MAP 565
Time series analysis : Lecture II

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December 9, 2015

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Outline of lecture II

Spectral measure

Given a function $\gamma : \mathbb{Z} \to \mathbb{C}$, does there exist a weakly stationary process $(X_t)_{t \in \mathbb{Z}}$ with autocovariance γ?

Herglotz Theorem

Let $\gamma : \mathbb{Z} \to \mathbb{C}$. Then the two following assertions are equivalent:

(i) γ is hermitian symmetric and non-negative definite.

(ii) There exists a finite non-negative measure ν on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ such that,

$$\text{for all } t \in \mathbb{Z}, \quad \gamma(t) = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t} \, \nu(\mathrm{d}\lambda) . \tag{1}$$

When these two assertions hold, ν is uniquely defined by (1).

Spectral density

If moreover $\gamma \in \ell^1(\mathbb{Z})$, these assertions are equivalent to

$$f(\lambda) := \frac{1}{2\pi} \sum_{t \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\lambda t} \gamma(t) \geq 0 \text{ for all } \lambda \in \mathbb{R} ,$$

and ν has density f (that is, $\nu(\mathrm{d}\lambda) = f(\lambda)\mathrm{d}\lambda$).

Definition : spectral measure and spectral density

If γ is the autocovariance of a weakly stationary process X, the corresponding measure ν is called the spectral measure of X. Whenever the spectral measure ν admits a density f, it is called the spectral density function.

Examples

- Let $X \sim \mathrm{WN}(\mu, \sigma^2)$. Then $f(\lambda) = \frac{\sigma^2}{2\pi}$.
- Let X be a weakly stationary process with covariance function γ/spectral measure ν. Define

$$Y = \sum_k \psi_k \, \mathrm{B}^k \circ X$$

for a finitely supported sequence ψ. Recall that Y is a weakly stationary process with covariance function

$$\gamma'(\tau) = \sum_{\ell,k} \psi_k \overline{\psi_\ell} \gamma(\tau + \ell - k) \; .$$

Then Y is a weakly stationary process with spectral measure ν' having density $\lambda \mapsto \left|\sum_k \psi_k \mathrm{e}^{-\mathrm{i}\lambda k}\right|^2$ with respect to ν,

$$\nu'(\mathrm{d}\lambda) = \left|\sum_k \psi_k \mathrm{e}^{-\mathrm{i}\lambda k}\right|^2 \; \nu(\mathrm{d}\lambda) \; .$$

A special one : the harmonic process

Let $(A_k)_{1\leq k\leq N}$ be N real valued L^2 random variables. Denote $\sigma_k^2 = \mathbb{E}\left[A_k^2\right]$. Let $(\Phi_k)_{1\leq k\leq N}$ be N i.i.d. random variables with a uniform distribution on $[0,\pi]$, and independent of $(A_k)_{1\leq k\leq N}$. Define

$$X_t = \sum_{k=1}^{N} A_k \cos(\lambda_k t + \Phi_k)\ , \tag{2}$$

where $(\lambda_k)_{1\leq k\leq N} \in [-\pi,\pi]$ are N frequencies. The process (X_t) is called a harmonic process. It satisfies $\mathbb{E}\left[X_t\right] = 0$ and, for all $s,t\in\mathbb{Z}$,

$$\mathbb{E}\left[X_s X_t\right] = \frac{1}{2}\sum_{k=1}^{N} \sigma_k^2 \cos(\lambda_k(s-t))\ .$$

Hence X is weakly stationary with autocovariance

$$\gamma(t) = \frac{1}{2}\sum_{k=1}^{N} \sigma_k^2 \cos(\lambda_k t)\ .$$

Spectral representation of the harmonic process

We deduce that X has spectral measure

$$\mu = \frac{1}{4} \sum_{k=1}^{N} \sigma_k^2 \left(\delta_{\lambda_k} + \delta_{-\lambda_k}\right) ,$$

where we denote by δ_λ the Dirac mass at point λ.

Similarly, we can write

$$\begin{aligned} X_t &= \frac{1}{2} \sum_{k=1}^{N} \left(A_k \mathrm{e}^{\mathrm{i}\Phi_k} \, \mathrm{e}^{\mathrm{i}\lambda_k t} + A_k \mathrm{e}^{-\mathrm{i}\Phi_k} \, \mathrm{e}^{-\mathrm{i}\lambda_k t} \right) \\ &= \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t} \, \mathrm{d}W(\lambda) , \end{aligned}$$

where W is the random (complex valued) measure

$$W = \frac{1}{2} \sum_{k=1}^{N} \left(A_k \mathrm{e}^{\mathrm{i}\Phi_k} \, \delta_{\lambda_k} + A_k \mathrm{e}^{-\mathrm{i}\Phi_k} \, \delta_{-\lambda_k} \right) .$$

Spectral representation

One can interpret the relation between X and W as saying that W is the Fourier transform of X, so we denote it by $\widehat{X}$:

$$X_t = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t}\, \mathrm{d}\widehat{X}(\lambda), \qquad t \in \mathbb{Z}\ .$$

This spectral representation of X can be extended to any weakly stationary processes with some remarkable properties on $\widehat{X}$.

But some work is necessary.

- The paths of X are random sequences, usually unbounded (no decay at infinity can be used!) so $\mathrm{d}\widehat{X}$ cannot be in the "nice" form $\widehat{X}(\lambda)\mathrm{d}\lambda$.
- Instead $\widehat{X}$ always is a random measure defined on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$.
- For the same reason, there is no simple formula for defining $\widehat{X}$ from X : we rely on Hilbert geometry.

Why is it useful?

Recall the backshift operator $\mathrm{B} : (x_t)_{t\in\mathbb{Z}} \mapsto (x_{t-1})_{t\in\mathbb{Z}}$.
Observe that from

$$X_t = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t}\, \mathrm{d}\widehat{X}(\lambda), \qquad t \in \mathbb{Z} ,$$

we get that

$$(\mathrm{B}\, X)_t = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t}\, \mathrm{e}^{-\mathrm{i}\lambda} \mathrm{d}\widehat{X}(\lambda) \Rightarrow \mathrm{d}\widehat{\mathrm{B}(X)}(\lambda) = \mathrm{e}^{-\mathrm{i}\lambda}\, \mathrm{d}\widehat{X}(\lambda) .$$

More generally, if $g = \sum_k \alpha_k\, \mathrm{B}^k$ for some finitely supported sequence $(\alpha_t)_{t\in\mathbb{Z}}$, we get

$$\mathrm{d}\widehat{g(X)}(\lambda) = \widehat{g}(\lambda)\, \mathrm{d}\widehat{X}(\lambda) \quad \text{with} \quad \widehat{g}(\lambda) = \sum_k \alpha_k \mathrm{e}^{-\mathrm{i}\lambda k} .$$

This will allow us to come up with linear operators g directly described by the function $\widehat{g}$ (under quite general conditions).

Random fields with orthogonal increments

In the following we let $(\mathbb{X}, \mathcal{X})$ be a measurable space.

Definition : Random fields with orthogonal increments

Let η be a finite non-negative measure on $(\mathbb{X}, \mathcal{X})$. Let $W = (W(A))_{A \in \mathcal{X}}$ be an L^2 process indexed by $\mathcal{X}$. It is called a random field with orthogonal increments and intensity measure η if it satisfies the following conditions.

(i) For all $A \in \mathcal{X}$, $\mathbb{E}[W(A)] = 0$.

(ii) For all $A, B \in \mathcal{X}$, $\mathrm{Cov}(W(A), W(B)) = \eta(A \cap B)$.

Consequence

For all $A, B \in \mathcal{X}$ such that $A \cap B = \emptyset$, $W(A)$ and $W(B)$ are uncorrelated and $W(A \cup B) = W(A) + W(B)$.

Example

We denote by δ_λ the Dirac mass at point λ.

Let λ_k, $k = 1, \ldots, n$ be fixed elements of $\mathbb{X}$. Let $Y_1, \ldots, Y_n$ be centered L^2 uncorrelated random variables with variances $\sigma_1^2, \ldots, \sigma_n^2$. Then

$$W = \sum_{k=1}^{n} Y_k \, \delta_{\lambda_k}$$

is a random field with orthogonal increments and intensity measure

$$\eta = \sum_{k=1}^{n} \sigma_k^2 \, \delta_{\lambda_k} \; .$$

Stochastic integral

Let W be a random field with orthogonal increments defined on $(\Omega, \mathcal{F}, \mathbb{P})$, with intensity measure η on $(\mathbb{X}, \mathcal{X})$.

The stochastic integral with respect to W is defined by the following steps.

- Step 1 We set $W(\mathbb{1}_A) = W(A)$, which defines a unitary operator from $\{\mathbb{1}_A,\, A \in \mathcal{X}\} \subset L^2(\mathbb{X}, \mathcal{X}, \eta)$ to $L^2(\Omega, \mathcal{F}, \mathbb{P})$.
- Step 2 Extend this unitary operator linearly on $\mathrm{Span}\,(\mathbb{1}_A,\, A \in \mathcal{X})$.
- Step 3 Extend this unitary operator continuously to the L^2-sense closure $\overline{\mathrm{Span}}\,(\mathbb{1}_A,\, A \in \mathcal{X}) = L^2(\mathbb{X}, \mathcal{X}, \eta)$.
- Step 4 One obtains a $L^2(\mathbb{X}, \mathcal{X}, \eta) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ unitary linear operator. We denote

$$W(g) = \int g \, \mathrm{d}W \;, \qquad g \in L^2(\mathbb{X}, \mathcal{X}, \eta) \;.$$

Conversely, any $L^2(\mathbb{X}, \mathcal{X}, \eta) \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ centered unitary linear operator defines a random field W with intensity measure η.

Application to the construction of weakly stationary processes

Let W be a random field with orthogonal increments with intensity measure η on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$.

Define, for all $t \in \mathbb{Z}$,

$$X_t = \int \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}W(\lambda) \; .$$

Then we have $\mathbb{E}\left[X_t\right] = 0$ and

$$\mathrm{Cov}\left(X_s, X_t\right) = \langle X_s, X_t \rangle = \left\langle \mathrm{e}^{\mathrm{i}s\cdot}, \mathrm{e}^{\mathrm{i}t\cdot} \right\rangle = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}(s-t)\lambda} \, \mathrm{d}\eta(\lambda) \; ,$$

We get a centered weakly stationary process with spectral measure η.

Construction of the spectral random field

Conversely, let $(X_t)_{t\in\mathbb{Z}}$ be a centered weakly stationary with spectral measure η.

Step 1 Define

$$\mathcal{H}_\infty^X = \overline{\mathrm{Span}}\,(X_t,\, t\in\mathbb{Z})\ .$$

Step 2 As previously, we can extend $X_t \mapsto \mathrm{e}^{\mathrm{i}t\cdot}$ linearly and continuously as a unitary linear operator from $\mathcal{H}_\infty^X$ to $L^2(\mathbb{T},\mathcal{B}(\mathbb{T}),\eta)$.

Step 3 Since $\overline{\mathrm{Span}}\left(\mathrm{e}^{\mathrm{i}t\cdot},\, t\in\mathbb{Z}\right) = L^2(\mathbb{T},\mathcal{B}(\mathbb{T}),\eta)$, this operator is bijective.

Step 4 Let $\widehat{X}$ be its inverse operator.

Then $\widehat{X}$ is a random field with orthogonal increments with intensity measure η on $(\mathbb{T},\mathcal{B}(\mathbb{T}))$.

Spectral representation

Moreover, by construction, every $Y \in \mathcal{H}_\infty^X$ can be represented as

$$Y = \int g(\lambda) \, \mathrm{d}\widehat{X}(\lambda) \; .$$

for a (unique) well chosen $g \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \eta)$.

In particular, for all $t \in \mathbb{Z}$,

$$X_t = \int \mathrm{e}^{\mathrm{i}t\lambda} \, \mathrm{d}\widehat{X}(\lambda) \; .$$

and $\widehat{X}$ is called the spectral representation of X.

Example: complex-valued Harmonic processes

The previous definition of harmonic processes can be extended as follows.

Definition : Harmonic processes

The process $(X_t)_{t\in\mathbb{Z}}$ is an harmonic process if its spectral representation $\widehat{X}$ is of the form

$$\widehat{X} = \sum_{k=1}^{n} Z_k \delta_{\lambda_k} ,$$

where $\lambda_1, \ldots, \lambda_n$ are deterministic frequencies in $\mathbb{T}$ and $Z_1, \ldots, Z_n$ are uncorrelated centered $\mathbb{C}$-valued random variables.

Example: real-valued Harmonic processes

To obtained a real valued process $\widehat{X}$ must satisfy an hermitian symmetry $\widehat{X}(-A) = \overline{\widehat{X}(A)}$.

Hence, for a real valued harmonic process, we obtain for $0 < \lambda_0 < \cdots < \lambda_n \leq \pi$,

$$\widehat{X} = Z_0 \delta_0 + \sum_{k=1}^{N} (Z_k \delta_{\lambda_k} + \overline{Z_k} \delta_{-\lambda_k}) \, ,$$

where $Z_0, Z_1, \ldots, Z_N, \overline{Z_1}, \ldots, \overline{Z_N}$ are uncorrelated centered $\mathbb{C}$-valued random variables and Z_0 is real valued.

(Recall our previous example where $Z_k = \frac{1}{2} A_k \mathrm{e}^{\mathrm{i}\Phi_k}$.)

Examples

Centered white noise

If $(X_t)_{t\in\mathbb{Z}} \sim \mathrm{WN}(0, \sigma^2)$ then $\widehat{X}$ satisfies

$$\mathrm{Var}\left(\widehat{X}((\lambda', \lambda])\right) = \frac{\sigma^2}{2\pi}\,(\lambda - \lambda')\;, \quad \lambda' < \lambda < \lambda' + 2\pi\;.$$

Linear filtering

Let $(X_t)_{t\in\mathbb{Z}}$ be centered, weakly stationary with spectral measure ν and spectral representation $\widehat{X}$. Then for any $\widehat{g} \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \nu)$, one can define a centered, weakly stationary process $(Y_t)_{t\in\mathbb{Z}}$ by its spectral representation $\widehat{Y}(\mathrm{d}\lambda) = \widehat{g}(\lambda)\,\widehat{X}(\mathrm{d}\lambda)$,

$$Y_t = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\,t\lambda}\,\widehat{Y}(\mathrm{d}\lambda) = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\,t\lambda}\,\widehat{g}(\lambda)\widehat{X}(\mathrm{d}\lambda)\;,$$

and $(Y_t)_{t\in\mathbb{Z}}$ is centered, weakly stationary with spectral measure $\nu'(\mathrm{d}\lambda) = |\widehat{g}(\lambda)|^2\,\nu(\mathrm{d}\lambda)$.

```
#############################################
# Harmonic proc. converging to white noise #
#############################################
# Settings
# lag between the numbers of frequencies
p <- 4
# number of plots
pp <- 100
# maximal number of frequencies
n <- p*pp
# length of the time series
l <- 2^9
# waiting time after each plot (seconds)
x <- 0.1
# random generators
# random phases and  amplititudes
phase <- runif(n)*2*pi; amp <- rnorm(n)
# set of frequencies picked randomly
lam <- sample((1:n)*pi/(n+1), n , replace = FALSE)

# grenerate signals by adding frequencies
# init. time and signal
tt <- 1:l; sig <- 0
op <- par(mfrow=c(2,1))
for (i in seq(from=1,to=n,by=p)){
  for (j in (i:(i+p-1))){
    sig <- sig+amp[j]*cos(lam[j]*tt+phase[j])
  }
  # plot spectral representation
  plot(lam[1:i],amp[1:i],type='h',
       xlim=c(0,pi),ylim=c(min(amp),max(amp)),
       xlab='Frequencies',ylab='Amplitudes',
       main=paste("Spectral representation with",
         i,"frequencies"))
  abline(h=0)
  # plot signal with normalized variance
```

```
  plot(tt,sig/sqrt(2*i),type='l',
       xlim=c(0,l),ylim=c(-2.5,2.5),
       xlab='time',ylab='Signal',
       main=paste("Harmonic process with",i,"frequencies"))
  Sys.sleep(x)
  }
par(op)

# have a look on the sample autocorrelations
acf(sig, lag.max=30)




############################################
#  White noise with Poisson spectral rep.  #
############################################

# length
n <- 2^9

# Poisson intensity
mu <- 5/pi

# generator of a process with Poisson spectral field
rpoispectral <- function(n=2^8,mu=1/pi)
{
  # grenerate Poisson processes (real and imag. part)
  N1 <- rpois(1,pi*mu)
  N2 <- rpois(1,pi*mu)

  # sum up the frequencies
  tt <- 0:(n-1)
  sig <- rep(0,n)

  if (N1>0) {
    lam1 <- runif(N1)*pi
```

```
    for (i in 1:N1){
      sig <- sig + 2 * cos(tt*lam1[i])
    }
  }
  if (N2>0) {
    lam2 <- runif(N2)*pi
    for (i in 1:N2){
      sig <- sig - 2 * sin(tt*lam2[i])
    }
  }

  ttt <- (1:(n-1))
  centering <- mu*c(2*pi,-4/ttt*(ttt %% 2))


  sc <- (sig-centering)/sqrt(4*pi*mu)
  return(sc)
}

sig <- rpoispectral(n=n,mu=mu)

# how it looks like
ts.plot(sig)

# Sample autocorrelations
l <- 30
acf(sig, lag.max=l)

# Monte Carlo Autocorrelations
NMC <- 2^12
allsig <- sig[1:l+1]
# simulate many processes
for (j in 2:NMC){
  allsig <- cbind(allsig,rpoispectral(n=l+1,mu=mu))
}
#deduce estimate of the cov. function
```

```
ref <- allsig[1,]-mean(allsig[1,])
acov <- numeric()
for (j in (0:l)){
  acov <- c(acov,mean(ref*allsig[j+1,]))
}

rho <- acov/acov[1]
plot(rho,type='h')
abline(h=0)
```