Identification and Estimation of Firms' Beliefs and Learning in Structural Models of Competition

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Firms' Information & Beliefs

- In oligopoly markets, a firm's behavior depends on its **beliefs about the behavior of other firms** in the market.
- Firms form their beliefs under **uncertainty**: about demand, costs, competitors' behavior.
- Firms are different in their **ability for collecting and processing information**, for similar reasons as they are heterogeneous in their costs of producing goods and services.
- We expect firms to be **heterogeneous in their beliefs**.
- This heterogeneity has implications on their performance and on market outcomes.

Models of Firms' Beliefs in Structural Econometrics

- The importance of firms' heterogeneity in their ability to form expectations and the possibility of **biased / non-rational beliefs** has been long recognized in economics (Herbert Simon, 1958).
- However, in different fields in economics, and in particular in empirical IO, the status quo has been to **assume rational expectations**.
- Models of rational expectations are not very useful to study **the value** (private or social) **of acquiring, processing, or sharing information**.
- In **this lecture**, I will review recent papers in empirical IO that have relaxed this assumption and allow for firms' asymmetric information, and heterogeneous and potentially biased beliefs.

OUTLINE

[1] **Examples: Empirical Applications**

[2] **Model(s)**

[3] **Identification results**
  (a) Static games
  (b) Dynamic games

[4] **Entry in the early years of the UK fast-food industry**

[5] **Conclusions**

... My work

- **"Identification and Estimation of Dynamic Discrete Games When Players' Beliefs are Not in Equilibrium"** (*REStud*, 2019) (with Arvind Magesan)
- **"Identification of Biased Beliefs in Games of Incomplete Information Using Experimental Data"** (2018) (with Erhao Xie)
- **"Testing for Equilibrium Beliefs in Procurement Auctions"** (2019) (with Yao Luo)

1. EXAMPLES: EMPIRICAL APPLICATIONS

US Telecommunication industry after deregulation

- **Goldfarb and Xiao (AER, 2011)** study entry decisions into local US telecommunication markets following the deregulatory Telecommunications Act of 1996, which allowed free competition.
- Holding other market characteristics constant, more experienced and better educated managers have a lower propensity to enter (and a lower propensity to exit after entry) into very competitive markets.
- This suggests that **better-educated managers are better at predicting competitors' behavior**.
- This hypothesis is confirmed from the estimation of a structural game of market entry with Cognitive Hierarchy beliefs.

Learning to bid after market deregulation

- **Doraszelski, Lewis, and Pakes (AER, 2018)** investigate firms' learning about competitors' bidding behavior just after the deregulation of the UK electricity market.
- In the first year after deregulation, **firms' bidding behavior was very heterogeneous and firms made frequent and sizable adjustments** in their bids.
- During the second year, there is a dramatic reduction in the range of bids. After three years, firms' bids become very stable.
- During these three phases, demand and costs were quite stable.
- The authors argue that the changes in firms' bidding strategies can be attributed to strategic uncertainty and learning..

Learning to price after market deregulation

- **Huang, Ellickson, and Lovett (2018)** study firms' price setting behavior in the Washington State liquor market following the privatization of the market in 2012.
- After liberalization, grocery chains newly entered the market. How did these new entrants learn about demand and learn to price optimally over time?
- The authors document **large and heterogeneous price movements in the first two years after the privatization**.
- The authors present evidence consistent with firms' learning about the idiosyncratic and common components of the demand shocks, and about the time persistence of these shocks.

Entry in the early years of UK fast-food restaurant industry

- **Aguirregabiria and Magesan (REStud, 2019)** study competition in store location between McDonalds (MD) and Burger King (BK) during the early years of the fast-food restaurant industry in the UK.
- Reduced form evidence shows that the number of own stores has a strong negative effect on the probability that BK opens a new store but **the effect of the competitor's number of stores is economically negligible**.
- This behavior cannot be rationalized by an equilibrium model of market entry where firms have equilibrium beliefs about the behavior of competitors.

Bidding behavior in the Texas electricity spot market

- **Hortacsu and Puller (RAND, 2008)** analyze firms' bidding behavior in the Texas electricity spot market.
- Their dataset contains detailed information not only on firms' bids but also on their marginal costs. Using these data, the authors construct the equilibrium bids of the game and compare them to the actual observed bids.
- They find statistically and **economically very significant deviations between equilibrium and actual bids**.
  Small firms don't supply much power even when is profitable to do so.
- This finding is consistent with low strategic ability in the bidding departments of small firms.
- This suboptimal behavior leads to significant efficiency losses.

Key Question

- A key issue in all these applications is how to find convincing evidence that (some) firms have non-equilibrium or biased beliefs, and this is not just **an artifact from the specification (or misspecification) of the model**.
- How can we be (more or less) confident that what we call bias beliefs cannot be explained by observable or unobservable variables affecting firms' demand or costs?
- To answer these questions, we need to study formally the identification of beliefs and structural parameters in profits in our model.
- This is the focus of most of the remaining part of this lecture.

2. MODEL

Static game: Profit function

- N firms competing in a market. The profit function of firm i:

$$\pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x})$$

a_i is is the action of firm i
$\mathbf{a}_{-i}$ is the vector with the actions of the other firms
$\mathbf{x}$ represents variables that are common knowledge
ε_i is private information of firm i

- Firms' types $(\varepsilon_1, \varepsilon_2, ..., \varepsilon_N)$ are drawn from a distribution F.

- Firms choose simultaneously their actions a_i to maximize their respective expected profits.

Static game: Beliefs

- A firm does not know the private information of its competitors and therefore it does not know their actions.

- Firms form probabilistic beliefs about the actions of competitors.

- Let $B_i(\mathbf{a}_{-i} \mid \varepsilon_i, \mathbf{x})$ be a probability density function that represents the belief of firm i.

Static game: Best response

- Given its beliefs, a firm's expected profit is:

$$\pi_i^e(a_i, \varepsilon_i, \mathbf{x}; B_i) \quad = \quad \int \pi_i(a_i, \mathbf{a}_{-i}, \varepsilon_i, \mathbf{x}) \; B_i(\mathbf{a}_{-i} | \varepsilon_i, \mathbf{x}) \; d\mathbf{a}_{-i}$$

- A firm chooses its action a_i, or its strategy function $\sigma_i(\varepsilon_i, \mathbf{x}; B_i)$, to maximize expected profits:

$$a_i = \sigma_i(\varepsilon_i, \mathbf{x}; B_i) = \arg\max_{a_i \in \mathcal{A}} \; \pi_i^e(a_i, \varepsilon_i, \mathbf{x}; B_i)$$

- We can represent a firm's strategy as a choice probability function.

$$P_i\left(a_i \mid \mathbf{x}\right) \equiv \int 1\left\{\sigma_i(\varepsilon_i, \mathbf{x}; B_i) = a_i\right\} \; dF_i(\varepsilon_i | \mathbf{x})$$

Static game: Restricted versions

- This framework contains as particular cases most games of competition with incomplete information in IO.

- We present here several cases that have received substantial attention in applications.

  o Bayesian Nash Equilibrium
  o Level-k and Cognitive Hierarchy model
  o Rationalizability

Bayesian Nash Equilibrium

- Under Bayesian Nash Equilibrium (with independent private values):

$$B_i(\mathbf{a}_{-i} \mid \mathbf{x}) = \Pr\left(\mathbf{a}_{-i} \mid \mathbf{x}\right)$$

- This is the most commonly used solution concept in games of incomplete information in IO.
- It has received particular attention in **auction games** and in discrete choice models of **market entry**, but it has been also applied to games of quantity or price competition.

Cognitive Hierarchy and Level-K models

- Propose equilibrium concepts where firms have biased beliefs, that is, $B_i(\mathbf{a}_{-i} \mid \mathbf{x}) \neq \Pr(\mathbf{a}_{-i} \mid \mathbf{x})$.

- There is a finite number K of belief types that correspond to different levels of strategic sophistication.

- Believes for Level-0 can be arbitrary, $B^{(0)}$. Therefore, their best response probability:

$$P^{(0)}(a_i \mid \mathbf{x}) = \int 1\left\{a_i = \arg\max_{a_i \in \mathcal{A}} \pi^e\left(a_i, \varepsilon_i, \mathbf{x}; B^{(0)}\right)\right\} dF(\varepsilon_i|\mathbf{x})$$

- Level-1 players believe that all the other players are level 0:

$$P^{(1)}(a_i \mid \mathbf{x}) = \int 1\left\{a_i = \arg\max_{a_i \in \mathcal{A}} \pi^e\left(a_i, \varepsilon_i, \mathbf{x}; P^{(1)}\right)\right\} dF(\varepsilon_i|\mathbf{x})$$

Cognitive Hierarchy and Level-K models [2]

- A level-k firm believes that the other firms come from a probability distribution over levels 0 to (k-1).

$$P^{(k)}\left(a_i \mid \mathbf{x}\right) = \int 1\left\{a_i = \arg\max_{a_i \in \mathcal{A}} \pi^e\left(a_i, \varepsilon_i, \mathbf{x};\ \textstyle\sum_{k'=0}^{k-1} \lambda_{k'}^{(k)}\ P^{(k')}\right)\right\}$$

- The model imposes restrictions on beliefs.

  o There is a finite number K of belief types (typically 2 or 3).

  o These belief functions satisfy a hierarchical equilibrium.

Rationalizability

- Aradillas-Lopez & Tamer (2008).
- Rationality is common knowledge.
- In a game **with multiple equilibria**, the solution concept of **Rationalizability allows for biased beliefs**.
- Each firm has beliefs that are consistent with a BNE, but these beliefs may not correspond to the same BNE.

Dynamic game: Profit and state variables

- Time is discrete and indexed by t. The profit function:

$$\pi_{it}(a_{it}, \mathbf{a}_{-it}, \varepsilon_{it}, \mathbf{x}_t)$$

- Two additional assumptions.

  $\mathbf{x}_t$ has transition density function $f_t\left(\mathbf{x}_{t+1} \mid a_{it},\ \mathbf{a}_{-it},\ \mathbf{x}_t\right)$
  ε_{it}'s are independently distributed over time.

- Every period t, firms select simultaneously their actions to maximize their respective values.

Dynamic game: Beliefs

- A firm does not know the private information of its competitors, now or in the future, and therefore it does not know their actions.

- To make its optimal choice at period t, a firm needs to form probabilistic **beliefs about the actions of competitors, now and in the future**.

- The probability function

$$B_{it+s}^{(t)}(\mathbf{a}_{-i,t+s} \mid \mathbf{x}_{t+s})$$

represents the beliefs of firm i at period t about the behavior of other players at period $t+s$.

Dynamic game: Beliefs

Sequence of Beliefs $B_{it+s}^{(t)}$

Beliefs formed (t)	Period of the opponents' behavior $(t+s)$: $t+s=1$	$t+s=2$	...	$t+s=T-1$	$t+s=T$
$t=1$	$B_{i1}^{(1)}$	$B_{i2}^{(1)}$	...	$B_{i,T-1}^{(1)}$	$B_{iT}^{(1)}$
$t=2$	-	$B_{i2}^{(2)}$	...	$B_{i,T-1}^{(2)}$	$B_{iT}^{(2)}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t=T$	-	-	...	-	$B_{iT}^{(T)}$

Dynamic game: Best Response

- Given its current beliefs, a firm's best response at period t is the solution of a single-agent DP problem.

$$V_{it}^{\mathbf{B}_i^{(t)}}(\mathbf{x}_t) = \max_{a_{it} \in \mathcal{A}} \left\{ \pi_{it}^{B_{it}^{(t)}}(a_{it}, \varepsilon_{it}, \mathbf{x}_t) + v_{it}^{\mathbf{B}_i(t)}(a_{it}, \mathbf{x}_t) \right\}$$

- where $\pi_{it}^{B_{it}^{(t)}}(a_{it}, \varepsilon_{it}, \mathbf{x}_t)$ is the current expected profit:

$$\pi_{it}^{B_{it}^{(t)}}(a_{it}, \varepsilon_{it}, \mathbf{x}_t) \equiv \int \pi_{it}(a_{it}, \mathbf{a}_{-it}, \varepsilon_{it}, \mathbf{x}_t) \; B_{it}^{(t)}(\mathbf{a}_{-it} | \mathbf{x}_t) \; d\mathbf{a}_{-it}$$

- And $v_{it}^{\mathbf{B}_i(t)}(a_{it}, \mathbf{x}_t)$ is the expected continuation value:

$$v_{it}^{\mathbf{B}_i(t)}(a_{it}, \mathbf{x}_t) \equiv \int V_{it+1}^{\mathbf{B}_i^{(t)}}(\mathbf{x}_{t+1}) \; f_t(\mathbf{x}_{t+1} | a_{it}, \mathbf{a}_{-it}, \mathbf{x}_t) \; B_{it}^{(t)}(\mathbf{a}_{-it} | \mathbf{x}_t) \; d\mathbf{x}_{t+1}$$

Dynamic game: Best Response [2]

- A firm chooses its action a_{it}, or its strategy function $\sigma_{it}(\varepsilon_{it}, \mathbf{x}_t)$, to maximize its expected value:

$$a_{it} = \sigma_{it}(\varepsilon_{it}, \mathbf{x}_t) = \arg\max_{a_{it} \in \mathcal{A}} \left\{ \pi_{it}^{B_{it}^{(t)}}(a_{it}, \varepsilon_{it}, \mathbf{x}_t) + v_{it}^{\mathbf{B}_i(t)}(a_{it}, \mathbf{x}_t) \right\}$$

- We can represent a firm's strategy as a choice probability function:

$$P_{it}(a_{it} \mid \mathbf{x}_t) \equiv \int 1\{\sigma_{it}(\varepsilon_{it}, \mathbf{x}_t) = a_{it}\} \; dF_i(\varepsilon_{it} | \mathbf{x}_t)$$

Dynamic game: Constrained versions of the model

- In this model, all the belief functions $\left\{B_{i,t+s}^{(t)}\right\}$ are unrestricted.

- This framework contains as particular cases most solution concepts in dynamics games of competition with incomplete information.

- **Markov Perfect Equilibrium.** This is the most commonly used solution concept in applications of dynamic games in empirical IO. This solution concept imposes the restriction of rational beliefs:

$$B_{i,t+s}^{(t)}(\mathbf{a}_{-i,t+s} \mid \mathbf{x}_{t+s}) = \Pr\left(\mathbf{a}_{-i,t+s} \mid \mathbf{x}_{t+s},\ t+s\right)$$

- **Dynamic equilibrium with Learning.** Bayesian, Adaptive, Experience-Based learning are all equilibrium concepts that are constrained versions of the model above: containts both on the heterogeneity and on the evolution of beliefs over time.

3. IDENTIFICATION

Data

- The researcher has a sample of M local markets, indexed by m, where she observes firms' actions and state variables (**firms' choice data**):

$$\{a_{imt},\ \mathbf{x}_{mt} : i = 1, 2, ..., N;\ t = 1, 2, ..., T^{data}\}$$

- In addition to these data, the researcher may have data on some components of the profit function.

- We distinguish three cases, from the best to the worst case scenario:

  (a) Only Choice Data
  (b) Choice data + Revenue function
  (c) Choice data + Revenue function + Cost function

Some Notation & Assumptions

- A firm's profit is equal to revenue minus cost, $\pi_i = R_i - C_i$.
- Let $MR_i \equiv \Delta R_i / \Delta a_i$ and $MC_i \equiv \Delta C_i / \Delta a_i$ be the marginal revenue and marginal cost, respectively [either continuous and discrete choice].
- (A.1) Marginal cost does not depend on the actions of competitors.
- (A.2) Private information is additive in the marginal cost.

$$MC_i = c_i(a_i, \mathbf{x}) + \varepsilon_i$$

- (A.3) The MR does not depend on the private information:

$$MR_i = r_i(a_i, \mathbf{a}_{-i}, \mathbf{x})$$

Identification Problem

- Under these conditions, the best response [continuos choice]:

$$r_i^{B_i}(a_i, \mathbf{x}) \quad = \quad c_i(a_i, \mathbf{x}) + \varepsilon_i$$

where $r_i^{B_i}(a_i, \mathbf{x}) \equiv \int r_i(a_i, \mathbf{a}_{-i}, \mathbf{x}) \; B_i(\mathbf{a}_{-i}|\mathbf{x}) \; d\mathbf{a}_{-i}$.

- This implies, for any value a^0:

$$\Pr(a_i \geq a^0 \mid \mathbf{x}) = F_i \left[r_i^{B_i}(a^0, \mathbf{x}) - c_i(a^0, \mathbf{x}) \right]$$

- This equation summarizes all the restrictions of the model.

Identification Problem [2]

$$\Pr(a_i \geq a^0 \mid \mathbf{x}) = F_i \left[r_i^{B_i}(a^0, \mathbf{x}) - c_i(a^0, \mathbf{x}) \right]$$

- $\Pr(a_i \geq a^0 \mid \mathbf{x})$ is nonparametrically identified from the data.
- The researcher is interested in the identification of:
  - firms' belief functions $\{B_i(\mathbf{a}_{-i}|\mathbf{x})\}$
  - MC functions $\{c_i(a_i, \mathbf{x})\}$, and MR functions $r_i(a_i, \mathbf{a}_{-i}, \mathbf{x})$
  - Distributions of the private information, F_i
- We are interested in identification results that do not rely on parametric assumptions, especially on beliefs, because beliefs are endogenous objects.

Binary choice – Two-player game

- Game of price competition where firms choose between a high and a low price.

$$P_i(\mathbf{x}) = F_i\left([r_i(0,\mathbf{x}) + B_i(\mathbf{x})\ \ [r_i(1,\mathbf{x}) - r_i(0,\mathbf{x})] - c_i(\mathbf{x})]\right)$$

$P_i(\mathbf{x})$ = probability for the choice of high price by firm i.
$B_i(\mathbf{x})$ = belief probability that competitor chooses high price.
$r_i(a_{-i},\mathbf{x}) \equiv R_i(1, a_{-i}, \mathbf{x}) - R_i(0, a_{-i}, \mathbf{x})$ is the marginal revenue.
$c_i(\mathbf{x}) \equiv C_i(1,\mathbf{x}) - C_i(0,\mathbf{x})$ is the marginal cost.

- Define the quantile function $Q_i(\mathbf{x}) \equiv F_i^{-1}\left(P_i(\mathbf{x})\right)$. The best response can be described as:

$$Q_i(\mathbf{x}) = r_i(0,\mathbf{x}) + B_i(\mathbf{x})\ \ [r_i(1,\mathbf{x}) - r_i(0,\mathbf{x})] - c_i(\mathbf{x})$$

Identification with revenue and cost data

- In the static case with two-players, beliefs are identified:

$$B_i(\mathbf{x}) = \frac{Q_i(\mathbf{x}) + c_i(\mathbf{x}) - r_i(0, \mathbf{x})}{r_i(1, \mathbf{x}) - r_i(0, \mathbf{x})}$$

- This belief function can be compared to the actual choice probability of the competitor to test unbiased / rational beliefs:

$$B_i(\mathbf{x}) - P_{-i}(\mathbf{x}) = 0 \quad ?$$

- We can also test other restrictions on beliefs such as level-K or Cognitive Hierarchy models.

- We can also test if these beliefs correspond to a collusive equilibrium.

Identification with revenue but not cost data

- MR functions $r_i(0, \mathbf{x})$ and $r_i(1, \mathbf{x})$ are known to the researcher but the MC cost $c_i(\mathbf{x})$ is not known.

- Without further restrictions, the system of equations

$$Q_i(\mathbf{x}) = r_i(0, \mathbf{x}) + B_i(\mathbf{x}) \ \left[r_i(1, \mathbf{x}) - r_i(0, \mathbf{x}) \right] - c_i(\mathbf{x})$$

  cannot identify the unknown functions $B_i(\mathbf{x})$ and $c_i(\mathbf{x})$.

- Without further restrictions, any Belief function (including the BNE belief) is consistent with observed behavior, $Q_i(\mathbf{x})$, given the appropriate MC function.

Identification: Firm-specific cost shifter

- **Exclusion Restriction (Firm specific cost shifter)**: The vector $\mathbf{x}$ has a firm-specific components that affect the MC of a firm but not MC (or MR) of other firms.

- That is, $\mathbf{x} = (\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i})$ such that:

  (a) the MC of firm i depends on $(\mathbf{w}, \mathbf{z}_i)$ but not on $\mathbf{z}_{-i}$:

$$c_i(\mathbf{x}) = c_i(\mathbf{w}, \mathbf{z}_i)$$

  (b) the MR functions do not depend on $(\mathbf{z}_i, \mathbf{z}_{-i})$:

$$r_i(a_{-i}, \mathbf{x}) = r_i(a_{-i}, \mathbf{w})$$

Identification: Firm-specific cost shifter [2]

- Let $\mathbf{z}^1_{-i}$ and $\mathbf{z}^2_{-i}$ be two values for $\mathbf{z}_{-i}$.

$$\begin{cases} Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i}) = r_i(0, \mathbf{w}) + B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i}) \; [r_i(1, \mathbf{w}) - r_i(0, \mathbf{w})] - c_i(\mathbf{w} \\ \\ Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) = r_i(0, \mathbf{w}) + B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) \; [r_i(1, \mathbf{w}) - r_i(0, \mathbf{w})] - c_i(\mathbf{w} \end{cases}$$

- Taking the difference between these best-response equations:

$$B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) - B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i}) = \frac{Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) - Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i})}{r_i(1, \mathbf{w}) - r_i(0, \mathbf{w})}$$

- Test for unbiased beliefs, or other restrictions on beliefs:

$$B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) - B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i}) = P_{-i}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) - P_{-i}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i}) \; ?$$

Identification using only firms' choice data

- This exclusion restriction can be applied to the identification of beliefs also when the researcher does not know the revenue function.

- Let $\mathbf{z}^1_{-i}$, $\mathbf{z}^2_{-i}$, and $\mathbf{z}^3_{-i}$ be three values for $\mathbf{z}_{-i}$.

$$\begin{cases} Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) - Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i}) \\ = \left[B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) - B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i})\right] \ \ [r_i(1, \mathbf{w}) - r_i(0, \mathbf{w})] \\ \\ Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^3_{-i}) - Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i}) \\ = \left[B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^3_{-i}) - B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i})\right] \ \ [r_i(1, \mathbf{w}) - r_i(0, \mathbf{w})] \end{cases}$$

- And taking the ratio between these two differences, we have that:

$$\frac{B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) - B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i})}{B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^3_{-i}) - B_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i})} = \frac{Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^2_{-i}) - Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i})}{Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^3_{-i}) - Q_i(\mathbf{w}, \mathbf{z}_i, \mathbf{z}^1_{-i})}$$

4. EXTENSIONS

(a) Multinomial or Continuos choice
(b) Nonparametric distribution of private info
(c) More than two players
(d) Dynamic game

Multinomial or continuous choice game

- All the previous result extend to a static two-player game where the decision variable is multinomial of continuous.

- The dimension of the beliefs function is still the same as the dimension of the choice probability function.

Identification with nonparametric distribution private info

- When the decision variable is continuous or discrete & ordered, there is identification of beliefs even if F_i is nonparametrically specified.

- Consider an ordered discrete choice model. We have:

$$F_i^{-1}\left[P_i(a_i \geq a^0 \mid \mathbf{x})\right] \quad = \quad r_i^{B_i}(a^0, \mathbf{x}) - c_i(a^0, \mathbf{x})$$

- Let $(a_i^1, \mathbf{z}_{-i}^1)$ and $(a_i^2, \mathbf{z}_{-i}^2)$ be two values of $(a_i, \mathbf{z}_{-i})$ such that $P_i(a_i^1 \mid \mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^1) = P_i(a_i^2 \mid \mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^2)$. This implies that:

$$F_i^{-1}\left[P_i(a_i^1 \mid \mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^1)\right] = F_i^{-1}\left[P_i(a_i^2 \mid \mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^2)\right]$$

- We can take the difference between the best responses for $P_i(a_i^1 \mid \mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^1)$ and for $P_i(a_i^2 \mid \mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^2)$ and obtain an expression that does not depend on the distribution of the private information.

More than two players

- Consider that there are $N > 2$ firms. The best response implies:

$$Q_i(\mathbf{x}) \quad = \quad \sum_{\mathbf{a}_{-i}} B_i(\mathbf{a}_{-i} \mid \mathbf{x}) \; r_i(\mathbf{a}_{-i}, \mathbf{x}) - c_i(\mathbf{x})$$

- Even if the researcher knows the MR and MC functions, there are infinite values of $B_i(\mathbf{a}_{-i} \mid \mathbf{x})$ that can rationalize the observed behavior $Q_i(\mathbf{x})$.

- However, the exclusion restriction of a firm-specific cost shifter still implies identification of beliefs with $N > 2$ players, and even when the MC and MR functions are not known to the researcher.

- All we need is that the space of the cost shifter $\mathbf{z}_{-i}$ has at least as many points of support as the points in the support of the competitors actions, $\mathbf{a}_{-i}$.

Dynamic games

- For the dynamic game, the best response implies:

$$Q_{it}(\mathbf{x}_t) = r_{it}^{B_{it}^{(t)}}(\mathbf{x}_t) - c_{it}(\mathbf{x}_t)$$

$$+ \left(1 - B_{it}^{(t)}(\mathbf{x}_t)\right) \ \Delta V_{it}^{\mathbf{B}_{i,t+1}^{(t)}}(0, \mathbf{x}_t) + B_{it}^{(t)}(\mathbf{x}_t) \ \Delta V_{it}^{\mathbf{B}_{i,t+1}^{(t)}}(1, \mathbf{x}_t)$$

- Even if the researcher knows the MR and MC at every period, it is not possible to identify the current belief function.
- This is because continuation values depend on beliefs about the opponents' behavior at future periods.
- Furthermore, without further conditions, the firm-specific cost shifter that provides identification in the static model is not enough to provide identification of beliefs in the dynamic game.
- This is because the continuation value may depend on the cost shifter.

Dynamic games: Shifter in Opponent's Adjustment cost

- A firm-specific cost shifter that provides identification of beliefs in the dynamic game is one that does not enter in the continuation value.
- This type of cost shifter often appears in dynamic games of oligopoly competition: the decision variable at previous period $t-1$ in a model with adjustment costs.
- In the model of price competition, suppose that there is a menu cost or changing the price. This implies that a firm's price at period $t-1$ affects its MC period t. However, given the price at period t, the price at $t-1$ does not have any effect on the continuation value.
- We can use variation in the lagged price of the competitors(s) to differentiate out the continuation value, as well as the current MR and MC, and identify beliefs.

Identification Assumptions

- **Exclusion Restriction 1**: $\mathbf{x}_t = (z_{it}, z_{jt}, \mathbf{w}_t)$ such that z_{it} enters in marginal cost of firm i but not in MC of the other firm.

$$c_{it}(z_{it}, z_{jt}, \mathbf{w}_t) = c_{it}(z_{it}, \mathbf{w}_t)$$

- **Exclusion Restriction 2**: The transition probability of the state variable z_{it} is such that, given a_{it} the value of z_{it+1} does not depend on z_{it}:

$$f_t(z_{it+1} \mid a_{it}, z_{it}) = f_t(z_{it+1} \mid a_{it})$$

Exclusion restriction in the transition probability (ID-3)

$$f_t\left(z_{it+1} \mid a_{it}, z_{it}\right) = f_t\left(z_{it+1} \mid a_{it}\right)$$

- An important class of models that satisfies this condition is when $z_{it} = a_{i,t-1}$, such that the transition rule is simply:

$$z_{it+1} = a_{it}$$

- Many dynamic games of oligopoly competition belong to this class, e.g., market entry/exit, technology adoption, and some dynamic games of quality or capacity competition, among others.

Example: Quality competition

- Quality ladder dynamic game (Pakes and McGuire, 1994).
- z_{it} is the firm's quality at $t-1$.
- The decision variable a_{it} is the firm's quality at period t, such that:

$$z_{it+1} = a_{it}$$

- The model is dynamic because the payoff function includes a cost of adjusting quality that depends on $a_{it} - z_{it}$:

$$AC_i \left(a_{it} - z_{it} \right)$$

- Given competitors quality at period t, a_{jt}, firm i's profit does not depend on competitors' qualities at $t-1$.

Dynamic games: Identification of beliefs

- $\mathbf{x}_t = (\mathbf{w}_t, \mathbf{z}_{it}, \mathbf{z}_{-it})$ with same properties as above, and the additional properties that the continuation values do not depend on $(\mathbf{z}_{it}, \mathbf{z}_{-it})$.

- Let $\mathbf{z}_{-i}^1$, $\mathbf{z}_{-i}^2$, and $\mathbf{z}_{-i}^2$ be three values of the shifter. We have:

$$\frac{B_{it}^{(t)}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^2) - B_{it}^{(t)}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^1)}{B_{it}^{(t)}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^3) - B_{it}^{(t)}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^1)} = \frac{Q_{it}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^2) - Q_{it}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^1)}{Q_{it}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^3) - Q_{it}(\mathbf{w}, \mathbf{z}_i, \mathbf{z}_{-i}^1)}$$

- We can identify the sequence of beliefs at period t about the opponents' contemporaneous behavior at period t. We cannot identify beliefs about the opponent's behavior in the future.

- However, identification of the evolution of contemporaneous beliefs is enough for testing almost any model of learning and beliefs formation.

Testing learning models

- Suppose that the researcher has identified the sequence of belief functions $B_{it}^{(t)}(a_{-it}|\mathbf{x}_t)$.

$$B_{it}^{(t)}(a_{-it}|\mathbf{x}_t) : t = 1, 2, ..., T^{data}$$

- Given these data on beliefs, we can test for different hypothesis about the evolution of beliefs.

- For notational simplicity, we represent these beliefs as if they were transition probabilities $B_{it}^{(t)}(\mathbf{x}_{t+1}|\mathbf{x}_t)$

Testing for Rational Beliefs

- Let $P_t(\mathbf{x}_{t+1}|\mathbf{x}_t)$ be the actual distribution of $\mathbf{x}_{t+1}$ conditional on $\mathbf{x}_t$ in the data. $P_t(\mathbf{x}_{t+1}|\mathbf{x}_t)$ is identified.
- Testing for Rational Beliefs is equivalent to testing for the restrictions:

$$B_{it}^{(t)}(\mathbf{x}_{t+1}|\mathbf{x}_t) = P_t(\mathbf{x}_{t+1}|\mathbf{x}_t)$$

Testing for Bayesian Learning

- Let $\mathcal{P}_i \equiv \{\psi_{\ell,i}(\mathbf{x}'|\mathbf{x}) : \ell = 1, 2, ..., L\}$ be a collection of L transition probabilities.

- The prior belief function for firm i at period $t = 0$ is a mixture of the distributions in $\mathcal{P}_i$, where $\{\lambda_{\ell,i}^{(0)}\}$ are the mixing probabilities.

- At any period $t \geq 1$, firms observe the new state $\mathbf{x}_t$ and use this information to update their respective beliefs using Bayes rule.

$$B_{it}^{(t)}(\mathbf{x}_{t+1}|\mathbf{x}_t) = \sum_{\ell=1}^{L} \lambda_{\ell,i}^{(t)}(\mathbf{x}', \mathbf{x}) \; \psi_{\ell,i}(\mathbf{x}'|\mathbf{x})$$

where Bayesian updating implies:

$$\lambda_{\ell,i}^{(t)}(\mathbf{x}', \mathbf{x}) = \frac{\psi_{\ell,i}(\mathbf{x}_t|\mathbf{x}_{t-1}) \; \lambda_{\ell,i}^{(t-1)}(\mathbf{x}', \mathbf{x})}{\sum_{\ell'=1}^{L} \psi_{\ell',i}(\mathbf{x}_t|\mathbf{x}_{t-1}) \; \lambda_{\ell',i}^{(t-1)}(\mathbf{x}', \mathbf{x})}$$

Testing for Adaptive Learning

- At period t:

$$B_{it}^{(t)}(\mathbf{x}'|\mathbf{x}) = (1-\delta_i)\ B_{it-1}^{(t-1)}(\mathbf{x}'|\mathbf{x}) + \delta_{it}\ K([\mathbf{x}_t, \mathbf{x}_{t-1}] - [\mathbf{x}', \mathbf{x}])$$

- $\delta_i \in (0,1)$ is a parameter that determines the speed of learning.

- $K(.)$ is a Kernel function that establishes whether the new information at period t is used to update beliefs only at that point or also at nearby values.

Testing for Fictitious Play

- Fictitious play is a learning rule where each firm believes that rivals' actions are sampled from the empirical distribution of their past actions.

- The belief function of firm i about the choice probability of firm j is:

$$B_{it}^{(t)}(a_j|\mathbf{x}) = \frac{\sum_{s=1}^{t} \omega_{(s,t)} \; 1\{[a_{jt-s}, \mathbf{x}_{t-s}] = [a_j, \mathbf{x}]\}}{\sum_{s=1}^{t} \omega_{(s,t)} \; 1\{\mathbf{x}_{t-s} = \mathbf{x}\}}$$

- $\{\omega_{(s,t)} : s \leq t\}$ are weights non-increasing in the lag index s.

- In its original version (Brown, 1951) the fictitious play model assumes that the weights $\omega_{(s,t)}$ are the same at every period s such that belief $B_{it}^{(t)}(a_j|\mathbf{x})$ is just the empirical frequency of action a_j conditional on state $\mathbf{x}$ during periods 1 to t.

Testing for Rationalizability

- *Rationalizability* (Bernheim, 1984; Pearce, 1984).

- The concept of rationalizability imposes two simple restrictions on firms' beliefs and behavior.
  - [A.1] Every firm is rational in the sense that it maximizes its own expected profit given beliefs.
  - [A.2] This rationality is common knowledge, i.e., every firms knows that all the firms know that it knows ... that all the firms are rational.

- We have impose [A.1] to identify beliefs, but we have not impose [A.2]. We can test for [A.2].

- The set of outcomes of the game that satisfy these conditions (the set of rationalizable outcomes) includes all the MPE Nash equilibria of the game, but it also includes many other outcomes too.

Testing for Level-K Rationality

- *Cognitive Hierarchy and Level-k Rationality*. These models assume that players have different levels of strategic sophistication.
- Every firm (player) maximizes its subjective expected profit given its beliefs.
- Firms are heterogeneous in their beliefs and there is a finite number of belief types.
- Beliefs for each type are determined by a hierarchical structure.
- Level-0 firms believe that strategic interactions are negligible and therefore they behave as in a single-agent model, i.e., as if they were monopolists.
- Level-1 firms believe that the rest of the firms are level-0, and they behave by best responding to these beliefs. And so on.

5. AN APPLICATION: ENTRY IN THE EARLY YEARS OF THE UK FAST FOOD INDUSTRY

EMPIRICAL APPLICATION

- Dynamic game of store location by McDonalds (MD) and Burger King (BK) using data for United Kingdom during the period 1990-1995.
- Panel of 422 local markets (districts) and six years, 1990-1995.
- Information on the number of stores of McDonalds (MD) and Burger King (BK) in United Kingdom.
- Information on local market characteristics such as population, density, income per capita, age distribution, average rent, local retail taxes, and distance to the headquarters of each firm in UK.

Table 1

Descriptive Statistics for the Evolution of the Number of Stores

Data: 422 markets, 2 firms, 5 years = 4,220 observations

	Burger King					**McDonalds**			
	1991	1992	1993	1994	1995	1991	1992	1993	1994
# Markets	98	104	118	131	150	213	220	237	248
Δ # Markets	17	6	14	13	19	7	7	17	11
# of stores	115	128	153	181	222	316	344	382	421
Δ # of stores	36	13	25	28	41	35	28	38	39
stores per mark	1.17	1.23	1.30	1.38	1.48	1.49	1.56	1.61	1.70

Model

- $K_{imt} \in \{0, 1, ..., |\mathcal{K}|\}$ number of stores of firm i in market m at period $t-1$.
- $Y_{imt} \in \{0, 1\}$ decision of firm i to open a new store.
- $Y_{imt} + K_{imt} = \#$ stores of firm i at period t.
- Firm i's total profit function is equal to:

$$\Pi_{imt} = VP_{imt} - EC_{imt} - FC_{imt}$$

Model (2)

- Variable profit function:

$$VP_{imt} = (\mathbf{W}_m \; \gamma) \; (Y_{imt} + K_{imt}) \left[\begin{array}{c} \theta_{0i}^{VP} + \theta_{can,i}^{VP}(K_{imt} + Y_{imt}) \\ +\theta_{com,i}^{VP}(K_{jmt} + Y_{jmt}) \end{array} \right]$$

- Entry cost:

$$EC_{imt} \;\; = \;\; 1\{Y_{imt} > 0\} \; \left[\theta_{0i}^{EC} + \theta_{K,i}^{EC} \; 1\{K_{imt} > 0\} + \theta_{S,i}^{EC} \; S_{imt} + \varepsilon_{it}\right]$$

- Fixed cost:

$$FC_{imt} \;\; = \;\; 1\{(K_{imt} + Y_{imt}) > 0\} \left[\begin{array}{c} \theta_{0i}^{FC} + \theta_{lin,i}^{FC}(K_{imt} + Y_{imt}) \\ +\theta_{qua,i}^{FC}(K_{imt} + Y_{imt})^2 \end{array} \right]$$

Tests of Unbiased Beliefs

Data: 422 markets, 5 years = 2,110 observations

BK: $\widehat{D}$ (p-value)	66.841	(0.00029)
MD: $\widehat{D}$ (p-value)	42.838	(0.09549)

- We can reject hypothesis that BK beliefs are unbiased (p-value 0.00029).
- Restriction is more clearly rejected for large values of the state variable (distance to chain network) S_{MD}.

Where to impose unbiased beliefs?

- We propose three different criteria:
  [1] Minimize distance $\|B_i - P_j\|$
  [2] Test for monotonicity of beliefs: if not rejected, impose unbiased beliefs in extreme values of S_j.
  [3] Most visited values of S_j.

- In this empirical application, the three criteria have the same implication: impose unbiased beliefs at the lowest values for the distance S_j.

Estimation of Dynamic Game

Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.95$ (not estimated)			
	Equilibrium Beliefs		Biased Beliefs	
	BK	MD	BK	MD
Var Profits:				
θ_0^{VP}	0.5413	0.8632	0.4017	0.8271
	$(0.1265)^*$	$(0.2284)^*$	$(0.2515)^*$	$(0.4278)^*$
θ_{can}^{VP} cannibalization	-0.2246	0.0705	-0.2062	0.0646
	$(0.0576)^*$	$(0.0304)^*$	$(0.1014)^*$	(0.0710)
θ_{com}^{VP} competition	**-0.0541**	-0.0876	**-0.1133**	-0.0856
	(0.0226)*	(0.0272)	**(0.0540)***	(0.0570)
Log-Likelihood	-848.4		-840.4	

Estimation of Dynamic Game				
Data: 422 markets, 2 firms, 5 years = 4,220 observations				
	$\beta = 0.95$ (not estimated)			
	Equilibrium Beliefs		Biased Beliefs	
	BK	MD	BK	MD
Fixed Costs:				
θ_0^{FC} fixed	0.0350	0.0374	0.0423	0.0307
	(0.0220)	(0.0265)	(0.0478)	(0.0489)
θ_{lin}^{FC} linear	0.0687	0.0377	0.0829	0.0467
	(0.0259)*	(0.0181)*	(0.0526)*	(0.0291)
θ_{qua}^{FC} quadratic	-0.0057	0.0001	-0.0007	0.0002
	(0.0061)	(0.0163)	(0.0186)	(0.0198)

Estimation of Dynamic Game

Data: 422 markets, 2 firms, 5 years = 4,220 observations

	$\beta = 0.95$ (not estimated)			
	Equilibrium Beliefs		Biased Beliefs	
	BK	MD	BK	MD
Entry Cost:				
θ_0^{EC} fixed	0.2378	0.1887	0.2586	0.1739
	(0.0709)*	(0.0679)*	(0.1282)*	(0.0989)*
θ_K^{EC} (K)	-0.0609	-0.107	-0.0415	-0.1190
	(0.043)	(0.0395)*	(0.096)	(0.0628)*
θ_S^{EC} (S)	0.0881	0.0952	0.1030	0.1180
	(0.0368)*	(0.0340)*	(0.0541)*	(0.0654)*

6. CONCLUSIONS

Conclusions

- The change in a firm's behavior as result of competitors' idiosyncratic shocks reveals information about the firm's beliefs.
- This idea can be used to relax the assumption of rational beliefs in models of firm competition.
- The identification of beliefs can be used to study:
  - How beliefs depend on firm/managers characteristics;
  - How firms learn over time
  - Implications of biased beliefs on firms' performance and market efficiency
- **Measuring beliefs using firms' surveys**