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Optimal Maintenance Time for Repairable Systems

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A repairable system operates under a maintenance strategy that calls for complete preventive repair actions at prescheduled times and minimal repair actions whenever a failure occurs. Under minimal repair, failures are modeled according to a nonhomogeneous Poisson process. When the intensity function is assumed to grow proportional to a power of time, this process is called the power law process. Assuming that the system will be in operation for an infinite time, we find the expected cost per unit of time for each preventive maintenance policy and hence obtain the optimal strategy as a function of the intensity function of the process. Large-sample procedures to estimate the optimal maintenance check points for the power law process are also discussed. The results are applied in a real data set concerning failures histories of power transformers.

Key Words: Maximum Likelihood Estimate; Minimal Repair; Nonhomogeneous Poisson Process; Power Law Process.

MANY real world systems are designed to be repaired rather than replaced after failure. Maintenance policies are fundamental under these conditions because an adequate preventive-maintenance strategy can save money and keep systems running longer.

A (perfect) preventive maintenance (PM) policy specifies the periodicity with which a system is maintained. Minimal repair (MR) at failures between these PM repairs only restores the function of the system to the condition just before the failure. That is, it does not change the general condition of the system. The combination of PM and MR has been of interest since the work of Barlow and Hunter (1960). Some of the developments for this situation can be found in Gertsbakh (1977), Block et al. (1990), Park et al. (2000), and Lai et al. (2001). However, statistical inference procedures for the maintenance stochastic models developed in the literature are still under consideration.

This work was motivated by a problem concerning maintenance of power transformers of an electrical power company in Brazil. Succinctly, after a major overhaul of the electrical power sector in the late 1990s in Brazil, power companies began to be fined heavily for nonscheduled repairs. Hence, the company wanted to adopt a maintenance policy that favored preventive maintenance, as opposed to repair actions taken after failures.

The nonhomogeneous Poisson process (NHPP) plays a key role in modeling the random occurrence of failures under minimal repair. Let $\mathrm{N}(t)$ be the number of failures in the time interval $(0,t]$. A process $\{N(t) : t \geq 0\}$ having independent increments and with $N(0) = 0$ is said to be a Poisson process with intensity $\rho(\cdot)$ if, for any t, the random variable $N(t)$ follows a Poisson distribution with mean $M(t) = \mathrm{E}(N(t)) = \int_0^t \rho(u)\,du$. The NHPP is a Poisson process for which the intensity function $\rho(\cdot)$ is nonconstant. The most popular parametric form for ρ is the power law process (PLP), $\rho(t) = \beta t^{\beta-1}/\theta^{\beta}$,

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proposed by Crow (1974). For the general theory of NHPP, see, for instance, Cox and Miller (1965). For a discussion of modeling issues and statistical inference for the NHPP and the PLP in the context of repairable systems, see Ascher and Feingold (1984), Bain and Engelhardt (1991), and more recently, Rigdon and Basu (2000) and references therein. Alternatives to the PLP are discussed by Pulcini (2001).

The goal of this work is to obtain optimal PM check points that minimize expected cost and develop large sample estimation procedures for the PLP. The rest of the paper is organized as follows. In the next section, we obtain the expected cost per unit of time for each PM policy, assuming that the system will be operated for an infinite time. This expected cost is then minimized to obtain the optimal policy. The section on Statistical Methodology discusses large-sample estimation of the optimal PM check points and corresponding expected cost. Finally, this methodology is then applied to the data set concerning failure times for power transformers.

Cost Function and Optimal PM

Consider a repairable system that will be operated during T units of time starting at $t_0 = 0$, and assume the following conditions:

- PM check points are scheduled after every τ units of time;
- at each PM check point, a repair action of fixed cost C_{PM} is executed, which instantly returns the system to a like-new condition (perfect PM);
- between successive PM check points, a minimal repair (MR) is done after each failure and the expected cost for each is C_{MR}. That is, for each period defined by successive PM check points, the expected total cost is equal to the expected cost per failure times the expected number of failures;
- repair costs and failure times are independent;
- repair times are neglected.

In order to compute the total expected cost $H_{(0,T]}(\tau)$ of this PM policy, decompose the time interval $(0, T]$ as $(0, \tau] \cup (\tau, 2\tau] \cup \cdots \cup ((m-1)\tau, m\tau] \cup (m\tau, T]$, where $m = [T/\tau]$ is the largest integer smaller than or equal to T/τ. Because at each PM check point the state of the system is restored to a like-new condition, the expected cost for each interval of the form $((i-1)\tau, i\tau]$ is given by

$$C_{PM} + C_{MR}\,\mathrm{E}(N(\tau)) = C_{PM} + C_{MR}\int_0^\tau \rho(u)\,du,$$

where the first and second terms are due, respectively, to the PM at the end of the interval and to the MRs at the failures that occur within the interval. Hence, the total expected cost will be given by

$$C_{(0,T]}(\tau) = m\left\{C_{PM} + C_{MR}\int_0^\tau \rho(u)\,du\right\} + R, \quad (1)$$

where

$$R = C_{MR}\,\mathrm{E}(N(T - m\tau))$$

is the expected cost for the interval $(m\tau, T]$.

Of course, for large values of T, $C_{(0,T]}(\tau)$ is also large. Therefore, it makes sense to work with the limiting expected cost *per unit of time*,

$$\begin{aligned} H(\tau) &= \lim_{T\to\infty} C_{(0,T]}(\tau)/T \\ &= \frac{1}{\tau}\left[C_{PM} + C_{MR}\int_0^\tau \rho(u)\,du\right], \end{aligned} \quad (2)$$

because $T \approx m\tau = [T/\tau]\tau$. An infinite horizon represented by $T \to \infty$ is a reasonable and convenient approximation when T is large.

The PM policy τ that minimizes $H(\tau)$ solves

$$\frac{dH}{d\tau}(\tau) = \frac{C_{MR}}{\tau^2}\left[\tau\rho(\tau) - \int_0^\tau \rho(u)\,du - \frac{C_{PM}}{C_{MR}}\right] = 0. \quad (3)$$

Using integration by parts, τ must satisfy

$$\frac{C_{PM}}{C_{MR}} = \tau\rho(\tau) - \int_0^\tau \rho(u)\,du = \int_0^\tau u\rho'(u)\,du. \quad (4)$$

Graphically, Equation (4) means that the darkened area in Figure 1, the rectangular area $\tau\rho(\tau)$ less the area under the curve $\rho(\tau)$, must be equal to the ratio C_{PM}/C_{MR}.

Equation (4) has no solution if the intensity function is nonincreasing ($\rho'(\cdot) \le 0$) and $C_{PM} > 0$ because, in this case, a PM could not improve the sys-

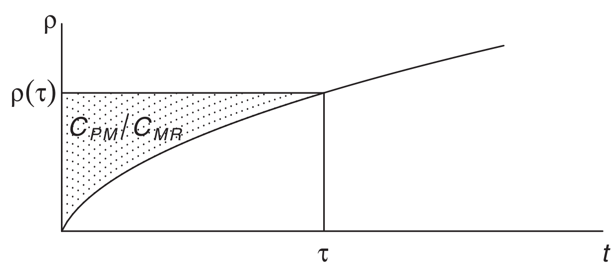

FIGURE 1. Optimal PM Times.

tem. For an increasing intensity function, the right side of Equation (4) is an increasing function of τ. Hence, in this case, Equation (4) will have a unique solution provided that $\int_0^\infty u\rho'(u)\,du > C_{PM}/C_{MR}$.

For the PLP, Equation (4) becomes $(\beta - 1)(\tau/\theta)^\beta = C_{PM}/C_{MR}$. The optimal PM policy calls for check points every

$$\tau = \theta \left[\frac{C_{PM}}{(\beta - 1)C_{MR}} \right]^{1/\beta} \tag{5}$$

units of time. For the log-linear intensity function $\rho(t) = \exp(\beta + \theta t)$, Equation (4) is equivalent to $\exp(\beta + \theta\tau)(\theta\tau - 1) + e^\beta = \theta C_{PM}/C_{MR}$, which must be solved numerically. Similarly, the intensity function $\rho(t) = \beta[1 - e^{-t/\theta}]$ of Pulcini (2001) gives $\beta[1 - e^{-\tau/\theta}(1 + \tau/\theta)] = C_{PM}/C_{MR}$, which also requires a numerical solution.

Statistical Methods

For practical purposes, it will be necessary to estimate the optimal maintenance time τ using data collected from the system under study. In the rest of this section, we discuss large-sample maximum likelihood estimation (MLE) of τ based on failure times observed for one or more identical systems under an MR policy.

There are basically two ways to observe data from a repairable system, depending on whether data collection is ceased after a specified number of failures k or at some predetermined time T. These sampling schemes are said to be *failure truncated* or *time truncated*, respectively. There are slight differences in the inference procedures depending on the sampling scheme considered.

Let $t_1 < t_2 < \cdots < t_m \leq T$ denote the times to failure observed until time T for an NHPP with intensity function $\rho(\cdot) = \rho(\cdot \mid \boldsymbol{\mu})$ depending on a vector of parameters $\boldsymbol{\mu}$. For instance, $\boldsymbol{\mu} = (\theta, \beta)$ for the PLP. Following Berman and Turner (1992), the likelihood function for $\boldsymbol{\mu}$ is given by

$$L(\boldsymbol{\mu}) = \exp\left(- \int_0^T \rho(u)\,du \right) \prod_{j=1}^m \rho(t_j). \tag{6}$$

For failure-truncated sampling, just take $T = t_m$ in (6). If n independent systems are observed, the likelihood function is the product of the n corresponding terms given by Equation (6), say

$$L(\boldsymbol{\mu}) = \exp\left(- \sum_{i=1}^{n_1} \int_0^{T_i} \rho(u)\,du - \sum_{i=n_1+1}^{n} \int_0^{t_{im_i}} \rho(u)\,du \right) \prod_{i,j} \rho(t_{ij}), \tag{7}$$

where t_{ij} is the jth failure time for the ith system and the ith system is time truncated at time $T = T_i$ for $i = 1, \ldots, n_1$ and failure truncated at the m_ith failure for $i = n_1 + 1, \ldots, n$.

When only one system is observed, the MLE of $\boldsymbol{\mu}$ will be asymptotically normally distributed provided that $\lim_{t\to\infty} M(t) = \int_0^\infty \rho(u)\,du = \infty$ and the truncation time T (or the number of failures m for failure truncated sampling) goes to ∞. See Cox and Hinkley (1974) for the general theory of MLE estimation and Zhao and Xie (1996) for the specific case of an NHPP. Zhao and Xie's results extend straightforwardly when n independent systems are observed. Hence, when at least one of the T_i's or m_i's in (7) is large, $\hat{\boldsymbol{\mu}} = \operatorname{argmax} L(\boldsymbol{\mu})$ follows approximately a normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ given by minus the inverse of the Hessian matrix of $\mathcal{L} = \log L$ evaluated at $\hat{\boldsymbol{\mu}}$. According to Equation (4), τ is given by a function of $\boldsymbol{\mu}$, say $\tau = g(\boldsymbol{\mu})$. So the MLE of τ will be $\hat{\tau} = g(\hat{\boldsymbol{\mu}})$. The delta method can now be used to obtain the approximate distribution of $\hat{\tau}$. This distribution will be normal with mean τ and variance $\sigma^2_{\hat{\tau}} = [\nabla g(\hat{\boldsymbol{\mu}})]'\boldsymbol{\Sigma}[\nabla g(\hat{\boldsymbol{\mu}})]$, where ∇g is the gradient of g. Hence, an approximate $100(1-\beta)\%$ confidence interval (CI) for τ is given by $\hat{\tau} \pm z_{\beta/2}\hat{\sigma}_{\hat{\tau}}$, where $z_{\beta/2}$ is the corresponding point from the standard normal curve and $\hat{\sigma}^2_{\hat{\tau}}$ estimates the variance of $\hat{\tau}$.

Even more useful than assessing the accuracy of $\hat{\tau}$ would be an error bound for $H(\hat{\tau}) - H(\tau)$, the difference between the expected cost attained when using $\hat{\tau}$ minus the minimum expected cost. In other words, even though the CI for τ may be short, the cost function H could be very peaked around its minimum, so that even small deviations from the optimal PM policy could have a large impact on the expected cost. For instance, in order to decide whether sampling information should be continued to obtain more precise estimates, the difference $H(\hat{\tau}) - H(\tau)$ gives an upper bound for the cost improvement that could be achieved. A confidence limit for $H(\hat{\tau}) - H(\tau)$ cannot be obtained by direct application of the delta method because $H'(\tau) = 0$ by definition of τ. Hence, up to a first-order approximation, $\hat{\sigma}^2_{H(\hat{\tau})}$ is zero. However, it is possible to overcome this problem by taking a second-order approximation,

$$H(\hat{\tau}) - H(\tau) \approx H'(\tau)(\hat{\tau} - \tau) + \frac{H''(\tau)}{2}(\hat{\tau} - \tau)^2$$

$$= \hat{\sigma}^2_{\hat{\tau}} \frac{H''(\tau)}{2} \left(\frac{\hat{\tau} - \tau}{\hat{\sigma}_{\hat{\tau}}} \right)^2.$$

Because the last factor in the last term has asymptotically a χ^2 distribution with one degree of freedom, it follows that $H(\hat{\tau}) - H(\tau) \leq \hat{\sigma}^2_{\hat{\tau}} H''(\tau) \chi^2_{1-\beta}(1)/2$ with approximate probability $(1 - \beta)$, where $\chi^2_{1-\beta}(1)$ denotes the $1 - \beta$ quantile of the χ^2 distribution with one degree of freedom. From Equation (3), we have

$$H''(\tau) = \left\{ C_{MR} \left[\tau^2 \rho'(\tau) - 2\tau\rho(\tau) + 2 \int_0^\tau \rho(u)\, du \right] + 2C_{PM} \right\} \div \tau^3.$$

Because τ satisfies Equation (4), it follows that $H''(\tau) = C_{MR}\rho'(\tau)/\tau \approx C_{MR}\hat{\rho}'(\hat{\tau})/\hat{\tau}$, where $\hat{\rho}'$ is obtained by replacing $\boldsymbol{\mu}$ by $\hat{\boldsymbol{\mu}}$ in the expression for ρ'. Hence, the $100(1-\beta)\%$ upper confidence limit for $H(\hat{\tau}) - H(\tau)$ becomes

$$C_{MR}\, \hat{\sigma}^2_{\hat{\tau}} \frac{\hat{\rho}'(\hat{\tau})}{2\,\hat{\tau}}\, \chi^2_{1-\beta}(1). \tag{8}$$

Numerical Example

Power transformers are the basic components of an electrical power-transmission system. They are complex, and most of their repairs involve the replacement of only a small fraction of their constituent parts. Hence, it is reasonable to assume that the system's reliability after a transformer repair is essentially the same as it was immediately before the failure.

Table 1 presents the repair and failure records between January 1999 and July 2001 for a group of 300- and 345-kilovolt transformers belonging to the electrical power company. There were 30 transformers, and 21 failure times were recorded. No action was taken and no failures occurred for 10 units, all of which were censored at 21,888 hours. There were 11 PM actions in the follow-up period, each treated as a perfect action, returning the system to like-new condition. Hence, in effect, there were $n = 41 = 30 + 11$ transformers. For instance, the third transformer failed at 10,445 hours, received PM at 13,533 hours, and was finally censored at 21,435 hours. This information was considered as two different systems: the first one failed at 10,445 and was censored at 13,533 hours, while the second one never failed and was censored at $7{,}902 = 21{,}435 - 13{,}533$ hours.

TABLE 1. Power Transformers Data Set: Censoring Times Are Enclosed Between Parentheses

Unit	Failures and PMs times (hours)			
1	8,839	17,057	(21,887)	
2	9,280	16,442	(21,887)	
3	10,445	(13,533)*	(21,435)	
4	(8,414)*	(21,745)		
5	17,156	(21,887)		
6	16,305	(21,887)		
7	16,802	(21,887)		
8	(4,881)*	(21,506)		
9	7,396	7,541	(19,590)*	(21,711)
10	15,821	19,746	(19,877)*	(21,804)
11	15,813	(21,886)		
12	15,524	(21,886)		
13	(21,440)*	(21,809)		
14	11,664	17,031	(21,857)	
15	(7,544)*	(13,583)*	15,751	(20,281)
16	18,840	(21,879)		
17	(2,288)*	(4,787)*		
18	10,668	(16,838)		
19	15,550	(21,887)		
20	(1,616)*	15,657	(21,620)	

Censoring due to performance of preventive maintenance are indicated by a * following the corresponding time.

Ten units censored at 21,888 hours had no failures and are not included in the table.

Figure 2 presents the cumulative number of failures versus time (in hours). This graph is useful in estimating the intensity-function tendency (Meeker and Escobar (1998)). There is a clear indication that the intensity function is increasing, therefore justifying preventive maintenance. As was said before, the company is working under a restrictive policy concerning nonscheduled maintenance. This policy makes the cost of an MR performed after a failure to be 15 times the cost of a scheduled PM, i.e., $C_{MR}/C_{PM} = 15$. Our analysis is based on the power law intensity function.

The MLEs of the intensity function parameters were $\hat{\beta} = 1.988$, $\hat{\theta} = 24{,}844$ and $\hat{\tau} = 6400$ hours (or about 267 days). Approximate standard deviations were $\hat{\sigma}_{\hat{\beta}} = 0.401$ and $\hat{\sigma}_{\hat{\theta}} = 2973.1$. The estimated correlation for $\hat{\beta}$ and $\hat{\theta}$ was -0.34. Hence, the estimated standard deviation of $\hat{\tau}$ was 1724 hours. The approximate 95% confidence limits for τ are 3021

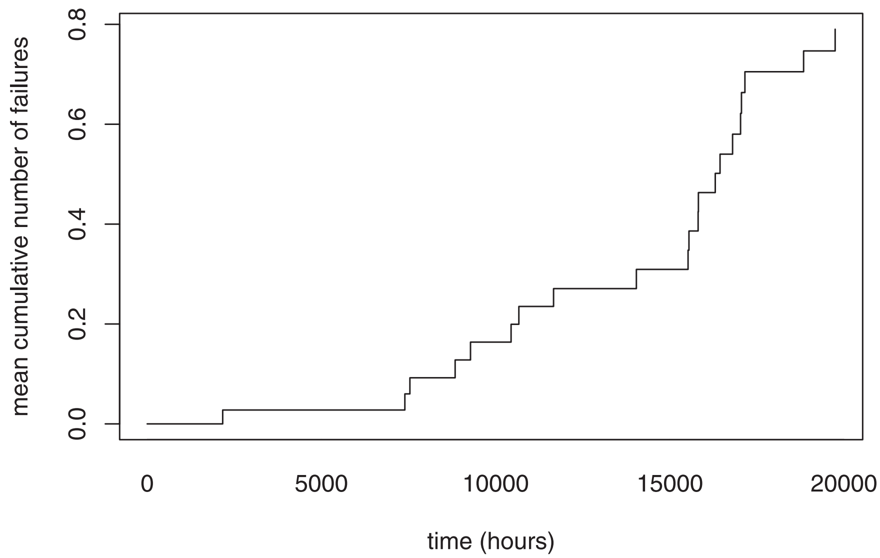

FIGURE 2. Mean Cumulative Number of Failures Versus Time.

and 9780 hours, obtained using the usual expression $\hat{\tau} \pm (1.96)\hat{\sigma}_{\hat{\tau}} = 6400 \pm (1.96)(1724)$.

In order to compute the approximate 95% confidence limit for the expected cost, note that, for the PLP, it follows from Equation (5) that $\rho'(\hat{\tau}) = \beta C_{PM}/C_{MR}\hat{\tau}^2$. Hence, Equation (8) becomes

$$C_{PM}\frac{(1724)^2(1.988)}{2(6400)^3}(1.96)^2 \doteq (4.6)10^{-5}C_{PM}$$

in monetary units per hour. In other words, with approximately 95% confidence, the company would lose at most $(365)(24)(0.000046)C_{PM} \doteq (0.40)C_{PM}$ monetary units per year per transformer by doing PM every $\hat{\tau} = 6400$ hours compared with the true optimal policy if perfect information about β and θ were available.

The confidence interval for τ seems to be quite large, suggesting that it could be convenient to continue observing the transformers. An indication that such a decision is economically reasonable could be obtained by considering the above upper limit for $H(\hat{\tau}) - H(\tau)$ along with the number of transformers operated by the company and the sampling costs.

Final Remarks

An optimal preventive maintenance check point was obtained for a repairable system that will be operated for an indefinitely long time and for which only two repair actions are available: perfect preventive maintenance returning the system to the good-as-new condition and minimal repair after a failure, returning the system to exactly the same condition as it was in just before the failure. A graphical interpretation was provided for the optimal preventive-maintenance policy. Furthermore, maximum likelihood estimation and confidence limits were discussed for the optimal check points. An error bound for the difference between the expected cost using the optimal strategy and the minimum cost expected was presented. This error bound might be more useful than an error bound for the optimal check point estimate. These results were applied for a real data set concerning repair and maintenance times for a set of electrical power transformers.

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