Fault Detection of an Inverted Pendulum on a Cart

Plant Model

- Control voltage to a DC motor delivers torque to the cart's drive wheel
- Position of the cart is measured using a circular coil potentiometer
- Velocity of the cart is measured using a tacho generator mounted on the drive wheel
- Angle of pendulum is determined via a potentiometer
- Angular velocity is not measured

System States: $x = \begin{bmatrix} cart\ \ position \\ cart\ \ velocity \\ pendulum\ \ angle \\ pendulum\ \ angular\ \ velocity \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Plant Parameters

```
M0 = 3.2;          % [kg]          Pendulum mass
M = 3.529;         % [kg]          Total mass pendulums and the cart
Theta = 0.072;     % [kg.m^2]      Moment of inertia of the pendulum rod
ls = 0.44;         % [m]           Rod length
C = 0.009;         % [kg.m^2/s]    Friction constant of the pendulum
g = 9.81;          % [m/s^2]       Acceleration of gravity
Fr = 6.2;          % [kg/s]        Static friction constant of the cart

% Kr = 2.6;

N = 0.1446;       % [kg.m]         Damping constant at the joints of the rod
N01_2 = 0.23315; % [kg^2.m^2]     Damping constant at the joints of the cart
```

Fault Model

We assume a possible fault in the actuator (DC motor drive) f_A, as well as a possible fault in each of the sensors f_{s1}, f_{s2}, and f_{s3}.

$$f = \begin{bmatrix} f_{A1} \\ f_{s1} \\ f_{s2} \\ f_{s3} \end{bmatrix}$$

Linearized Discrete-Tme Model

The system is linearized with a sampling time of $0.03s$, at the operating point:

$$x = \begin{bmatrix} r \\ \dot{r} \\ \phi \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The linearized model is valid under the following conditions:

- Control signal, $|F| \leq 20N$
- Cart position, $|r| \leq 0.5m$
- Cart velocity, $|\dot{r}| \leq 1m/s$
- Pendulum angle, $|\phi| \leq \frac{1}{18}\pi \ \ rad$

System Model

$$\dot{x} = Ax + Bu + E_d d + E_f f$$
$$y = Cx + F_f f$$

with:

- state vector, x
- input vector, u
- disturbance vector, d
- fault vector, f

```
A= [
    1, -0.0569,  0.0010, 0.0000;
    0,  0.9442, -0.0038, 0.0000;
    0,  0.0116,  1.0097, 0.0300;
    0,  0.7688,  0.6442,  1.0056
   ];

B= [ 0.0053; -0.1789; 0.0373; 2.4632];

E_d= B;

E_f= [B, zeros(4,3) ];

C= [diag([1,1,1]), zeros(3,1)];

D= zeros(3,1);

F_d= zeros(3,1);

F_f= [zeros(3,1), diag([1,1,1]) ];

% Sampling time
T_s= 0.03;
```

State Feedback Control

Because the system is unstable at the linearizaiton point $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, we stabilize using a state feedback controller.

We design the gain matrix $K$by placing the poles of $A + BK$ as $(0.21, 0.22, 0.23, 0.24)$.

```
% State feedback controller
feed_poles= [ 0.21; 0.22; 0.23; 0.24 ];
K= place(A, B, feed_poles);
```

Simulation

```
% No of states
n= size(A,1);
% No of inputs
m= size(B,2);
% No of disturbance inputs
n_d= size(E_d,2);
% No of fault inputs
n_f= size(E_f,2);
% No of outputs
p= size(C,1);

% Number of simulation steps
sim_steps= 15+1;

% Simulatipn time
sim_t= (0:sim_steps)*T_s;

% Pre-allocate matrices to hold simulation data
% Output matrix
y_k= zeros(p, sim_steps);
% State matrix
x_k= zeros(n, sim_steps+1);
% Input matrix
u_k= zeros(m, sim_steps);

% Set initial state into state matrix
x_k(:, 1)= [0; 0; 0.02; 0];


for idx= 1:sim_steps
    % Input
    u_k(:, idx)= -K*x_k(:,idx);
    % Next state
    x_k(:, (idx+1))= A*x_k(:,idx) + B*u_k(:,idx);
```

```
    % Output
    y_k(:, idx)= C*x_k(:,idx);
end
```

Plotting

We plot the state estimation error to visualize the accuracy of the observed states

```
Fig_sfc= figure;

subplot(1,2,1);
plot(sim_t, x_k, 'LineWidth', 1);
grid on;
title('Fig. 1- State Feedback Control (States)');
ylabel('States', 'Interpreter', 'latex');
xlabel('time, sec', 'Interpreter', 'latex');
leg= legend(['$x_1$', "$x_2$", '$x_3$', '$x_4$']);
leg.Interpreter='latex';
leg.Location= 'southeast';

subplot(1,2,2);
plot(sim_t(1:end-1), u_k, 'LineWidth', 1);
grid on;
ylabel('Control Input', 'Interpreter', 'latex');
xlabel('time, sec', 'Interpreter', 'latex');
leg= legend(['$u$']); leg.Interpreter='latex';
leg.Location= 'southeast';
```

Full Order Observer Design

We want to design a full-order obeserver such that:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

$$\hat{y} = C\hat{x}$$

The observer error dynamics is then governed by:

$$\dot{e} = (A - LC)e + E_d d + (E_f + LF_f)f$$

with:

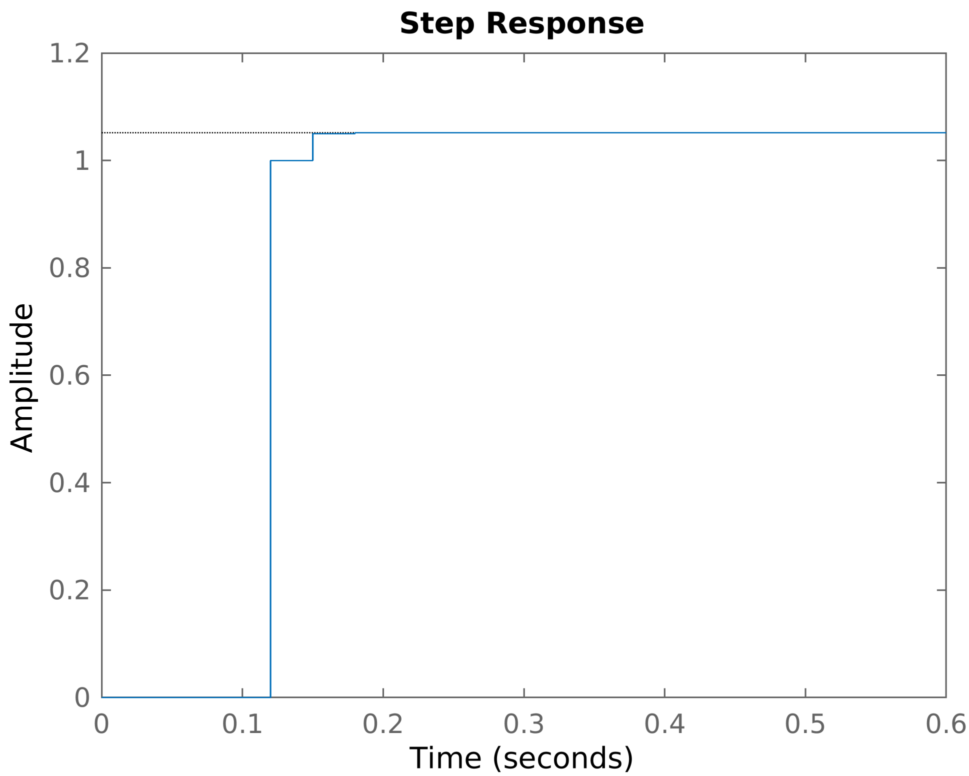
- $e = x - \hat{x}$

Pole Placement

If we select our desired ole positions to be: $(0.011, 0.012, 0.013, 0.014)$, we can examine the possible system response as:

```
obs_poles= [0.011, 0.012, 0.013, 0.014];
sys_temp= zpk([],obs_poles,1, T_s);
```

```
figure
step(sys_temp)
```



As can be seen, the response is quite satisfactory, with a response time of about 2×10^{-3}.

Now we compute the gain matrix L for our full order observer as:

```
L= place(A', C', obs_poles)'
```

```
L = 4×3
     0.9870   -0.0569    0.0010
          0    0.9302   -0.0038
     0.0000    0.0116    1.9923
     0.0000    0.7688   33.5854
```

Full Order Observer Simulation

Here, we compare system states and output with the full order observer under zero disturbance and an initial state of $x = \begin{bmatrix} 0 \\ 0 \\ 0.02 \\ 0 \end{bmatrix}$.

Because the system is unstable at the linearizaiton point $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, we stabilize using a state feedback controller.

We design the gain matrix $K$by placing the poles of $A + BK$ as $(0.21, 0.22, 0.23, 0.24)$.

```
% State feedback controller
feed_poles= [ 0.21; 0.22; 0.23; 0.24 ];
K= place(A, B, feed_poles);
```

Simulation

```
% Number of simulation steps
sim_steps= 15+1;

% Simulatipn time
sim_t= (0:sim_steps)*T_s;

% Pre-allocate matrices to hold simulation data
% Output matrix
y_k= zeros(p, sim_steps);
% State matrix
x_k= zeros(n, sim_steps+1);
% Input matrix
u_k= zeros(m, sim_steps);

% Observed state matrix
x_k_hat= zeros(n, sim_steps+1);
% Observed output matrix
y_k_hat= zeros(p, sim_steps);

% State estimation error
e_k= zeros(n, sim_steps+1);

% Set initial state into state matrix
x_k(:, 1)= [0; 0; 0.02; 0];

% Initial  state estimation error
e_k(:, 1)= x_k(:, 1) - x_k_hat(:, 1);

for idx= 1:sim_steps
    % Input
    u_k(:, idx)= -K*x_k(:,idx);
    % Next state
    x_k(:, (idx+1))= A*x_k(:,idx) + B*u_k(:,idx);
    % Output
    y_k(:, idx)= C*x_k(:,idx);

    % Observed output
    y_k_hat(:, idx)= C*x_k_hat(:,idx);
    % Observed state
    x_k_hat(:, (idx+1))= A*x_k_hat(:,idx) + B*u_k(:,idx) + ...
        L*( y_k(:, idx) - y_k_hat(:, idx) );

    % State estimation error
    e_k(:, idx+1)= x_k(:, idx+1) - x_k_hat(:, idx+1);
```

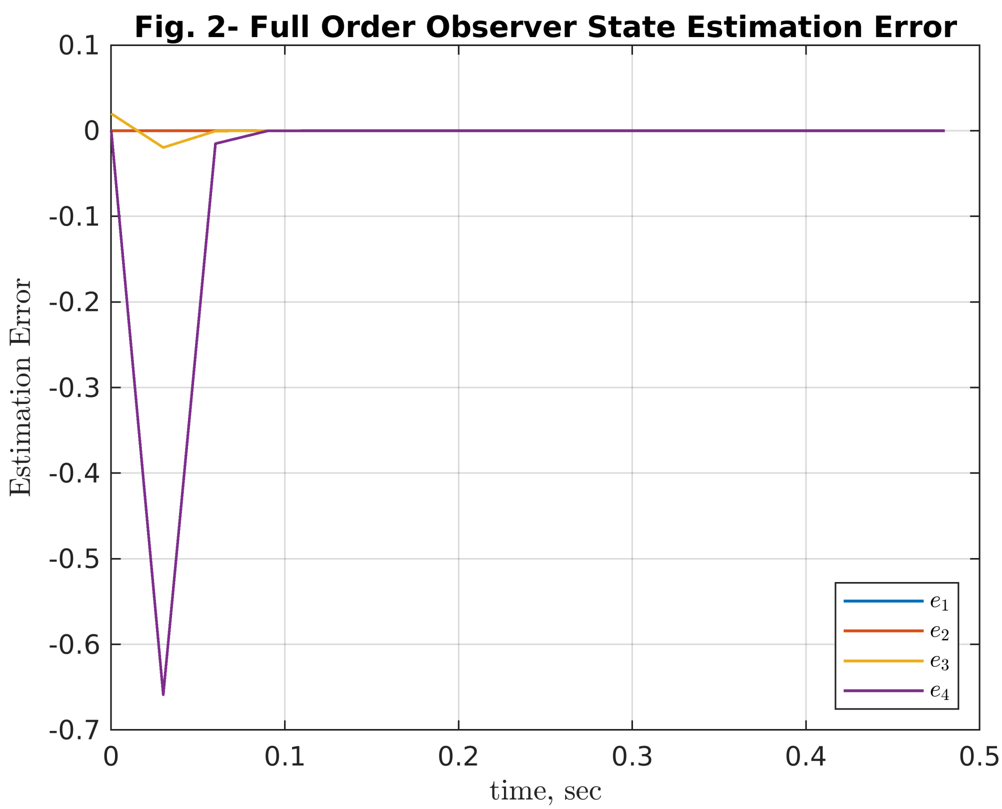
```
end
```

Plotting

We plot the state estimation error to visualize the accuracy of the observed states

```
Fig_obs= figure;

plot(sim_t, e_k, 'LineWidth', 1);
grid on;
title('Fig. 2- Full Order Observer State Estimation Error');
ylabel('Estimation Error', 'Interpreter', 'latex');
xlabel('time, sec', 'Interpreter', 'latex');
leg= legend(['$e_1$', "$e_2$", '$e_3$', '$e_4$']); leg.Interpreter='latex';
leg.Location= 'southeast';
```



Fault Injection and Residual Generation

Now we inject faults into the system. We simulate the faults to happen at $(3.5s,\ 5s,\ 7.5s,\ 12s)$ respectively.

```
fault_times= [3.5; 5; 7.5; 12];
fault_points= round(fault_times./T_s);
```

Simulation

```
% Number of simulation steps
sim_steps= fault_points(end)+100;
```

```
% Simulatipn time
sim_t= (0:sim_steps)*T_s;

% Pre-allocate matrices to hold simulation data
% Output matrix
y_k= zeros(p, sim_steps);
% State matrix
x_k= zeros(n, sim_steps+1);
% Input matrix
u_k= zeros(m, sim_steps);
% Disturbance Matrix
d_k= 0.5*rand(n_d, sim_steps);
% Fault Matrix
f_k= zeros(n_f, sim_steps);
% Residual Matrix
r_k= zeros(p, sim_steps);
% Reisudual Filter to improve residual dynamics
W= diag([3,2,1]);

% Fault injection
for idx=1:n_f
    f_k(idx, fault_points(idx):end)= ones(1,sim_steps-fault_points(idx)+1);
end


% Observed state matrix
x_k_hat= zeros(n, sim_steps+1);
% Observed output matrix
y_k_hat= zeros(p, sim_steps);

% Set initial state into state matrix
x_k(:, 1)= [0; 0; 0.02; 0];

% Initial  state estimation error
e_k(:, 1)= x_k(:, 1) - x_k_hat(:, 1);

for idx= 1:sim_steps
    % Input
    u_k(:, idx)= -K*x_k(:,idx);
    % Next state
    x_k(:, (idx+1))= A*x_k(:,idx) + B*u_k(:,idx) + E_d*d_k(:,idx) + E_f*f_k(:,idx);
    % Output
    y_k(:, idx)= C*x_k(:,idx) + F_f*f_k(:,idx);

    % Observed output
    y_k_hat(:, idx)= C*x_k_hat(:,idx);
    % Observed state
    x_k_hat(:, (idx+1))= A*x_k_hat(:,idx) + B*u_k(:,idx) + ...
        L*( y_k(:, idx) - y_k_hat(:, idx) );

    % Residual Generation
    r_k(:, idx)= W*( y_k(:,idx) - y_k_hat(:,idx) );
```

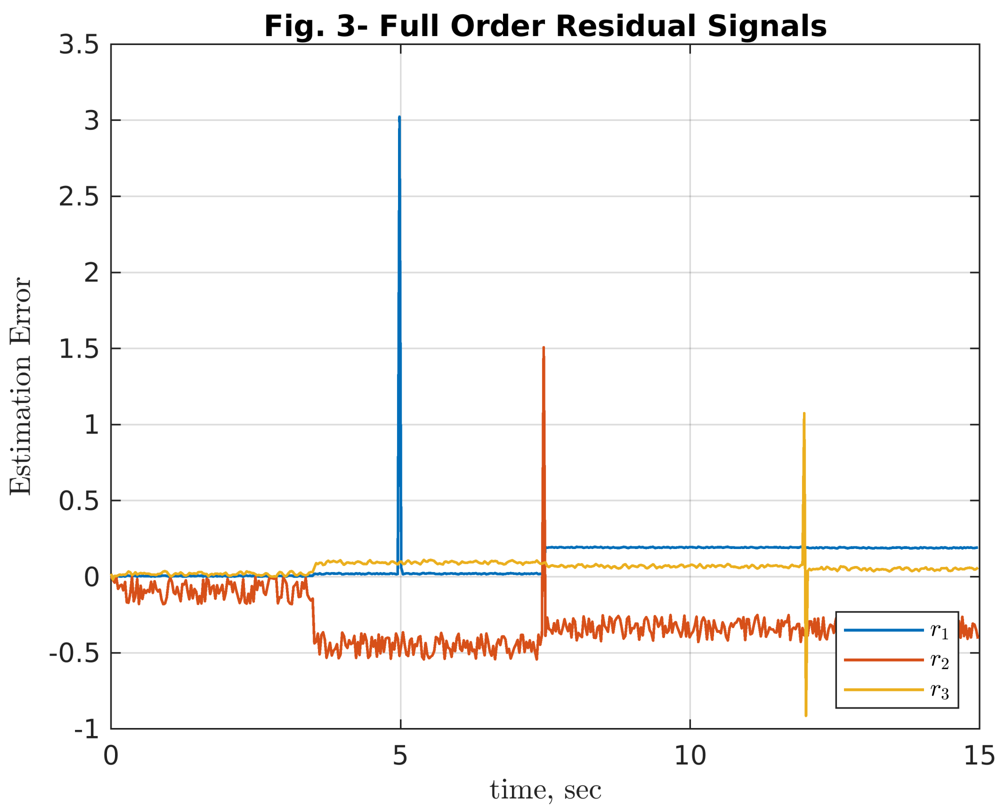
```
end
```

Plotting

We plot the state estimation error to visualize the accuracy of the observed states

```
Fig_res= figure;

plot(sim_t(1:end-1), r_k, 'LineWidth', 1);
grid on;
title('Fig. 3- Full Order Residual Signals');
ylabel('Estimation Error', 'Interpreter', 'latex');
xlabel('time, sec', 'Interpreter', 'latex');
leg= legend(['$r_1$', "$r_2$", '$r_3$']); leg.Interpreter='latex';
leg.Location= 'southeast';
```



Full Decoupling using Unknown Input Observer

We try to eliminate the influence of the disturbance signals on the residual by usung an Unknown Input Observer.

Assuming a fault-free operation, we have:

$$\dot{x} = Ax + Bu + E_d d$$
$$y = Cx$$

By taking the derivative of y, we have:

$\dot{y} = C(Ax + Bu + E_d d)$

We need CE_d to be left-invertible (of full column rank), such that:

$d = CE_d^{-1}(\dot{y} - CAx - CBu)$

With $M = CE_d^{-1}$ we have:

$\dot{x} = (A - E_d MCA)x + (B - E_d MCB)u + E_d M\dot{y}$

To avoid using the derivative of the ooutput signal, we can perform a state transformation:

$z = x - E_d My$

Such that:

$\dot{z} = (A - E_d MCA)x + (B - E_d MCB)u$

for which we can then construct an observer.

```
% Checking if CE_d is left-invertible
C_E_d= C*E_d;
size(C_E_d)
```

```
ans = 1×2
     3     1
```

```
% Not square- i.e. not invertible
```

Hoewever, for this plant, CE_d is not invertible, thus this approach cannot be used.

Parity Space Approach

With the parity space approach, we explore the inherent characteristics of the system to generate residual signals.

First, we examine the behaviour of the system over a window of length s

$y(k-s) = Cx(k-s) + Du(k-s) + F_d d(k-s) + F_f f(k-s)$

$y(k-s+1) = Cx(k-s+1) + Du(k-s+1) + F_d d(k-s+1) + F_f f(k-s+1)$
$y(k-s+1) = CAx(k-s) + CBu(k-s) + CE_d d + CE_f f + Du(k-s+1) + F_d d(k-s+1) + F_f f(k-s+1)$

.
.
.

$$y(s) = CA^s x(k-s) + CA^{s-1}Bu(k-s) + \cdots + CBu(k-1) + Du(k) + CA^{s-1}E_d d(k-s) + \cdots + CE_d d(k-1) + F_d d(k) + CA^{s-1}E_f f(k-s) + \cdots + CE_f f(k-1) + F_f f(k)$$

If we stack up all the equations into matrix form, we have

$$\begin{bmatrix} y(k-s) \\ y(k-s+1) \\ y(k-s+2) \\ \cdot \\ \cdot \\ \cdot \\ y(k-1) \\ y(k) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \cdot \\ \cdot \\ \cdot \\ CA^{s-1} \\ CA^s \end{bmatrix} x(k-s) + \begin{bmatrix} D & 0 & 0 & \cdots & 0 & 0 \\ CB & D & 0 & \cdots & 0 & 0 \\ CAB & CB & D & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ CA^{s-2}B & CA^{s-3}B & CA^{s-4}B & \cdots & D & 0 \\ CA^{s-1}B & CA^{s-2}B & CA^{s-3}B & \cdots & CB & D \end{bmatrix}$$

$$+ \begin{bmatrix} F_d & 0 & 0 & \cdots & 0 & 0 \\ CE_d & F_d & 0 & \cdots & 0 & 0 \\ CAE_d & CE_d & F_d & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ CA^{s-2}E_d & CA^{s-3}E_d & CA^{s-4}E_d & \cdots & F_d & 0 \\ CA^{s-1}E_d & CA^{s-2}E_d & CA^{s-3}E_d & \cdots & CE_d & F_d \end{bmatrix} + \begin{bmatrix} F_f & 0 & 0 & \cdots & 0 & 0 \\ CE_f & F_f & 0 & \cdots & 0 & 0 \\ CAE_f & CE_f & F_f & \cdots & 0 & 0 \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ CA^{s-2}E_f & CA^{s-3}E_f & CA^{s-4}E_f & \cdots & F_f & 0 \\ CA^{s-1}E_f & CA^{s-2}E_f & CA^{s-3}E_f & \cdots & CE_f & F_f \end{bmatrix}$$

We can then compress this into

$$y_s(k) = H_{o,s}x(k-s) + H_{u,s}u_s(k) + H_{d,s}d_s(k) + H_{f,s}f_s(k)$$

Rearranging yields

$$y_s(k) - H_{u,s}u_s(k) = H_{o,s}x(k-s) + H_{d,s}d_s(k) + H_{f,s}f_s(k)$$

Since $y_s(k)$ and $u_s(k)$ are available online, we generate the residual signal as

$$r(k) = v_s(y_s(k) - H_{u,s}u_s(k))$$

The dynamics of the residual are now governed by

$$r(k) = v_s(H_{o,s}x(k-s) + H_{d,s}d_s(k) + H_{f,s}f_s(k))$$

The vector v_s is called the parity vector and must satisfy:

- $v_s H_{o,s} = 0$
- $v_s H_{f,s} \neq 0$
- $v_s H_{d,s} = 0$ (for full decoupling)

For full decoupling to be possible, the following rank condition must be satisfied:

$rank[H_{o,s}\ \ H_{d,s}] < rank[H_{o,s}\ \ H_{d,s}\ \ H_{f,s}]$

Build up data matrices

```
% Window size
s= 5;

% H_os, H_us, H_ds, H_fs
H_os = zeros(s*p,n);
H_us= zeros(s*p,s*m);
H_ds= zeros(s*p,s*n_d);
H_fs= zeros(s*p,s*n_f);

for r = 1:s
    H_os((r-1)*p+1:r*p,1:n) = C*(A^(r-1));
    for c = 1:r
        if r == c
            H_us((r-1)*p+1:r*p,(c-1)*m+1:c*m) = D;
            H_ds((r-1)*p+1:r*p,(c-1)*n_d+1:c*n_d) = F_d;
            H_fs((r-1)*p+1:r*p,(c-1)*n_f+1:c*n_f) = F_f;
        else
            H_us((r-1)*p+1:r*p,(c-1)*m+1:c*m) = C*A^(r-c-1)*B;
            H_ds((r-1)*p+1:r*p,(c-1)*n_d+1:c*n_d) = C*A^(r-c-1)*E_d;
            H_fs((r-1)*p+1:r*p,(c-1)*n_f+1:c*n_f) = C*A^(r-c-1)*E_f;
        end
    end
end


% Check rank condition for full decoupling
rank([H_os H_ds]) < rank([H_os H_ds H_fs])
```

```
ans = logical
   1
```

Since the rank condition is satisfied, full decoupling can be achieved.

Parity Vector

```
% Left null space of H_os H_ds
N_basis= null([H_os H_ds]')';

% Parity vector (can be selected as any of the rows)
v_s= N_basis(1, :)
```

```
v_s = 1×15
   -0.0727    0.0363   -0.1677    0.1874    0.1251   -0.2757   -0.0701    0.0076 ···
```

Pre-computation of $v_s H_{u,s}$ to speed up online computation

```
rho_s= v_s*H_us
```

```
rho_s = 1×5
10^-16 ×
    0.1388    0.3469    0.1735    0.0694         0
```

Simulation

```
% Number of simulation steps
sim_steps= fault_points(end)+100;

% Simulatipn time
sim_t= (0:sim_steps)*T_s;

% Pre-allocate matrices to hold simulation data
% Output matrix
y_k= zeros(p, sim_steps);
% State matrix
x_k= zeros(n, sim_steps+1);
% Input matrix
u_k= zeros(m, sim_steps);
% Disturbance Matrix
d_k= 0.5*rand(n_d, sim_steps);
% Fault Matrix
f_k= zeros(n_f, sim_steps);
% Residual Matrix
r_k= zeros(1, (sim_steps-s));

% Fault injection
for idx=1:n_f
    f_k(idx, fault_points(idx):end)= ones(1,sim_steps-fault_points(idx)+1);
end


% Set initial state into state matrix
x_k(:, 1)= [0; 0; 0.02; 0];

% Before time k
% We need prior data for k-s to be valid
for idx= 1:(s+1)
    % Input
    u_k(:, idx)= -K*x_k(:,idx);
    % Next state
    x_k(:, (idx+1))= A*x_k(:,idx) + B*u_k(:,idx) + E_d*d_k(:,idx);
    % Output
    y_k(:, idx)= C*x_k(:,idx) + D*u_k(:,idx) + F_d*d_k(:,idx);
end

% After Time k
for idx= (s+1):sim_steps
    % Input
    u_k(:, idx)= -K*x_k(:,idx);
    % Next state
    x_k(:, (idx+1))= A*x_k(:,idx) + B*u_k(:,idx) + E_d*d_k(:,idx) + E_f*f_k(:, idx);
    % Output
    y_k(:, idx)= C*x_k(:,idx) + D*u_k(:,idx) + F_d*d_k(:,idx) + F_f*f_k(:, idx);
```

```
    % Residual Generation
    % y_s
    y_s_k= y_k(:, (idx-s+1):(idx));
    y_s_k= y_s_k(:);
    % u_s
    u_s_k= u_k(:, (idx-s+1):(idx));
    u_s_k= u_s_k(:);
    % Residual
    r_k(:, (idx-s))= v_s*y_s_k - rho_s*u_s_k;

end
```

Plotting

We plot the state estimation error to visualize the accuracy of the observed states

```
Fig_ps_res= figure;

plot(sim_t(s+1:end-1), r_k, 'LineWidth', 1);
grid on;
title('Fig. 4- Parity Space Residual Signal');
ylabel('Residual Signal', 'Interpreter', 'latex');
xlabel('time, sec', 'Interpreter', 'latex');
leg= legend(['$r_1$']); leg.Interpreter='latex';
leg.Location= 'southeast';
```

As can be seen in the figure above, the generated residual is insensitive to the actuator fault, this is probably because $B = E_d$.

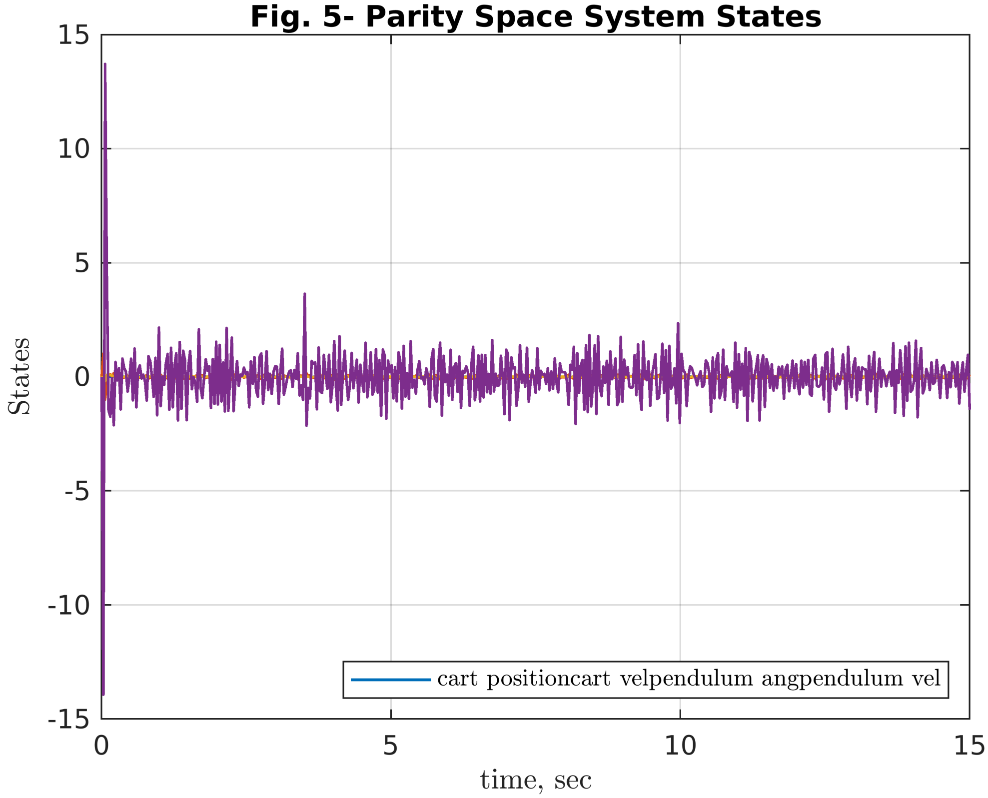
If the residual signal is decoupled from the disturbance (via E_d), then it is effectively decoupled from the input (via B).

System States

```
Fig_ps_states= figure;

plot(sim_t, x_k, 'LineWidth', 1);
grid on;
title('Fig. 5- Parity Space System States');
ylabel('States', 'Interpreter', 'latex');
xlabel('time, sec', 'Interpreter', 'latex');
leg= legend(['cart position', 'cart vel', 'pendulum ang', 'pendulum vel']); leg.Interpr
leg.Location= 'southeast';
```

Fig. 5- Parity Space System States



References

Fault Diagnosis and Fault Tolerant Systems Exercises, TU Kaiserslautern- Dina Martynova