Examples of Joint Estimation of a Non-Parametric Pay-off function and Discount Factor in DDC models

Hari's Awesome Class

October 10, 2019

Overview

We develop two numerical exercises for highly stylized dynamic discrete choice models. Our goal here is not to develop a rich empirical model, but to illustrate a few points about identification in dynamic discrete choice models.

1. A point identified Zurcher-like model
   - Solve for the value functions at the true primitives using full solution methods.
   - Generate the choice probabilities
   - Recover β and $\boldsymbol{c}_1$ given the choice probabilities $\boldsymbol{p}$ and states $\boldsymbol{x}$.
2. A general, set identified model
   - Recover all β and $\boldsymbol{c}_1$ that rationalize given choice probabilities $\boldsymbol{p}$ and states $\boldsymbol{x}$.

Set up

Single agent problem

- Discrete time, infinite horizon.
- Binary choice $d \in \mathcal{D} = \{0, 1\}$.
- Observable states $\mathcal{X} = \{x_1, \ldots, x_J\}$.
- Markov state transitions $\boldsymbol{f}_d\,(x) = \{Pr(x_1|x, d), \ldots, Pr(x_J|x, d)\}$ for all $d \in D$.
- Unobservable states $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_K\} \sim g$, independent of x.
- Pay-offs $c_d(x, \varepsilon) = c_d(x) + \varepsilon_d$.
- Normalize the replacement costs $c_0(x, \varepsilon) = \varepsilon_0$
- Observed choices $p_d(x)$ for all d and x.

We consider a non-parametric pay-off function which $\boldsymbol{c}_1 = [c_1(x_1), \ldots, c_1(x_J)]'$. For the first exercise, it can be useful to think of $c_1(x)$ as a maintenance cost function in a Zurcher-type problem. For the second exercise, $c_1(x)$ is a general pay-off function. We let the $J \times J$ matrix $\boldsymbol{f}_d$ stack the probability distributions $\boldsymbol{f}_d(x)$.

The primitives of the problems are $\{\boldsymbol{c}_1, \beta, \boldsymbol{f}, g\}$, where $\boldsymbol{c}_1$. We are concerned with the joint identification of $\boldsymbol{c}_1$ and β, taking $\boldsymbol{f}$ as identified from the transitions in the data and g as identified as

$EV1$ by assumption.

We consider the exclusion restriction

$$c_1(x_1) - c_1(x_2) = 0 \tag{1}$$

for some pair of states $x_1, x_2 \in \mathcal{X}, x_1 \neq x_2$. For the Zurcher problem, the restriction could hold if the maintenance cost is known to be flat between mileages x_1 and x_2.

Exercise 1: Point identification

1.1 Solve for the value functions using NFXP

The first exercise sets the primitives $\boldsymbol{c}_1, \boldsymbol{f}$, and β at some true values and solves for the value functions using nested fixed point. This gives the alternative specific value functions $\boldsymbol{v}_d$.

1.2 Construct the (true) choice probabilities from the reduced form

Use the value function $\boldsymbol{v}$ in the Hotz-Miller conditions

$$\ln(p_1(x)) - \ln(p_0(x))) = v_1(x) - v_0(x) \tag{2}$$

to get the choice probabilities for all $x \in \mathcal{X}$. We can now construct $\Psi(x)$, the corrected value function, from the choice probabilities.

$$\Psi(x) = -\ln(p_0(x)) \tag{3}$$

1.3 Recover the discount factor

With Ψ in hand, we construct the identifying moment condition

$$\begin{aligned}\ln(p_1(x_1)) - \ln(p_0(x_1))) - \ln(p_1(x_2)) + \ln(p_0(x_2))) = \\ \beta\left[\boldsymbol{f}_1(x_1) - \boldsymbol{f}_0(x_1) - \boldsymbol{f}_1(x_2) + \boldsymbol{f}_0(x_2)\right]\left[\boldsymbol{I} - \beta \boldsymbol{f}_0\right]^{-1}\boldsymbol{\Psi}.\end{aligned} \tag{4}$$

The state transitions $\boldsymbol{f}$ in this example have the regenerative optimal stopping structure of the Zurcher model: choosing to replace resets the state distribution. The Zurcher model is one example from the class models with the one-period finite dependence property. We know that the discount factor is point identified for this class. We verify point identification by plotting the moment and check that the moment condition has a unique zero, which is the recovered β.

1.4 Invert out the cost function from the HM conditions given β

With β known, we can recover $\boldsymbol{c}_1$ from the Hotz-Miller conditions in (**??**) (this recovery uses the function *rationalizingc*). For each $x \in \mathcal{X}$, we have one unknown parameter $c_1(x)$ recovered from

$$\ln(p_1(x)) - \ln(p_0(x))) - \beta\left[\boldsymbol{f}_1(x) - \boldsymbol{f}_0(x)\right]\left[\boldsymbol{I} - \beta \boldsymbol{f}_0\right]^{-1}\boldsymbol{\Psi} = c_1(x). \tag{5}$$

We see that the recovered $\boldsymbol{c}_1$ satisfies the exclusion restrictions.

This exercise demonstrates point identification: if the exclusion restriction holds and we know the choice probabilities and the transitions, then we know β and $\boldsymbol{c}$, and vice versa. In technical terms, this model is a one-to-one mapping from the data space to the parameter space.

1.5 Rationalize the data for $\beta = 0$

If we instead set $\beta = 0$, we can still rationalize the same choice probabilities as above. We however see that at $\beta = 0$, the pay-off function does not satisfy the exclusion restriction, i.e $c_1(x_1) \neq c_1(x_2)$. We see that without the exclusion restriction, we can equally well rationalize the data with $\beta = 0$ as any other $\beta \in [0, 1)$.

1.6 No primitives that rationalize the data

We next change the data (choice probabilities) and plot the moment condition. We see that there is no $\beta \in [0, 1)$ that solves the moment condition for these data. This implies that the model is rejected by these data.

Exercise 2: Set identification

In the previous two examples, we either had point identification, i.e. a unique discount factor and non-parametric normalized cost function that rationalized the data, or there were no primitives that could jointly rationalize the data. Point identification followed from the particular structure of the optimal stopping problem. We now consider a more general case where we only have set identification: there may be more than one discount factor that can rationalize the data under the exclusion restriction. We will see that we get *local point identification*: a zero to the moment in (**??**) is an isolated point in a small neighbourhood around the solution, and not an interval.

2.1 Plot the moment condition over the domain of β

This time, we do not start with the primitives, but use the choice probabilities directly. We now see that the moment condition has two zeros. Both discount factors that satisfy the moment condition are equally consistent with the data.

2.2 Solve for all β that satisfy the moment condition

W solve for the two β that satisfy (**??**) given the choice probabilities. For each of the two βs, we solve for the unique $\boldsymbol{c}_1$ that is consistent with that β.

2.3 Solve for the rationalizing costs for all $\beta \in [0, 1)$

In the final exercise, we plot $\boldsymbol{c}_1$ for a range of β. We see that the exclusion restrictions are satisfied in exactly two points: the solutions β' and β^*.