

Puzzle Problem

The letters below correspond to digits. Find the digit that corresponds to each letter.

$$\begin{array}{r} \text{F O R T Y} \\ \text{T E N} \\ + \text{T E N} \\ \hline \text{S I X T Y} \end{array}$$

Solution. Starting with the last column, we have

$$\begin{aligned} Y + N + N &= 10a + Y \\ N &= 5a \end{aligned}$$

where $a = 1$ or $a = 0$, since N is a single digit number.

Case 1. $a = 1$. Plugging this into the previous equation, we get

$$N = 5a = 5(1) = 5.$$

Going to the fourth column, since we have to carry a 1, we have

$$\begin{aligned} 1 + T + E + E &= 10b + T \\ 2E &= 10b - 1 \end{aligned}$$

where $b \in \mathbb{Z}$. However, this is a contradiction since $10b - 1$ is not even. Therefore, $a \neq 1$.

Case 2. $a = 0$. Plugging this into the same equation as in *Case 1*, we get

$$N = 5a = 5(0) = 0.$$

Going to the fourth column, since we do not carry a 1, we have

$$\begin{aligned} T + E + E &= 10b + T \\ E &= 5b \end{aligned}$$

where $b = 1$ or $b = 0$, since E is a single digit number. If $b = 0$, we get

$$E = 5b = 5(0) = 0.$$

However, we already have $N = 0$. Therefore, $b = 1$ so

$$E = 5b = 5(1) = 5.$$

Hence, $\boxed{N = 0}$ and $\boxed{E = 5}$.

Since $F \neq S$, we must carry a number to the first column. To find this number, notice that

$$\max(ORTY + TEN + TEN) = 9999 + 999 + 999 < 20000.$$

This means we must carry a 1 to the first column. Therefore,

$$S = F + 1.$$

Similarly, since $O \neq I$, we must carry a number to the second column. To find this number, notice that

$$\max(RTY + TEN + TEN) = 999 + 999 + 999 < 3000.$$

This means we must carry a 1 or a 2 to the second column. Therefore, there are two cases we must look at.

Case 1. Carry a 1. This yields

$$\begin{aligned} 1 + O &= 10 + I \\ O &= 9 + I. \end{aligned}$$

The 10 came from the fact that we are carrying a 1 to the first column. Now, since O is a single digit number, it must be the case that $I = 0$. However, we already have $N = 0$, so this is a contradiction. Therefore, we must carry a 2.

Case 2. Carry a 2. This yields

$$\begin{aligned} 2 + O &= 10 + I \\ O &= 8 + I. \end{aligned}$$

Now, since O is a single digit number, it must be the case that $I = 0$ or $I = 1$. However, we already have $N = 0$. Thus, $I = 1$. Therefore

$$O = 8 + I = 8 + 1 = 9.$$

Hence, $\boxed{I = 1}$ and $\boxed{O = 9}$.

Looking at the fourth column, since $E = 5$, we have

$$\max(T + E + E) = \max(T + 5 + 5) = 9 + 5 + 5 = 19.$$

Therefore, we must carry a 1 to the third column. This yields

$$\begin{aligned} 1 + R + T + T &= 20 + X \\ 1 + R + 2T &= 20 + X \\ R &= X - 2T + 19. \end{aligned}$$

The 20 came from the fact that we are carrying a 2 to the second column. Now we have an equation for R . To digest this equation, let us look at all of the possibilities for T . First of all, it must be the case that

$$1 + R + T + T = 1 + R + 2T \geq 20.$$

This is since the sum must be at least 20 to carry a 2 to the second column.

Case 1. $T = 0$, $T = 1$, $T = 5$, or $T = 9$. Any of these T values violate the values of N , E , I , and O . Therefore, $T \neq 0$, $T \neq 1$, $T \neq 5$, and $T \neq 9$.

Case 2. $T \in \mathbb{Z}$ such that $2 \leq T \leq 4$. We have

$$\max(1 + R + 2T) = 1 + 9 + 2(4) = 18 < 20.$$

This means the sum is not big enough to carry a 2 to the second column. Therefore, T cannot be $2 \leq T \leq 4$.

Case 3. $T = 6$. We have

$$\max(1 + R + 2T) = \max(1 + R + 2(6)) = \max(R + 13) = 9 + 13 = 22 \geq 20.$$

This means the sum is big enough to carry a 2 to the second column. Since this inequality holds for $T = 6$, it will also hold for any $T \in \mathbb{Z}$ such that $T > 6$. Thus, this inequality holds for *Case 4* and *Case 5*. Now, using the equation that we derived for R , we get

$$\begin{aligned} R &= X - 2T + 19 \\ R &= X - 2(6) + 19 \\ R &= X + 7. \end{aligned}$$

First of all, $X \neq 0$ and $X \neq 1$ since we already have $N = 0$ and $I = 1$. Next, if $X = 2$, we get

$$R = X + 7 = 2 + 7 = 9.$$

However, $R \neq 9$ since we already have $O = 9$. Finally, if $X \in \mathbb{Z}$ such that $X \geq 3$, we get

$$R = X + 7 \geq 3 + 7 = 10.$$

However, R must be a single digit number. Therefore, by way of contradiction, we have $T \neq 6$.

Case 4. $T = 7$. Using the equation that we derived for R , we get

$$\begin{aligned} R &= X - 2T + 19 \\ R &= X - 2(7) + 19 \\ R &= X + 5. \end{aligned}$$

First of all, $X \neq 0$ and $X \neq 1$ since we already have $N = 0$ and $I = 1$. Next, if $X = 2$, we get

$$R = X + 5 = 2 + 5 = 7.$$

However, $R \neq 7$ since we already have $T = 7$. Next, if $X = 4$, we get

$$R = X + 5 = 4 + 5 = 9.$$

However, $R \neq 9$ since we already have $O = 9$. Next, if $X \in \mathbb{Z}$ such that $X \geq 5$, we get

$$R = X + 5 \geq 5 + 5 = 10.$$

However, R must be a single digit number. Finally, if $X = 3$, we get

$$R = X + 5 = 3 + 5 = 8.$$

Thus, $R = 8$. Now, to see if these values of T , X , and R work, we look at the list of values that we found out for each variable. We have $N = 0$, $E = 5$, $I = 1$, $O = 9$, $T = 7$, $X = 3$, and $R = 8$. Ordering these values, we get

$$0, 1, 3, 5, 7, 8, 9.$$

From the equation that we derived earlier,

$$S = F + 1,$$

we must have a gap of 3. This is since, for instance, if $F = 2$, we would get

$$S = F + 1 = 2 + 1 = 3.$$

However, 3 is already occupied by another variable. Therefore, by way of contradiction, we have $T \neq 7$.

Case 5. $T = 8$. Using the equation that we derived for R , we get

$$\begin{aligned} R &= X - 2T + 19 \\ R &= X - 2(8) + 19 \\ R &= X + 3. \end{aligned}$$

First of all, $X \neq 0$, $X \neq 1$, and $X \neq 5$ since we already have $N = 0$, $I = 1$, and $E = 5$. Next, if $X = 2$, we get

$$R = X + 3 = 2 + 3 = 5.$$

However, $R \neq 5$ since we already have $E = 5$. Next, if $X = 6$, we get

$$R = X + 3 = 6 + 3 = 9.$$

However, $R \neq 9$ since we already have $O = 9$. Next, if $X \in \mathbb{Z}$ such that $X \geq 7$, we get

$$R = X + 3 \geq 7 + 3 = 10.$$

However, R must be a single digit number. Finally, there are two more cases: $X = 3$ and $X = 4$.

Case 5a. $X = 3$. We have

$$R = X + 3 = 3 + 3 = 6.$$

Thus, $R = 6$. Now, to see if these values of T , X , and R work, we look at the list of values that we found out for each variable. We have $N = 0$, $E = 5$, $I = 1$, $O = 9$, $T = 8$, $X = 3$, and $R = 6$. Ordering these values, we get

$$0, 1, 3, 5, 6, 8, 9.$$

From the equation that we derived earlier,

$$S = F + 1,$$

we must have a gap of 3. However, there is no gap of 3. Therefore, by way of contradiction, we have $X \neq 3$.

Case 5b. $X = 4$. We have

$$R = X + 3 = 4 + 3 = 7.$$

Thus, $R = 7$. Now, to see if these values of T , X , and R work, we look at the list of values that we found out for each variable. We have $N = 0$, $E = 5$, $I = 1$, $O = 9$, $T = 8$, $X = 4$, and $R = 7$. Ordering these values, we get

$$0, 1, 4, 5, 7, 8, 9.$$

From the equation that we derived earlier,

$$S = F + 1,$$

we must have a gap of 3. The gap of 3 happens between 1 and 4. Thus, $F = 2$. This implies

$$S = F + 1 = 2 + 1 = 3.$$

Therefore, $\boxed{T = 8}$, $\boxed{X = 4}$, $\boxed{R = 7}$, $\boxed{F = 2}$, and $\boxed{S = 3}$.

The last variable we must find is Y . The variables that we found are $N = 0$, $E = 5$, $I = 1$, $O = 9$, $T = 8$, $X = 4$, $R = 7$, $F = 2$, and $S = 3$. Ordering these values, we get

$$0, 1, 2, 3, 4, 5, 7, 8, 9.$$

The missing digit is 6. Thus, $Y = 6$. Therefore, the answer is

$$\boxed{N = 0}, \boxed{I = 1}, \boxed{F = 2}, \boxed{S = 3}, \boxed{X = 4}, \boxed{E = 5}, \boxed{Y = 6}, \boxed{R = 7}, \boxed{T = 8}, \boxed{O = 9}.$$

To see if this is correct, we plug in the values,

$$FORTY = 29786,$$

$$TEN = 850,$$

$$SIXTY = 31486,$$

and check the sum,

$$\begin{aligned} FORTY + TEN + TEN &= 29786 + 850 + 850 \\ &= 31486 \\ &= SIXTY. \end{aligned}$$

□