-\vec{j}+2\vec{k},\quad \vec{b}=2\vec{i}-\vec{k},\quad \vec{c}=\vec{i}-3\vec{j}+10\vec{k}$

ANSWERS

1. $\overrightarrow{AB} = \langle -3, 6, -2 \rangle$; $|\overrightarrow{AB}| = 7$; $\overrightarrow{AB^0} = \left\langle -\frac{3}{7}, \frac{6}{7}, -\frac{2}{7} \right\rangle$

2.
   (a) $\vec{a} + \vec{b} = 2\vec{i} + 7\vec{j} - 3\vec{k}$; (b) $\vec{a} - \vec{b} = 2\vec{i} - \vec{j} - 9\vec{k}$; (c) $2\vec{a} - 3\vec{b} = 4\vec{i} - 6\vec{j} - 21\vec{k}$;

   (d) $|\vec{a}| = 7$; (e) $|\vec{b}| = 5$; (f) $|\vec{a} + \vec{b}| = \sqrt{62}$.

3. $\vec{a} = -\vec{i} + 2\vec{j} + 2\vec{k}$

4. $\cos^2 \gamma = 1 - \cos^2 60° - \cos^2 120° = \frac{1}{2}$; $\cos \gamma = \pm \frac{\sqrt{2}}{2}$

   $a_x = 4\cos 60° = 2$; $a_y = 4\cos 120° = -2$; $a_y = 4\cos \gamma = \pm 2\sqrt{2}$; $\vec{a} = \langle 2, -2, \pm 2\sqrt{2} \rangle$

5. $\frac{6}{4} = \frac{a}{10} = \frac{-9}{b}$; $a = 15$, $b = -6$

6. $\overrightarrow{AB} = \langle 4, -6, 2 \rangle$, $\overrightarrow{BC} = \langle -6, 9, -3 \rangle$, $\overrightarrow{CD} = \langle 14, -21, 7 \rangle$; all vectors are collinear.

7. $\overrightarrow{AB} = \langle 1, 4, -5 \rangle = \overrightarrow{DC}$; then $D(1, -5, 7)$.

   $\overrightarrow{AM} = \frac{1}{2}\overrightarrow{AC} = \langle 0, 1, -1 \rangle$ and $M\ (2, -2, 3)$

8.
   (a) $\begin{vmatrix} 5 & 2 & 1 \\ 1 & -1 & 2 \\ 0 & 1 & -3 \end{vmatrix} = 12 \neq 0$; independent

   (b) $\begin{vmatrix} 3 & -1 & 2 \\ 2 & 0 & -1 \\ 1 & -3 & 10 \end{vmatrix} = 0$; dependent; $\vec{c} = 3\vec{a} - 4\vec{b}$.