3. Solving Linear Systems by Gauss' Method (by elimination)

The Gauss' Method allows to solve linear systems of **any shape *m*×*n*.**

Definition 11. The matrix of all coefficients of a linear system (including the free coefficient column which is usually separated by a vertical line) is called the ***augmented*** system matrix . This matrix is used to solve the linear system using gradual elimination of variables.

Definition 12. A matrix is said to be in a ***row echelon form*** (REF) if the first nonzero element (from the left), called the ***leading coefficient***, in every row is located strictly to the right of the leading coefficient of the previous row, and all rows containing nonzero elements are above all full zero rows; this means that all elements in the column below the leading coefficient are zeros. Note that the REF is **not unique** and a matrix can be transformed into REF in different ways.

For example, the augmented matrix $\left(\begin{array}{cccccc|c} 1 & 0 & 2 & 4 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 & 4 & 2 & 1 \\ 0 & 0 & 0 & 4 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ is in a row echelon form.

The Gauss' Method uses ***row operations*** to transform the augmented matrix to row-echelon form; it is advisable to ensure that **all leading coefficients equal 1.** After the transformation the system can be rewritten using the new coefficients (discarding any full zero rows) and then solved starting with the last row and moving upwards.

In particular, **considering a 3×3 system**:

We write the augmented system matrix $\left(\begin{array}{ccc|c} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{array}\right)$; then transform the matrix to obtain the identity matrix on the left-hand side. The column on the right-hand side gives the solution values.

The form $\left(\begin{array}{ccc|c} 1 & 0 & 0 & A \\ 0 & 1 & 0 & B \\ 0 & 0 & 1 & C \end{array}\right)$ is called the ***reduced row-echelon form*** (RREF) of the augmented matrix.

In the RREF **all leading coefficients equal 1**, and **in all columns containing a leading coefficient all other elements equal 0**. There may be some other columns with non-leading coefficients.

(It is not strictly necessary to change matrix into RREF; while the RREF is more efficient, REF is sufficient for solving the system.)

It may ***not always*** be possible to get the identity matrix on the left-hand side. In this case some of the coefficients might also not be eliminated.

If we end up with **a row consisting only of 0-s,** then

$\left(\begin{array}{ccc|c} 1 & 0 & k_1 & A \\ 0 & 1 & k_2 & B \\ 0 & 0 & 0 & 0 \end{array}\right)$ means there is ***an infinite number*** of solutions: the last row means $0x+0y+0z=0$.

$\left(\begin{array}{ccc|c} 1 & 0 & k_1 & A \\ 0 & 1 & k_2 & B \\ 0 & 0 & 0 & C \end{array}\right)$ where $C \neq 0$ means there is **NO** solution; $0x+0y+0z=C$ - impossible.

The following ***elementary row operations*** are allowed in the augmented matrix (and easily understood if you remember that each row represents an equation):

1. Interchanging two rows – the answer does not depend on the order in which the equations are written.

2. Multiplying (or dividing) any row by a non-zero number – this does not change the solutions of that equation.

3. Replacing a row by the sum of itself and another row or its multiple – if (x_0, y_0, z_0) is a solution for each of the equations, then it is also a solution for the sum of the equations.

Example 1. Solve $\begin{cases} 2x-y+5z=0 \\ x-2y-3z=4 \\ 3x+4y+6z=7 \end{cases}$ by Gauss' method.

To do that, write the augmented matrix and transform it into the REF:

$$\left(\begin{array}{ccc|c} 2 & -1 & 5 & 0 \\ 1 & -2 & -3 & 4 \\ 3 & 4 & 6 & 7 \end{array}\right) \sim [R_1 \leftrightarrow R_2] \sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 4 \\ 2 & -1 & 5 & 0 \\ 3 & 4 & 6 & 7 \end{array}\right) \sim \left[\begin{array}{c} \\ R_2+(-2)R_1 \\ R_3+(-3)R_1 \end{array}\right] \sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 4 \\ 0 & 3 & 11 & -8 \\ 0 & 10 & 15 & -5 \end{array}\right) \sim \left[\begin{array}{c} \\ \\ R_3 \cdot \frac{1}{5} \end{array}\right] \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 4 \\ 0 & 3 & 11 & -8 \\ 0 & 2 & 3 & -1 \end{array}\right) \sim \left[\begin{array}{c} \\ R_2+(-1)R_3 \\ \\ \end{array}\right] \sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 4 \\ 0 & 1 & 8 & -7 \\ 0 & 2 & 3 & -1 \end{array}\right) \sim \left[\begin{array}{c} \\ \\ R_3+(-2)R_2 \end{array}\right] \sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 4 \\ 0 & 1 & 8 & -7 \\ 0 & 0 & -13 & 13 \end{array}\right) \sim \left[\begin{array}{c} \\ \\ R_3 \cdot \left(-\frac{1}{5}\right) \end{array}\right] \sim \left(\begin{array}{ccc|c} 1 & -2 & -3 & 4 \\ 0 & 1 & 8 & -7 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

Now either (A) rewrite the system with these coefficients and solve:

$$\begin{cases} x-2y-3z=4 \\ \quad y+8z=-7 \\ \quad\quad z=-1 \end{cases} \Rightarrow \begin{cases} x-2y-3z=4 \\ \quad y-8=-7 \\ \quad\quad z=-1 \end{cases} \Rightarrow \begin{cases} x-2+3=4 \\ \quad y=1 \\ \quad\quad z=-1 \end{cases} \Rightarrow \begin{cases} x=3 \\ y=1 \\ z=-1 \end{cases}$$

or (B) continue to RREF:

$$\left(\begin{array}{ccc|c} 1 & -2 & -3 & 4 \\ 0 & 1 & 8 & -7 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left[\begin{array}{c} R_1+3R_3 \\ R_2+(-8)R_3 \\ \\ \end{array}\right] \sim \left(\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left[\begin{array}{c} R_1+2R_2 \\ \\ \\ \end{array}\right] \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \Rightarrow \begin{cases} x=3 \\ y=1 \\ z=-1 \end{cases}$$

Non-Unique Solutions

If a linear system has infinitely many solutions, it is possible to write **a general solution** which gives formulae for finding all the **particular solutions**. One (or more) variable is assumed to be free (can take any real value) and the other variables are expressed in terms of the free variable(-s).

Example 2. $\begin{cases} 2x-y+2z=1 \\ 3y+z=-2 \\ 4x+7y+7z=-1 \end{cases}$:

Solution: $\left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & 1 & -2 \\ 4 & 7 & 7 & -1 \end{array}\right) \underset{R_3-2R_1}{\sim} \left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 9 & 3 & -3 \end{array}\right) \underset{R_3-3R_3}{\sim} \left(\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & 3 & 1 & -2 \\ 0 & 0 & 0 & 3 \end{array}\right)$.

Tthe system has no solutions .

Example 3. $\begin{cases} 3x-2y+z=7 \\ 7x+y=18 \\ x+5y-2z=4 \end{cases}$:

$$\left(\begin{array}{ccc|c} 3 & -2 & 1 & 7 \\ 7 & 1 & 0 & 18 \\ 1 & 5 & -2 & 4 \end{array}\right) \overset{R_1 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 5 & -2 & 4 \\ 7 & 1 & 0 & 18 \\ 3 & -2 & 1 & 7 \end{array}\right) \begin{array}{c} \\ R_2-7R_1 \\ R_3-3R_1 \end{array} \sim \left(\begin{array}{ccc|c} 1 & 5 & -2 & 4 \\ 0 & -34 & 14 & -10 \\ 0 & -17 & 7 & -5 \end{array}\right) \sim$$

$$R_2-2R_3 \sim \left(\begin{array}{ccc|c} 1 & 5 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -17 & 7 & -5 \end{array}\right) \underset{R_2 \leftrightarrow R_3}{\sim} \left(\begin{array}{ccc|c} 1 & 5 & -2 & 4 \\ 0 & -17 & 7 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right) - \tfrac{1}{17} R_2 \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 5 & -2 & 4 \\ 0 & 1 & -\frac{7}{17} & \frac{5}{17} \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \begin{array}{c} R_1-5R_2 \\ \\ \\ \end{array} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{17} & \frac{43}{17} \\ 0 & 1 & -\frac{7}{17} & -5 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{– infinitely many solutions}$$

Let $z=c$ (any number); then from Row 1:

$x+\frac{1}{17}c=\frac{43}{17} \Rightarrow x=\frac{43}{17}-\frac{1}{17}c$ and from Row 2:

$y-\frac{7}{17}c=\frac{5}{17} \Rightarrow y=\frac{5}{17}+\frac{7}{17}c$;

general solution is $\boxed{x=\dfrac{43-c}{17},\quad y=\dfrac{5+7c}{17},\quad z=c-\text{any number}}$

Particular solutions are, for example,

$x=\dfrac{43}{17},\ y=\dfrac{5}{17},\ z=0$ (if c = 0); $\quad x=2,\ y=4,\ z=9$ (if c = 9).

Similarly, in augmented matrix of any size, a row of zero coefficients may occur. If the number of rows (equations) is less than the number of columns (unknown variables) in the unaugmented matrix (that is, ***m***<***n***), the same method is used to find the general solution (if it exists; the system may still be inconsistent).

Example 4.

$$\begin{cases} 3x_1+2x_2+x_3-x_4=5 \\ 4x_1+3x_2+5x_4=2 \\ x_1+x_2-x_3+6x_4=-3 \end{cases}$$

$$\left(\begin{array}{cccc|c} 3 & 2 & 1 & -1 & 5 \\ 4 & 3 & 0 & 5 & 2 \\ 1 & 1 & -1 & 6 & -3 \end{array}\right) \sim [R_1 \leftrightarrow R_3] \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 6 & -3 \\ 4 & 3 & 0 & 5 & 2 \\ 3 & 2 & 1 & -1 & 5 \end{array}\right) \sim \begin{bmatrix} \\ R_2-4R_1 \\ R_3-3R_1 \end{bmatrix} \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 6 & -3 \\ 0 & -1 & 4 & -19 & 14 \\ 0 & -1 & 4 & -19 & 14 \end{array}\right) \sim$$

$$\begin{bmatrix} \\ \\ R_3-R_2 \end{bmatrix} \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & 6 & -3 \\ 0 & -1 & 4 & -19 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \begin{bmatrix} R_1+R_2 \\ \\ \\ \end{bmatrix} \sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & -13 & 11 \\ 0 & -1 & 4 & -19 & 14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 3 & -13 & 11 \\ 0 & 1 & -4 & 19 & -14 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim$$

The matrix is now in RREF. Let $x_3=a, x_4=b$ (any numbers); then $x_2=4a-19b-14, x_1=-3a+13b+11,$

and the general solution $x_1=-3a+13b+11, x_2=4a-19b-14, x_3=a, x_4=b$.

A particular solution is, e.g. $x_1=-2, x_2=5, x_3=0, x_4=-1$ (if $a=0, b=-1$).

Rank Of A Matrix

Definition 1. Rows or columns of a matrix are are **linearly dependent** if one row resp. column is (a) a multiple of another, or (b) a linear combination of two or more other rows/columns; e.g. $R_3=5R_1$ or $C_1=C_2+3C_3$.

Definition 2. The **rank** of a matrix **A** is defined as **the maximum number of linearly independent row vectors** (or the maximum number of of linearly independent **column vectors** of **A**; both numbers are the same.).
The rank of a matrix can be found by reducing it to a row-echelon form using elementary row operations; the rank equals to the number of non-zero rows.

Example. To find the rank of the matrix $\mathbf{A}=\begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix}$, apply elementary row operations similar to Gaussian method:

$$\begin{pmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{pmatrix} \begin{matrix} \\ R_2+2R_1 \\ R_3-3R_1 \end{matrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix} \begin{matrix} \\ \\ R_3+R_2 \end{matrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \begin{matrix} R_1-2R_2 \\ \\ \\ \end{matrix} \to \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

The final matrix is in the RREF form and has two non-zero rows, therefore the rank of the matrix is $r(\mathbf{A})=2$.

The number of solutions of a system of linear equations can be determined by finding the rank of the system coefficient (inaugmented) matrix **A** and the augmented system matrix $\mathbf{A}'$ in the following way:

1. If $r(\mathbf{A})=r(\mathbf{A}')=n$ where n is the number of variables then the system is consistent and has **a unique solution.**
2. If $r(\mathbf{A})=r(\mathbf{A}')<n$ then the system is consistent and has **infinitely many solutions.** The number of independent constants in the general solution is $n-r$.
3. If $r(\mathbf{A})<r(\mathbf{A}')$ then there are **no solutions.**

In **Example 1**, $r(\mathbf{A})=r(\mathbf{A}')=n=3$ and the system has a unique solution.
In **Example 2**, $r(\mathbf{A})=2$ and $r(\mathbf{A}')=3$, and $r(\mathbf{A})<r(\mathbf{A}')$; system is inconsistent.
In **Example 3**, $r(\mathbf{A})=r(\mathbf{A}')=2$ but the number of variables is 3; consistent with infinitely many solutions and 3–2 =1 independent constant; similarly, in **example 4**, $r(\mathbf{A})=r(\mathbf{A}')=2$, number of variables is 4, number of independent constants is 4–2 = 2.

Exercise 6A. Solve the given systems of equations using Gauss' method.

1. $\begin{cases} 3x-y+z=0 \\ 2x-5y-3z=-5 \\ x+y-z=4 \end{cases}$ **2.** $\begin{cases} x-4z=13 \\ 2x+3z=-5 \\ -2x+6y-5z=0 \end{cases}$ **3.** $\begin{cases} 2x-3y+z=-3 \\ -4x+3y+2z=-11 \\ x-y-z=3 \end{cases}$

Exercise 6B. Solve by Gauss' method. Use the ranks of the relevant matrices. If there are infinitely many solutions, write the general solution and one particular solution.

1. $\begin{cases} 6x-4y=-8 \\ -15x+10y=20 \end{cases}$ **2.** $\begin{cases} x-4y+3z=3 \\ 2x-y-2z=-3 \\ x+3y-5z=-6 \end{cases}$ **3.** $\begin{cases} 2x+9y=1 \\ x+2y+3z=2 \\ 5y-6z=4 \end{cases}$

4. $\begin{cases} x_1+2x_2-2x_3+x_4=4 \\ 3x_1+2x_2-2x_3=3 \\ 4x_1+2x_3-x_4=5 \\ x_1-2x_2+4x_3-x_4=4 \end{cases}$ **5.** $\begin{cases} 5x_1+x_2-2x_3+3x_4=1 \\ x_1-x_2-x_3-x_4=-3 \\ 8x_1-2x_2-5x_3=-8 \\ 3x_1+3x_2+5x_4=7 \end{cases}$

ANSWERS

Exercise 6A: **1.** $\begin{cases} x=1 \\ y=2 \\ z=-1 \end{cases}$ **2.** $\begin{cases} x=5 \\ y=0 \\ z=-2 \end{cases}$ **3.** $\begin{cases} x=4 \\ y=3 \\ z=-2 \end{cases}$

Exercise 6B:

1. REF e.g. $\left(\begin{array}{cc|c} 3 & -2 & -4 \\ 0 & 0 & 0 \end{array}\right)$, $r(\mathbf{A})=r(\mathbf{A}')=1$, $n=2$, general solution $\begin{cases} x=\frac{2c-4}{3} \\ y=c \end{cases}$

2. RREF $\left(\begin{array}{ccc|c} 1 & 0 & -\frac{11}{7} & -\frac{15}{7} \\ 0 & 1 & -\frac{8}{7} & -\frac{9}{7} \\ 0 & 0 & 0 & 0 \end{array}\right)$, $r(\mathbf{A})=r(\mathbf{A}')=2$, $n=3$, general solution $\begin{cases} x=\frac{11c-15}{7} \\ y=\frac{8c-9}{7} \\ z=c \end{cases}$

3. REF e.g. $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 5 & -6 & -3 \\ 0 & 0 & 0 & 7 \end{array}\right)$, $r(\mathbf{A})=2, r(\mathbf{A}')=3$ - no solution

4. REF e.g. $\left(\begin{array}{cccc|c} 1 & 2 & -2 & 1 & 4 \\ 0 & -4 & 4 & -3 & -9 \\ 0 & 0 & 2 & 1 & 7 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right)$, $r(\mathbf{A})=3, r(\mathbf{A}')=4$ - no solution

5. REF e.g. $\left(\begin{array}{cccc|c} 1 & -1 & -1 & -1 & -3 \\ 0 & 6 & 3 & 8 & 16 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$, $r(\mathbf{A})=2, r(\mathbf{A}')=2$, $n=4$, general solution with 2 free constants

$$\begin{cases} x_1=-3+a+b+\frac{16-3a-8b}{6}=\frac{3a-2b-18}{6} \\ x_2=\frac{16-3a-8b}{6} \\ x_3=a \\ x_4=b \end{cases}$$