Puzzle Problem

The letters below correspond to digits. Find the digit that corresponds to each letter.

$$
\begin{array}{r}
\text{F O R T Y} \\
\text{T E N} \\
+ \quad \text{T E N} \\
\hline
\text{S I X T Y}
\end{array}
$$

Solution. Starting with the last column, we have

$$
\begin{aligned}
Y + N + N &= 10a + Y \\
N &= 5a
\end{aligned}
$$

where $a = 1$ or $a = 0$, since N is a single digit number.

Case 1. $a = 1$. Plugging this into the previous equation, we get

$$N = 5a = 5(1) = 5.$$

Going to the fourth column, since we have to carry a 1, we have

$$
\begin{aligned}
1 + T + E + E &= 10b + T \\
2E &= 10b - 1
\end{aligned}
$$

where $b \in \mathbb{Z}$. However, this is a contradiction since $10b - 1$ is not even. Therefore, $a \neq 1$.

Case 2. $a = 0$. Plugging this into the same equation as in *Case 1*, we get

$$N = 5a = 5(0) = 0.$$

Going to the fourth column, since we do not carry a 1, we have

$$
\begin{aligned}
T + E + E &= 10b + T \\
E &= 5b
\end{aligned}
$$

where $b = 1$ or $b = 0$, since E is a single digit number. If $b = 0$, we get

$$E = 5b = 5(0) = 0.$$

However, we already have $N = 0$. Therefore, $b = 1$ so

$$E = 5b = 5(1) = 5.$$

Hence, $\boxed{N = 0}$ and $\boxed{E = 5}$.

Since $F \neq S$, we must carry a number to the first column. To find this number, notice that

$$\max(ORTY + TEN + TEN) = 9999 + 999 + 999 < 20000.$$

This means we must carry a 1 to the first column. Therefore,

$$S = F + 1.$$

Similarly, since $O \neq I$, we must carry a number to the second column. To find this number, notice that

$$\max(RTY + TEN + TEN) = 999 + 999 + 999 < 3000.$$

This means we must carry a 1 or a 2 to the second column. Therefore, there are two cases we must look at.

Case 1. Carry a 1. This yields

$$\begin{aligned} 1 + O &= 10 + I \\ O &= 9 + I. \end{aligned}$$

The 10 came from the fact that we are carrying a 1 to the first column. Now, since O is a single digit number, it must be the case that $I = 0$. However, we already have $N = 0$, so this is a contradiction. Therefore, we must carry a 2.

Case 2. Carry a 2. This yields

$$\begin{aligned} 2 + O &= 10 + I \\ O &= 8 + I. \end{aligned}$$

Now, since O is a single digit number, it must be the case that $I = 0$ or $I = 1$. However, we already have $N = 0$. Thus, $I = 1$. Therefore

$$O = 8 + I = 8 + 1 = 9.$$

Hence, $\boxed{I = 1}$ and $\boxed{O = 9}$.

Looking at the fourth column, since $E = 5$, we have

$$\max(T + E + E) = \max(T + 5 + 5) = 9 + 5 + 5 = 19.$$

Therefore, we must carry a 1 to the third column. This yields

$$\begin{aligned} 1 + R + T + T &= 20 + X \\ 1 + R + 2T &= 20 + X \\ R &= X - 2T + 19. \end{aligned}$$

The 20 came from the fact that we are carrying a 2 to the second column. Now we have an equation for R. To digest this equation, let us look at all of the possibilities for T. First of all, it must be the case that

$$1 + R + T + T = 1 + R + 2T \geq 20.$$

This is since the sum must be at least 20 to carry a 2 to the second column.

Case 1. $T = 0$, $T = 1$, $T = 5$, or $T = 9$. Any of these T values violate the values of N, E, I, and O. Therefore, $T \neq 0$, $T \neq 1$, $T \neq 5$, and $T \neq 9$.

Case 2. $T \in \mathbb{Z}$ such that $2 \leq T \leq 4$. We have

$$\max(1 + R + 2T) = 1 + 9 + 2(4) = 18 < 20.$$

This means the sum is not big enough to carry a 2 to the second column. Therefore, T cannot be $2 \leq T \leq 4$.

Case 3. $T = 6$. We have

$$\max(1 + R + 2T) = \max(1 + R + 2(6)) = \max(R + 13) = 9 + 13 = 22 \geq 20.$$

This means the sum is big enough to carry a 2 to the second column. Since this inequality holds for $T = 6$, it will also hold for any $T > 6$. Thus, this inequality holds for *Case 4* and *Case 5*. Now, using the equation that we derived for R, we get

$$\begin{aligned} R &= X - 2T + 19 \\ R &= X - 2(6) + 19 \\ R &= X + 7. \end{aligned}$$

First of all, $X \neq 0$ and $X \neq 1$ since we already have $N = 0$ and $I = 1$. Next, if $X = 2$, we get

$$R = X + 7 = 2 + 7 = 9.$$

However, $R \neq 9$ since we already have $O = 9$. Finally, if $X \geq 3$, we get

$$R = X + 7 \geq 3 + 7 = 10.$$

However, R must be a single digit number. Therefore, by way of contradiction, we have $T \neq 6$.

Case 4. $T = 7$. Using the equation for R, we get

$$R = X - 2T + 19$$
$$R = X - 2(7) + 19$$
$$R = X + 5.$$

First of all, $X \neq 0$ and $X \neq 1$ since we already have $N = 0$ and $I = 1$. Next, if $X = 4$, we get

$$R = X + 5 = 4 + 5 = 9.$$

However, $R \neq 9$ since we already have $O = 9$. Next, if $X \geq 5$, we get

$$R = X + 5 \geq 5 + 5 = 10.$$

However, R must be a single digit number. Next, if $X = 2$, we get

$$R = X + 5 = 2 + 5 = 7.$$

However, $R \neq 7$ since we already have $T = 7$. Finally, if $X = 3$, we get

$$R = X + 5 = 3 + 5 = 8.$$

Thus, $R = 8$. Now, to see if these values of T, X, and R work, we look at the list of values that we found out for each variable. We have $N = 0$, $E = 5$, $I = 1$, $O = 9$, $T = 7$, $X = 3$, and $R = 8$. Ordering these values, we get

$$0, 1, 3, 5, 7, 8, 9.$$

From the equation that we derived earlier,

$$S = F + 1,$$

so we must have a gap of 3. This is since, for instance, if $F = 2$, we would get

$$S = F + 1 = 2 + 1 = 3.$$

However, 3 is already occupied by another variable. Therefore, by way of contradiction, we have $T \neq 7$.

Case 5. $T = 8$. Using the equation for R, we get

$$R = X - 2T + 19$$
$$R = X - 2(8) + 19$$
$$R = X + 3.$$

First of all, $X \neq 0$, $X \neq 1$, and $X \neq 5$ since we already have $N = 0$ and $I = 1$, and $E = 5$. Next, if $X = 2$, we get

$$R = X + 3 = 2 + 3 = 5.$$

However, $R \neq 5$ since we already have $E = 5$. Next, if $X = 6$, we get

$$R = X + 3 = 6 + 3 = 9.$$

However, $R \neq 9$ since we already have $O = 9$. Next, if $X \geq 7$, we get

$$R = X + 3 \geq 7 + 3 = 10.$$

However, R must be a single digit number. Finally, there are two more cases: $X = 3$ and $X = 4$.

Case 5a. $X = 3$. We have

$$R = X + 3 = 3 + 3 = 6.$$

Thus, $R = 6$. Now, to see if these values of T, X, and R work, we look at the list of values that we found out for each variable. We have $N = 0$, $E = 5$, $I = 1$, $O = 9$, $T = 8$, $X = 3$, and $R = 6$. Ordering these values, we get

$$0, 1, 3, 5, 6, 8, 9.$$

From the equation that we derived earlier,

$$S = F + 1,$$

so we must have a gap of 3. However, there is no gap of 3. Therefore, by way of contradiction, we have $X \neq 3$

Case 5b. $X = 4$. We have

$$R = X + 3 = 4 + 3 = 7.$$

Thus, $R = 7$. Now, to see if these values of T, X, and R work, we look at the list of values that we found out for each variable. We have $N = 0$, $E = 5$, $I = 1$, $O = 9$, $T = 8$, $X = 4$, and $R = 7$. Ordering these values, we get

$$0, 1, 4, 5, 7, 8, 9.$$

From the equation that we derived earlier,

$$S = F + 1,$$

so we must have a gap of 3. The gap of 3 happens between 1 and 4. Thus, $F = 2$. This implies

$$S = F + 1 = 2 + 1 = 3.$$

Therefore, $\boxed{T = 8}$, $\boxed{X = 4}$, $\boxed{R = 7}$, $\boxed{F = 2}$, and $\boxed{S = 3}$.

The last variable we must find is Y. The variables that we found are $N = 0$, $E = 5$, $I = 1$, $O = 9$, $T = 8$, $X = 4$, $R = 7$, $F = 2$, and $S = 3$. Ordering these values, we get

$$0, 1, 2, 3, 4, 5, 7, 8, 9.$$

The missing digit it 6. Thus, $Y = 6$. Therefore, the answer is

$$\boxed{N = 0}, \boxed{I = 1}, \boxed{F = 2}, \boxed{S = 3}, \boxed{X = 4}, \boxed{E = 5}, \boxed{Y = 6}, \boxed{R = 7}, \boxed{T = 8}, \boxed{O = 9}.$$

To see if this is correct, we plug in the values,

$$\begin{aligned} FORTY &= 29786 \\ TEN &= 850 \\ SIXTY &= 31486, \end{aligned}$$

and check the sum,

$$\begin{aligned} FORTY + TEN + TEN &= 29786 + 850 + 850 \\ &= 31486 \\ &= SIXTY. \end{aligned}$$

$\square$