Matrices. Basic Operations of Matrix Algebra

Definition 1. A ***matrix*** (plural: *matrices*) is a rectangular table (array) of numbers (***elements***) in *m* rows and *n* columns; its ***dimension*** (also ***size, shape, order***) is said to be $m \times n$. (***Note***: *the number of* ***rows*** *is always written* ***first****, and the number of* ***columns****,* ***second****.)*

Consider the matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots a_{2n} \\ & \dots\dots & & \\ a_{m1} & a_{m2} & a_{m3} & \dots a_{mn} \end{pmatrix}$

Here a_{ij} denotes the element in the *i*-th row, *j*-th column. (Notice that the first index always refers to rows, and the second to columns.) We can indicate the dimension of the matrix : $\mathbf{A}_{m\times n}$.

Definition 2. The matrix with dimension $m \times 1$ is called a ***vector*** (a ***column vector***); the matrix with dimensions $1 \times n$ is called a ***row vector.***

Definition 3. If $m = n$, the matrix is called a ***square matrix***.

Definition 4. As opposed to matrices and vectors, plain numbers are called ***scalars***.

Definition 5. Two matrices are called ***equal*** if
(1) they have exactly the **same shape** (order) and
(2) **all the elements in corresponding positions are equal**, that is,
A = **B** if and only if $a_{ij} = b_{ij}\ \forall i \in \{1, 2, \dots, m\},\ \forall j \in \{1, 2, \dots, n\}$

The symbol $\forall$ means *all, every, any.*

Definition 6. A ***zero matrix*** (sometimes called the ***null matrix***), denoted by $\boldsymbol{O}_{m\times n}$, is a matrix in which all elements are 0.

Definition 7. In a square matrix all elements a_{ij} where $i = j$ (that is, all from the top left corner to the bottom right corner) form the ***main diagonal*****.** The other diagonal (from top right to bottom left) is called the ***secondary*** diagonal.

Definition 8. A square matrix is called a ***diagonal matrix*** if all its elements except for the main diagonal are zero**.**

Definition 9. A diagonal matrix where all elements on the main diagonal equal 1 is called the ***identity matrix*** (also the ***unit matrix***) **I** (sometimes denoted **E**).

For example,
$\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$ is a diagonal matrix, $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are identity matrices.

Matrix Algebra

1. Transposition of a matrix. If all the rows of a $m \times n$ matrix **A** are switched with the corresponding column, its ***transposed matrix*** or ***transpose*** $\mathbf{A}^T$ is obtained; $a_{ij}^T = a_{ji}$ and the transpose has dimension $n \times m$**.**

2. The product of a scalar *s* and the matrix A is matrix *s***A** = **B** obtained by multiplying all elements of **A** by *s*: $\forall b_{ij} = s \cdot a_{ij}$.

3. Addition and subtraction. If two or more matrices have ***the same dimension,*** then their ***sum is a matrix*** **C = A + B** ***also of the same dimension***, where the corresponding elements are added: $\forall c_{ij} = a_{ij} + b_{ij}$. Similarly, the ***difference is a matrix*** **D = A – B** and $\forall d_{ij} = a_{ij} - b_{ij}$.

4. The product of two matrices: AB = X *such that* $x_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$. That is, the elements of the **i-th row from the first matrix** and elements of the **j-th column in the second matrix** are multiplied in corresponding pairs and then all the products are added.

Note that the **number of elements in each row of A must equal the number of elements in each column of B,** and if the two matrices have dimensions $\mathbf{A}_{m\times n}$ and $\mathbf{B}_{n\times k}$, then the product is the matrix $\mathbf{X}_{m\times k}$: that is, matrices can be **multiplied only** if the **number of columns in the first matrix equals the number of rows in the second matrix.**

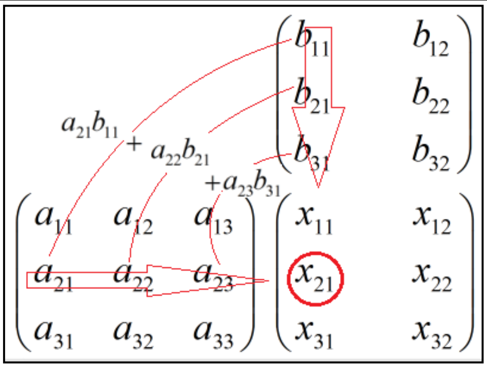
The numbers in the middle must be **equal** ($n = n$), and the outer numbers will be the dimensions $m \times k$ of **X** :

$$\boldsymbol{m} \times \boxed{\boldsymbol{n}\quad \boldsymbol{n}} \times \boldsymbol{k}.$$

E.g., for $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$,

to find the element in the 2nd row, 1st column, we use 2nd row of A and 1st column of B:

$$x_{21}=a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31}$$



5. The power of a matrix: similarly to number powers, **for square matrices only** $\mathbf{A}^n=\underbrace{\mathbf{A}\cdot\mathbf{A}\cdot\ldots\mathbf{A}}_{n\ factors}$.

Examples

Consider the matrices $\mathbf{A}=\begin{pmatrix} 2 & 4 \\ -3 & 1 \\ 0 & -5 \end{pmatrix}, \mathbf{B}=\begin{pmatrix} -6 & -1 \\ 3 & 8 \\ 2 & -5 \end{pmatrix}, \mathbf{C}=\begin{pmatrix} x & y \\ -3 & 1 \\ 0 & 2z+1 \end{pmatrix}, \mathbf{D}=\begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}$

Then $\mathbf{A}^T=\begin{pmatrix} 2 & -3 & 0 \\ 4 & 1 & -5 \end{pmatrix}$; $-3\mathbf{B}=\begin{pmatrix} 18 & 3 \\ -9 & -24 \\ -6 & 15 \end{pmatrix}$; $\mathbf{A}+\mathbf{B}=\begin{pmatrix} -4 & 3 \\ 0 & 9 \\ 2 & -10 \end{pmatrix}$, $\mathbf{A}-\mathbf{B}=\begin{pmatrix} 8 & 5 \\ -6 & -7 \\ -2 & 0 \end{pmatrix}$

A = C if and only if $x=2,\ y=4,\ 2z+1=-5 \Rightarrow z=-3$

The product $\mathbf{AD}=\begin{pmatrix} 2\cdot3+4\cdot2 & 2\cdot(-4)+4\cdot(-1) \\ -3\cdot3+1\cdot2 & -3\cdot(-4)+1\cdot(-1) \\ 0\cdot3-5\cdot2 & 0\cdot(-4)-5\cdot(-1) \end{pmatrix}=\begin{pmatrix} 14 & -12 \\ -7 & 11 \\ -10 & 5 \end{pmatrix}$

The square $\mathbf{D}^2=\begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}\cdot\begin{pmatrix} 3 & -4 \\ 2 & -1 \end{pmatrix}=\begin{pmatrix} 3\cdot3-4\cdot2 & 3\cdot(-4)-4\cdot(-1) \\ 3\cdot2-1\cdot2 & 2\cdot(-4)-1\cdot(-1) \end{pmatrix}=\begin{pmatrix} 1 & -8 \\ 4 & -7 \end{pmatrix}$

Note that **some operations** are **not possible**;

e.g. the sums **A+D** and **A+A**$^{\mathrm{T}}$ are impossible because the matrices have different dimensions and the product **DA** is impossible because the number of columns in **D** is not equal to the number of rows in **A**.

Properties of Matrix Algebra

The following **properties of matrix algebra** are true for all square matrices of the same order.

1. Multiplication by a scalar:

(i) $(a+b)\mathbf{A}=a\mathbf{A}+b\mathbf{A}$

(ii) $a(\mathbf{A}+\mathbf{B})=a\mathbf{A}+a\mathbf{B}$

2. Matrix addition:

(i) zero element: A + O = O + A = A.

(ii) commutative law: $\mathbf{A}+\mathbf{B}=\mathbf{B}+\mathbf{A}$

(iii) associative law: $(\mathbf{A}+\mathbf{B})+\mathbf{C}=\mathbf{A}+(\mathbf{B}+\mathbf{C})$

3. Matrix multiplication:

(i) If **O** is the zero matrix, then **AO = OA = O** for all **A.**

(ii) identity law: AI = IA = A for all **A.**

(iii) in general, the commutative property **does not hold** for multiplication: $\mathbf{AB}\neq\mathbf{BA}$.

(iv) associative law: $(\mathbf{AB})\mathbf{C}=\mathbf{A}(\mathbf{BC})$

(v) distributive law: $\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC}$ and $(\mathbf{B}+\mathbf{C})\mathbf{A}=\mathbf{BA}+\mathbf{CA}$.

Practice

Exercise 1.A

1. If $\mathbf{A}=\begin{pmatrix}3&4\\5&2\end{pmatrix}, \mathbf{B}=\begin{pmatrix}6&-3\\-2&1\end{pmatrix}, \mathbf{C}=\begin{pmatrix}-3&7\\-4&-2\end{pmatrix}$, find (a) $\mathbf{A}+\mathbf{B}$ (b) $\mathbf{A}+\mathbf{B}+\mathbf{C}$ (c) $\mathbf{B}+\mathbf{C}$ (d) $\mathbf{C}+\mathbf{B}-\mathbf{A}$

2. If $\mathbf{P}=\begin{pmatrix}3&5&-11\\10&2&6\\-2&-1&7\end{pmatrix}$, $\mathbf{Q}=\begin{pmatrix}17&-4&3\\-2&8&-8\\3&-4&11\end{pmatrix}$, find (a) $\mathbf{P}+\mathbf{Q}$ (b) $\mathbf{P}-\mathbf{Q}$ (c) $\mathbf{Q}-\mathbf{P}$

3. Find the scalars x and y if (a) $\begin{pmatrix}x&x^2\\3&-1\end{pmatrix}=\begin{pmatrix}y&4\\3&y+1\end{pmatrix}$ (b) $\begin{pmatrix}x&y\\y&x\end{pmatrix}=\begin{pmatrix}-y&x\\x&-y\end{pmatrix}$

4. If $\mathbf{B}=\begin{pmatrix}6&12\\24&6\end{pmatrix}$ find: (a) $2\mathbf{B}$ (b) $\frac{1}{3}\mathbf{B}$ (c) $\frac{1}{12}\mathbf{B}$ (d) $-\frac{1}{2}\mathbf{B}$

5. If $\mathbf{A}=\begin{pmatrix}2&3&5\\1&6&4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1&2&1\\1&2&3\end{pmatrix}$ find: (a) $\mathbf{A}+\mathbf{B}$ (b) $\mathbf{A}-\mathbf{B}$ (c) $2\mathbf{A}+\mathbf{B}$ (d) $3\mathbf{A}-\mathbf{B}$

6. Given the matrices **M** and/or **N**, find the matrix X:

(a) $\mathbf{M}=\begin{pmatrix}1&2\\3&6\end{pmatrix}$ and $\frac{1}{3}\mathbf{X}=\mathbf{M}$

(b) $\mathbf{N}=\begin{pmatrix}2&-1\\3&5\end{pmatrix}$ and $4\mathbf{X}=\mathbf{N}$

(c) $\mathbf{M}=\begin{pmatrix}1&0\\-1&2\end{pmatrix}$, $\mathbf{N}=\begin{pmatrix}1&4\\-1&1\end{pmatrix}$ and $\mathbf{M}-2\mathbf{X}=\mathbf{N}$

7. Transpose the matrices given: $\boldsymbol{D}=\begin{pmatrix}3&-4&0\\-1&5&2\\7&0&-5\end{pmatrix}$ $\boldsymbol{K}=\begin{pmatrix}12&10&-8&1\\-3&65&2&42\end{pmatrix}$ $\boldsymbol{P}=\begin{pmatrix}9&8\\6&-5\\-6&1\end{pmatrix}$

8. Find, if possible, the products **AB** and **BA**: (a) $\mathbf{A}=\begin{pmatrix}2&1\\3&4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}5&6\end{pmatrix}$ (b) $\mathbf{A}=\begin{pmatrix}2&0&3\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}1\\4\\2\end{pmatrix}$

9. Multiply the matrices: (a) $\begin{pmatrix}1&2&1\end{pmatrix}\begin{pmatrix}2&3&1\\0&1&0\\1&0&2\end{pmatrix}$ (b) $\begin{pmatrix}1&0&-1\\-1&1&0\\9&-1&1\end{pmatrix}\begin{pmatrix}2\\3\\4\end{pmatrix}$

10. Find the matrix products: (a) $\begin{pmatrix}3&-2\\5&8\end{pmatrix}\begin{pmatrix}-5&1\\0&-2\end{pmatrix}$ (b) $\begin{pmatrix}1&2\\-1&0\\5&-2\end{pmatrix}\begin{pmatrix}2&-3\\4&10\end{pmatrix}$ (c) $\begin{pmatrix}5&2\\1&0\\3&-2\end{pmatrix}\begin{pmatrix}-1&0&2\\4&1&-3\end{pmatrix}$

(d) $\begin{pmatrix}-1&0&2\\4&1&-3\end{pmatrix}\begin{pmatrix}5&2\\1&0\\3&-2\end{pmatrix}$ (e) $\begin{pmatrix}5&-11\\-4&3\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}$ (f) $\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}4&0&7\\-1&5&6\\2&3&-3\end{pmatrix}$

Exercise 1.B

1. If $\mathbf{K}=\begin{pmatrix}-2 & 1\\ 0 & 5\\ 3 & 4\end{pmatrix}$, $\mathbf{L}=\begin{pmatrix}-1 & 3\\ 5 & -5\\ 2 & 6\end{pmatrix}$, $\mathbf{M}=\begin{pmatrix}0 & 4\\ 11 & -5\\ 7 & 6\end{pmatrix}$, find

(a) 2**K**+3**L**; (b) 3**K**-**L**+2**M**.

2. Find **G**+**G**^{T} if $\mathbf{G}=\begin{pmatrix}2 & 5 & 4\\ -3 & 0 & -1\\ 4 & 2 & 0\end{pmatrix}$.

3. If $\mathbf{A}=\begin{pmatrix}2 & -1\\ 3 & 4\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-1 & 5 & 0\\ 3 & 2 & 4\end{pmatrix}$, $\mathbf{C}=\begin{pmatrix}-3 & 2\\ 5 & 6\\ 7 & 8\end{pmatrix}$ and $\mathbf{D}=\begin{pmatrix}1\\ 3\\ 5\\ 7\end{pmatrix}$,

find, if possible, (a) **AB** (b) **BA** (c) **CA** (d) **CB**
(e) **AD** (f) **BAD** (g) **BCA**.

If the product is not possible, explain why.

4.

(a) Find $\mathbf{A}^2$ if $\mathbf{A}=\begin{pmatrix}2 & 1\\ 3 & -2\end{pmatrix}$ (b) find $\mathbf{M}^3$ if $\mathbf{M}=\begin{pmatrix}5 & -1\\ 2 & 4\end{pmatrix}$ (c) find the square of $\mathbf{C}=\begin{pmatrix}1 & 2 & 0\\ -2 & 1 & -1\\ 0 & 3 & 0\end{pmatrix}$

5. Find constants *a* and *b* such that $\mathbf{A}^2 = a\mathbf{A}+b\mathbf{I}$ given that

(a) $\mathbf{A}=\begin{pmatrix}1 & 2\\ -1 & 2\end{pmatrix}$ (b) $\mathbf{A}=\begin{pmatrix}3 & 1\\ 2 & -2\end{pmatrix}$

6. If $\mathbf{A}=\begin{pmatrix}2 & 3\\ -1 & 0\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}1 & -2\\ 1 & 1\end{pmatrix}$, compute and compare $(\mathbf{A}+\mathbf{B})^2$ and $\mathbf{A}^2+2\mathbf{AB}+\mathbf{B}^2$.

7. Find the matrix **X** given that **AX=B** where $\mathbf{A}=\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}-1 & -2\\ 5 & 3\end{pmatrix}$. *Hint:* Let $\mathbf{X}=\begin{pmatrix}x & y\\ z & t\end{pmatrix}$

8. Given $\mathbf{A}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$ find all possible matrices **M** such that **MA = AM.**

ANSWERS

Exercise 1.A

1. (a) $\begin{pmatrix} 9 & 1 \\ 3 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 6 & 8 \\ -1 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 4 \\ -6 & -1 \end{pmatrix}$ (d) $\begin{pmatrix} 0 & 0 \\ -11 & -3 \end{pmatrix}$

2. (a) $\begin{pmatrix} 20 & 1 & -8 \\ 8 & 10 & -2 \\ 1 & -5 & 18 \end{pmatrix}$ (b) $\begin{pmatrix} -14 & 9 & -14 \\ 12 & -6 & 14 \\ -5 & 3 & -4 \end{pmatrix}$ (c) $\begin{pmatrix} 14 & -9 & 14 \\ -12 & 6 & -14 \\ 5 & -3 & 4 \end{pmatrix}$

3. (a) $x = y = -2$ (b) $x = y = 0$

4. (a) $\begin{pmatrix} 12 & 24 \\ 48 & 12 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 4 \\ 8 & 2 \end{pmatrix}$ (c) $\begin{pmatrix} \frac{1}{2} & 1 \\ 2 & \frac{1}{2} \end{pmatrix}$ (d) $\begin{pmatrix} -3 & -6 \\ -12 & -3 \end{pmatrix}$

5. (a) $\begin{pmatrix} 3 & 5 & 6 \\ 2 & 8 & 7 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 5 & 8 & 11 \\ 3 & 14 & 11 \end{pmatrix}$ (d) $\begin{pmatrix} 5 & 7 & 14 \\ 2 & 16 & 9 \end{pmatrix}$

6. (a) $\begin{pmatrix} 3 & 6 \\ 9 & 18 \end{pmatrix}$ (b) $\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix}$ (c) $\begin{pmatrix} 0 & -2 \\ 0 & \frac{1}{2} \end{pmatrix}$

7. $\mathbf{D}^T = \begin{pmatrix} 3 & -1 & 7 \\ -4 & 5 & 0 \\ 0 & 2 & -5 \end{pmatrix}$ $\mathbf{K}^T = \begin{pmatrix} 12 & -3 \\ 10 & 65 \\ -8 & 2 \\ 1 & 42 \end{pmatrix}$ $\mathbf{P}^T = \begin{pmatrix} 9 & 6 & -6 \\ 8 & -5 & 1 \end{pmatrix}$

8. (a) $\mathbf{AB}$ *impossible*, $\mathbf{BA} = \begin{pmatrix} 28 & 29 \end{pmatrix}$ (b) $\mathbf{AB} = \begin{pmatrix} 8 \end{pmatrix}$ $\mathbf{BA} = \begin{pmatrix} 2 & 0 & 3 \\ 8 & 0 & 12 \\ 4 & 0 & 6 \end{pmatrix}$

9. (a) $\begin{pmatrix} 3 & 5 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} -2 \\ 1 \\ 19 \end{pmatrix}$

10. (a) $\begin{pmatrix} -15 & 7 \\ -25 & -11 \end{pmatrix}$ (b) $\begin{pmatrix} 10 & 17 \\ -2 & 3 \\ 2 & -35 \end{pmatrix}$ (c) $\begin{pmatrix} 3 & 2 & 4 \\ -1 & 0 & 2 \\ -11 & -2 & 12 \end{pmatrix}$ (d) $\begin{pmatrix} 1 & -6 \\ 12 & 14 \end{pmatrix}$

(e) $\begin{pmatrix} 5 & -11 \\ -4 & 3 \end{pmatrix}$ (f) $\begin{pmatrix} 4 & 0 & 7 \\ -1 & 5 & 6 \\ 2 & 3 & -3 \end{pmatrix}$

Exercise 1.B

1. (a) $\begin{pmatrix} -7 & 11 \\ 15 & -5 \\ 12 & 26 \end{pmatrix}$ (b) $\begin{pmatrix} -5 & 8 \\ 17 & 10 \\ 21 & 18 \end{pmatrix}$

2. $\begin{pmatrix} 4 & 2 & 8 \\ 2 & 0 & 1 \\ 8 & 1 & 0 \end{pmatrix}$

3. (a) $\begin{pmatrix} -5 & 8 & -4 \\ 9 & 23 & 16 \end{pmatrix}$ (b) not possible (c) $\begin{pmatrix} 0 & 11 \\ 28 & 19 \\ 38 & 25 \end{pmatrix}$ (d) $\begin{pmatrix} 9 & -11 & 8 \\ 13 & 37 & 24 \\ 17 & 51 & 32 \end{pmatrix}$

(e) not possible (f) not possible (g) $\begin{pmatrix} 140 & 84 \\ 208 & 171 \end{pmatrix}$

(b), (e), (f) the number of columns of the first matrix does not equal the number of columns of the second matrix.

4. (a) $\mathbf{A}^2 = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$ (b) $\mathbf{M}^2 = \begin{pmatrix} 23 & -9 \\ 18 & 14 \end{pmatrix}, \mathbf{M}^3 = \begin{pmatrix} 97 & -59 \\ 118 & 38 \end{pmatrix}$ (c) $\mathbf{C}^2 = \begin{pmatrix} -3 & 4 & -2 \\ -4 & -6 & -1 \\ -6 & 3 & -3 \end{pmatrix}$

5. (a) $a = 3, b = -4$ (b) $a = 1, b = 8$

6. $(\mathbf{A}+\mathbf{B})^2 = \begin{pmatrix} 9 & 4 \\ 0 & 1 \end{pmatrix}$ $\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2 = \begin{pmatrix} 10 & 0 \\ -2 & 0 \end{pmatrix}$, not equal

7. $\mathbf{X} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

8. $\mathbf{M} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ for $\forall a, b \in \mathbb{R}$