Solving Linear Systems

Definition 9. A ***linear system of m equations in n variables*** is defined as $\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{cases}$

The ***solution*** of the system is an ***ordered set of values*** $(x_1, x_2, \ldots, x_n)$ that satisfy all equations.

The linear system can be rewritten in the matrix form **AX=B** where

$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & . \; . \; . & & \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$ is system coefficient matrix, $\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ \\ x_n \end{pmatrix}$ is matrix of unknown variables and $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$ is the matrix of free coefficients.

Definition 10. A linear system is called ***consistent*** if it has at least one solution, and ***inconsistent*** (contradictory) if it has no solutions.

1. Cramer's Rule

Cramer's method can be used for systems where ***m* = *n***.

If $\det\mathbf{A} \neq \mathbf{0}$, a **unique solution** for the ***n*×*n* system** $\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$, exists and can be found by

$x_i = \dfrac{\Delta_{x_i}}{\Delta}$, where $\Delta = \det \mathbf{A}$ and Δ_{x_i} are determinants obtained from det**A** by **replacing the column of the coefficients for the corresponding variable by the free coefficients**; $i = 1, 2, 3, \ldots, n$.

For example, the solution of a **3×3 system** $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$ can be found by $x = \dfrac{\Delta_x}{\Delta}$; $y = \dfrac{\Delta_y}{\Delta}$; $z = \dfrac{\Delta_z}{\Delta}$,

where $\Delta = \det \mathbf{A} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}$, $\Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$, if $\det\mathbf{A} \neq \mathbf{0}$.

Example 1. To solve: $\begin{cases} x - 2y - z = -8 \\ 2x + y + z = 2 \\ 3y - 5z = 4 \end{cases}$.

Find the system determinant $\det \mathbf{A} = \begin{vmatrix} 1 & -2 & -1 \\ 2 & 1 & 1 \\ 0 & 3 & -5 \end{vmatrix} = -34 \neq 0$; therefore the system has a unique solution and the determinants $\Delta_x = \begin{vmatrix} -8 & -2 & -1 \\ 2 & 1 & 1 \\ 4 & 3 & -5 \end{vmatrix} = 34$; $\Delta_y = \begin{vmatrix} 1 & -8 & -1 \\ 2 & 2 & 1 \\ 0 & 4 & -5 \end{vmatrix} = -102$, $\Delta_z = \begin{vmatrix} 1 & -2 & -8 \\ 2 & 1 & 2 \\ 0 & 3 & 4 \end{vmatrix} = -34$

(these are obtained by putting the free coefficient column instead of the respective column in det **A**).

Then $x = \dfrac{\Delta_x}{\det A} = \dfrac{34}{-34} = -1$, $y = \dfrac{\Delta_y}{\det A} = \dfrac{-102}{-34} = 3$, and $z = \dfrac{\Delta_z}{\det A} = \dfrac{-34}{-34} = 1$.

Exercise 5A.

1. $\begin{cases} 2x-7y=-5 \\ -x+5y=1 \end{cases}$

2. $\begin{cases} 12x-7y=8 \\ 8x-5y=5 \end{cases}$

3. $\begin{cases} 3x+4y+2z=8 \\ x+5y+2z=5 \\ 2x+3y+4z=3 \end{cases}$

4. $\begin{cases} 2x-y-3z=3 \\ 3x+4y-5z=-8 \\ 2x-7y=17 \end{cases}$

5. $\begin{cases} -3x+y-z=1 \\ x+4y+4z=-3 \\ 2x-5y-3z=2 \end{cases}$

6. $\begin{cases} x+y+z=2 \\ 2x-6y-z=-1 \\ 3x-2z=8 \end{cases}$

7. $\begin{cases} 2x-3y+z=-3 \\ -4x+3y+2z=-11 \\ x-y-z=3 \end{cases}$

2. The Matrix Method

This method can be used only for systems where ***m* = *n***.

Consider the system in the matrix form **AX=B**. Matrix properties can be used to solve it for **X** as follows: Suppose **A** is **not singular**, that is, if $\det \mathbf{A} \neq 0$; then its inverse matrix $\mathbf{A}^{-1}$ exists.
Both sides of the matrix equation are **multiplied from the left** by the inverse of **A**:

$$\begin{aligned} \mathbf{AX} &= \mathbf{B} \\ \mathbf{A}^{-1}\mathbf{AX} &= \mathbf{A}^{-1}\mathbf{B} \quad ; \text{ remembering that } \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}, \\ \mathbf{IX} &= \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{X} &= \mathbf{A}^{-1}\mathbf{B} \end{aligned}$$

Note that this method produces a solution only if the inverse matrix exists, that is, if $\det \mathbf{A} \neq 0$.

Example 2.

Solve $\begin{cases} 3x+2y+2z=1 \\ x-2y+z=0 \\ -x+y-z=-2 \end{cases}$; then $\mathbf{A}=\begin{pmatrix} 3 & 2 & 2 \\ 1 & -2 & 1 \\ -1 & 1 & -1 \end{pmatrix}$; $\mathbf{B}=\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$; det**A** = 1; the inverse $\mathbf{A}^{-1}=\begin{pmatrix} 1 & 4 & 6 \\ 0 & -1 & -1 \\ -1 & -5 & -8 \end{pmatrix}$.

$$\mathbf{X}=\mathbf{A}^{-1}\mathbf{P}=\begin{pmatrix} 1 & 4 & 6 \\ 0 & -1 & -1 \\ -1 & -5 & -8 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}=\begin{pmatrix} 1\cdot1+4\cdot0+6\cdot(-2) \\ 0\cdot1-1\cdot0-1\cdot(-2) \\ -1\cdot1-5\cdot0-8\cdot(-2) \end{pmatrix}=\begin{pmatrix} -11 \\ 2 \\ 15 \end{pmatrix}; \quad \underline{x=-11; \quad y=2; \quad z=15.}$$

Exercise 5B.

1. Find the inverse matrix for each of the following:
A. $\begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$ B. $\begin{pmatrix} 2 & -5 \\ -4 & 9 \end{pmatrix}$ C. $\begin{pmatrix} 4 & -9 \\ 8 & 7 \end{pmatrix}$
and solve the systems
(a) $\begin{cases} 5x+3y=4 \\ 3x+2y=-1 \end{cases}$ (b) $\begin{cases} 2x-5y=-2 \\ -4x+9y=6 \end{cases}$ (c) $\begin{cases} 4x-9y=50 \\ 8x+7y=24 \end{cases}$
using the inverse matrix method.

2. (i) Find the inverse matrices of (a) $\begin{pmatrix} 1 & -3 & 0 \\ 0 & 3 & 1 \\ 2 & -1 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 0 & -1 \\ 2 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

and **(ii)** use them to solve the systems (a) $\begin{cases} x-3y=2 \\ 3y+z=-2 \\ 2x-y+2z=1 \end{cases}$ (b) $\begin{cases} x-3y=-2 \\ 3y+z=7 \\ 2x-y+2z=7 \end{cases}$ (c) $\begin{cases} x-z=4 \\ 2x-y=8 \\ x+y+z=0 \end{cases}$

3. Solve (a) $\begin{cases} x-y+3z=3 \\ 2x-3y+2z=1 \\ 4x-5y+7z=6 \end{cases}$ (b) $\mathbf{AX}=\mathbf{B}$ where $\mathbf{A}=\begin{pmatrix} 2 & -1 & 2 \\ 1 & -1 & 1 \\ 3 & 2 & 4 \end{pmatrix}, \mathbf{B}=\begin{pmatrix} 16 \\ 9 \\ 21 \end{pmatrix}$

ANSWERS

Exercise 5A

1.
$\det \mathbf{A} = 3;\ \Delta_x = -18,\ \Delta_y = -3,$

$x = -6;\ y = -1$

2.
$\det \mathbf{A} = -4;\ \Delta_x = -5,\ \Delta_y = -4,$

$x = 1.25;\ y = 1$

3.
$\det \mathbf{A} = 28;\ \Delta_x = 56,\ \Delta_y = 28,\ \Delta_z = -28,$

$x = 2;\ y = 1;\ z = -1$

4.
$\det \mathbf{A} = 27;\ \Delta_x = 16,\ \Delta_y = -61,\ \Delta_z = 4,$

$x = \frac{16}{27};\ y = -\frac{61}{27};\ z = \frac{4}{27}$

5.
$\det \mathbf{A} = 0;$ *there is no unique solution*

6.
$\det \mathbf{A} = 31;\ \Delta_x = 62,\ \Delta_y = 31,\ \Delta_z = -31,$

$x = 2;\ y = 1;\ z = -1$

7.
$\det \mathbf{A} = 5;\ \Delta_x = 20,\ \Delta_y = 15,\ \Delta_z = -10,$

$x = 4;\ y = 3;\ z = -2$

Exercise 5B

1.

$$\mathbf{A}^{-1} = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix} \quad \mathbf{B}^{-1} = \begin{pmatrix} 9 & 5 \\ 4 & 2 \end{pmatrix} \quad \mathbf{C}^{-1} = \frac{1}{100}\begin{pmatrix} 7 & 9 \\ -8 & 4 \end{pmatrix}$$

(a) $x = 11, y = -17$ (b) $x = 12, y = 4$ (c) $x = 5.66, y = -3.04$

2.

(i) (a) $\begin{pmatrix} 7 & 6 & -3 \\ 2 & 2 & -1 \\ -6 & -5 & 3 \end{pmatrix}$ (b) $-\frac{1}{4}\begin{pmatrix} -1 & -1 & -1 \\ -2 & 2 & -2 \\ 3 & -1 & -1 \end{pmatrix}$

(ii) (a) (-1, -1, 1) (b) (7, 3, -2) (c) (3, -2, -1)

3.

(a) $\mathbf{A}^{-1} = \begin{pmatrix} -11 & -8 & 7 \\ -6 & -5 & 4 \\ 2 & 1 & -1 \end{pmatrix};\ \begin{cases} x = 1 \\ y = 1 \\ z = 1 \end{cases}$ (b) $\mathbf{A}^{-1} = \begin{pmatrix} 6 & -8 & -1 \\ 1 & -2 & 0 \\ -5 & 7 & 1 \end{pmatrix};\ \begin{cases} x = 3 \\ y = -2 \\ z = 4 \end{cases}$