The Determinant: General Definition.

Definition 5. The ***cofactor*** C_{ij} of the element a_{ij} of a square matrix is given by $C_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of a_{ij}.

$$C_{ij} = \begin{cases} M_{ij} & \text{if } i+j = \text{even integer} \\ -M_{ij} & \text{if } i+j = \text{odd integer} \end{cases}$$

A simpler way of memorising the cofactor signs is the principle of the chessboard:

$$\begin{vmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \dots & & & & \end{vmatrix}$$

Definition 6. The nth order ***determinant*** ($n \geq 2$) is **the sum of the products of all elements of any row or column and their cofactors**. That is, by the rth row: $\det \mathbf{A} = a_{r1}C_{r1} + a_{r2}C_{r2} + a_{r3}C_{r3} + \dots + a_{rn}C_{rn}$ and by the cth column, $\det \mathbf{A} = a_{1c}C_{1c} + a_{2c}C_{2c} + a_{3c}C_{3c} + \dots + a_{nc}C_{nc}$.

In this manner, the determinant of order ***n*** is defined ***recursively*** – that means, using previously defined determinants of order ***n-1***.

Properties Of Determinants.

Effects of Elementary Row/Column Operations on the Determinant

If **A** is a square matrix of order n, then :

- Interchanging any two rows/columns of **A** ***changes the sign*** of det**A**.
- Multiplying a row/column of **A** by a constant k ***multiplies*** det**A by *k*;** that is, a **common factor of all elements in a row or a column can be taken in front of the determinant.**
- Adding a multiple of a row/column of **A** to another row/column ***does not change*** the value of det**A.**

Conditions for a zero determinant. Sometimes it is convenient to notice that the determinant equals zero, without having to evaluate it.

- A row (column) consists only of zeros.
- Two rows (columns) are identical.
- One row (column) is a constant multiple of another row (column) (for example, $C_3 = -2C_1$).
- One row (column) is a linear combination of two or more other rows (columns) (that is, $R_k = aR_i + bR_j$).

More Properties

1. $\det \mathbf{A}\cdot\mathbf{B} = \det \mathbf{A}\cdot\det \mathbf{B}$ if **A** and **B** are square matrices of the same order.
2. $\det k\mathbf{A} = k^n \det \mathbf{A}$ where **A** is the n-th order square matrix.
3. If all elements of a row (column) of a determinant are written as sums of two numbers then the determinant equals the sum of two determinants where only one of the terms is taken in each element of that row (column) as follows:

$$\begin{vmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{31} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{31} \end{vmatrix} + \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ a_{21} & a_{22} & a_{21} \\ a_{31} & a_{32} & a_{31} \end{vmatrix}.$$

Determinant properties can often be used to simplify the calculations.

Exercise 3.A

1. Explain why each of the following determinants equals zero.

(a) $\begin{vmatrix} x & 3x \\ -7 & -21 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 4 & 1 \\ 0 & 0 & 0 \\ 3 & 5 & -1 \end{vmatrix}$ (c) $\begin{vmatrix} 1 & -2 & 3 \\ 3 & 1 & -2 \\ 1 & -2 & 3 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & 3 & -4 \\ 0 & -2 & 0 \\ 3 & 5 & -6 \end{vmatrix}$ (e) $\begin{vmatrix} 2 & 3 & 5 \\ -1 & 2 & 0 \\ 1 & 5 & 5 \end{vmatrix}$ (f) $\begin{vmatrix} 2 & -4 & 3 & 6 \\ 3 & 6 & 1 & -9 \\ 0 & 2 & -1 & -3 \\ -4 & 0 & 3 & 0 \end{vmatrix}$ (g) $\begin{vmatrix} 1 & 0 & -2 & 2 \\ -1 & 3 & 0 & 5 \\ -1 & 9 & -4 & 19 \\ 5 & 6 & -3 & 0 \end{vmatrix}$

2. Evaluate (a) $\begin{vmatrix} 2 & 3 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 5 \end{vmatrix}$ (b) $\begin{vmatrix} -1 & 1 & 0 & -2 \\ 0 & 3 & -1 & 1 \\ 1 & 0 & 0 & -2 \\ -2 & 4 & -3 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} -2 & 10 & 6 & 5 \\ 0 & -4 & 7 & -3 \\ 0 & 0 & 1 & 21 \\ 0 & 0 & 0 & 3 \end{vmatrix}$

3. The matrix $\mathbf{M} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ -1 & 1 & 0 \end{pmatrix}$. Evaluate $|\mathbf{M}|$ and **hence** find (a) $|-2\mathbf{M}|$ (b) $\begin{vmatrix} 3 & 6 & -3 \\ 0 & 1 & -2 \\ -1 & 1 & 0 \end{vmatrix}$ (c) $\begin{vmatrix} 4 & 8 & -4 \\ 0 & 4 & -8 \\ -4 & 4 & 0 \end{vmatrix}$.

Determinant of a Triangular Matrix

Definition 7. A square matrix **A** is said to be ***triangular*** if all the elements below the main diagonal (or above it) equal 0: $\mathbf{A}=\begin{pmatrix}a_{11}&a_{12}&a_{13}&\dots&a_{1n}\\0&a_{22}&a_{23}&\dots&a_{2n}\\0&0&a_{33}&\dots&a_{3n}\\\dots&\dots&\dots&&\\0&0&0&\dots&a_{nn}\end{pmatrix}$.

The determinant of a triangular matrix is the ***product of elements on the main diagonal***:
$\det\mathbf{A}=a_{11}\cdot a_{22}\cdot a_{33}\cdot\ldots\cdot a_{nn}$.

This is the best method for finding determinants of higher order, e.g. when n=4 or n=5. To change a determinant into triangular form, elementary row (column) operations should be applied.

Example 1.

$$\begin{vmatrix}2&4&1\\-4&0&-2\\3&5&-1\end{vmatrix}\begin{matrix}\\R_2+2R_1\\ \\\end{matrix}\to\begin{vmatrix}2&4&1\\0&8&0\\3&5&-1\end{vmatrix}\begin{matrix}\\ \\R_3-1.5R_1\end{matrix}\to$$

$$\to\begin{vmatrix}2&4&1\\0&8&0\\0&-1&-2.5\end{vmatrix}\begin{matrix}\\ \\R_3+\frac{1}{8}R_2\end{matrix}\to\begin{vmatrix}2&4&1\\0&8&0\\0&0&-2.5\end{vmatrix}=2\cdot8\cdot(-2.5)=-40$$

Example 2.

$$\begin{vmatrix}2&-1&0&1\\3&1&-2&-1\\0&2&1&1\\-2&0&2&1\end{vmatrix}\begin{matrix}\\R_2-R_1\\ \\R_4+R_1\end{matrix}=\begin{vmatrix}2&-3&0&1\\1&2&-2&-2\\0&2&1&1\\0&-1&2&2\end{vmatrix}\begin{matrix}\\R_2\leftrightarrow R_1\\ \\ \\\end{matrix}=-\begin{vmatrix}1&2&-2&-2\\2&-3&0&1\\0&2&1&1\\0&-1&2&2\end{vmatrix}\begin{matrix}\\R_2-2R_1\\ \\ \\\end{matrix}=$$

$$=-\begin{vmatrix}1&2&-2&-2\\0&-7&4&5\\0&2&1&1\\0&-1&2&2\end{vmatrix}\begin{matrix}\\R_2\leftrightarrow R_4\\ \\ \\\end{matrix}=+\begin{vmatrix}1&2&-2&-2\\0&-1&2&2\\0&2&1&1\\0&-7&4&5\end{vmatrix}\begin{matrix}\\ \\R_3+2R_2\\R_3-7R_2\end{matrix}=\begin{vmatrix}1&2&-2&-2\\0&-1&2&2\\0&0&5&5\\0&0&-10&-9\end{vmatrix}\begin{matrix}\\ \\ \\R_4+2R_3\end{matrix}=$$

$$=\begin{vmatrix}1&2&-2&-2\\0&-1&2&2\\0&0&5&5\\0&0&0&1\end{vmatrix}=1\cdot(-1)\cdot5\cdot1=-5$$

Exercise 3.B

1. $\begin{vmatrix}3&1&-2\\1&2&-3\\2&-1&4\end{vmatrix}$
2. $\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix}$
3. $\begin{vmatrix}119&125&122\\428&431&429\\579&582&580\end{vmatrix}$
4. $\begin{vmatrix}2&3&4&5\\3&4&5&6\\4&6&8&10\\2&4&3&5\end{vmatrix}$
5. $\begin{vmatrix}3&-2&1&1\\0&1&2&0\\-1&2&4&2\\1&-3&0&0\end{vmatrix}$
6. $\begin{vmatrix}3&-1&3&4\\0&2&5&-3\\0&0&-5&3\\0&0&0&-1\end{vmatrix}$
7. $\begin{vmatrix}1&5&-2&3\\0&2&7&1\\2&10&-1&5\\-3&-15&-6&13\end{vmatrix}$
8. $\begin{vmatrix}1&2&3&-1\\6&5&9&8\\2&4&12&-1\\1&2&6&-1\end{vmatrix}$
9. $\begin{vmatrix}1&2&0&-2\\-1&1&3&5\\2&1&4&0\\-2&5&2&6\end{vmatrix}$
10. $\begin{vmatrix}2&4&6\\1&2&1\\3&8&6\end{vmatrix}$
11. $\begin{vmatrix}3&0&10\\3&-2&7\\2&-1&5\end{vmatrix}$
12. $\begin{vmatrix}3&-8&7\\2&-3&6\\1&-3&2\end{vmatrix}$
13. $\begin{vmatrix}4&9&-11\\2&6&-3\\3&7&-8\end{vmatrix}$
14. $\begin{vmatrix}1&-1&2&1\\2&-1&6&3\\3&-1&8&7\\3&0&9&9\end{vmatrix}$
15. $\begin{vmatrix}1&2&-2&3\\3&7&-3&11\\2&3&-5&11\\2&6&1&8\end{vmatrix}$