MAP 565
Time series analysis : Lecture I

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Telecom ParisTech – École Polytechnique

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Schedule

- Lectures
  - When : Wednesday 10:45–12:45 (check the dates).
  - Faculty : François Roueff (Telecom ParisTech)
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- Class works (PC)
  - When : Wednesday 14:00–16:00 or 16:15–18:15 (same dates).
  - Faculty : Olivier Cappé (CNRS), Marc Lavielle (INRIA)
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Exams

- A written exam
  - When : Wednesday, February 17, check the schedule.
  - Mark between 0 and 20.
- An optional short project.
  - One or two students.
  - A typed report including numerical experiments.
  - Provides a bonus between 0 and 3 added to the written exam mark.
  - Subjects available soon.
- A literal mark (from A to F) is deduced.

Outline of the course

- ▷ Stochastic modeling ⇐
  - I Random processes.
  - II Spectral representation.
- ▷ Linear models
  - III Innovation process.
  - IV ARMA processes.
  - V Linear forecasting.
- ▷ Statistical inference
  - VI Overview of goals and methods
  - VII Asymptotic statistics in a dependent context.
- ▷ Non-linear models
  - VIII Standard models for financial time series.
  - IX Complements.

⇐ : we are here.

Outline of lecture I

1. A brief introduction
2. Stochastic modelling of Time series
   - Random processes
   - Examples
   - Complement
3. Stationarity
   - Strict Stationarity
   - L^2 processes
   - An illustrative example with R
   - Weak stationarity

Examples of applications

Time series analysis based on stochastic modeling is applied in various fields :

- Health : physiological signal analysis (image analysis).
- Engineering : monitoring, anomaly detection, localizing/tracking.
- Audio data : analysis, synthesis, coding.
- Ecology : climatic data, hydrology.
- Econometrics : economic/financial data.
- Insurance : risk analysis.

Heartbeats

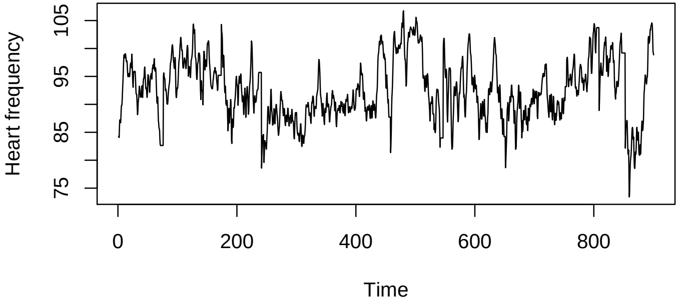

Figure: Heart rate of a resting person over a period of 900 seconds. This rate is defined as the number of heartbeats per unit of time. Here the unit is the minute and is evaluated every 0.5 seconds.

Internet traffic

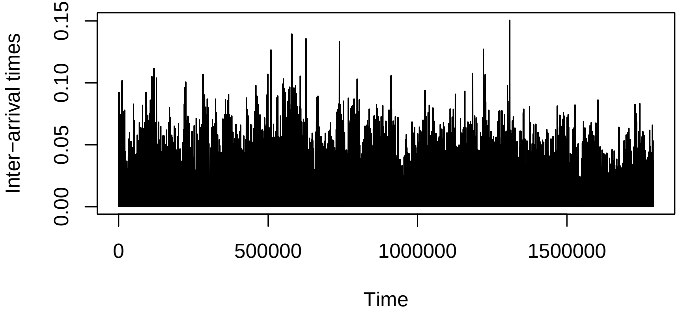

Figure: Inter-arrival times of TCP packets, expressed in seconds, obtained from a 2 hours record of the traffic going through an Internet link. http://ita.ee.lbl.gov/.

Speech audio data

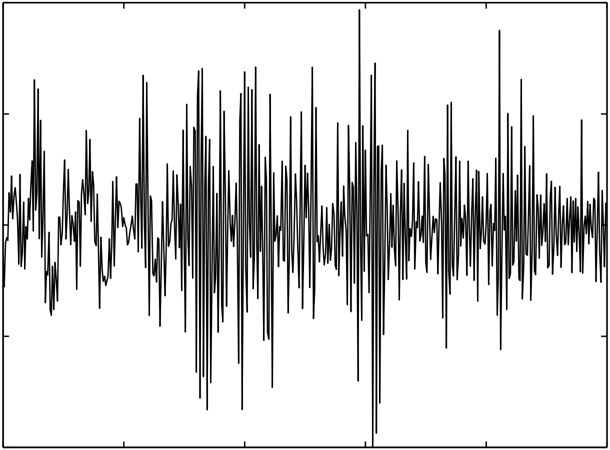

Figure: A speech audio signal with a sampling frequency equal to 8000 Hz. Record of the unvoiced fricative phoneme *sh* (as in *sh*arp).

Climatic data: wind speed

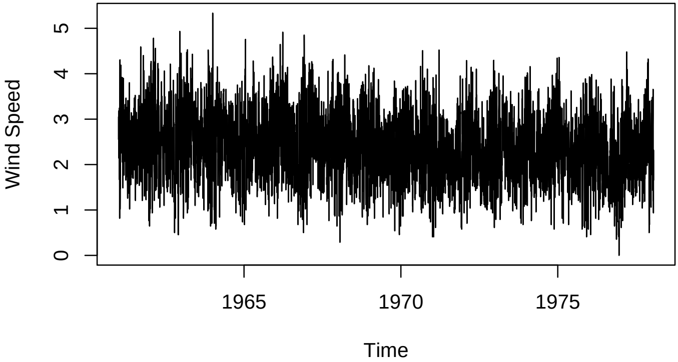

Figure: Daily record of the wind speed at Kilkenny (Ireland) in knots (1 knot = 0.5148 metres/second).

Climatic data: temperature changes

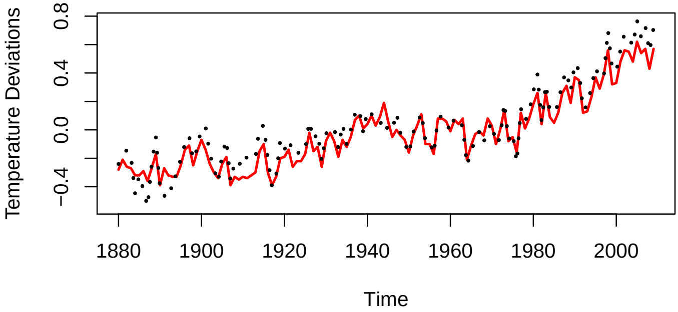

Figure: Global mean land-ocean temperature index (solid red line) and surface-air temperature index (dotted black line).
http://data.giss.nasa.gov/gistemp/graphs/.

Gross National Product of the USA

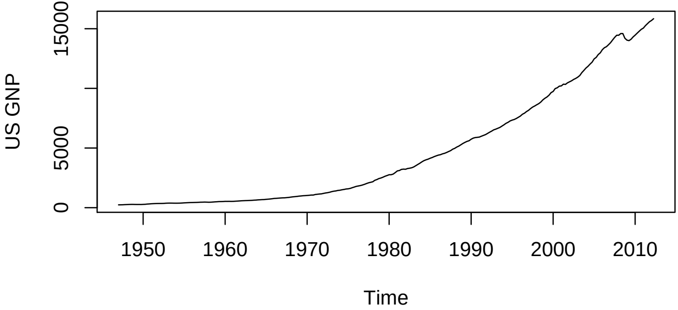

Figure: Growth national product (GNP) of the USA in Billions of $s. http://research.stlouisfed.org/fred2/series/GNP.

GNP quarterly rate

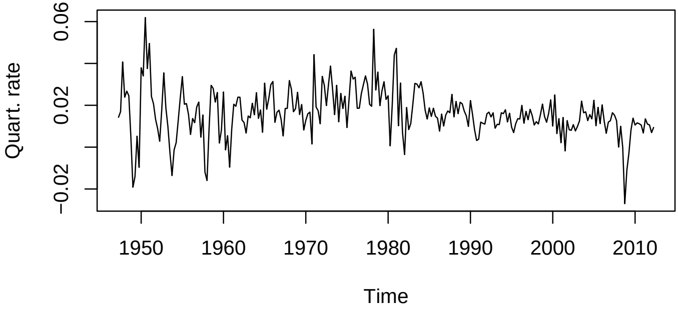

Figure: Quarterly rate of the US GNP.

Financial index

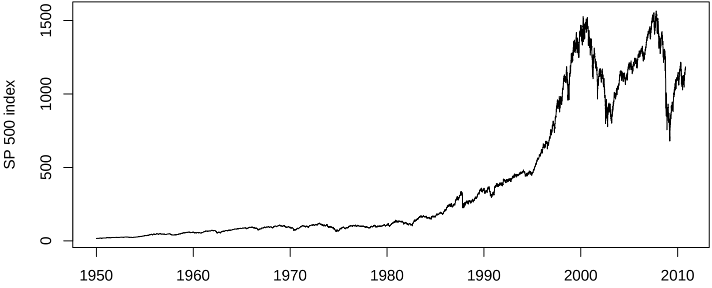

Figure: Daily open value of the Standard and Poor 500 index. This index is computed as a weighted average of the stock prices of 500 companies traded at the New York Stock Exchange (NYSE) or NASDAQ.

Financial index: log returns

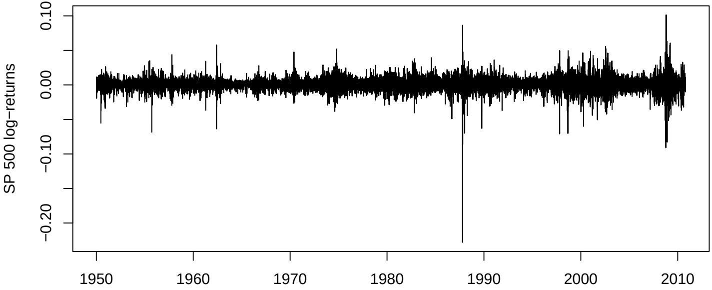

Figure: SP500 log-returns.

Main goals of time series analysis

- Stochastic modelling : trend (seasonal,linear, ...) + noise (with some "structural properties").
- Statistical inference : estimate the parameters of the model, test hypotheses (detect the presence of a signal, classify signals).
- Forecasting : based on a stochastic model, use historical data to predict future values.
- Filtering and tracking : estimate hidden (indirectly observed) variables and track them.
- Change point detection : find out as soon as possible whether the time series evolve through statistically significant changes (anomaly detection).

Stochastic modelling

Definition : random processes

A random or stochastic process valued in $(E, \mathcal{E})$ and indexed on T is a collection of random variables $(X_t)_{t\in T}$ defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

In the following , we generally consider t as a time index, in which case $T = \mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$... A spatial index can also be considered, say $T = \mathbb{R}^d$.

Note that a random vector of length n can be seen as a random process $(X_t)_{t\in T}$ with $T = \{1, \ldots, n\}$.

Random path

Definition : path

Let $(X_t)_{t\in T}$ be a random process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The path of the random experiment $\omega \in \Omega$ is defined as $(X_t(\omega))_{t\in T}$ viewed as an element of E^T.

Let $\mathcal{E}^{\otimes T}$ is the smallest σ-field of E^T containing the cylinder sets

$$\prod_{t\in I} A_t \times E^{T\setminus I}, \qquad \text{with } I \in \mathcal{I}(T), \quad \text{and } A_t \in \mathcal{E} \text{ for all } t \in I \ ,$$

where

$$\mathcal{I}(T) = \{I \subset T, I \text{ finite}\} \ ,$$

It is also also the smallest σ-field on E^T which makes ξ_t measurable for all $t \in T$, where ξ_t is the canonical projection $\xi_t : (x_s)_{s\in T} \mapsto x_t$ from E^T to $(E, \mathcal{E})$.

Law of X

Lemma

The mapping $\omega \mapsto (X_t(\omega))_{t\in T}$ is measurable from $(\Omega, \mathcal{F})$ to $(E^T, \mathcal{E}^{\otimes T})$. We denote this random variable by $X = (X_t)_{t\in T}$.

Definition : law in the sense of fidi distributions

Let $(X_t)_{t\in T}$ be a random process. The law of the process in the sense of fidi distributions is defined as the image probability measure $\mathbb{P}^X = \mathbb{P} \circ X^{-1}$ on $(E^T, \mathcal{E}^{\otimes T})$.

We denote

$$X \overset{\text{fidi}}{=} Y \ ,$$

when X and Y have the same law in the sense of fidi distributions.

Finite dimensional (fidi) distributions

For all $I \in \mathcal{I}(T)$,

(i) denote by Π_I is the canonical projection $(x_t)_{t \in T} \mapsto (x_t)_{t \in I}$,

(ii) denote by X_I the random vector $(X_t)_{t \in I} = \Pi_I \circ X$,

(iii) denote by $\mathbb{P}^{X_I}$ the distribution of X_I, which is defined by

$$\mathbb{P}^{X_I}\left(\prod_{t \in I} A_t\right) = \mathbb{P}\left(X_t \in A_t,\, t \in I\right), \quad \text{where } A_t \in \mathcal{E} \text{ for all } t \in I\,.$$

Observe that, for all $I \in \mathcal{I}(T)$,

$$\mathbb{P}^{X_I} = \mathbb{P} \circ X^{-1} \circ \Pi_I^{-1}$$

By definition of $\mathcal{E}^{\otimes T}$, $\mathbb{P}^{X}$ is characterized by the collection of fidi distributions $(\mathbb{P}_I)_{I \in \mathcal{I}(T)}$.

Back and forth

One can go back and forth from/to :

(a) A collection of r.v.'s $(X_t)_{I \in T}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

(b) The law $(E^T, \mathcal{E}^{\otimes T}, \mathbb{P}^X)$.

(c) The fidi distributions $\left((E^I, \mathcal{E}^{\otimes I}, \mathbb{P}^{X_I})\right)_{I \in \mathcal{I}(T)}$.

We already mentioned: (a)$\rightarrow$(b)$\rightarrow$ (c).

We can also go the other way around:

- (b)$\rightarrow$ (a) is obtained by setting $\Omega = E^T$, $\mathcal{F} = \mathcal{E}^{\otimes T}$, $\mathbb{P} = \mathbb{P}^X$ and defining the process X as the canonical process $X_t = \xi_t$
- (c)$\rightarrow$ (b) follows from Kolmogorov's Theorem (See below for details).

The usual steps

Step 1 Start with a collection of fidi distributions $\left((E^I, \mathcal{E}^{\otimes I}, \nu_I)\right)_{I \in \mathcal{I}(T)}$.

Step 2 Deduce the probability space $(E^T, \mathcal{E}^{\otimes T}, \nu_T)$.

Step 3 Deduce $(\Omega, \mathcal{F}, \mathbb{P})$ and X. Hence we get a process X on $(\Omega, \mathcal{F}, \mathbb{P})$ with the desired fidi distributions.

Step 4 Define new processes by filtering X, for instance $Y_t = g_t(X)$ where $g_t : E^T \to F$ is measurable for all t, or equivalently, $Y = g(X)$ where $g : E^T \to F^T$ is measurable.

Complementary facts

- Let $T = \mathbb{N}, \mathbb{Z}, \mathbb{R}_+$ or $\mathbb{R}$. The process X is adapted to a given filtration $(\mathcal{F}_t)_{t \in T}$ if, for all $t \in T$, X_t is $\mathcal{F}_t$-measurable. Example: natural filtration $\mathcal{F}_t = \sigma(X_s,\ s \leq t)$.
- The filtering step can be implicit : Let Y be the unique solution of the equation $g(X, Y) = 0$ such that $(\dots)$.
- In some cases, one can endow E^T with a metric, and define X as a random element of E^T endowed with the corresponding Borel σ-field.

Independent and i.i.d. processes

Let $(\nu_t)_{t\in T}$ be a collection of probability measures on $(E,\mathcal{E})$.

Then there exists a process $X = (X_t)_{t\in T}$ such that, for all $I \in \mathcal{I}(T)$,

$$\mathbb{P}^{X_I} = \bigotimes_{t\in I} \nu_t \ ,$$

that is, for all $(A_t)_{t\in I} \in \mathcal{E}^I$, we have

$$\mathbb{P}\left(X_t \in A_t \text{ for all } t \in I\right) = \mathbb{P}\left(X_I \in \prod_{t\in I} A_t\right) = \prod_{t\in I} \nu_t(A_t)$$

It is called an independent process with marginal distributions $(\nu_t)_{t\in T}$.

If $\nu_t = \nu$ for all $t \in T$ we say that $(X_t)_{t\in T}$ is a an i.i.d. (independent and identically distributed) process with marginal distribution ν.

Gaussian processes

Let T be an arbitrary set of indices. Let $\mu = (\mu_t)_{t \in T}$ be real-valued and $(\gamma_{s,t})_{s,t \in T}$ be such that, for all $I \in \mathcal{I}(T)$

$$\Gamma_I = [\gamma_{s,t}]_{s,t \in I} \quad \text{is symmetric non-negative definite .}$$

Then there exists a process $(X_t)_{t \in T}$ on a probability space $(\Omega, \mathcal{F}, \xi)$ such that, for all $I \in \mathcal{I}(T)$

$$\mathbb{P}^{X_I} = \mathcal{N}\left((\mu_t)_{t \in I}, \Gamma_I\right) \ .$$

We denote $X \sim \mathcal{N}(\mu, \gamma)$ and say that X is a Gaussian process with mean μ and covariance γ.

Kolmogorov's Theorem

Let $\nu_I := \mathbb{P}^{X_I}$. Note that if $J \subset I$ are in $\mathcal{I}(T)$, then, for all $A \in \mathcal{E}^{\otimes J}$, $\mathbb{P}(X_J \in A) = \mathbb{P}(X_I \in A \times E^{I \setminus J})$, hence

$$\nu_J(A) = \nu_I \left(A \times E^{I \setminus J} \right) . \tag{1}$$

Theorem : Kolmogorov

Let $(E, \mathcal{E})$ be a measurable space, T an arbitrary set of indices and $(\nu_I)_{I \in \mathcal{I}(T)}$ such that each ν_I is a probability on $(E^I, \mathcal{E}^{\otimes I})$. The two following assertions are equivalent.

- (i) $(\nu_I)_{I \in \mathcal{I}(T)}$ satisfies the compatibility condition (1) for all $J \subseteq I$.
- (ii) There is a unique probability ν_T on $(E^T, \mathcal{E}^{\otimes T})$ such that $\nu_I = \nu_T \circ \Pi_I^{-1}$ for all $I \in \mathcal{I}(T)$.

Shift and backshift operators

Suppose that $T = \mathbb{Z}$ or $T = \mathbb{N}$.

Definition : Shift and backshift operators

Let the shift operator $\mathrm{S} : E^T \to E^T$ be defined by

$$\mathrm{S}(x) = (x_{t+1})_{t \in T} \quad \text{for all} \quad x = (x_t)_{t \in T} \in E^T .$$

For all $\tau \in T$, we define S^τ by

$$\mathrm{S}^\tau(x) = (x_{t+\tau})_{t \in T} \quad \text{for all} \quad x = (x_t)_{t \in T} \in E^T .$$

The operator S^{-1} is called the backshift operator, denoted by B.

Strict stationarity

Definition : Strict stationarity

Let $X = (X_t)_{t \in T}$ be a random process defined on $(\Omega, \mathcal{F}, \xi)$ with $T = \mathbb{Z}$ or $T = \mathbb{N}$. We say that X is stationary in the strict sense if

$$X \stackrel{\text{fidi}}{=} \mathrm{S} \circ X \ ,$$

which is equivalent to

$$\mathbb{P} \circ X^{-1} = \mathbb{P} \circ X^{-1} \circ \mathrm{S}^{-1} \ .$$

Examples based on finite distributions

- A constant process,

$$X_t = X_0 \qquad \text{for all } t \in T$$

  is stationary.
- A sequence of independent random variables is strictly stationary if and only if they are indentically distributed. (Thus it is an i.i.d. process).
- Gaussian processes : $X \sim \mathcal{N}(\mu, \Gamma)$ is stationary if and only if $\mu_t = \mu_0$ and $\gamma_{s,t} = \gamma_{s-t,0}$ for all $s, t \in T$.

Examples based on stationarity preserving filters

Suppose that X is stationary. Is $g(X)$ stationary ?

Examples of filters $g : E^T \to F^T$ preserving stationarity :

- Let ψ be a finitely supported sequence and define

$$g = \sum_k \psi_k \, \mathrm{B}^k : x \mapsto \psi \star x \; .$$

- More generally, if

$$g \circ \mathrm{S} = \mathrm{S} \circ g \; ,$$

  then $g(X)$ is also stationary.
- Time reversing operator: $g : (x_t)_{t \in \mathbb{Z}} \mapsto (x_{-t})_{t \in \mathbb{Z}}$. Here

$$g \circ \mathrm{S} = \mathrm{S}^{-1} \circ g \; .$$

L^2 space

We set $E = \mathbb{C}^d$. We denote

$$L^2(\Omega, \mathcal{F}, \mathbb{P}) = \left\{ X \ \mathbb{C}^d\text{-valued r.v. such that } \mathbb{E}\left[|X|^2\right] < \infty \right\} .$$

$(L^2, \langle , \rangle)$ is a Hilbert space with

$$\langle X, Y \rangle = \mathbb{E}\left[X^T \overline{Y}\right] .$$

Definition : L^2 Processes

The process $\mathbf{X} = (\mathbf{X}_t)_{t \in T}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{C}^d$ is an L^2 process if $\mathbf{X}_t \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ for all $t \in T$.

Mean and covariance functions

Let $\mathbf{X} = (\mathbf{X}_t)_{t \in T}$ be an L^2 process.

- Its mean function is defined by $\mu(t) = \mathbb{E}[\mathbf{X}_t]$,
- Its covariance function is defined by

$$\Gamma(s,t) = \mathrm{cov}(\mathbf{X}_s, \mathbf{X}_t) = \mathbb{E}\left[\mathbf{X}_s \mathbf{X}_t^H\right] - \mathbb{E}[\mathbf{X}_s]\,\mathbb{E}[\mathbf{X}_t]^H \ .$$

Linear combinations $\rightarrow$ scalar case

Let $\mathbf{X} = (\mathbf{X}_t)_{t \in T}$ be an L^2 process with mean function μ and covariance function Γ. This is equivalent to say that for all $\mathbf{u} \in \mathbb{C}^d$, $\mathbf{u}^H \mathbf{X}$ is a scalar L^2 process with mean function $\mathbf{u}^H \mu$ and covariance function $\mathbf{u}^H \Gamma \mathbf{u}$.

Scalar case $E = \mathbb{C}$, examples

Hermitian symmetry, non-negative definiteness

For all $I \in \mathcal{I}(T)$, $\Gamma_I = \mathrm{Cov}([X(t)]_{t \in I}) = [\gamma(s,t)]_{s,t \in I}$ is a hermitian non-negative definite matrix.

Examples

- L^2 independent random variables $(X_t)_{t \in \mathbb{Z}}$ have mean $\mu(t) = \mathbb{E}(X_t)$ and covariance

$$\Gamma(s,t) = \begin{cases} \mathrm{var}(X_t) & \text{if } s = t, \\ 0 & \text{otherwise.} \end{cases}$$

- A Gaussian process is an L^2 process whose law is entirely determined by its mean and covariance functions.

```
## HEART BEAT    ###
hrdata='~/data/dataset/sante/heartbeat/hr11839.dat'

# as ts data
hr <- ts(read.table(hrdata), frequency=2)

# plot the time series
plot.ts(hr,ylab='Heart frequency')


# randomly ordering
hr_shuffled <- sample(hr, length(hr) , replace = FALSE)
op <- par(mfrow=c(2,1))
plot.ts(hr,ylab='Heart frequency')
plot.ts(hr_shuffled,ylab='Heart frequency randomly ordered')
par(op)

# marginal distribution via histogram
hist(hr,30)

# two-dimensional distribution
graphics.off()
T <- length(hr)
dev.new(width=10, height=5)
op <- par(mfrow=c(1,2))
hrc <- t(rbind(hr[1:T-1],hr[2:T]))
plot(hrc, xlab='Heart beat at time t',
     ylab='Heart beat at time t+1')
hrc_shuffled <- t(rbind(hr_shuffled[1:T-1],hr[2:T]))
plot(hrc_shuffled, xlab='Shuffled heart beat at time t',
     ylab='Shuffled heart beat at time t+1')

# Keep the first one and add the regression line
lmar <- lsfit(hrc[,1],hrc[,2])
par(op)
plot(hrc,ylab=expression(X[t]),xlab=expression(X[t-1]))
```

```
x <- c(min(hrc[,1]),max(hrc[,1]))
lines(x, lmar$coefficients[1]+lmar$coefficients[2]*x,
      col=2, lty=2,lwd=2)

# pair-wise distributions
graphics.off()
n <- 8
hrcs <- hr[1:(T-(n-1))]
for (i in 1:(n-1)){
  hrcs <- rbind(hrcs,hr[(1+i):(T+i-(n-1))])
}
pairs(t(hrcs))

# correlations up to n
n <- 60
graphics.off()
dev.new(width=10, height=5)
op <- par(mfrow=c(1,2))
acf(hr, lag.max=n, main='Heart beat')
bm <-  ts(rnorm(n=length(hr), mean=0, sd=1), frequency=2)
acf(bm, lag.max=n, main='White noise')
par(op)
```

Weakly stationary processes

Let $T = \mathbb{Z}$. Let X be an L^2 strictly stationary process with mean function μ and covariance function Γ.

Then $\mu(t) = \mu(0)$ and $\gamma(s,t) = \gamma(s-t,0)$ for all $s,t \in T$.

Definition : Weak stationarity

We say that a random process X is weakly stationary with mean μ and autocovariance function $\gamma : \mathbb{Z} \to \mathbb{C}$ if it is L^2 with mean function $t \mapsto \mu$ and covariance function $(s,t) \mapsto \gamma(s-t)$.

The autocorrelation function is defined (when $\gamma(0) > 0$) by

$$\rho(t) = \frac{\gamma(t)}{\gamma(0)} .$$

Examples

An L^2 strictly stationary process is weakly stationary.

- The constant L^2 process has constant autocovariance function.

Strong and weak white noise

- A sequence of L^2 i.i.d. random variables is called a strong white noise, denoted by $X \sim \mathrm{IID}(\mu, \sigma^2)$.
- An L^2 process X with constant mean μ and constant diagonal covariance function equal to σ^2 is called a weak white noise. It is denoted by $X \sim \mathrm{WN}(\mu, \sigma^2)$. (It does not have to be i.i.d.)

Examples based on stationarity preserving linear filters

Let X be weakly stationary with mean μ and autocovariance γ.

In the following examples, $Y = g(X)$ is weakly stationary with mean μ' and autocovariance γ'.

- Let g be the time reversing operator. Then

$$\mu' = \mu \qquad \text{and} \qquad \gamma' = \overline{\gamma}\,.$$

- Let $g = \sum_k \psi_k \, \mathrm{B}^k : x \mapsto \psi \star x$ for a finitely supported sequence ψ.

  Then

$$\mu' = \mu \sum_k \psi_k$$

$$\gamma'(\tau) = \sum_{\ell,k} \psi_k \overline{\psi_\ell} \gamma(\tau + \ell - k) \tag{2}$$

Heartbeats : autoregression

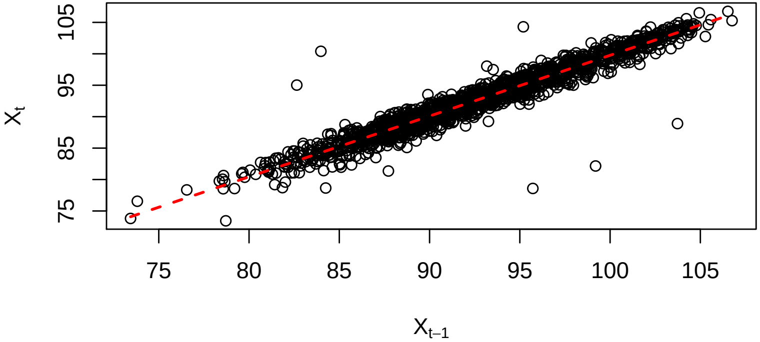

Figure: Illustration of $\gamma(1)$: X_t VS X_{t-1} for the heartbeats data (see Figure 8). The red dashed line is the best linear fit.

Empirical estimates

Suppose you want to estimate the mean and the autocovariance from a sample $X_1, \ldots, X_n$. Define the empirical mean as

$$\widehat{\mu}_n = \frac{1}{n} \sum_{k=1}^{n} X_k \ ,$$

and the empirical autocovariance and autocorrelation functions as

$$\widehat{\gamma}_n(h) = \frac{1}{n} \sum_{k=1}^{n-|h|} (X_k - \widehat{\mu}_n)(X_{k+|h|} - \widehat{\mu}_n) \quad \text{and}$$

$$\widehat{\rho}_n(h) = \frac{\widehat{\gamma}_n(h)}{\widehat{\gamma}_n(0)} \ .$$

Heartbeats : autocorrelation (empirical)

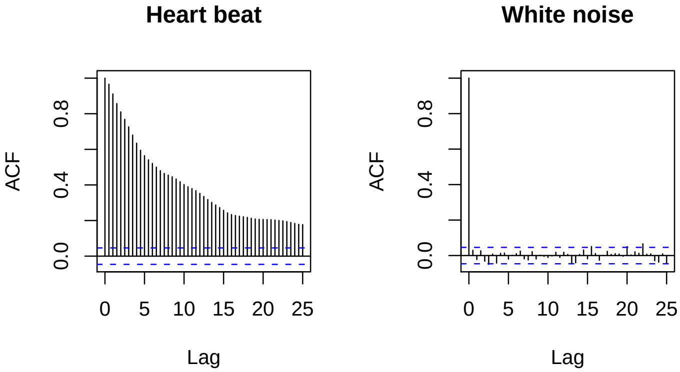

Figure: Left : empirical autocorrelation $\widehat{\rho}_n(h)$ of heartbeat data for $h = 0, \ldots, 100$. Right : the same from a simulated white noise sample with same length.

Spectral measure

Given a function $\gamma : \mathbb{Z} \to \mathbb{C}$, does there exist a weakly stationary process $(X_t)_{t\in\mathbb{Z}}$ with autocovariance γ?

Herglotz Theorem

Let $\gamma : \mathbb{Z} \to \mathbb{C}$. Then the two following assertions are equivalent:

(i) γ is hermitian symmetric and non-negative definite.

(ii) There exists a finite non-negative measure ν on $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ such that, for all $t \in \mathbb{Z}$, $\gamma(t) = \int_{\mathbb{T}} \mathrm{e}^{\mathrm{i}\lambda t}\, \nu(\mathrm{d}\lambda)$.

Spectral density

If moreover $\gamma \in \ell^1(\mathbb{Z})$, these assertions are equivalent to

$$f(\lambda) := \frac{1}{2\pi} \sum_{t \in \mathbb{Z}} \mathrm{e}^{-\mathrm{i}\lambda t} \gamma(t) \geq 0 \text{ for all } \lambda \in \mathbb{R} ,$$

and ν has density f (that is, $\nu(\mathrm{d}\lambda) = f(\lambda)\mathrm{d}\lambda$).

Definition : spectral measure and spectral density

If γ is the autocovariance of a weakly stationary process X, the corresponding measure ν is called the spectral measure of X. Whenever the spectral measure ν admits a density f, it is called the spectral density function.

Examples

- Let $X \sim \mathrm{WN}(\mu, \sigma^2)$. Then $f(\lambda) = \frac{\sigma^2}{2\pi}$.

- Let X be a weakly stationary process with spectral measure ν. Define

$$Y = \sum_k \psi_k \, \mathrm{B}^k \circ X$$

for a finitely supported sequence ψ.

Then, by (2), Y is a weakly stationary process with spectral measure ν' having density $\lambda \mapsto \left|\sum_k \psi_k \mathrm{e}^{-\mathrm{i}\lambda k}\right|^2$ with respect to ν,

$$\nu'(\mathrm{d}\lambda) = \left|\sum_k \psi_k \mathrm{e}^{-\mathrm{i}\lambda k}\right|^2 \nu(\mathrm{d}\lambda) \; .$$

A special one : the harmonic process

Let $(A_k)_{1\leq k\leq N}$ be N real valued L^2 random variables. Denote $\sigma_k^2 = \mathbb{E}\left[A_k^2\right]$. Let $(\Phi_k)_{1\leq k\leq N}$ be N i.i.d. random variables with a uniform distribution on $[-\pi,\pi]$, and independent of $(A_k)_{1\leq k\leq N}$. Define

$$X_t = \sum_{k=1}^{N} A_k \cos(\lambda_k t + \Phi_k) \ , \tag{3}$$

where $(\lambda_k)_{1\leq k\leq N} \in [-\pi,\pi]$ are N frequencies. The process (X_t) is called a harmonic process. It satisfies $\mathbb{E}\left[X_t\right] = 0$ and, for all $s,t\in\mathbb{Z}$,

$$\mathbb{E}\left[X_s X_t\right] = \frac{1}{2}\sum_{k=1}^{N} \sigma_k^2 \cos(\lambda_k(s-t)) \ .$$