MAP 565
Time series analysis : Lecture IV

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Outline of the course

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$\leftarrow$: we are here.

Outline of Lecture IV

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1. Composition and inversion using ℓ^1 convolution filters
   - Example
   - General results
   - Inversion of a FIR filter

2. ARMA processes

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Inverting a convolution filter, an example

Consider the process X defined by

$$X_t = \sum_{k \geq 0} 2^{-k} Z_{t-k}, \qquad t \in \mathbb{Z} ,$$

where $Z \sim \mathrm{WN}(0, \sigma^2)$. This can be rewritten as

$$X = \mathrm{F}_\psi(Z) \quad \text{with} \quad \psi_k = \begin{cases} 2^{-k} & \text{if } k \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

This process satisfies the equation

$$X_t = X_{t-1}/2 + Z_t, \qquad t \in \mathbb{Z} .$$

Or, equivalently, $Z = (\mathrm{I} - \frac{1}{2}\,\mathrm{B})(X)$, so we can obtain Z back from X.

In fact, for any weakly stationary process Z, we have

$$(\mathrm{I} - \frac{1}{2}\,\mathrm{B}) \circ \mathrm{F}_\psi(Z) = \mathrm{F}_\psi \circ (\mathrm{I} - \frac{1}{2}\,\mathrm{B})(Z) = Z .$$

Composition

The convolution product $\star$ is commutative and associative in ℓ^1. So if $\psi, \phi \in \ell^1$, then for all $x \in \ell^1$,

$$\mathrm{F}_{\psi} \circ \mathrm{F}_{\phi}(x) = \psi \star (\phi \star x) = (\psi \star \phi) \star x = \mathrm{F}_{\psi \star \phi}(x) \ .$$

Theorem : composition

Let $\psi, \phi \in \ell^1$. Then, for all random process $X = (X_t)_{t \in \mathbb{Z}}$ such that

$$\sup_{t \in \mathbb{Z}} \mathbb{E}|X_t| < \infty \ ,$$

we have

$$\mathrm{F}_{\psi} \circ \mathrm{F}_{\phi}(X) = \mathrm{F}_{\phi} \circ \mathrm{F}_{\psi}(X) = \mathrm{F}_{\psi \star \phi}(X) \quad \text{a.s.}$$

Important remark

For any $\psi, \phi \in \ell^1$, we have

$$(\psi \star \phi)^* = \psi^* \times \phi^*, \text{ where } \psi^*(\lambda) = \sum_{k \in \mathbb{Z}} \psi_k \mathrm{e}^{-\mathrm{i}\lambda k}.$$

Inversion

Definition : invertible linear representations

Let $\psi \in \ell^1$ and X be a centered weakly stationary process. Consider a process Y which admits the linear representation

$$Y = \mathrm{F}_\psi(X) \iff Y_t = \sum_k \psi_k X_{t-k} , \quad \text{for all } t \in \mathbb{Z} .$$

We will say that this linear representation is invertible if there exists $\phi \in \ell^1$, such that $X = \mathrm{F}_\phi(Y)$.

Sufficient condition : Inverse filter

By the composition theorem, to inverse the linear representation $Y = \mathrm{F}_\psi(X)$ as $X = \mathrm{F}_\phi(Y)$, it is sufficient to have

$$\psi \star \phi = \mathrm{e}_0 \ , \iff \psi^* \times \phi^* = 1 \ ,$$

where e_0 is the impulse sequence, $\mathrm{e}_{0,k} = \mathbb{1}_{\{0\}}(k)$.

Setting

A Finite Impulse Response (FIR) filter is a convolution filter of the form

$$\mathrm{F}_{\psi} = \sum_{k\in\mathbb{Z}} \psi_k \, \mathrm{B}^k \quad \text{for a finitely supported sequence } \psi = (\psi_k)_{k\in\mathbb{Z}} \; .$$

We consider without loss of generality the polynomial case where

$$\mathrm{F}_{\psi} = \Theta(\mathrm{B})$$

with

$$\Theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k \; .$$

Inversion of a FIR filter

In this setting we have

$$\psi^*(\lambda) = \sum_{k\in\mathbb{Z}} \psi_k \, \mathrm{e}^{-\mathrm{i}\lambda k} = \Theta(\mathrm{e}^{-\mathrm{i}\lambda}), \qquad \lambda \in \mathbb{R} \; .$$

We are looking for $\phi \in \ell^1(\mathbb{Z})$ such that $\psi^* \times \phi^* = 1$, hence

$$\phi^*(\lambda) = \frac{1}{\Theta(\mathrm{e}^{-\mathrm{i}\lambda})}, \qquad \lambda \in \mathbb{R} \; , \tag{1}$$

which has a unique solution $\phi \in \ell^1$ if[1] and only if[2] Θ does not vanish on the

unit circle $\quad \Gamma_1 = \left\{\mathrm{e}^{-\mathrm{i}\lambda} \;:\; \lambda \in \mathbb{R}\right\} = \{z \in \mathbb{C} \;:\; |z| = 1\} \; .$

[1] For any 2π-periodic $\mathcal{C}^\infty$ function g, we have $g(\lambda) = \sum_k c_k(g)\mathrm{e}^{-\mathrm{i}\lambda k}$ with $c_k(g) = (2\pi)^{-1} \int_{\mathbb{T}} g(\lambda)\mathrm{e}^{\mathrm{i}\lambda k}\mathrm{d}\lambda = O(|k|^{-p})$ for any $p > 0$ as $k \to \pm\infty$.

[2] ϕ^* has to be bounded if $\phi \in \ell^1(\mathbb{Z})$.

Inversion of a FIR filter: practical computation.

Rewrite (1) as

$$\frac{1}{\Theta(z)} = \sum_{k\in\mathbb{Z}} \phi_k \, z^k, \qquad z \in \Gamma_1 \; .$$

Let us consider the case $\Theta(z) = 1 - \alpha z$.

- ▷ If $|\alpha| < 1$ we have, for all $z \in \Gamma_1$,

$$\frac{1}{1-\alpha z} = \sum_{k\geq 0} \alpha^k z^k \qquad \text{(Causal inverse filter)} \; .$$

- ▷ If $|\alpha| > 1$ we have, for all $z \in \Gamma_1$,

$$\frac{1}{1-\alpha z} = -\sum_{k\leq -1} \alpha^k z^k \qquad \text{(Anticausal inverse filter)} \; .$$

Note that $\phi_k = O(\delta^{|k|})$ as $k \to \pm\infty$ for $\delta = |\alpha| \wedge |\alpha|^{-1}$.

Inversion of a FIR filter: practical computation, cont.

Let us denote by $\alpha_1^{-1}, \ldots, \alpha_d^{-1}$ the distinct roots of Θ with multiplicity orders $m_1, \ldots, m_d$. Note that

$$\Theta \text{ does not vanish on } \Gamma_1 \text{ iff } \alpha_1^{-1}, \ldots, \alpha_d^{-1} \in \mathbb{C} \setminus \Gamma_1 .$$

Then the partial-fraction decomposition of $\dfrac{1}{\Theta}$ gives that, for some coefficients $(\beta_{j,\ell})_{1 \leq \ell \leq m_j}$,

$$\frac{1}{\Theta(z)} = \sum_{j=1}^{d} \sum_{\ell=1}^{m_j} \frac{\beta_{j,\ell}}{(1-\alpha_j z)^\ell} .$$

Noting that

$$\frac{1}{(1-\alpha_j z)^\ell} = \frac{1}{\ell!\,\alpha_j^\ell} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\ell \frac{1}{1-\alpha_j z} ,$$

we easily obtain the general case from the previous case $P(z) = 1 - \alpha z$.

MA(q) process

MA(q) equation

Let $Z \sim \mathrm{WN}(0, \sigma^2)$ and Θ be a polynomial of degree q of the form

$$\Theta(z) = 1 + \sum_{k=1}^{q} \theta_k z^k \ .$$

The associated MA(q) equation is defined by

$$X = [\Theta(\mathrm{B})](Z) \Leftrightarrow X_t = Z_t + \sum_{k=1}^{p} \theta_k Z_{t-k} \text{ for all } t \in \mathbb{Z}. \tag{2}$$

Definition : MA(q) process

The process X of the MA(q) equation (2) is called an MA(q) process with MA coefficients $\theta_1, \ldots, \theta_q$ and white noise Z. Then X is a centered weakly stationary process, and has spectral density $f(\lambda) = \left|\Theta(\mathrm{e}^{-\mathrm{i}\lambda})\right|^2 \dfrac{\sigma^2}{2\pi}$.

MA(q) processes: autocovariance function

The autocovariance function γ of

$$X = [\Theta(\mathrm{B})](Z) = \mathrm{F}_\theta(Z)$$

satisfies

$$\gamma(h) = \begin{cases} \sigma^2 \sum_{t=0}^{q-h} \theta_k \overline{\theta}_{k+h}, & \text{if } 0 \leq h \leq q \;, \\ \sigma^2 \sum_{t=0}^{q+h} \overline{\theta}_k \theta_{k-h}, & \text{if } -q \leq h \leq 0 \;, \\ 0, & \text{otherwise} \;, \end{cases}$$

Important remark

Observe that $\gamma(h) = 0$ for all $|h| > q$. We will show that this property provides a characterization of MA(q) processes.

AR(p) processes

Definition : AR(p) processes

Let $Z \sim \mathrm{WN}(0, \sigma^2)$ and Φ be a polynomial of degree p of the form $\Phi(z) = 1 - \sum_{k=1}^{q} \phi_k z^k$. The associated AR($p$) equation is defined by

$$[\Phi(\mathrm{B})](X) = Z \Leftrightarrow X_t = \sum_{k=1}^{q} \phi_k X_{t-k} + Z_t \text{ for all } t \in \mathbb{Z}.$$

If X is weakly stationary, it is called an AR(p) process.

Theorem

The AR(p) equation admits a weakly stationary solution if and only if Φ does not vanish on Γ_1, in which case it is the unique one and, moreover, it is centered and admits a spectral density $f(\lambda) = \frac{\sigma^2}{2\pi} \frac{1}{\left|\Phi(\mathrm{e}^{-\mathrm{i}\lambda})\right|^2}$.

ARMA(p,q) processes

Definition : ARMA(p,q) processes

Let $Z \sim \mathrm{WN}(0,\sigma^2)$ and Θ, Φ be two coprime polynomials of degree q and p of the same forms as above. The associated ARMA(p,q) equation is defined by

$$[\Phi(\mathrm{B})](X) = [\Theta(\mathrm{B})](Z) \Leftrightarrow X_t = \sum_{k=1}^{q} \phi_k X_{t-k} + Z_t + \sum_{k=1}^{q} \theta_k Z_{t-k} \text{ for all } t \in \mathbb{Z}.$$

If moreover X is weakly stationary, it is called an ARMA(p,q) process.

Theorem

The ARMA(p,q) equation admits a weakly stationary solution if and only if Φ does not vanish on Γ_1, in which case it is the unique one and, moreover, it is centered and admits a spectral density $f(\lambda) = \frac{\sigma^2}{2\pi} \frac{\left|\Theta(\mathrm{e}^{-\mathrm{i}\lambda})\right|^2}{\left|\Phi(\mathrm{e}^{-\mathrm{i}\lambda})\right|^2}$.

ARMA(p, q) representations

Consider an ARMA(p, q) process X solution to

$$[\Phi(\mathrm{B})](X) = [\Theta(\mathrm{B})](Z) \ .$$

Then X admits a linear representation $X = \mathrm{F}_\psi(Z)$ for a well chosen $\psi \in \ell^1$.

We say that the ARMA(p, q) representation is

- causal if F_ψ is causal. (iff Φ does not vanish on the unit closed disk Δ_1).
- invertible if X is causally invertible with respect to Z. (iff Θ does not vanish on the unit closed disk Δ_1).
- canonical if it is causal and invertible.

Existence of a canonical representation

ARMA representations are not unique!

Theorem

Consider an ARMA(p,q) process X solution to

$$[\Phi(\mathrm{B})](X) = [\Theta(\mathrm{B})](Z) \ .$$

Suppose that neither Φ nor Θ vanishes on the unit circle Γ_1. Then X admits a canonical representation

$$[\tilde{\Phi}(\mathrm{B})](X) = [\tilde{\Theta}(\mathrm{B})](\tilde{Z}) \ .$$

($\tilde{\Phi}$ and $\tilde{\Theta}$ do not vanish on Δ_1 and $\tilde{Z}$ is a white noise).

Idea of the proof

Consider the anticausal AR(1) case $\Theta = 1$ and $\Phi(z) = 1 - \alpha z$ with $|\alpha| > 1$. Define the polynomial $\tilde{\Phi}(z) = 1 - \overline{\alpha}^{-1} z$ so that $\tilde{\Phi}(\mathrm{B})$ is a causally invertible filter. Let $\psi \in \ell^1$ such that

$$\psi^*(\lambda) = \frac{\tilde{\Phi}(\mathrm{e}^{-\mathrm{i}\lambda})}{\Phi(\mathrm{e}^{-\mathrm{i}\lambda})} , \quad \lambda \in \mathbb{R} .$$

Then, applying F_ψ on both sides,

$$[\Phi(\mathrm{B})](X) = Z \Leftrightarrow [\tilde{\Phi}(\mathrm{B})](X) = \tilde{Z} ,$$

where $\tilde{Z} = \mathrm{F}_\psi(Z)$. Now observe that, for all $\lambda \in \mathbb{R}$,

$$|\psi^*(\lambda)|^2 = \left| \frac{1 - \overline{\alpha}^{-1} \mathrm{e}^{-\mathrm{i}\lambda}}{1 - \alpha \mathrm{e}^{-\mathrm{i}\lambda}} \right|^2 = |\alpha|^{-2} .$$

Hence $\tilde{Z}$ is a white noise and we obtain a canonical representation.

Application : innovations of an ARMA process

Theorem

Let X be an ARMA(p, q) process with canonical representation

$$[\Phi(\mathrm{B})](X) = [\Theta(\mathrm{B})](Z) .$$

Then Z is the innovation process of X.

Proof

The proof is in 3 steps

Step 1 Since $\Theta(\mathrm{B})$ is causally invertible, $Z_t \in \mathcal{H}_t^X$ for all $t \in \mathbb{Z}$.

Step 2 Since $\Phi(\mathrm{B})$ is causally invertible, $X_t \in \mathcal{H}_t^Z$ for all $t \in \mathbb{Z}$.

Step 3 Hence we have

$$Z_t \perp \mathcal{H}_{t-1}^Z = \mathcal{H}_{t-1}^X ,$$

and thus $\mathrm{proj}\left(X_t \middle| \mathcal{H}_{t-1}^X \right) = \sum_{k=1}^p \phi_k X_{t-k} + \sum_{k=1}^q \theta_k Z_{t-k}$.

Necessary condition

Recall that, if X is an MA(q) process,

$$X_t = Z_t + \sum_{k=1}^{q} \theta_k Z_{t-k}$$

with $Z \sim \mathrm{WN}(0, \sigma^2)$, then its autocovariance γ satisfies

$$\gamma(k) = 0 \quad \text{for all} \quad |k| \geq q+1 \;,$$

and

$$\gamma(q) = \overline{\gamma(-q)} = \theta_q \sigma^2 \;.$$

Does this characterize MA(q) processes ?

Characterization of MA processes

Theorem

Let X be a centered weakly stationary process with autocovariance γ and let $q \geq 1$. Then the two following assertions are equivalent.

(i) X is an MA process of minimal order q.

(ii) $\gamma(q) \neq 0$ and $\gamma(k) = 0$ for all $k \geq q+1$.

We already know that (i) implies (ii).

Proof of the converse implication

Let $(\epsilon_t)_{t\in\mathbb{Z}}$ denote the innovation process of X and σ^2 be the variance of $(\epsilon_t)_{t\in\mathbb{Z}}$.

Step 1 Recall that $\mathcal{H}_t^X = \mathcal{H}_{t-q-1}^X \overset{\perp}{\oplus} \mathrm{Span}\,(\epsilon_{t-q}, \dots, \epsilon_t)$.

Step 2 Observe that (ii) implies $X_t \perp \mathcal{H}_{t-q-1}^X$.

Step 3 Hence $X_t \in \mathrm{Span}\,(\epsilon_{t-q}, \dots, \epsilon_t)$ and we deduce that

$$X_t = \epsilon_t + \sum_{k=0}^{q} \psi_k \epsilon_{t-k} \ ,$$

where $\psi_1, \dots, \psi_q$ are defined by

$$\psi_k = \frac{\langle X_t, \epsilon_{t-k} \rangle}{\sigma^2}$$

(recall that these coefficients do not depend on t)

Consequence

As a consequence, we have the following result.

Proposition

The sum of two uncorrelated MA(q) processes is an MA(q) process.

And also :

Proposition

The sum of two uncorrelated ARMA(p,q) processes is an ARMA(2p,q+p) process.

1. Composition and inversion using ℓ^1 convolution filters

2. ARMA processes
   - ARMA equations, stationary solutions
   - Innovations of ARMA processes
   - Characterization of MA processes
   - Characterization of AR processes

3. Complement : innovation of a non-invertible MA process

4. An illustrative example with R

Necessary condition

Let X be an AR(p) process, that is, a weakly stationary solution of

$$X_t = \sum_{k=1}^{p} \phi_k X_{t-k} + Z_t \;,$$

with $Z \sim \mathrm{WN}(0, \sigma^2)$.
Recall that we can assume without loss of generality that $(Z_t)_{t\in\mathbb{Z}}$ is the innovation process of X, so that

$$Z_t \perp \mathcal{H}_{t-1}^X \;.$$

Denote

$$\mathcal{H}_{t-1,p}^X = \mathrm{Span}\left(X_{t-k},\ k = 1, \ldots, p\right) \;.$$

Since $\sum_{k=1}^p \phi_k X_{t-k} \in \mathcal{H}_{t-1,p}^X$, we get that, for all $q \geq p$, for all $t \in \mathbb{Z}$,

$$\mathrm{proj}\left(X_t \middle| \mathcal{H}_{t-1,q}^X\right) = \sum_{k=1}^{p} \phi_k X_{t-k} = \mathrm{proj}\left(X_t \middle| \mathcal{H}_{t-1}^X\right) \;.$$

Necessary condition (cont.)

Using the notation

$$\mathrm{proj}\left(X_t \middle| \mathcal{H}^X_{t-1,q} \right) = \sum_{k=1}^{q} \phi^+_{k,q} X_{t-k} \;,$$

and $\boldsymbol{\phi}^+_q = [\phi^+_{1,q}, \ldots, \phi^+_{q,q}]^T$, we get that , for all $q \geq p$,

$$\boldsymbol{\phi}^+_q = \begin{bmatrix} \phi_1 & \phi_2 & \ldots & \phi_p & 0 \ldots & 0 \end{bmatrix}^T \;.$$

In particular, we have $\phi^+_{p,p} = \phi_p$ and

$$\phi^+_{q,q} = 0 \quad \text{for all} \quad q > p \;.$$

Characterization of AR processes

Definition : partial autocorrelation function

Let X be a centered weakly stationary process and denote by $\boldsymbol{\phi}_p^+ = [\phi_{1,p}^+, \ldots, \phi_{p,p}^+]^T$ its forward prediction coefficients (as above). Then the sequence $(\phi_{p,p}^+)_{p\geq 1}$ is called the partial autocorrelation function of X.

Theorem

Let X be a centered weakly stationary process with partial autocorrelation $(\kappa(j))_{j\geq 1}$ and let $p \geq 1$. Then the two following assertions are equivalent.

(i) X is an AR process of minimal order p.

(ii) $\kappa(p) \neq 0$ and $\kappa(j) = 0$ for all $j \geq p+1$.

We already know that (i) implies (ii).

Proof of the converse implication

Suppose that (ii) holds.

Step 1 By definition of κ, $\kappa(j) = 0$ implies that $\mathrm{proj}\left(X_t \middle| \mathcal{H}^X_{t-1,j} \right) \in \mathcal{H}^X_{t-1,j-1}$ and thus

$$\mathrm{proj}\left(X_t \middle| \mathcal{H}^X_{t-1,j} \right) = \mathrm{proj}\left(X_t \middle| \mathcal{H}^X_{t-1,j-1} \right) \ .$$

Step 2 With (ii), iterating, we obtain that, for all $j \geq p$,

$$\mathrm{proj}\left(X_t \middle| \mathcal{H}^X_{t-1,j} \right) = \mathrm{proj}\left(X_t \middle| \mathcal{H}^X_{t-1,p} \right) \ .$$

Step 3 Letting $j \to \infty$, we get

$$\mathrm{proj}\left(X_t \middle| \mathcal{H}^X_{t-1} \right) = \mathrm{proj}\left(X_t \middle| \mathcal{H}^X_{t-1,p} \right) = \sum_{j=1}^{p} \phi_j X_{t-j} \quad \text{(say)} \ .$$

Step 4 Since the innovation process is a white noise, this concludes the proof.

Composition and inversion in the Fourier domain

Let $(X_t)_{t\in\mathbb{Z}}$ be a centered weakly stationary process with an arbitrary spectral measure ν.

We have

$$\widehat{\mathrm{F}}_g \circ \widehat{\mathrm{F}}_h(X) = \widehat{\mathrm{F}}_{g\times h}(X) \;,$$

provided some natural restriction on X, namely,

$$h \quad \text{and} \quad g \times h \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \nu) \;.$$

In particular, if $h \neq 0$ ν-a.e., then h^{-1} is well defined ν-a.e. and, provided that $h \in L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \nu)$, we have

$$\widehat{\mathrm{F}}_{h^{-1}} \circ \widehat{\mathrm{F}}_h(X) = X \;.$$

Application : inversion of a non-invertible MA filter

Let

$$\theta = -\mathrm{e}^{\mathrm{i}\lambda_0} ,$$

with $\lambda_0 \in \mathbb{T}$ and let $Z \sim \mathrm{WN}(0, \sigma^2)$. Define the MA(1) process

$$X_t = Z_t + \theta\, Z_{t-1} , \quad t \in \mathbb{Z} .$$

What is the innovation process of X?

If $Z_t \in \mathcal{H}_t^X$ for all $t \in \mathbb{Z}$, then the answer is Z.

In the invertible case, we can write, for all $|z| = 1$,

$$(1 + \theta z)^{-1} = \sum_{k \geq 0} \psi_k z^k \cancel{\sum_{k \geq 0} \psi_k z^k} ,$$

But for the above θ, $\Theta(z) = 1 + \theta z$ vanishes on the unit circle!

Spectral representation of the noise

The spectral representation of X reads

$$X_t = \int e^{i\lambda t}\, \Theta(e^{-i\lambda})\, d\widehat{Z}(\lambda)\ .$$

Define h on $\mathbb{T}$ by

$$h(\lambda) = \begin{cases} \frac{1}{\Theta(e^{-i\lambda})} & \text{if } \lambda \neq \lambda_0 \\ 0 & \text{otherwise.} \end{cases}$$

We note that $h(\lambda) = 1/\Theta(e^{-i\lambda})$ for Lebesgue a.e. λ.

Hence, using the inversion of general time invariant linear filters,

$$Z_t = \int e^{i\lambda t}\, \frac{1}{\Theta(e^{-i\lambda})}\, d\widehat{X}(\lambda)\ .$$

Expression of the noise as a limit

Can we deduce that $Z_t \in \mathcal{H}_t^X$?

Let $0 < a < 1$. Then, since $|\theta| = 1$, we have

$$h_a(\lambda) := \frac{1}{\Theta(a\mathrm{e}^{-\mathrm{i}\lambda})} = \sum_{k \geq 0} a^k \theta^k \, \mathrm{e}^{-\mathrm{i}k\lambda} \;,$$

and the convergence holds uniformly.

We deduce that for all $0 < a < 1$,

$$Z_t^a := \int \mathrm{e}^{\mathrm{i}\lambda t} \, \frac{1}{\Theta(a\mathrm{e}^{-\mathrm{i}\lambda})} \, \mathrm{d}\widehat{X}(\lambda) \in \mathcal{H}_t^X \;.$$

Conclusion : noise = innovation

It can be shown that

$$\lim_{a \uparrow 1} h_a = h \quad \text{in } L^2(\mathbb{T}, \mathcal{B}(\mathbb{T}), \nu) ,$$

where ν denotes the spectral measure of X, $\nu(\mathrm{d}\lambda) = \Theta(\mathrm{e}^{-\mathrm{i}\lambda})\,\mathrm{d}\lambda$.

Since the spectral representation is a unitary operator, we obtain

$$\lim_{a \uparrow 1} Z_t^a = Z_t \quad \text{in } L^2(\Omega, \mathcal{F}, \mathbb{P}) ,$$

We conclude that $Z_t \in \mathcal{H}_t^X$ for all $t \in \mathbb{Z}$ and thus

$$(Z_t)_{t\in\mathbb{Z}} \text{ is the innovation process of } (X_t)_{t\in\mathbb{Z}}.$$

```
# some settings
set.seed(1)
# nb of simulations
N <- 2^8
# length of each time series
n=2^7
# noise
z <- rnorm(n,0,1)
# frequencies
lam <- (-200:200)/200*pi
# wait after each plot
dx <-  0.01 # seconds
############################################
#             AR(1)  simulation            #
############################################
aronesp <- function(lam=0,phi=1/2,s=1){
  # compute the spectral dens of AR(1)
  # Input : lam :  freq. at which sp. density computed
  #         phi : AR coeff.
  #         s : innovations std dev
  # Output : AR(1) sp. dens.
  return(s**2/(2*pi)/(abs(1-phi*exp(-1i*lam))**2))
}
aronevar <- function(phi=1/2,s=1){
  # compute the variance of AR(1)
  # Input : phi : AR coeff.
  #         s : innovations std dev
  # Output : AR(1) variance
  return(s**2/(1-phi**2))
}
arone <- function(n=2**8,phi=1/2,x=0,z=rnorm(n,0,1),lam=0){
  # iterate AR(1) equation
  # Input : n : nb of iterations
  #         phi : AR coeff.
  #         x : initial value
  #         z : innovations (with length n)
```

```
  # Output : samp: AR(1) iterates
  #          s : AR(1) std dev/innov std dev
  #          sp : AR(1) sp. dens. at lam / innov var
  y <- x
  for (t in 2:(n+1)){
    y <- c(y,y[t-1]*phi+ z[t-1])
  }
  return(list( samp = y[2:n+1], s = sqrt(aronevar(phi)),
               sp = aronesp(lam,phi) ) )
}
# simulate AR(1) processes with AR coeff getting close to 1
# set of values for phi
phiset <- (0:N)/(N+1)
# minimal and maximal spectral density
spmin <- min((1-abs(phiset))/(2*pi)/(1+abs(phiset)))
spmax <-  max((1+abs(phiset))/(2*pi)/(abs(1-abs(phiset))))
# start plotting
graphics.off()
dev.new(width=10, height=5)
op <- par(mfrow=c(1,2))
for (phi in phiset){
  simu <- arone(n,phi,0,z,lam)
  # plot the simulated sample on the left
  ts.plot(simu$samp/simu$s, ylim=c(-2.5,2.5), ylab='',
          main=paste('phi=',sprintf(fmt="%.4f",phi),sep=''))
  # plot the log-spectral density on the right
  plot(lam,simu$sp/(simu$s**2),type='l',
       xlab='Frequency',log='y',ylab='',
       main='Log spectral density', ylim=c(spmin,spmax))
  grid(col='black')
  Sys.sleep(dx)
}
par(op)
############################################
#             AR(2)  simulation            #
############################################
```

```
artwovar <- function(fr=0,r=0,s=1){
  # variance of AR(2) equation with complex roots
  # Input : fr : main freq.
  #         r : norm of root inverse
  #         s : std dev of innovations
  # Output : AR(2) variance
  g <- solve(rbind(c(1,-2.0*r*cos(fr),r**2),
                   c(-2.0*r*cos(fr),1+r**2,0),
                   c(r**2,-2.0*r*cos(fr),1)),c(1,0,0))
  return(g[1])
}
artwosp<- function(lam=0,fr=0,r=0,s=1){
  # sp. dens. of AR(2) equation with complex roots
  # Input : lam :  freq. at which sp. density computed
  #         fr : main freq.
  #         r : norm of root inverse
  #         s : std dev of innovations
  # Output : AR(2) sp. dens.
  return(s**2/(2*pi)/(abs(1-2*r*cos(fr)*exp(-1i*lam)+
                         (r**2)*exp(-1i*2*lam))**2))
}
artwo <- function(n=2**8,fr=0,r=0,x=c(0,0),z=rnorm(n,0,1),
                  lam=0){
  # iterate AR(2) equation  with complex roots
  # Input : n : nb of iterations
  #         fr : main freq.
  #         r : norm of root inverse
  #         x : initial values
  #         z : innovations (with length n)
  #         lam :  freq. at which sp. density computed
  #
  # Output : samp: AR(2) iterates
  #          s : AR(2) std dev/innov std dev
  #          sp : AR(2) sp. dens. at lam / innov var
  y <- x
  for (t in 3:(n+2)){
```

```
    y <- c(y,2.0*r*cos(fr)*y[t-1]-(r**2)*y[t-2] + z[t-2])
  }
  return( list( samp=y[3:(n+2)] , s=sqrt(artwovar(fr,r)) ,
               sp=artwosp(lam,fr,r) ) )
}
# simulate AR(2) processes with coeff getting close to 1
# main frequency
fr <- 2*pi/8
# frequencies
lam <- (-200:200)/200*pi
# wait after each plot
dx <-  0.01 # seconds
# set of values for r
rset <- (0:N)/(N+1)
# minimal and maximal spectral density
rmax <- max(rset)
sp <- artwosp(lam=lam,fr=fr,r=rmax)/artwovar(fr=fr,r=rmax)
spmin <- min(sp);spmax <-  max(sp)
# start plotting
graphics.off()
dev.new(width=10, height=5)
op <- par(mfrow=c(1,2))
for (r in rset){
  simu <- artwo(n,fr,r,c(0,0),z,lam)
  # plot the simulated sample on the left
  ts.plot(simu$samp/simu$s,ylim=c(-2.5,2.5), ylab='',
          main=paste('fr=',sprintf(fmt="%.3f",fr),'; r=',
            sprintf(fmt="%.4f",r),sep=''))
  # plot the log-spectral density on the right
  plot(lam,simu$sp/(simu$s**2),type='l',xlab='Frequency',
       log='y',ylab='',main='Log spectral density',
       ylim=c(spmin,spmax))
  abline(v=-fr, lty = "dotted",col=2)
  abline(v=fr, lty = "dotted",col=2)
  grid(col='gray')
  Sys.sleep(dx)
```

```
}
# Now move the main frequency
frset <- (fr + pi * (1:N)/N) %% pi
for (fr in frset){
  simu <- artwo(n,fr,r,c(0,0),z,lam)
  # plot the simulated sample on the left
  ts.plot(simu$samp/simu$s,ylim=c(-2.5,2.5), ylab='',
          main=paste('fr=',sprintf(fmt="%.3f",fr),'; r=',
            sprintf(fmt="%.4f",r),sep=''))
  # plot the log-spectral density on the right
  plot(lam,simu$sp/(simu$s**2),type='l',xlab='Frequency',
       log='y',ylab='',main='Log spectral density',
       ylim=c(spmin,spmax))
  abline(v=-fr, lty = "dotted",col=2)
  abline(v=fr, lty = "dotted",col=2)
  grid(col='gray')
  Sys.sleep(dx)
}
par(op)
```