

Monotone Stochastic Choice Models: The Case of Risk and Time Preferences

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Suppose that, when evaluating two alternatives x and y by means of a parametric utility function, low values of the parameter indicate a preference for x and high values indicate a preference for y . We say that a stochastic choice model is monotone whenever the probability of choosing x is decreasing in the preference parameter. We show that the standard use of random utility models in the context of risk and time preferences may sharply violate this monotonicity property and argue that their use in preference estimation may be problematic. They may pose identification problems and could yield biased estimations. We then establish that the alternative random parameter models are always monotone.

I. Introduction

In the most standard *random utility model* (RUM), the individual chooses the alternative that provides maximal utility, which is assumed to be addi-

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tively composed of two terms: (i) the representative utility, $U_\omega(x)$, based on the characteristics of the alternative x and the relevant preference parameter ω ; and (ii) a random independent and identically distributed (i.i.d.) term, $\epsilon(x)$, that follows a continuous and strictly increasing cumulative distribution Ψ on the reals. In particular, when facing a choice between two alternatives x and y , the RUM probability assigned to choosing x is simply the probability that $U_\omega(x) + \epsilon(x)$ is greater than $U_\omega(y) + \epsilon(y)$. By far the most widely used error distributions are the type I extreme value and the normal, which lead to the well-known logit (or Luce) and probit models, respectively.

RUMs represent a natural way of introducing variability in choice and have proven critical in the microeconometrics of discrete choice analysis.¹ Over the last years, RUMs have been intensively used to estimate preference parameters such as risk aversion or delay aversion. We argue that the RUMs employed in these contexts may be theoretically flawed and, presumably, problematic for use in the estimation exercise.

For purposes of illustration consider the setting in which ω represents the level of risk aversion. Gamble x gives \$1 with probability .9 and \$60 with probability .1, and gamble y gives \$5 for sure. As in standard practice, let these gambles be evaluated by constant relative risk aversion (CRRA) expected utilities, that is,

$$U_\omega^{\text{crta}}(x) = .9 \frac{1^{1-\omega}}{1-\omega} + .1 \frac{60^{1-\omega}}{1-\omega}$$

and

$$U_\omega^{\text{crta}}(y) = \frac{5^{1-\omega}}{1-\omega}.$$

Notice that gamble x is riskier than gamble y , since it is only for low levels of risk aversion that x is preferred to y . Then any stochastic model should fulfill the following monotonicity condition: higher levels of risk aversion should be associated with lower probabilities of choosing the riskier gamble x . Consider, in particular, the logit model that has the closed-form probability of choosing gamble x equal to $e^{\lambda U_\omega^{\text{crta}}(x)} / [e^{\lambda U_\omega^{\text{crta}}(x)} + e^{\lambda U_\omega^{\text{crta}}(y)}]$.² Figure 1 shows that, unfortunately, the logit model is nonmonotone.³ There

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¹ Section II reviews the relevant literature.

² Parameter λ is inversely related to the variance of the initial distribution Ψ and is typically interpreted as a rationality parameter. The larger λ , the more rational the individual.

³ The figure also reports the RUM probability of choosing x for constant absolute risk aversion (CARA) expected utilities; all the formal definitions are given in Sec. III. The fig-

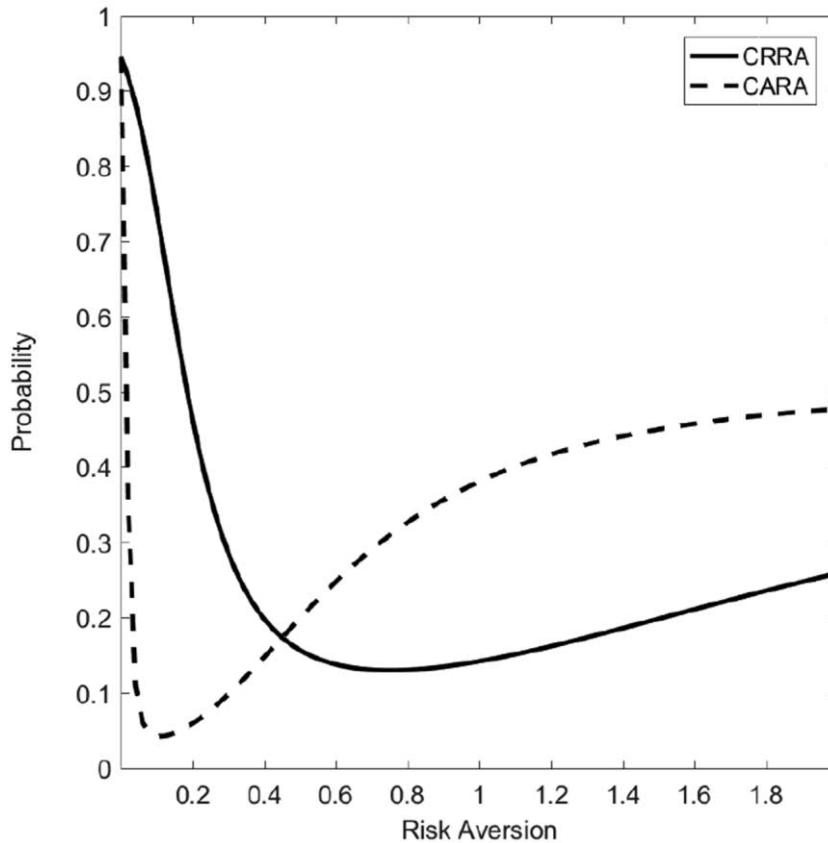


FIG. 1.—Logit RUM probabilities of choosing gamble x , for CRRA and CARA expected utilities.

is a large range of risk aversion parameters for which the probability of choosing the riskier gamble x is increasing with the level of risk aversion. The practical implications of this inconsistency in the empirical estimation of risk aversion are apparent: (i) there is an identification problem arising from the fact that the same choice probabilities may be associated with two different levels of risk aversion, and (ii) there is an upper limit to the level of risk aversion that can be estimated when using maximum likelihood techniques, even for extremely risk-averse individuals.

The intuition for this nonmonotonicity is simple and pertains to the very core structure of RUMs. In these models, since the two errors affect

ure uses $\lambda = 1.5$. Appendix B shows that another popular presentation of the logit model, which uses the log transformation of utilities, presents the same type of problems.

utilities independently, the probability of choosing x is completely determined by the cardinal difference in utilities $U_\omega(x) - U_\omega(y)$. Therefore, for the choice probability of x to decrease with ω , the utility difference must also decrease with ω . For low levels of the parameter, the model works well; higher levels of risk aversion cause the difference to diminish, yielding the desired preference for the safe gamble. However, once the utility difference is already negative, and hence the probability of choosing x is less than one-half, there is a risk aversion level above which problems arise. The reason is that both $U_\omega^{\text{crra}}(x)$ and $U_\omega^{\text{crra}}(y)$ approach zero with higher levels of risk aversion. Hence, $U_\omega^{\text{crra}}(x) - U_\omega^{\text{crra}}(y)$ also approaches zero, allowing the noise to acquire a prominent role and making the choice probabilities converge to one-half.

In this paper we address a number of issues surrounding the problem described in the previous paragraphs. First, we show in corollary 1 that every RUM using either CRRA or CARA utilities, and not exclusively the logit RUM, violates monotonicity, and this happens for every pair of gambles in which one is riskier than the other, showing that the problem is ubiquitous. Second, proposition 2 shows that, in consonance with the above example, for every such RUM and every such pair of gambles, there is always a level of risk aversion beyond which the probability of choosing the riskier gamble increases. Third, proposition 3 shows that the problem is heightened by increasing the payoffs involved in the gambles. Fourth, we show that the nonmonotonicity problem is not restricted to risk preferences but also affects other important preference dimensions, such as time preferences. In particular, proposition 4 shows that, under the standard discount functions, for every RUM, and for basically every pair of streams that can be ordered in terms of delay aversion, the probability of choosing the longer-delay stream starts to increase beyond a certain level of delay aversion. Fifth, proposition 1 provides easy-to-check conditions to determine the monotonicity of RUMs based on any preference parameter, not necessarily involving risk or time.

Once we have precisely established the extent of the nonmonotonicity problems, we show that an alternative random choice model, the *random parameter model* (RPM), is free from these problems. In RPMs, the random error distorts the agent's preference parameter. Hence, the agent opts for the alternative that maximizes $U_{\omega+\epsilon}$, where the random error ϵ follows a continuous and strictly increasing cumulative distribution Φ on the space of the preference parameters. Then, given a pair of alternatives x and y , the RPM probability of choosing x is simply the probability mass of realizations ϵ such that $U_{\omega+\epsilon}(x)$ is greater than $U_{\omega+\epsilon}(y)$. By way of illustration, consider the gambles described above, with CRRA expected utilities and logistically distributed errors. In this case, the RPM probability of choosing x has the closed form of $e^{\lambda\omega^{(x,y)}} / [e^{\lambda\omega^{(x,y)}} + e^{\lambda\omega}]$, where $\omega^{(x,y)}$ is the value such

that $U_{\omega^{(x,y)}}^{\text{crra}}(x) = U_{\omega^{(x,y)}}^{\text{crra}}(y)$.⁴ Notice that an increase in ω increases the denominator of the former expression, guaranteeing the desired monotonicity.

The intuition for the monotonicity of RPMs follows from the fact that the functional form is preserved, which guarantees monotonicity by construction. That is, there is a single random error on the utility function that transforms it into another utility function within the original space of utilities, and the alternatives are evaluated according to the same distorted utility. Then, the higher the level of risk aversion ω , the more improbable it is for the shock ϵ to result in a risk aversion $\omega + \epsilon$ below $\omega^{(x,y)}$, where the riskier gamble is preferred. Therefore, the mass of shocks for which the riskier option x is chosen is decreasing in ω .

Finally, we use the experimental data of Andersen et al. (2008) to evaluate differences between estimates from standard RUMs and RPMs. In our empirical analysis of risk preferences we show that, in line with our theoretical results, the standard RUM significantly underestimates the population risk aversion level, and the severity of the estimation bias associated with the RUM increases with more risk-averse individuals. For the full sample of subjects, the RUM gives a CRRA risk aversion level of 0.66, while that given by the RPM is 0.75, which is about 14 percent higher. Taking a subsample of the most risk-averse subjects, the RUM risk aversion estimate is 1.46, while the RPM estimate is 1.87, which is about 28 percent higher. We consider these results a clear indication of the importance of making the right choice of random model for the estimation of risk preferences.

We close this introduction by reviewing the study closest to our own, Wilcox (2008, 2011). Wilcox first shows that, using our terminology, RUMs may be nonmonotone. His results are for risk preferences using three-outcome gambles related by mean-preserving spreads and use the logit RUM. This paper establishes general theoretical findings that show the problem to be pervasive. Starting with the context of risk preferences, we show that, not only the logit but basically every i.i.d. RUM is nonmonotone for every pair of gambles ordered by risk aversion within a range of parameters that we characterize and show to be of practical importance. Importantly, we also show that this problem extends to other key preference parameters, such as time preferences. We also establish the conditions guaranteeing the monotonicity of RUMs based on any preference parameter. These conditions are practical when contemplating the im-

⁴ The exact values of $\omega^{(x,y)}$ for the gambles in the example are .19 for the CRRA case and .02 for the CARA case. Again, λ is related to the inverse of the variance of the distribution Φ .

plementation of a RUM. Finally, we show that RPMs are always monotone and therefore safe for use in applications.⁵

The rest of the paper is organized as follows. Section II reviews the remaining relevant literature. Section III lays down some basic definitions. Section IV is devoted to the study of RUMs and Section V to that of RPMs. Section VI contains the empirical application, and Section VII presents the conclusions. All the proofs of the theoretical results established in the main body of the paper are given in appendix A. Several extensions of the theoretical and empirical parts are reported in appendices B and C, respectively.

II. Related Literature

Thurstone (1927) and Luce (1959) are two of the first key contributions, within the field of mathematical psychology, to stochastic choice theory. Some stochastic choice models that have appeared recently in the theoretical literature are Gul, Natenzon, and Pesendorfer (2014), Manzini and Mariotti (2014), Caplin and Dean (2015), and Fudenberg, Iijima, and Strzalecki (2015). Discrete choice models in general settings are surveyed in McFadden (2001). See also Train (2009) for a detailed textbook introduction.

For theoretical papers recommending the use of random utility models in risk settings, see Becker, DeGroot, and Marschak (1963) and Busemeyer and Townsend (1993). Wilcox (2008, 2011), as reviewed in the introduction, criticizes the use of these models in risky settings. In addition, Blavatsky (2011) shows that there is always one comparison of gambles, where the safer gamble is degenerate, for which random utility models based on expected utility differences are nonmonotone. The literature using random utility models in the estimation of risk aversion is immense and certainly too large to be exhaustively cited here. We therefore cite only a few of the most influential pieces of work, such as Friedman (1974), Cicchetti and Dubin (1994), Hey and Orme (1994), Holt and Laury (2002), Harrison, List, and Towe (2007), Andersen et al. (2008), Post et al. (2008), Von Gaudecker, Van Soest, and Wengström (2011), Toubia et al. (2013), and Noussair, Trautmann, and van de Kuilen (2014). Our results for risk preferences immediately extend to situations in which strategic uncertainty causes the individual to replace objective probabilities with beliefs. A prominent example of this approach in game theory is the quantal response equilibrium of McKelvey and Palfrey (1995), which assumes a random

⁵ Wilcox proposes a novel model, contextual utility, which is monotone for his choice of gambles. We discuss contextual utility in Sec. A of app. B, showing that it does not, alas, solve the problem for the case of gambles involving more than three possible outcomes.

utility model using (subjective) expected utility. Hence, for given beliefs, our results show that there is a level of risk aversion beyond which more risk-averse individuals may have a higher probability of choosing the riskier action. The random utility model is also the most commonly used approach in the estimation of time preferences (see, e.g., Andersen et al. 2008; Chabris et al. 2008; Ida and Goto 2009; Tanaka, Camerer, and Nguyen 2010; Toubia et al. 2013; Meier and Sprenger 2015).

Starting with the seminal papers by Wolpin (1984) and Rust (1987), dynamic discrete choice models have been used to tackle issues such as fertility (Ahn 1995), health (Gilleskie 1998; Crawford and Shum 2005), labor (Berkovec and Stern 1991; Rust and Phelan 1997), or political economy (Diermeier, Keane, and Merlo 2005). Our results may be relevant for this literature, for two reasons. The first is that some of these settings involve risk and are modeled by means of random utility models with errors over expected utility. The second is the dynamic nature of the setting, which makes our results also relevant with respect to time preferences.

Finally, the use of random parameter models in settings involving gambles has been theoretically discussed in Eliashberg and Hauser (1985), Loomes and Sugden (1995), and Gul and Pesendorfer (2006). For papers using such models to estimate risk aversion, see Barsky et al. (1997), Fullenkamp, Tenorio, and Battalio (2003), Cohen and Einav (2007), and Kimball, Sahm, and Shapiro (2008, 2009). Coller and Williams (1999) and Warner and Pleeter (2001) are two examples of the use of this approach in the context of time preferences. Our results guarantee that one can confidently use random parameter models to estimate preference parameters without running into the kind of inconsistencies studied in this paper.

III. Preliminaries

Let X be a set of alternatives and consider a collection of utility functions $\{U_\omega\}_{\omega \in \Omega}$ defined on X . The term Ω represents the space of possible values of a given preference parameter, which consists of the set of all real numbers, unless explicitly stated otherwise. The preference parameter represents aversion to choosing some alternatives over others. For instance, higher values of ω may represent greater risk aversion or delay aversion causing the individual to be less inclined toward riskier gambles or monetary streams involving more distant payoffs.

Some pairs of alternatives (x, y) are clearly ordered with respect to the preference parameter. Formally, we say that x and y are Ω -ordered whenever $U_{\omega^L}(y) > U_{\omega^L}(x)$ implies that $U_{\omega^H}(y) > U_{\omega^H}(x)$, for every $\omega^L, \omega^H \in \Omega$ such that $\omega^L < \omega^H$. That is, x and y are Ω -ordered if, when the low type ω^L prefers alternative y over x , so does the high type ω^H . In other words, option x

generates more aversion than y and hence can be chosen only by individuals with low aversion levels, that is, low values of the preference parameter ω .⁶ The notion of Ω -ordered pairs of alternatives is natural and applicable, as the following two examples show. In a risk context, consider a gamble x involving risk and a degenerate gamble y , which gives the expected payoff of x with certainty. It is immediate that if $\{U_\omega\}_{\omega \in \Omega}$ is a collection of expected utility functions ordered by the Arrow-Pratt coefficient, x and y are Ω -ordered. Similarly, in a context of intertemporal decision making, consider a payoff stream x with a later bonus payout and a stream y with an earlier bonus payout. Clearly, if $\{U_\omega\}_{\omega \in \Omega}$ is a collection of standard discounted utility functions, x and y are again Ω -ordered. Sections IV.A and IV.B further illustrate the generality of the definition of Ω -ordered pairs.

In Section I we introduced the two main stochastic choice models used in the literature: random utility models (RUMs) and random parameter models (RPMs).⁷ Let us briefly discuss the different nature of these models. First, notice that the main difference between the two stochastic choice models lies in where the disturbance occurs. In RUMs, the error distorts the utility evaluation of each alternative independently, whereas in RPMs, the error distorts the preference parameter, thereby implying that the evaluation of each alternative is not distorted independently of the other. Furthermore, in RPMs, transformed utilities always belong to the family $\{U_\omega\}_{\omega \in \Omega}$, but this is not necessarily the case in RUMs. Finally, therefore, choice probabilities are determined by substantially different mechanisms. In RUMs, they are determined by the cardinal utility difference of the two alternatives at the considered value of the parameter. In the case of RPMs, they are related to the entire collection of utilities, since they are determined by the mass of utility functions ranking one option over the other, conditional on the distribution generated by the considered value of the parameter ω .

Denote by ρ^{rum} the generic choice probabilities associated with a RUM and by $\rho_\omega^{\text{rum}}(x, y)$ the RUM choice probability of choosing x over y given ω . We are now in a position to introduce the main notion in this paper. We say that the stochastic model ρ^{rum} is *monotone for the Ω -ordered pair* (x, y) whenever $\rho_\omega^{\text{rum}}(x, y)$ is decreasing in ω . Define ρ^{rpm} , $\rho_\omega^{\text{rpm}}(x, y)$, and monotonicity of ρ^{rpm} analogously.

⁶ Notice that the definition of Ω -ordered pairs of alternatives is related to the influential single-crossing condition of Milgrom and Shannon (1994). Basically, a pair of alternatives (x, y) is Ω -ordered if the collection of utilities $\{U_\omega\}_{\omega \in \Omega}$ satisfies the single-crossing condition with respect to (x, y) . For a discussion on the consideration of more than two alternatives, see Sec. C of app. B.

⁷ There is no consensus in the literature as to the denomination of these models. They are variously referred to as random preference models, random utility functions, or random utility models.

IV. Random Utility Models

The following proposition specifies simple conditions for checking whether a RUM is monotone for the Ω -ordered pair (x, y) .

PROPOSITION 1. Let (x, y) be an Ω -ordered pair. Then

1. ρ^{rum} is monotone for (x, y) if and only if the function $U_\omega(x) - U_\omega(y)$ is decreasing in ω ;
2. if $\lim_{\omega \rightarrow \infty} [U_\omega(x) - U_\omega(y)] = 0$ and there exists $\omega^* \in \Omega$ such that $U_{\omega^*}(y) > U_{\omega^*}(x)$, then ρ^{rum} is nonmonotone for (x, y) .

The first part of proposition 1 establishes a sufficient and necessary condition for monotonicity. The intuition is very simple. We know that in a RUM the probability of choosing x is the probability of $U_\omega(x) + \epsilon(x)$ being greater than $U_\omega(y) + \epsilon(y)$, which can be expressed as the probability of the difference in the errors being smaller than the difference in the utilities. The i.i.d. nature of the errors allows computation of this probability by means of the cumulative distribution function of the random variable defined by the difference of the two original errors, evaluated at the difference of the utilities. Hence, the probability of choosing x is decreasing in ω if and only if the difference in utilities is decreasing in ω . The second part of proposition 1 provides an even simpler necessary condition based on the limiting behavior of utilities and follows immediately from the first part. Notice that whenever the utility difference is negative at ω^* and goes to zero with growing ω , the utility difference, and consequently the choice probability of x , cannot be decreasing.

In the following sections, we show the relevance of this result in the context of risk and time preferences.⁸ In particular, the second part of proposition 1 enables us to show immediately that most of the RUMs used in these contexts are nonmonotone for basically every Ω -ordered pair of alternatives. Meanwhile, the first part of proposition 1 allows us to exploit the functional structure of these models to obtain results strong enough to characterize the extent of the problem for every Ω -ordered pair.

A. Risk Preferences

A gamble $x = [x_1, \dots, x_N; p(x_1), \dots, p(x_N)]$ consists of a finite collection of monetary outcomes with $x_i \in \mathbb{R}_+$ and associated probabilities such that $p(x_i) > 0$ and $\sum_i p(x_i) = 1$.⁹ In most standard analysis, utility functions over gambles take the form of expected utility $U_\omega(x) = \sum_i p(x_i) u_\omega(x_i)$,

⁸ Apart from risk and time, another potential preference parameter of interest is the one governing the degree of complementarity between two different inputs, say, the monetary payoffs to oneself and to another subject, such as in a distributive problem with social preferences, or different consumption goods in general. In these settings, the use of constant elasticity of substitution utilities could give rise to the same problems.

⁹ The finiteness of the support of the gambles could be relaxed. We retain it for expositional purposes.

where u_ω is a monetary utility function that is strictly increasing and continuous in outcomes. In this section, we focus on the CARA and CRRA families of monetary utility functions that are by far the most widely used specifications in real applications.¹⁰ The following are standard definitions. CARA utility functions are such that the utility of monetary outcome m is $u_\omega^{\text{cara}}(m) = (1 - e^{-\omega m})/\omega$ for $\omega \neq 0$ and $u_0^{\text{cara}}(m) = m$, while CRRA utility functions are defined by $u_\omega^{\text{crra}}(m) = m^{1-\omega}/(1-\omega)$ for $\omega \neq 1$ and $u_1^{\text{crra}}(m) = \log m$.¹¹ We write U_ω^{cara} and U_ω^{crra} for the corresponding expected utilities and $\rho^{\text{rum(cara)}}$ and $\rho^{\text{rum(crra)}}$ for the corresponding RUM choice probabilities.

We focus on the interesting case of Ω -ordered pairs of gambles that are not stochastic dominance related. This implies that, for some values of risk aversion, x is above y , while for others y is above x . We now mention three examples of classes of Ω -ordered pairs of gambles often used in applications, which serve to illustrate the large size of the class of Ω -ordered pairs of gambles. The standard textbook treatment of risk aversion uses pairs of gambles (x, y) in which y involves no risk at all, that is, $y = [y_1; 1]$. The monetary value y_1 is sometimes taken to be the expected value of x , but any y_1 in the interval $(\min\{x_1, \dots, x_N\}, \max\{x_1, \dots, x_N\})$ forms an Ω -ordered pair with x . Another widely used comparison involves pairs (x, y) in which x is a mean-preserving spread of y .¹² Finally, experimental studies often use simple nested pairs of gambles in which $x = [x_1, x_2; p, 1-p]$ and $y = [y_1, y_2; p, 1-p]$, with $x_1 < y_1 < y_2 < x_2$ and $p \in (0, 1)$.¹³ As in the first case, although the mean of x is sometimes assumed to be larger than that of y , no further assumptions are required for them to constitute an Ω -ordered pair of gambles.

We start the analysis by illustrating the usefulness of proposition 1 in identifying the existence of nonmonotonicity problems in RUMs. First, notice that the absence of stochastic dominance between the gambles guarantees that there is always a risk aversion level ω^* such that both the CARA and CRRA expected utilities from the riskier gamble are lower than those from the safer gamble. Second, it is straightforward to see that the CARA and CRRA expected utilities from every gamble go to zero as the risk aversion grows, which forces the difference to also go to zero. These two observations immediately imply the following result.

¹⁰ In Sec. A of app. B we extend the results of this section in a number of dimensions.

¹¹ For ease of exposition, when dealing with CARA and CRRA we assume that $m \geq 1$.

¹² Gamble x is a mean-preserving spread of gamble y through outcome y_j^* and gamble z , if x can be expressed as a compound gamble that replaces outcome y_j^* in gamble y with gamble z , which has y_j^* as its expected value. Then x is a mean-preserving spread of y if there is a sequence of such spreads from y to x .

¹³ See, e.g., the gambles used in the elicitation procedure of Holt and Laury (2002).

COROLLARY 1. Let (x, y) be an Ω -ordered pair of gambles. Then $\rho^{\text{rum(cara)}}$ and $\rho^{\text{rum(crra)}}$ are nonmonotone for (x, y) .

We devote the remainder of the section to further explaining the structure of the nonmonotonicity problem. We start by extending the previous result, showing that the choice probabilities always adopt the form described in figure 1.

PROPOSITION 2. Let (x, y) be an Ω -ordered pair of gambles. Then there exists $\bar{\omega}_{(x,y)}$ such that $\rho_{\omega}^{\text{rum(cara)}}(x, y)$ and $\rho_{\omega}^{\text{rum(crra)}}(x, y)$ are strictly increasing in ω whenever $\omega \geq \bar{\omega}_{(x,y)}$.

Proposition 2 establishes the structure of the nonmonotonicity problem. It exploits the differentiability of U_{ω}^{cara} and U_{ω}^{crra} to show that utility differences are always increasing in ω beyond a certain risk aversion level. We then use the first part of proposition 1 to conclude the result.

Proposition 2 shows that the extent and the economic relevance of the problem are characterized by the value of $\bar{\omega}(x, y)$. We now briefly expand on this issue. First, we note that the proof of proposition 2 shows how to obtain this critical value. Importantly, it is shown that $\bar{\omega}(x, y)$ depends exclusively on the pair of gambles and the representative utilities. In particular, it is independent of the probability distribution Ψ . Moreover, the proof helps us to understand how $\bar{\omega}(x, y)$ varies with the relative quality of the gambles involved. Consider, for instance, two Ω -ordered pairs of gambles, (x, y) and (x, y') , where y' first-order stochastically dominates y . Then it is evident from the proof that $\bar{\omega}_{(x,y)} < \bar{\omega}_{(x,y')}$. Hence, the better the safer option, the wider the range of problems. In our next result, we further discuss the economic relevance of the problem in relation to the stakes involved. The following notation is needed for the result. Consider an Ω -ordered pair of gambles (x, y) , and let (x_{+t}, y_{+t}) , $(x_{\times t}, y_{\times t})$, and $(x_{\wedge t}, y_{\wedge t})$ denote the Ω -ordered pairs of gambles in which all the payoffs in gambles x and y are increased by, multiplied by, and raised to the power of $t > 0$, respectively.

PROPOSITION 3. Let (x, y) be an Ω -ordered pair of gambles.

- CARA: (i) $\lim_{t \rightarrow \infty} \bar{\omega}_{(x_{+t}, y_{+t})} = \omega^{(x,y)}$, and (ii) for every $t > 0$, $\bar{\omega}_{(x_{+t}, y_{+t})} = \bar{\omega}_{(x,y)} / t$.
- CRRA: (i) $\lim_{t \rightarrow \infty} \bar{\omega}_{(x_{\times t}, y_{\times t})} = \omega^{(x,y)}$, and (ii) for every $t > 0$, $1 - \bar{\omega}_{(x_{\wedge t}, y_{\wedge t})} = [1 - \bar{\omega}_{(x,y)}] / t$.

In part i of the results, we take a pair of Ω -ordered gambles (x, y) and increase the stakes by keeping constant the relative quality of the gambles involved.¹⁴ Proposition 3 shows that larger payoffs extend the range of problematic parameters, by pushing $\bar{\omega}_{(x,y)}$ toward the risk aversion level that makes the gambles indifferent, $\omega^{(x,y)}$. Consequently, with high stakes the model is monotone, essentially, only when the choice probability of

¹⁴ Notice that the CARA (respectively, CRRA) risk aversion level that makes the gambles (x_{+t}, y_{+t}) (respectively, $(x_{\times t}, y_{\times t})$) indifferent is always $\omega^{(x,y)}$.

the riskier gamble is greater than one-half, that is, only when the utility difference is positive. Part ii of proposition 3 provides further results by applying a different set of transformations. Under CARA, if the monetary payoffs are multiplied by a positive scalar, the critical value of risk aversion $\bar{\omega}_{(x,y)}$ diminishes, eventually converging to zero. Hence, the model becomes monotone only for the risk-loving range of parameters. Under CRRA, if the monetary payoffs are raised to the power of a positive scalar, the critical risk aversion value eventually converges to one. This implies that when $\bar{\omega}_{(x,y)}$ is above (below) one, increases (decreases) in payoffs become more problematic, since the critical value diminishes. It is worth stressing, however, that one does not need to use large payoffs to obtain small critical values $\bar{\omega}_{(x,y)}$, as illustrated by the example in the introduction.

B. Time Preferences

A monetary stream $x = (x_0, x_1, \dots, x_T)$ describes the amount of money $x_t \in \mathbb{R}_+$ realized at every time period t .¹⁵ The standard approach uses discounted utility $U_\omega(x) = \sum_t D_\omega(t) u(x_t)$, with discount functions for which $D_\omega(0) = 1$ and $\lim_{t \rightarrow \infty} D_\omega(t) = 0$ and with a parameter space $\Omega = \mathbb{R}_+$.¹⁶ The utility function over monetary outcomes u is strictly increasing and continuous. The most commonly used discount functions are the power function $D_\omega^{\text{pow}}(t) = 1/(1 + \omega)^t$ and recently, the hyperbolic discount function $D_\omega^{\text{hyp}}(t) = 1/(1 + \omega t)$, and the $\beta - \delta$ discount function, where $D_\omega^{\text{beta}}(0) = 1$ and $D_\omega^{\text{beta}}(t) = \beta D_\omega^{\text{pow}}(t)$ whenever $t > 0$, with $\beta \in (0, 1]$. We write U_ω^{pow} , U_ω^{hyp} , and U_ω^{beta} for the corresponding discounted utilities and $\rho^{\text{rum(pow)}}$, $\rho^{\text{rum(hyp)}}$, and $\rho^{\text{rum(beta)}}$ for the corresponding RUM probabilities.

As in the case of risk preferences, we consider Ω -ordered pairs of streams for which some values of delay aversion rank x over y and others rank y over x . A simple version of these comparisons used in common practice is one in which there is a unique conflict between waiting a shorter period for a given monetary payoff and waiting a longer period for a larger monetary payoff.¹⁷ Proposition 4 reproduces proposition 2 for the case of time preferences.

PROPOSITION 4. Let (x, y) be an Ω -ordered pair of streams with $x_0 = y_0$. Then there exists $\bar{\omega}_{(x,y)}$ such that $\rho_\omega^{\text{rum(pow)}}(x, y)$, $\rho_\omega^{\text{rum(beta)}}(x, y)$, and $\rho_\omega^{\text{rum(hyp)}}(x, y)$ are strictly increasing in ω whenever $\omega \geq \bar{\omega}_{(x,y)}$.

¹⁵ Whether streams are finite or infinite is immaterial to this analysis.

¹⁶ Notice that proposition 1 follows immediately with $\Omega = \mathbb{R}_+$.

¹⁷ These have been shown to be key streams in the treatment of time preferences (see Benoît and Ok 2007; see also Horowitz 1992).

Having established the nonmonotonicity problem, we now argue that the practical relevance of the problem may be, in fact, more limited. First, note that whenever the Ω -ordered streams x and y already differ in terms of present payoffs, that is, $x_0 \neq y_0$, the choice probabilities $\rho^{\text{rum(pow)}}$, $\rho^{\text{rum(hyp)}}$, and $\rho^{\text{rum(beta)}}$ may be monotone. Importantly, this is the class of streams used in Chabris et al. (2008) and Tanaka et al. (2010). Next, when the streams differ in the near future, as is typically the case in the experimental literature, the critical values $\bar{\omega}_{(x,y)}$ happen to be too large to be economically relevant. For instance, the lowest critical value with the pairs of streams used in Andersen et al. (2008) is a yearly discount rate of 4.25, which is clearly absurdly high in empirical terms. However, the use of RUMs in the context of time preferences should be executed with great caution, since streams involving longer time horizons diverging in the future, as can occur in field studies, may be associated with low critical values $\bar{\omega}_{(x,y)}$.

V. Random Parameter Models

The following result establishes that RPMs are immune to the problems discussed above.

PROPOSITION 5. ρ^{rpm} is monotone for every Ω -ordered pair of alternatives.

The monotonicity of ρ^{rpm} follows trivially. Consider two parameters $\omega^L < \omega^H$. For every shock ϵ such that $\omega^H + \epsilon \leq \omega^{(x,y)}$, which makes x more attractive than y , it is also the case that $\omega^L + \epsilon \leq \omega^{(x,y)}$. Proposition 5 implies that RPMs can be safely used for the estimation of preference parameters. Furthermore, they are easily implementable. That is, given an Ω -ordered pair of alternatives (x, y) , all that is required to obtain $\rho_{\omega}^{\text{rpm}}(x, y)$ is to compute the value $\omega^{(x,y)}$ and the corresponding probability $\Phi(\omega^{(x,y)} - \omega)$.

Note that a distinguishing feature of RPMs is that when, for a given pair of options (x, y) , every utility function regards one option as better than the other, then the probability of choosing the former is one. This is sometimes seen as a limitation of the model, as in the case of stochastic dominance–related gambles, for instance, where the observed probability of choosing the dominated gamble is typically greater than zero. One way to deal with this in the context of RPMs is to add a trembling stage, in the spirit of the trembling hand approach used in game theory. This would work as follows. After a particular utility has been realized, with a large probability $1 - \kappa$ the choice is made according to the realized utility, and with probability κ there is a tremble and the reverse choice is made. It is easy to see that such a model is also monotone for every Ω -ordered pair of alternatives.

VI. An Empirical Application

In an influential paper, Andersen et al. (2008) implement a field experiment to jointly estimate risk and time preferences, using a representative sample of the adult Danish population comprising 253 subjects. In this section we use their data to obtain separate risk and time preference estimates, using both RUMs and RPMs. The purpose of this exercise is not to attempt to replicate the original results of Andersen et al., but rather to illustrate the difference in the estimations obtained by using the two random models under scrutiny. As for similarities with their use of RUMs, we adopt a maximum likelihood approach and treat indifferences in the same way. In addition, we also use the same families of utility functions and distributions of errors. We depart from their analysis, mainly, in separately estimating risk and time preferences. Also, given that there are dominated alternatives, we introduce a tremble parameter, as defined in the previous section. Further details are given below.

A. Estimation of Risk Preferences

There were four different risk aversion choice tasks in the style of the multiple-price lists of Holt and Laury (2002). Each task comprised 10 pairs of nested gambles, as described in appendix C. For every pair of gambles, subjects could either choose one of the gambles or express indifference between the two. In the latter case, they were told that the experimenter would settle indifferences by tossing a fair coin. Accordingly, every expression of indifference is transformed by assigning a half choice to each of the two gambles.¹⁸

We use CRRA utilities and assume the error distribution to be type I extreme value in the RUM estimations and logistic in the RPM ones.¹⁹ This gives the closed-form probabilities of selecting the riskier gamble x over the safer one y described in the introduction. We then use standard maximum likelihood procedures to estimate the population risk aversion level ω , the tremble parameter κ , and the precision parameter λ .²⁰

Table 1 presents the estimates. When considering the entire population of 253 subjects, we see that the RPM risk aversion estimate is about 14 percent higher than that of the RUM. We see this as a considerable bias in the RUM estimation. We test the significance of this difference using

¹⁸ Five percent of all choices were expressions of indifference. With indifferences omitted, the estimates are practically identical.

¹⁹ Given the large values of the payoffs involved in the gambles, we avoid the use of CARA utilities. Also, using the probit model basically provides the same RUM and RPM estimations.

²⁰ Appendix C details the RUM and RPM log likelihood functions.

TABLE 1
RUM AND RPM ESTIMATIONS OF RISK AVERSION

	RUM			RPM			RPM – RUM		
	ω	λ	κ	ω	λ	κ	$\Delta\omega$	CI($\Delta\omega$)	% ω
All subjects	.661 (.032)	.275 (.068)	.034 (.011)	.752 (.043)	2.495 (.137)	.051 (.008)	.091 (.014)	.065 .12	.137
z% more risk-averse individuals:									
z = 50	1.031 (.036)	4.595 (1.279)	.016 (.006)	1.231 (.055)	2.793 (.188)	.029 (.006)	.2 (.029)	.149 .262	.194
z = 45	1.076 (.038)	6.5 (1.904)	.01 (.005)	1.283 (.059)	2.891 (.201)	.025 (.005)	.206 (.032)	.15 .277	.192
z = 40	1.127 (.042)	9.563 (3.128)	.01 (.005)	1.358 (.068)	2.871 (.219)	.024 (.006)	.231 (.039)	.165 .318	.205
z = 35	1.198 (.049)	16.29 (6.166)	.011 (.007)	1.468 (.075)	2.818 (.22)	.025 (.006)	.27 (.044)	.191 .365	.225
z = 30	1.249 (.056)	23.24 (9.667)	.004 (.005)	1.577 (.084)	2.853 (.231)	.023 (.006)	.328 (.054)	.229 .44	.263
z = 25	1.366 (.068)	62.1 (42.7)	.011 (.006)	1.699 (.101)	2.857 (.26)	.022 (.007)	.333 (.066)	.222 .48	.244
z = 20	1.465 (.101)	128.8 (112.4)	.013 (.006)	1.869 (.148)	2.676 (.282)	.026 (.008)	.403 (.096)	.238 .606	.275

NOTE.—Block bootstrap standard errors clustered at the individual level, shown in parentheses, are calculated using 10,000 resamples. The variables ω , λ , and κ are the population risk aversion level, precision parameter, and tremble parameter, respectively; $\Delta\omega$ reports the difference between the RPM and RUM estimated risk aversion levels; CI($\Delta\omega$) is the 95 percent bootstrap confidence interval for $\Delta\omega$; and % ω reports the percentage increase in the estimated risk aversion level when using RPM as opposed to RUM.

the bootstrap method. Bootstrapping provides a robust and simple technique to obtain estimates for parameters that may exhibit complex patterns. Given our theoretical results on the nonmonotonicity of RUMs, we expect this to be the case, and hence the bootstrap method is particularly appropriate here. We perform 10,000 resamples at the individual level, where we estimate the corresponding RUM and RPM parameters and compute the difference between them, which is our statistic.²¹ When we compute the bootstrap confidence interval of our statistic at standard confidence levels, we note that it never includes zero.

Moreover, our theoretical results suggest that the higher the degree of risk aversion in the subjects, the greater the potential bias in the RUM estimates. In order to test this prediction empirically, we rank subjects in terms of their revealed risk aversion, using a simple method that relies neither on RUMs nor on RPMs. The method focuses on the 36 Ω -ordered

²¹ Figure C1 in app. C reports the density functions of the RUM and RPM risk aversion estimates (panel A) and the density of the difference in the bootstrap estimations (panel B).

pairs of gambles that are not related by stochastic domination and computes, for each individual, the proportion of pairs in which they opted for the riskier gamble x' , with indifferences counting as half a choice to each gamble.²² Now, given the ranking of the individuals provided by this method, table 1 reports the estimates for the z percent more risk-averse individuals, where $z \in \{50, 45, 40, 35, 30, 25, 20\}$.²³ First, we see that the risk-averse estimates for both RUMs and RPMs are increasing, suggesting the appropriateness of the selected ranking method. Second, as predicted by our theoretical results, we see that the gap between the two methods is increasing, up to a sizable 28 percent, for the 20 percent most risk-averse individuals. Following the same bootstrap method explained above, we obtain that all these differences are statistically significant. Finally, it is worth noting that, while the RPM estimates of the precision parameter λ are very robust, the RUM λ estimates increase substantially as the estimation progresses toward more risk-averse individuals.²⁴

B. Estimation of Time Preferences

There were six delay aversion choice tasks of the multiple-price list type, each comprising 10 pairs of streams differing in only two periods, as described in appendix C. Indifferences were again allowed and dealt with in the same way as in the case of risk preferences. We use power and hyperbolic discounted utility functions, including a tremble parameter.²⁵ In the RUM estimations, we assume the error distribution to be type I extreme value, and in the RPM estimations we use a log logistic instead of a logistic distribution. Using the appropriate utility representations, the log likelihood functions are similar to those employed in the risk aversion analysis. We then use standard maximum likelihood procedures to estimate the population delay aversion level ω , the tremble parameter κ , and the precision parameter λ . In consonance with our theoretical results, the RUM and RPM estimations of the delay aversion parameter are very

²² In order to break possible ties between individuals, we consider all values $\omega^{(x',y')}$ that make the gambles x' and y' indifferent and focus on the first value in which the individual chooses the riskier gamble over the safer one. Moreover, in app. C, we explore another possible simple method, with similar results.

²³ We stop at the 20 percent mark to ensure some choice variability.

²⁴ Taking the estimated precision parameters for the full sample of 253 individuals (0.275 in the RUM case and 2.495 in the RPM case) and estimating RUM and RPM risk aversion coefficients for the 50 percent more risk-averse individuals, we obtain 0.687 and 1.268, respectively, which represents a bias of 84 percent. Reproducing this exercise for the 20 percent more risk-averse individuals, we obtain 0.712 and 1.915, respectively, which is a difference of 169 percent.

²⁵ Discounted utility of $\beta - \delta$ is indistinguishable from power discounted utility, since the relevant payoffs take place in the future.

TABLE 2
RUM AND RPM ESTIMATIONS OF DELAY AVERSION

	RUM			RPM			RPM – RUM		
	ω	λ	κ	ω	λ	κ	$\Delta\omega$	CI($\Delta\omega$)	% ω
Power	.274 (.001)	.103 (.015)	.228 (.009)	.265 (.011)	9.336 (.728)	.064 (.018)	–.009 (.018)	–.012 .03	–.033
Hyperbolic	.245 (.016)	.04 (.01)	.221 (.011)	.262 (.011)	11.36 (.802)	.107 (.015)	.018 (.01)	–.024 .009	.067

NOTE.—Block bootstrap standard errors clustered at the individual level, shown in parentheses, are calculated using 10,000 resamples. The variables ω , λ , and κ are the population delay aversion level, precision parameter, and tremble parameter, respectively; $\Delta\omega$ reports the difference between the RPM and RUM estimated delay aversion levels; CI($\Delta\omega$) is the 95 percent bootstrap confidence interval for $\Delta\omega$; and % ω reports the percentage increase in the estimated delay aversion level when using RPM as opposed to RUM.

close, as can be appreciated in table 2. In particular, note that the bootstrap confidence intervals now include zero.

VII. Conclusions

In this paper, we have introduced the notion of monotonicity of a stochastic choice model with respect to a preference parameter. Specifically, consider a pair of alternatives (x, y) with a utility evaluation by which x is preferred to y for low values of the parameter and y is preferred to x for larger values. That is, the preference parameter represents an aversion to x with respect to y . Monotonicity then implies that the probability of selecting x should decrease as the aversion to choosing x increases. We argue that this is a minimal property for a stochastic model.

We have focused on two popular stochastic models, random utility models and random parameter models. After establishing the conditions for these models to be monotone, we have focused on the particular cases of risk and delay aversion. We have shown that the standard application of random utility models to risk or time settings is subject to serious theoretical inconsistencies. In the main results we have shown that there is a level of risk aversion (respectively, of delay aversion) beyond which the probability of choosing the riskier gamble (respectively, the more delayed stream) increases with the level of risk aversion (respectively, of delay aversion). We have then established that random parameter models are free from all these inconsistencies.

To conclude, we note that random utility models have a long history in microeconomic theory and microeconometrics and have proved to be of great relevance in enhancing our understanding of a large variety of economic problems. Our findings should constitute an alert to exercise caution when directly applying sound stochastic choice models to settings other than those originally contemplated.

Appendix A

Proofs of the Results in Sections IV and V

Proof of Proposition 1

Consider a RUM and an Ω -ordered pair (x, y) . Notice that

$$\begin{aligned}\rho_{\omega}^{\text{rum}}(x, y) &= \Pr\{U_{\omega}(x) + \epsilon(x) \geq U_{\omega}(y) + \epsilon(y)\} \\ &= \Pr\{\epsilon(y) - \epsilon(x) \leq U_{\omega}(x) - U_{\omega}(y)\}.\end{aligned}$$

Hence, denoting by Ψ^* the cumulative distribution of the difference between two i.i.d. error terms with distribution Ψ , it is the case that $\rho_{\omega}^{\text{rum}}(x, y) = \Psi^*(U_{\omega}(x) - U_{\omega}(y))$. Since Ψ^* is a continuous strictly increasing cumulative distribution, $\rho_{\omega}^{\text{rum}}(x, y)$ is decreasing in ω if and only if $U_{\omega}(x) - U_{\omega}(y)$ is decreasing in ω , which proves the first part.

For the second part, suppose that there exists ω^* such that $U_{\omega^*}(y) > U_{\omega^*}(x)$. If the RUM is monotone for (x, y) , we know from the first part of the proposition that $U_{\omega}(x) - U_{\omega}(y)$ must be decreasing in ω . Clearly, therefore, either $\lim_{\omega \rightarrow \infty} [U_{\omega}(x) - U_{\omega}(y)]$ does not exist or it must be the case that $\lim_{\omega \rightarrow \infty} [U_{\omega}(x) - U_{\omega}(y)] < 0$. In both cases we reach a contradiction, thus proving the result. QED

Proof of Corollary 1

The proof is an immediate implication of the second part of proposition 1. Notice that

$$\begin{aligned}\lim_{\omega \rightarrow \infty} U_{\omega}^{\text{cara}}(x) &= \lim_{\omega \rightarrow \infty} U_{\omega}^{\text{cara}}(y) = \lim_{\omega \rightarrow \infty} U_{\omega}^{\text{crra}}(x) \\ &= \lim_{\omega \rightarrow \infty} U_{\omega}^{\text{crra}}(y) = 0.\end{aligned}$$

Furthermore, since gamble x does not stochastically dominate y , there are levels of risk aversion for CARA and CRRA utilities for which gamble y is preferred. Hence, the second part of proposition 1 leads to the result. QED

Proof of Proposition 2

Consider an Ω -ordered pair of gambles (x, y) , with $x = [x_1, \dots, x_N; p(x_1), \dots, p(x_N)]$ and $y = [y_1, \dots, y_M; q(y_1), \dots, q(y_M)]$. With reasoning analogous to that used in proposition 1, we need to show that there exists a risk aversion level $\bar{\omega}_{(x,y)}$ such that the difference between the utility values of x and y is strictly increasing in values of ω above $\bar{\omega}_{(x,y)}$. Since, by assumption, x does not stochastically dominate y , the two gambles are different. Let \underline{m} be the minimum monetary payoff to which gambles x and y assign different probabilities. Since (x, y) is an Ω -ordered pair of gambles and x does not stochastically dominate y , it must be that $q(\underline{m}) < p(\underline{m})$. This is so because, for sufficiently large values of ω , the utility evaluation of the gambles is determined by the first payoff where they differ, \underline{m} . We now prove that we can consider, without loss of generality, that $y_j > \underline{m} = \min\{x_i\}$ for all payoffs in gamble y . To see this, suppose that $q^* = \sum_{j: y_j \leq \underline{m}} q(y_j) > 0$. Since $q(\underline{m}) < p(\underline{m})$, the definition of \underline{m} guarantees that it is also the case that $q^* < 1$, and hence we can

express gamble y as a compound gamble that assigns probability q^* to gamble y' and probability $1 - q^*$ to gamble \hat{y} . Gamble y' contains payoffs in y that are below \underline{m} , with associated probability $q'(y_j) = q(y_j)/q^*$. Gamble \hat{y} contains payoffs in y that are strictly above \underline{m} , with associated probability $\hat{q}(y_j) = q(y_j)/(1 - q^*)$. We can also express x as a compound gamble that assigns probability q^* to gamble y' and probability $1 - q^*$ to gamble \hat{x} . Gamble \hat{x} contains all payoffs in x above \underline{m} , with associated probabilities $\hat{p}(\underline{m}) = [p(\underline{m}) - q(m)]/(1 - q^*)$ and $\hat{p}(x_i) = p(x_i)/(1 - q^*)$ whenever $x_i > \underline{m}$. By the additive nature of expected utility, we know that the utility difference between gambles x and y is proportional to the utility difference between gambles \hat{x} and \hat{y} , which proves the claim.

We start by considering the case of CARA and focus on $\omega \neq 0$, where the family is differentiable.²⁶ In this domain, $\rho_\omega^{\text{rum(cara)}}(x, y)$ is strictly increasing in ω if and only if $\partial[U_\omega^{\text{cara}}(x) - U_\omega^{\text{cara}}(y)]/\partial\omega > 0$, which, by expected utility, is equivalent to

$$\sum p(x_i) \frac{\partial u_\omega^{\text{cara}}(x_i)}{\partial\omega} - \sum q(y_j) \frac{\partial u_\omega^{\text{cara}}(y_j)}{\partial\omega} > 0.$$

Since

$$-\frac{\partial u_\omega^{\text{cara}}(m)}{\partial\omega} = -\frac{e^{-\omega m}(1 + \omega m) - 1}{\omega^2}$$

is a strictly increasing and continuous utility function over monetary outcomes, $\rho_\omega^{\text{rum(cara)}}(x, y)$ is strictly increasing in ω if and only if $V_\omega^{\text{cara}}(y) > V_\omega^{\text{cara}}(x)$, where V_ω^{cara} is the expected utility using $-\partial u_\omega^{\text{cara}}(m)/\partial\omega$. Denoting by $CE(x, V_\omega^{\text{cara}})$ and $CE(y, V_\omega^{\text{cara}})$ the certainty equivalents of V_ω^{cara} for gambles x and y , it follows that $\rho_\omega^{\text{rum(cara)}}(x, y)$ is strictly increasing in ω if and only if $CE(y, V_\omega^{\text{cara}}) > CE(x, V_\omega^{\text{cara}})$.²⁷ Now, notice that the Arrow-Pratt coefficient of risk aversion for $-\partial u_\omega^{\text{cara}}(m)/\partial\omega$ is simply $\omega - (1/m)$.²⁸ When ω grows, the Arrow-Pratt coefficient goes to infinity, thereby guaranteeing that

$$\lim_{\omega \rightarrow \infty} CE(x, V_\omega^{\text{cara}}) = \underline{m} < \min\{y_1, \dots, y_M\} = \lim_{\omega \rightarrow \infty} CE(y, V_\omega^{\text{cara}}).$$

Hence, we can find a value, which we denote by $\bar{\omega}_{(x,y)}$, such that for every $\omega \geq \bar{\omega}_{(x,y)}$, $CE(y, V_\omega^{\text{cara}}) > CE(x, V_\omega^{\text{cara}})$, which proves the result.

The proof of the CRRA case can be obtained analogously by considering that, for any $\omega \neq 1$, CRRA utility functions are differentiable,

$$-\frac{\partial u_\omega^{\text{crra}}(m)}{\partial\omega} = -\frac{m^{1-\omega}[1 - (1 - \omega) \log m]}{(1 - \omega)^2}$$

is a continuous and strictly monotone utility function over monetary outcomes, and the corresponding Arrow-Pratt coefficient is $(\omega \log m - 1)/(m \log m)$. QED

²⁶ Note that the discontinuity of the CARA family at this point is not relevant for the result.

²⁷ In general, the certainty equivalent of a gamble x for some utility function U is the amount of money $CE(x, U)$ such that $U(x) = U([CE(x, U); 1])$.

²⁸ The coefficient has a strictly positive derivative with respect to ω , and thus, from the classic result of Pratt (1964), it follows that the certainty equivalent of a nondegenerate gamble is strictly decreasing in ω .

Proof of Proposition 3

Since the Arrow-Pratt coefficient of risk aversion for $-\partial u_\omega^{\text{cara}}(m)/\partial \omega$ is $\omega - (1/m)$, when the monetary outcomes grow, the Arrow-Pratt coefficient becomes as close to ω as desired. Then $V_\omega^{\text{cara}}(x_{+t}) \geq V_\omega^{\text{cara}}(y_{+t})$ becomes essentially $U_\omega^{\text{cara}}(x_{+t}) \geq U_\omega^{\text{cara}}(y_{+t})$. Given that (x_{+t}, y_{+t}) is an Ω -ordered pair for U_ω^{cara} , the latter inequality holds if and only if $\omega \leq \omega^{(x_{+t}, y_{+t})}$. By noting that CARA utilities imply that $\omega^{(x_{+t}, y_{+t})} = \omega^{(x, y)}$, we show the first claim with respect to CARA. Now, given that

$$-\frac{\partial u_\omega^{\text{cara}}(m)}{\partial \omega} = -\frac{e^{-\omega m}(1 + \omega m) - 1}{\omega^2},$$

it is immediate that $V_\omega^{\text{cara}}(x) = V_\omega^{\text{cara}}(y)$ if and only if $V_{\omega/t}^{\text{cara}}(x_{\times t}) = V_{\omega/t}^{\text{cara}}(y_{\times t})$, and then, by proposition 2, $\bar{\omega}_{(x_t, y_t)} = \bar{\omega}_{(x, y)}/t$.

In the CRRA case, note that the relative Arrow-Pratt coefficient for $-\partial u_\omega^{\text{crta}}(m)/\partial \omega$ is $\omega - (1/\log m)$, and, using the same argument as in the case of CARA, we obtain that, as t grows, $\bar{\omega}_{(x_t, y_t)}$ converges toward $\omega^{(x, y)}$. Finally, since

$$-\frac{\partial u_\omega^{\text{crta}}(m)}{\partial \omega} = -\frac{m^{1-\omega}[1 - (1 - \omega) \log m]}{(1 - \omega)^2} = -\frac{e^{(1-\omega) \log m}[1 - e^{(1-\omega) \log m}]}{(1 - \omega)^2},$$

the same argument as used in the CARA case leads to $1 - \bar{\omega}_{(x_t, y_t)} = [1 - \bar{\omega}_{(x, y)}]/t$. QED

Proof of Proposition 4

Let (x, y) be an Ω -ordered pair of streams with $x_0 = y_0$, and denote by $t^* > 0$ the first period for which streams x and y differ. Consider, first, the case of the power discount function. Our first claim is that $u(x_{t^*}) - u(y_{t^*}) < 0$. To see this, notice that, since (x, y) is an Ω -ordered pair, when ω is sufficiently large, stream y must be preferred to stream x or, equivalently, the sign of

$$\sum_t D_\omega^{\text{pow}}(t)[u(x_t) - u(y_t)] = D_\omega^{\text{pow}}(t^*) \sum_{t: t \geq t^*} (1 + \omega)^{t^*-t} [u(x_t) - u(y_t)]$$

must be negative. By standard arguments, for sufficiently large ω , the latter sign is equivalent to the sign of $u(x_{t^*}) - u(y_{t^*})$, thus proving the claim.

Now $\rho^{\text{sum}(\text{pow})}$ is strictly increasing in ω if and only if $\sum_t D_\omega^{\text{pow}}(t)[u(x_t) - u(y_t)]$ is strictly increasing in ω . Given the differentiability of D_ω^{pow} , the latter condition is equivalent to

$$\begin{aligned} \sum_t \frac{\partial D_\omega^{\text{pow}}(t)}{\partial \omega} [u(x_t) - u(y_t)] &= \sum_{t: t \geq t^*} -t(1 + \omega)^{-t-1} [u(x_t) - u(y_t)] \\ &= \sum_{t: t \geq t^*} -t D_\omega^{\text{pow}}(t+1) [u(x_t) - u(y_t)]. \end{aligned}$$

When ω grows, the sign of the previous expression coincides with the sign of $-[u(x_{t^*}) - u(y_{t^*})]$, which we have shown to be positive. Hence, there exists $\bar{\omega}_{(x, y)}$ such that $\rho^{\text{sum}(\text{pow})}$ is strictly increasing in ω for every $\omega \geq \bar{\omega}_{(x, y)}$, as desired.

The additivity of discounted utility, the fact that $x_0 = y_0$, and that $D^{\text{beta}}(t) = \beta D^{\text{pow}}(t)$ whenever $t > 0$, makes the proof of the $\beta - \delta$ case analogous. For the hy-

perbolic case, we start by claiming that $\sum_{t: \mathcal{E}t^*} (1/t)[u(x_t) - u(y_t)]$ is negative. To see this, notice that since (x, y) is an Ω -ordered pair, when ω is sufficiently large, stream y must be preferred to stream x or, equivalently, the sign of

$$\sum_t D_\omega^{\text{hyp}}(t)[u(x_t) - u(y_t)] = D_\omega^{\text{hyp}}(t^*) \sum_{t: \mathcal{E}t^*} \frac{1 + \omega t^*}{1 + \omega t} [u(x_t) - u(y_t)]$$

must be negative. As ω increases, the limit of $(1 + \omega t^*)/(1 + \omega t)$ is t^*/t , and hence $\sum_{t: \mathcal{E}t^*} (1/t)[u(x_t) - u(y_t)]$ must be negative. Now, notice that

$$\sum_t \frac{\partial D_\omega^{\text{hyp}}(t)}{\partial \omega} [u(x_t) - u(y_t)] = \sum_{t: \mathcal{E}t^*} -t(1 + \omega t)^{-2} [u(x_t) - u(y_t)],$$

which in turn is equal to

$$[D^{\text{hyp}}(t^*)]^2 \sum_{t: \mathcal{E}t^*} -t \left(\frac{1 + \omega t^*}{1 + \omega t} \right)^2 [u(x_t) - u(y_t)].$$

Clearly, when ω grows, the sign is equal to the sign of $\sum_{t: \mathcal{E}t^*} (-1/t)[u(x_t) - u(y_t)]$, which we know to be positive, thus concluding the proof. QED

Proof of Proposition 5

Let $\omega^L, \omega^H \in \Omega$, with $\omega^L < \omega^H$. Consider a realization ϵ of Φ such that $U_{\omega^L + \epsilon}(y) > U_{\omega^L + \epsilon}(x)$. Since (x, y) is an Ω -ordered pair of alternatives, it must be the case that $U_{\omega^H + \epsilon}(y) > U_{\omega^H + \epsilon}(x)$. Consequently, the set of realizations for which x is preferred to y shrinks with the preference parameter, implying that $\rho_\omega^{\text{rpm}}(x, y)$ is decreasing in ω , as desired. QED

Appendix B

Theoretical Extensions

A. Random Utility Models and Risk Preferences

We now comment on several extensions to proposition 2.

Logarithmic transformation of the representative utility.—This approach starts by assuming that the representative utility of every option is strictly positive. Thus, the probability of selecting option x over option y is

$$\Pr\{\log(U_\omega(x)) + \epsilon(x) \geq \log(U_\omega(y)) + \epsilon(y)\} = \Psi^*(\log(U_\omega(x)) - \log(U_\omega(y))),$$

where Ψ^* is the distribution function of the difference of the i.i.d. errors. In parallel to the first part of proposition 1, it is now immediate that a RUM based on the logs of the utilities, LRUM, is monotone for the Ω -ordered pair (x, y) if and only if the ratio $U_\omega(x)/U_\omega(y)$ is decreasing in ω . The second part of proposition 1 can also be directly reproduced considering $\lim_{\omega \rightarrow \infty} [\log(U_\omega(x)) - \log(U_\omega(y))] = 0$. Denote by $\rho^{\text{lrum(cara)}}$ the CARA probabilities.²⁹

²⁹ Notice that CRRA functions are not entirely appropriate in this context because, for values of ω above one, the utilities become negative, which is incompatible with the use of log transformations. The function $x^{1-\omega}$, without the normalization $1/(1-\omega)$, is positive for values of $\omega > 1$, but in this case it is not monotone in outcomes and thus is also problematic.

PROPOSITION 6. Let (x, y) be an Ω -ordered pair of gambles. Then there exists $\bar{\omega}_{(x,y)}$ such that $\rho_{\omega}^{\text{lrum(cara)}}(x, y) > \rho_{\bar{\omega}_{(x,y)}}^{\text{lrum(cara)}}(x, y)$ whenever $\omega > \bar{\omega}_{(x,y)}$.

Proof of proposition 6. Consider any Ω -ordered pair of gambles (x, y) . Given the structure of gambles and that of the CARA family, there always exists $\hat{\omega} > 0$ such that $U_{\hat{\omega}}^{\text{cara}}(x) < U_{\hat{\omega}}^{\text{cara}}(y)$, or $\log(U_{\hat{\omega}}^{\text{cara}}(x)) < \log(U_{\hat{\omega}}^{\text{cara}}(y))$. It is also immediate to see that the limits of $\log(\omega U_{\omega}^{\text{cara}}(x))$ and $\log(\omega U_{\omega}^{\text{cara}}(y))$ as ω increases are both zero; hence

$$\lim_{\omega \rightarrow \infty} [\log(U_{\omega}(x)) - \log(U_{\omega}(y))] = \lim_{\omega \rightarrow \infty} [\log(\omega U_{\omega}(x)) - \log(\omega U_{\omega}(y))] = 0.$$

Therefore, there exists $\tilde{\omega} > \hat{\omega}$ such that

$$\log(U_{\omega}^{\text{cara}}(x)) - \log(U_{\omega}^{\text{cara}}(y)) > \log(U_{\tilde{\omega}}^{\text{cara}}(x)) - \log(U_{\tilde{\omega}}^{\text{cara}}(y))$$

for every $\omega \geq \tilde{\omega}$. This, together with the logarithmic counterpart of proposition 1, implies that $\rho_{\omega}^{\text{lrum(cara)}}(x, y) > \rho_{\tilde{\omega}}^{\text{lrum(cara)}}(x, y)$ for every $\omega \geq \tilde{\omega}$. Now, the function $\rho_{\omega}^{\text{lrum(cara)}}(x, y)$ is continuous on $[\hat{\omega}, \infty)$ and hence achieves a minimum ω^* in the closed interval $[\hat{\omega}, \tilde{\omega}]$, and, by the above reasoning, we know that ω^* is also a minimum in $[\hat{\omega}, \infty)$. Given continuity, we need only to consider $\bar{\omega}_{(x,y)}$ to be the largest value of ω for which $\rho_{\omega_{(x,y)}}^{\text{lrum(cara)}}(x, y) = \rho_{\omega^*}^{\text{lrum(cara)}}(x, y)$, and the result follows. QED

Generalized expected utility.—Proposition 2 works under the assumption of expected utility. Clearly, generalizations of expected utility, such as cumulative prospect theory, rank-dependent expected utility, disappointment aversion, and so forth, are susceptible to the problems identified above, since they include expected utility as a special case. More importantly, however, the additive nature of these models makes them vulnerable to similar anomalies, even when considering only non-expected utilities. Formally, consider a function π that associates every gamble $x = [x_1, \dots, x_N; p(x_1), \dots, p(x_N)]$ with another gamble $\pi(x) = [x_1, \dots, x_N; q(x_1), \dots, q(x_N)]$ over the same set of outcomes. We assume that, for any given vector of outcomes, the distortion of probabilities is a one-to-one, continuous, and monotone function over each argument. Then the generalized CARA expected utility is $U_{\omega}^{\text{gcara}}(x) = U_{\omega}^{\text{cara}}(\pi(x))$, while the generalized CRRA expected utility is $U_{\omega}^{\text{gcrra}}(x) = U_{\omega}^{\text{crra}}(\pi(x))$, and the corresponding RUM choice probabilities are denoted by $\rho^{\text{lrum(gcara)}}$ and $\rho^{\text{lrum(gcrra)}}$. It is straightforward to see that, whenever $(\pi(x), \pi(y))$ is an Ω -ordered pair, the logic behind proposition 2 can be applied directly. Without assuming that the transformed gambles are Ω -ordered, proposition 7 nevertheless establishes analogous results.³⁰

PROPOSITION 7. Let (x, y) be an Ω -ordered pair of gambles such that $\min\{x_1, \dots, x_N\} \neq \min\{y_1, \dots, y_M\}$. Then there exists $\bar{\omega}_{(x,y)}$ such that $\rho_{\omega}^{\text{lrum(gcara)}}(x, y)$ and $\rho_{\omega}^{\text{lrum(gcrra)}}(x, y)$ are strictly increasing in ω whenever $\omega \geq \bar{\omega}_{(x,y)}$.

Proof of proposition 7. Consider an Ω -ordered pair of gambles (x, y) such that $\min\{x_1, \dots, x_N\} \neq \min\{y_1, \dots, y_M\}$. Since x and y are Ω -ordered and not stochastic dominance related, it must be the case that $\min\{x_1, \dots, x_N\} < \min\{y_1, \dots, y_M\}$. Following the same logic as in the proof of proposition 2, it follows that

³⁰ For the result, we assume that gambles x and y in the original Ω -ordered pair do not share the same minimum outcome. Notice, however, that when the gambles do share the same minimum monetary outcome, the second part of proposition 1 becomes immediately applicable, thereby showing the model to be nonmonotone for such gambles.

$\rho_{\omega}^{\text{rum}(\text{gcara})}(x, y)$ is strictly increasing in ω if and only if $CE(\pi(y), V_{\omega}^{\text{cara}}) > CE(\pi(x), V_{\omega}^{\text{cara}})$. These certainty equivalents converge, with increasing ω , toward the corresponding minimum outcomes in $\pi(x)$ and $\pi(y)$, which are, by construction, the corresponding minimum outcomes in x and y . Now the rest of the proof proceeds as in the proof of proposition 2. The case of CRRA utilities is completely analogous and hence is omitted. QED

As already discussed, CARA and CRRA utility specifications are by far the most commonly used in the literature. In our previous results, the use of CARA and CRRA utilities enables the characterization of the structure of the choice probabilities involved in the problem and thereby the identification of the global minimum $\bar{\omega}_{(x,y)}$. Notice, however, that one can show that every RUM based on generalized expected utilities using any monetary utility function that is strictly increasing and continuous in outcomes is nonmonotone for some Ω -ordered pairs of gambles.³¹

Certainty equivalents.—The certainty equivalent is sometimes used to replace the expected utility as the representative utility. The main intuition behind this approach is that the certainty equivalent is a monetary representation of preferences, where the use of a common scale facilitates interpersonal comparisons. Thus, it is not beyond reason that, by creating a common scale, this method could provide a solution to the problem under discussion, as is indeed the case whenever, for instance, the Ω -ordered pair (x, y) involves a degenerate gamble. This can be appreciated by noticing that the certainty equivalent of the nondegenerate gamble x decreases with the level of risk aversion, while the certainty equivalent of the degenerate gamble y is constant across risk aversion levels. Thus, the difference between the certainty equivalents of the two gambles decreases with the level of risk aversion and, by proposition 1, the probability of choosing the risky gamble decreases, as desired. However, caution is required when using certainty equivalents, because problems may arise with other comparisons. We illustrate this point by considering Ω -ordered pairs (x, y) such that $\min\{x_1, \dots, x_N\} = \min\{y_1, \dots, y_M\}$. We denote by $\rho^{\text{rum}(\text{ccara})}$ and $\rho^{\text{rum}(\text{ccrra})}$ the choice probabilities associated with this model when using the certainty equivalent representation of CARA and CRRA expected utilities, respectively.

COROLLARY 2. $\rho^{\text{rum}(\text{ccara})}$ and $\rho^{\text{rum}(\text{ccrra})}$ are nonmonotone for every Ω -ordered pair of gambles (x, y) such that $\min\{x_1, \dots, x_N\} = \min\{y_1, \dots, y_M\}$.

Proof of corollary 2. Consider an Ω -ordered pair of gambles (x, y) such that $\min\{x_1, \dots, x_N\} = \min\{y_1, \dots, y_M\}$. Since the Arrow-Pratt coefficients of u_{ω}^{cara} and u_{ω}^{crra} are ω and ωm , respectively, it follows that

$$\begin{aligned} & \lim_{\omega \rightarrow \infty} [CE(x, U_{\omega}^{\text{cara}}) - CE(y, U_{\omega}^{\text{cara}})] \\ &= \lim_{\omega \rightarrow \infty} [CE(x, U_{\omega}^{\text{crra}}) - CE(y, U_{\omega}^{\text{crra}})] \\ &= \min\{x_1, \dots, x_N\} - \min\{y_1, \dots, y_M\} = 0. \end{aligned}$$

Now, since by assumption x and y are not stochastic dominance related, there is a level of risk aversion for which y is preferred to x . Hence, proposition 1 is immediately applicable, and the claim is proved. QED

³¹ Details are available on request.

Mean-variance utilities.—Let us now consider mean-variance utilities, which are much used in portfolio theory and macroeconomics. Markowitz (1952) was the first to propose a mean-variance evaluation of risky asset allocations. Roberts and Urban (1988) and Barseghyan et al. (2013) provide examples of the use of mean-variance utilities in a RUM, for the estimation of risk preferences. Formally, given a gamble x , let us denote the expected value and variance of x by $\mu(x) = \sum_i p_i x_i$ and $\sigma^2(x) = \sum_i p_i [x_i - \mu(x)]^2$, respectively. Mean-variance utilities are then described by $U_\omega^{mv}(x) = \mu(x) - \omega \sigma^2(x)$. We now argue that the corresponding RUM choice probabilities $\rho^{\text{rum}(mv)}$ are always monotone. This follows from the linear dependence of the utility function with respect to the parameter.

PROPOSITION 8. $\rho^{\text{rum}(mv)}$ is monotone for every Ω -ordered pair of gambles (x, y) .

Proof of proposition 8. Consider an Ω -ordered pair of gambles (x, y) . Notice that

$$U_\omega(x) - U_\omega(y) = \mu(x) - \mu(y) - \omega[\sigma^2(x) - \sigma^2(y)].$$

Since (x, y) are Ω -ordered, it cannot be that $\sigma^2(x) < \sigma^2(y)$. If the case were otherwise, individuals with an ω that goes to $-\infty$ would prefer gamble y to x , while those with an ω that goes to ∞ would prefer gamble x to y , thereby contradicting that the pair (x, y) is Ω -ordered. Hence, it must be that $\sigma^2(x) \geq \sigma^2(y)$. In this case, $U_\omega(x) - U_\omega(y)$ is decreasing in ω , and proposition 1 is directly applicable. QED

Contextual utility.—To conclude, Wilcox (2011) suggests normalizing the utility difference between the gambles by the difference between the utilities of the best and worst of all the outcomes involved in the two gambles under consideration. This variation of a RUM goes under the name of contextual utility. The author shows that the suggested normalization solves the problem for cases in which both gambles are defined over the same three outcomes (thus covering the important Marschak-Machina triangles) and related through the notion of mean-preserving spreads. However, this normalization does not solve the problem beyond the case mentioned. We illustrate this point by contemplating the Ω -ordered pair of gambles (x, y) , with $x = [0, 10, 50, 90, 100; .1, .4, 0, .4, .1]$ and $y = [0, 10, 50, 90, 100; .05, 0, .9, 0, .05]$, where x is a mean-preserving spread of y . It can be seen that the RUM probability of choosing x using CRRA expected utility is lower for the risk aversion coefficient $\omega_1 = .7$ than for $\omega_2 = .9$.

B. Random Utility Models and Time Preferences

We now consider the LRUM case in the context of time preferences, as introduced in Section A. In order to impose the condition that the discounted utilities must be strictly positive for Ω -ordered pairs, we assume that $u(0) = 0$. Denote by $\rho^{\text{lrum}(pow)}$, $\rho^{\text{lrum}(\beta)}$, and $\rho^{\text{lrum}(hyp)}$ the LRUM probabilities for the power, $\beta - \delta$, and hyperbolic utilities. Proposition 9 establishes for the LRUM case results analogous to those of proposition 4. For the hyperbolic case denote by \hat{m}_x and \hat{m}_y the monetary payoffs such that

$$u(\hat{m}_x) = \sum_{t \geq t^*} \frac{t^* + 1}{t} u(x_t)$$

and

$$u(\hat{m}_y) = \sum_{t > t^*} \frac{t^* + 1}{t} u(y_t).$$

PROPOSITION 9.

1. Let (x, y) be an Ω -ordered pair of streams with $t^* > 0$ and $y_t > 0$ for some $t < t^*$. Then there exists $\bar{\omega}_{(x,y)}$ such that $\rho_{\omega}^{\text{lrum}(\text{pow})}(x, y)$ and $\rho_{\omega}^{\text{lrum}(\text{beta})}(x, y)$ are strictly increasing in ω whenever $\omega \geq \bar{\omega}_{(x,y)}$.
2. Let (x, y) be an Ω -ordered pair of streams with

$$\frac{u(y_{t^*}) - u(x_{t^*})}{u(\hat{m}_x) - u(\hat{m}_y)} > \frac{t^*}{t^* + 1} > 0$$

and $y_0 > 0$. Then there exists $\bar{\omega}_{(x,y)}$ such that $\rho_{\omega}^{\text{lrum}(\text{hyp})}(x, y)$ is strictly increasing in ω whenever $\omega \geq \bar{\omega}_{(x,y)}$.

Proof of proposition 9. From the logarithmic version of proposition 1 and the differentiability of U_{ω}^{α} , $\alpha \in \{\text{pow}, \text{hyp}, \text{beta}\}$, $\rho^{\text{lrum}(\alpha)}$ is strictly increasing in ω if and only if the derivative of $\sum_t D_{\omega}^{\alpha}(t) u(x_t) / [\sum_t D_{\omega}^{\alpha}(t) u(y_t)]$ with respect to ω is strictly positive. Clearly, the sign of this derivative is the same as that of

$$\sum_t \frac{\partial D_{\omega}^{\alpha}(t)}{\partial \omega} u(x_t) \left[\sum_t D_{\omega}^{\alpha}(t) u(y_t) \right] - \sum_t \frac{\partial D_{\omega}^{\alpha}(t)}{\partial \omega} u(y_t) \left[\sum_t D_{\omega}^{\alpha}(t) u(x_t) \right].$$

Now, in the case of $\alpha = \text{pow}$, since $\partial D_{\omega}^{\alpha}(0) / \partial \omega = 0$, the relevant sign is equivalent to that of

$$- \sum_t t D_{\omega}^{\text{pow}}(t) u(x_t) \left[\sum_t D_{\omega}^{\text{pow}}(t) u(y_t) \right] + \sum_t t D_{\omega}^{\text{pow}}(t) u(y_t) \left[\sum_t D_{\omega}^{\text{pow}}(t) u(x_t) \right].$$

The latter expression is equivalent to

$$\sum_r \sum_s (s - r) D_{\omega}^{\text{pow}}(r) D_{\omega}^{\text{pow}}(s) u(x_r) u(y_s)$$

or, simply,

$$\sum_r \sum_s (s - r) D_{\omega}^{\text{pow}}(r + s) u(x_r) u(y_s).$$

To analyze the sign of the previous expression when ω is sufficiently large, we need only to consider the smallest integer k for which the term $\sum_r \sum_{s:r+s=k} (s - r) u(x_r) u(y_s)$ is different from zero. Now, let \bar{t} be the smallest integer such that $\bar{t} < t^*$ with $y_{\bar{t}} > 0$, which exists by assumption. Any sum in which $k < \bar{t} + t^*$ is equal to zero, while the sum $\sum_r \sum_{s:r+s=\bar{t}+t^*} (s - r) u(x_r) u(y_s)$ is equal to $(t^* - \bar{t}) u(y_{\bar{t}}) [u(y_{t^*}) - u(x_{t^*})]$, which is strictly positive by the assumptions on x and y . This makes the desired derivative strictly positive above a certain value $\bar{\omega}_{(x,y)}$, and hence $\rho_{\omega}^{\text{lrum}(\text{pow})}(x, y)$ is strictly increasing above $\bar{\omega}_{(x,y)}$.

When $\alpha = \text{beta}$, the relevant expression becomes

$$\begin{aligned} & \beta \sum_s s D_{\omega}^{\text{pow}}(s) u(x_0) u(y_s) - \beta \sum_r r D_{\omega}^{\text{pow}}(r) u(x_r) u(y_0) \\ & + \beta^2 \sum_{r>0} \sum_{s>0} (s - r) D_{\omega}^{\text{pow}}(r + s) u(x_r) u(y_s). \end{aligned}$$

Since $\beta > 0$, for sufficiently high values of ω , the sign is equivalent to the sign of $(t^* - \bar{t})u(y_i)[u(y_i^*) - u(x_i^*)]$, and the result follows.

Now consider the case of $\alpha = \text{hyp}$, where the relevant expression becomes

$$\sum_r \sum_s (s - r) [D_\omega^{\text{hyp}}(r) D_\omega^{\text{hyp}}(s)]^2 u(x_r) u(y_s).$$

When ω goes to infinity, the expression converges to zero and the dominant terms are all terms in which either r or s is zero, that is, those of the forms $s[D_\omega^{\text{hyp}}(s)]^2 u(x_0) u(y_s)$ and $-r[D_\omega^{\text{hyp}}(r)]^2 u(x_r) u(y_0)$. To study the sign of their sum, simply notice that, as ω increases, the limit of $D_\omega^{\text{hyp}}(a)/D_\omega^{\text{hyp}}(b)$ is b/a . Hence, the determining expression is

$$\sum_s \frac{1}{s} u(x_0) u(y_s) - \sum_r \frac{1}{r} u(x_r) u(y_0),$$

which is equal to

$$\frac{1}{t^*} [u(x_0) u(y_{i^*}) - u(x_{i^*}) u(y_0)] + \sum_{t > t^*} \frac{1}{t} [u(x_0) u(y_t) - u(x_t) u(y_0)],$$

with $x_0 = y_0 > 0$. Note that the summation in the former expression is strictly negative, since (x, y) is an Ω -ordered pair of streams. Then the extra condition assumed in the $\alpha = \text{hyp}$ guarantees that the expression is strictly positive and the result follows. QED

We close the treatment of time preferences by noticing that this problem pervades beyond the usual parametric functions used in the literature. One can show that for every discounted utility RUM there is always an Ω -ordered pair of streams for which the model is not well defined.³²

C. More than Two Alternatives

We now show that the use of an Ω -ordered pair of alternatives causes no loss of generality.

First, consider the case in which stochastic choice is defined over a menu A involving more than two options. Suppose that there is an alternative $x \in A$ such that the pair (x, y) is Ω -ordered for every $y \in A \setminus \{x\}$. In RUMs, our results show that, in the most standard applications, there is a preference parameter $\bar{\omega}_{(x,y)}$ such that $\rho_\omega^{\text{rum}}(x, y)$ is strictly increasing whenever $\omega \geq \bar{\omega}_{(x,y)}$, for every $y \in A \setminus \{x\}$. We now contemplate the probability of choosing x from A , formally,

$$\rho_\omega^{\text{rum}}(x, A) = \Pr\{U_\omega(x) + \epsilon(x) \geq U_\omega(y) + \epsilon(y) \text{ for every } y \in A\}.$$

It is easy to see that this probability is also strictly increasing whenever $\omega \geq \max_{y \in A \setminus \{x\}} \{\bar{\omega}_{(x,y)}\}$. To see this, notice that the probability of choosing x from A is simply equal to the probability of $\epsilon(y) - \epsilon(x)$ being smaller than $U_\omega(x) - U_\omega(y)$ for every $y \in A \setminus \{x\}$. Our results show that all the utility differences $U_\omega(x) - U_\omega(y)$ are increasing, at least for preference parameters $\omega \geq \max_{y \in A \setminus \{x\}} \{\bar{\omega}_{(x,y)}\}$, and hence

³² Again, the details are omitted here but are available on request.

the result follows. Nevertheless, it is obvious that RPMs are still monotone when considering the choice probability of x from a menu A ,

$$\rho_{\omega}^{\text{rpm}}(x, A) = \Pr\{\epsilon: U_{\omega+\epsilon}(x) \geq U_{\omega+\epsilon}(y) \text{ for every } y \in A\}.$$

This follows from observing that the set of realizations from Ψ for which x is maximal in A shrinks with the value of ω , for exactly the same reason as given in proposition 5.

Now consider the case in which there is a collection of Ω -ordered pairs of alternatives $\{(x^i, y^i)\}_{i=1}^K$, and the exercise revolves around the selection of one alternative from each pair. We comment on two conceptual problems that arise when $K > 1$. These two problems collapse into the one we have studied in this paper when there is just one pair of alternatives. We first contemplate the conditional probability of choosing vector \mathbf{x} with respect to \mathbf{y} , where \mathbf{x} and \mathbf{y} differ in only one pair, say pair j , where x^j is in \mathbf{x} and y^j is in \mathbf{y} . It is obvious that, in the case of RUMs, this conditional probability is increasing above $\omega_{(x^j, y^j)}$. Now consider the joint probability of selecting each x^i from the corresponding pair (x^i, y^i) , $i = 1, \dots, K$. It is immediate that this joint probability is prey to the same type of problem. Specifically, since the joint probability is simply the product probability of the different choices, it strictly increases at least for values $\omega \geq \max_i \{\bar{\omega}_{(x^i, y^i)}\}$. However, any conditional or joint probability of an RPM is monotone because of the product nature of the considered probability.

Appendix C

Empirical Application: Further Considerations

The classes of pairs of gambles used in the four different risk aversion tasks are, respectively, $([3,850, 100; p, 1-p], [2,000, 1,600; p, 1-p])$, $([4,000, 500; p, 1-p], [2,250, 1,500; p, 1-p])$, $([4,000, 150; p, 1-p], [2,000, 1,750; p, 1-p])$, and $([4,500, 50; p, 1-p], [2,500, 1,000; p, 1-p])$, with $p \in \{.1, .2, .3, .4, .5, .6, .7, .8, .9, 1\}$. The last pair of gambles in each of the four tasks is not Ω -ordered, since one gamble dominates the other in all four cases.

All 253 subjects were confronted with the four risk aversion tasks, but 116 of them were presented with all 40 pairs of gambles, 67 were presented with pairs 3, 5, 7, 8, 9, and 10 in every task, and the remaining 70 subjects were presented with pairs 1, 2, 3, 5, 7, and 10, again in every task. Subjects had to make a choice from each pair of gambles presented, which made a total of 7,928 choices.

The RUM log likelihood function used for the estimates reported in Section VI.A, contingent on the three parameters, is

$$\sum_{i=1}^{i=40} \left\{ \left(X^i + \frac{I^i}{2} \right) \log \left[\frac{(1-2\kappa)e^{\lambda U_{\omega}^{\text{rpm}}(x^i)}}{e^{\lambda U_{\omega}^{\text{rpm}}(x^i)} + e^{\lambda U_{\omega}^{\text{rpm}}(y^i)}} + \kappa \right] \right. \\ \left. + \left(Y^i + \frac{I^i}{2} \right) \log \left[\frac{(1-2\kappa)e^{\lambda U_{\omega}^{\text{rpm}}(y^i)}}{e^{\lambda U_{\omega}^{\text{rpm}}(x^i)} + e^{\lambda U_{\omega}^{\text{rpm}}(y^i)}} + \kappa \right] \right\},$$

where $i = 1, \dots, 40$ denotes the i th pair of gambles and X^i , Y^i , and I^i the number of subjects expressing a preference for the riskier gamble, for the safer gamble,

or indifference between the two gambles in pair i , respectively. Analogously, the RPM log likelihood function is

$$\sum_{i=1}^{i=36} \left\{ \left(X^i + \frac{I^i}{2} \right) \log \left[\frac{(1-2\kappa)e^{\lambda\omega^{(i,j')}}}{e^{\lambda\omega^{(i,j')}} + e^{\lambda\omega}} + \kappa \right] + \left(Y^i + \frac{I^i}{2} \right) \log \left[\frac{(1-2\kappa)e^{\lambda\omega}}{e^{\lambda\omega^{(i,j')}}} + \kappa \right] \right\} \\ + \sum_{i=37}^{i=40} \left(X^i + \frac{I^i}{2} \right) \log[1-\kappa] + \sum_{i=37}^{i=40} \left(Y^i + \frac{I^i}{2} \right) \log[\kappa],$$

where $i = 1, \dots, 36$ denotes the 36 Ω -ordered pairs of gambles and $i = 37, \dots, 40$ the four pairs in which x^i dominates y^i .

Figure C1 reports the density functions of the RUM and RPM risk aversion estimates (panel A) and the density of the difference in the bootstrap estimations (panel B). It is immediate to see that the RPM risk aversion estimates are systematically greater than those obtained by the RUM, as predicted by our theoretical results.

Table C1 reports, for each of the 40 nested pairs of lotteries, the actual choices, the predicted RUM and RPM probabilities given the estimated parameters, and the contribution of each observation to the log likelihood, enabling us to trace the impact of the nonmonotonicity problem on the estimations. Take, for example, task 2 and note that, for values of p lower than or equal to one-half, the RUM probabilities of the riskier gambles are greater than the actual choice probabilities. If we were working with a monotone model, the impulse would be to increase the level of risk aversion. However, the critical risk aversion values at which the mono-

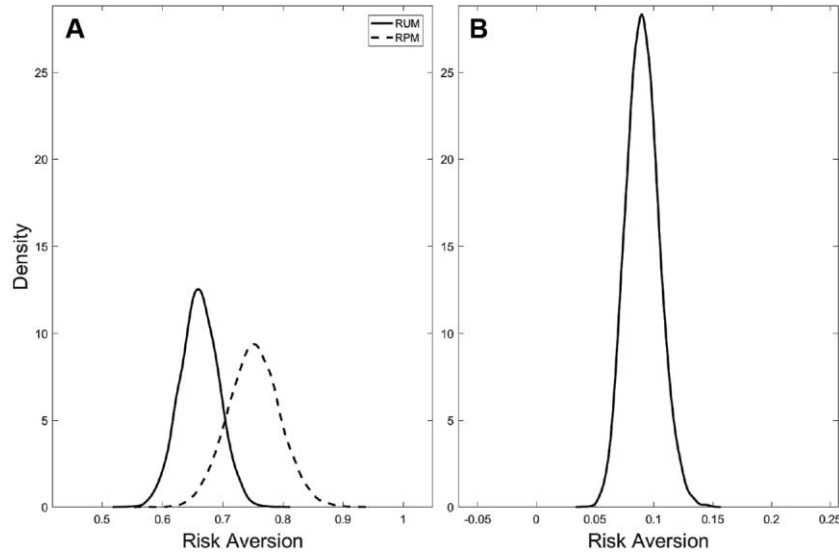


FIG. C1.—A, Density functions of the RUM and RPM bootstrap risk aversion estimates. B, Density function of the difference between the RUM and RPM bootstrap risk aversion estimates.

TABLE C1
ACTUAL CHOICES, RUM AND RPM THEORETICAL PROBABILITIES,
AND LOG LIKELIHOODS BY PAIR OF GAMBLES

	$p = .1$	$p = .2$	$p = .3$	$p = .4$	$p = .5$	$p = .6$	$p = .7$	$p = .8$	$p = .9$	$p = 1$
Task 1:										
Actual choice	.04	.05	.1	.11	.27	.44	.53	.71	.87	.94
RUM probability	.04	.05	.06	.1	.18	.32	.51	.7	.83	.9
RUM likelihood	-.17	-.2	-.33	-.35	-.62	-.71	-.69	-.6	-.39	-.22
RPM probability	.05	.06	.09	.14	.21	.32	.46	.62	.79	.95
RPM likelihood	-.17	-.2	-.32	-.35	-.6	-.71	-.7	-.62	-.41	-.21
Task 2:										
Actual choice	.06	.09	.14	.22	.32	.49	.61	.78	.9	.96
RUM probability	.11	.15	.22	.31	.43	.55	.67	.77	.84	.89
RUM likelihood	-.24	-.32	-.43	-.55	-.65	-.7	-.68	-.53	-.34	-.2
RPM probability	.05	.07	.12	.21	.35	.53	.71	.85	.93	.95
RPM likelihood	-.23	-.31	-.42	-.52	-.63	-.7	-.7	-.54	-.33	-.17
Task 3:										
Actual choice	.04	.06	.08	.13	.21	.4	.48	.63	.79	.93
RUM probability	.04	.05	.07	.12	.21	.36	.56	.73	.85	.91
RUM likelihood	-.17	-.23	-.29	-.39	-.52	-.68	-.7	-.69	-.53	-.25
RPM probability	.05	.06	.09	.14	.22	.34	.5	.67	.83	.95
RPM likelihood	-.17	-.23	-.29	-.39	-.52	-.68	-.69	-.67	-.52	-.25
Task 4:										
Actual choice	.05	.06	.09	.11	.24	.43	.5	.61	.78	.94
RUM probability	.04	.05	.08	.13	.22	.36	.54	.71	.83	.9
RUM likelihood	-.19	-.22	-.31	-.34	-.55	-.69	-.7	-.69	-.54	-.24
RPM probability	.07	.11	.16	.22	.29	.37	.47	.59	.73	.95
RPM likelihood	-.19	-.23	-.33	-.38	-.56	-.69	-.69	-.67	-.53	-.23

NOTE.—Each task uses different monetary payoffs, as detailed in the text, and comprises 10 different pairs of nested gambles, one for each value of p . Actual choice reports the proportion of x choices and half the proportion of indifferences. RUM probability (RPM probability) reports the theoretical RUM (RPM) probability of choosing x , given the population estimated parameters. RUM likelihood (RPM likelihood) reports the contribution of the corresponding nested pair of gambles to the total RUM (RPM) log likelihood, normalized by the number of observations. There are 253 observations for the nested pairs involving $p \in \{.3, .5, .7, 1\}$, 186 observations for $p \in \{.1, .2\}$, 183 observations for $p \in \{.8, .9\}$, and 116 observations for $p \in \{.4, .6\}$.

tonicity problem arises for these pairs of gambles are always lower than the estimated population risk aversion level of .661. This implies that the estimated risk aversion level is located in the increasing part of the corresponding RUM probability distribution. Hence, further increasing the level of risk aversion would, indeed, further increase the corresponding RUM probabilities and thereby lead to a worse likelihood value.

In Section VI.A we propose a method with which to rank individuals in terms of their revealed risk aversion. We now study another simple method that also serves this purpose, method B, and show that it replicates the results reported in Section VI.A. Method B ranks individuals according to the average of the $\omega^{(x,y)}$ corresponding to the first pair in which the riskier option was taken or indifference was expressed and the $\omega^{(x,y)}$ corresponding to the last pair in which the safer option was taken or indifference was expressed. We can now repeat

TABLE C2
RUM AND RPM ESTIMATIONS FOR THE Z PERCENT MORE RISK-AVERSE
INDIVIDUALS CLASSIFIED BY METHOD B

	RUM			RPM			RPM – RUM		
	ω	λ	κ	ω	λ	κ	$\Delta\omega$	CI($\Delta\omega$)	% ω
$z = 50$	1.019 (.032)	4.572 (1.072)	.008 (.004)	1.179 (.045)	3.254 (.207)	.023 (.005)	.16 (.021)	.122 .205	.157
$z = 45$	1.058 (.032)	6.408 (1.49)	.009 (.004)	1.219 (.047)	3.345 (.227)	.023 (.005)	.161 (.023)	.121 .212	.152
$z = 40$	1.109 (.034)	9.08 (2.242)	.005 (.004)	1.287 (.051)	3.367 (.257)	.023 (.006)	.178 (.027)	.131 .238	.161
$z = 35$	1.166 (.035)	14.22 (3.416)	.006 (.004)	1.371 (.057)	3.364 (.266)	.022 (.006)	.205 (.031)	.151 .274	.176
$z = 30$	1.229 (.037)	23.57 (5.765)	.005 (.005)	1.447 (.065)	3.417 (.315)	.021 (.006)	.218 (.039)	.15 .302	.178
$z = 25$	1.306 (.041)	40.85 (11.76)	.008 (.007)	1.578 (.079)	3.223 (.334)	.024 (.007)	.272 (.051)	.181 .382	.208
$z = 20$	1.402 (.048)	76.83 (23.86)	.000 (.001)	1.774 (.095)	3.13 (.263)	.024 (.008)	.372 (.067)	.253 .512	.266

NOTE.—Block bootstrap standard errors clustered at the individual level, shown in parentheses, are calculated using 10,000 resamples. The variables ω , λ , and κ are the population risk aversion level, precision parameter, and tremble parameter, respectively; $\Delta\omega$ reports the difference between the RPM and RUM estimated delay aversion levels; CI($\Delta\omega$) is the 95 percent bootstrap confidence interval for $\Delta\omega$; and % ω reports the percentage increase in the estimated delay aversion level when using RPM as opposed to RUM.

the estimation analysis described in Section VI.A, conditional on the rankings of individuals given by method B. Table C2 reports the results. It is immediately apparent that we reach the same main conclusions as in Section VI.A. Notably, RPM risk aversion estimates are always significantly greater than those provided by the RUM, and the differences show an increasing trend, with the magnitude of bias reaching as high as 27 percent.

Table C3 reports the six delay aversion choice tasks. All 253 subjects were confronted with all the binary choices in the tasks, which made a total of 15,180 choices.

TABLE C3
STREAMS OF PAYOFFS USED IN ANDERSEN ET AL. (2008)

	x_t									
$t_2 = 2$	3,012	3,025	3,037	3,049	3,061	3,073	3,085	3,097	3,109	3,120
$t_2 = 5$	3,050	3,100	3,151	3,202	3,253	3,304	3,355	3,407	3,458	3,510
$t_2 = 7$	3,075	3,152	3,229	3,308	3,387	3,467	3,548	3,630	3,713	3,797
$t_2 = 13$	3,153	3,311	3,476	3,647	3,823	4,006	4,196	4,392	4,595	4,805
$t_2 = 19$	3,232	3,479	3,742	4,020	4,316	4,630	4,962	5,315	5,687	6,082
$t_2 = 25$	3,313	3,655	4,027	4,432	4,873	5,350	5,869	6,431	7,039	7,697

NOTE.—Every pair of streams involves a comparison of x_t , as detailed in the table, with $y_1 = 3,000$.

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