Estimation of Endogenously Sampled Time Series:
The Case of Commodity Price Speculation in the Steel Market

George Hall, *Brandeis University*
John Rust, *Georgetown University**

May 18, 2018

Abstract: We consider the problem of estimating stochastic processes that are *endogenously sampled.* We observe a discrete-time stochastic process $\{p_t\}$ at a subset of times $\{t_1,\ldots,t_n\}$ that depend on the outcome of a probabilistic sampling rule that depends on the history of the process as well as other observed covariates x_t. We focus on a particular example where p_t is the daily wholesale price of a standardized steel product. The endogenous sampling problem arises from the fact that the firm only records p_t on the days that it purchases steel. We present a parametric analysis of this problem under the assumption that the timing of steel purchases is part of an optimal trading strategy that maximizes expected discounted profits. We show that estimation of this model is both tractable and simple using the *method of simulated moments* (MSM) where censored simulations of the price process are used instead of high dimensional numerical integration of a likelihood function. We use the MSM estimator to estimate a truncated lognormal AR(1) model of the wholesale price processes for particular types of steel plate but find the model is rejected by specification tests. We provide evidence that suggests that one reason for this rejection may be that one of the key assumptions underlying our model is invalid, namely that the firm we study maximizes expected discounted profits.

Keywords: endogenous sampling, Markov processes, simulation estimation, method of simulated moments (MSM), commodity price speculation, inventory investment, (S,s) strategies

JEL classification: C13-15

1 Introduction

This paper studies the problem of estimating a stochastic process that is *endogenously sampled.* Our objective is to estimate the parameters of a discrete-time stochastic process $\{p_t\}$ that is observed only at a subset of times $\{t_1, \ldots, t_n\}$ that are a realization of a probabilistic sampling rule that depends on the history of the process as well as other observed covariates x_t. Knowledge of this process is critical to the behavior of traders in the steel market, who attempt to "buy low and sell high" — to acquire inventories of steel at low prices for subsequent resale to retail customers at a markup.

We focus on a particular example where p_t denotes the daily wholesale price of a standardized steel product. There are no formal markets or centralized exchanges where steel is traded. Instead nearly all steel transaction prices are a result of private bilateral negotiations between buyers and sellers, typically intermediated by traders or "middlemen" that are called *steel service centers* in this industry.[1] Even though there is no central record of daily transactions prices in the steel market, we do observe transaction prices for a particular firm — a steel service center that purchases large quantities of steel in the wholesale market for subsequent resale in the retail market. The endogenous sampling problem arises from the fact that the firm only records p_t on the days that it purchases steel.

We introduce the endogenous sampling problem in the context of price speculation in the steel market in order to provide a concrete example, though endogenous sampling problems arise in many other contexts. Examples include financial applications where transaction prices are observed at randomly spaced intervals (see Aït-Sahalia and Mykland (2001), Duffie and Glynn (2004), and Engle and Russell (1998)), and in marketing applications where the prices of goods that a household purchases are generally only recorded for the items the household purchased and on the dates it purchased them (see Allenby, McCulloch and Rossi (1996), and Erdem and Keane (1996)). However we are not aware of any econometric literature that is directly relevant for handling endogenous sampling problems in a time series context. The most directly related work is the literature on likelihood-based methods for correcting for endogenous sampling in cross-sectional and panel contexts (Heckman (1981), Manski and McFadden (1981), and Mc-Fadden (1997)). However in our time-series context, a likelihood-based approach is intractable since it

---

[1]It is a puzzle why centralized exchanges exist for some commodities such as pork bellies, but not for steel. Rust and Hall (2003) develop a theory of intermediation in which the microstructure of trade in a commodity or asset is endogenously determined. Depending on the parameters of this model there are equilibria consistent with all trade occurring via a *market maker* on a centralized exchange, or all trade occurring via decentralized transactions with *middlemen,*, or trade segmenting between middlemen and market makers. This theory can explain the variety of different trading institutions that we see in different markets, including the nonexistence of centralized exchanges for steel.

requires high dimensional numerical integration to form a marginal likelihood, since for the vast majority of days the firm does not purchase this product, so we must "integrate out" the unobserved values of p_t.

We show how inference can be conducted using a consistent, less efficient, but computationally simpler *method of simulated moments* (MSM) estimator that avoids high dimensional numerical integrations required by likelihood-based approaches. The MSM estimator was introduced by McFadden 1989 to avoid the burden of numerical integration in maximum likelihood estimation of discrete choice models. He showed that it is possible to consistently estimate parameters of random utility models using simulations of unobserved utilities by finding parameters that *best fit simulated choices to actual choices.* McFadden's key insight was that simulation error averages out in large samples just as sampling error does, so MSM enables consistent estimation even with a single simulation draw of random utilities per observation.

The general applicability of McFadden's idea to other contexts was quickly appreciated, and the asymptotic properties of MSM were shown to hold for time series as well as the cross-sectional case that McFadden first analyzed. MSM is also known as "simulated minimum distance estimator" (SMD) or a simulated moments estimator (SME) (Lee and Ingram (1991) and Duffie and Singleton, (1993)), but we are unaware of time series applications where MSM has been applied to solve the endogenous sampling problem. Using the MSM estimator, we estimate the parameters of a truncated lognormal AR(1) model of the wholesale price processes for particular types of steel plate. We use these estimates to infer the share of the firm's discounted profits that are due to markups paid by its retail customers, and the share due to price speculation. The latter measures the firm's success in forecasting steel prices and in timing its purchases in order to "buy low and sell high". The more successful the firm is in speculation (i.e. in strategically timing its purchases), the more serious are the potential biases that would result from failing to account for the endogeneity of the sampling process.

This paper originated from previous theoretical work (Hall and Rust (1999, 2003, 2007)) on modeling the speculative trading and inventory investment decisions of a particular steel wholesaler. This firm does minimal production processing: its main activity is to stockpile quantities of various types of steel via bulk purchases at wholesale prices from steel producers and other large intermediaries in order to profit from subsequent resale to retail customers at a mark-up. This firm has provided us with a unique new data set with daily observations on purchases and sales of the more than 8,900 products it carries. While these data are unique in their level of detail and quality, the firm does not record any prices in its computerized data base unless a purchase, sale, or adjustment occurs.

Let $\{p_t\}$ denote the stochastic process representing the *lowest price* offered by any seller of a particular

steel product on day t. We assume that the firm observes p_t at each day t, but it only records p_t when it decides to place an order. Let q_t^o denote the quantity orders (purchased) on day t. The endogenous sampling rule can be stated as follows:

$$p_t \quad \text{is observed} \quad \Longleftrightarrow \quad q_t^o > 0.$$

It is notationally convenient to treat the endogenous sampling problem as a censored sampling problem: i.e., we set p_t to some arbitrary value such as $p_t = 0$ when $q_t^o = 0$, and let p_t equal the observed purchase price when $q_t^o > 0$. Note that we also observe the retail sales prices $\{p_t^r\}$ that the firm charges its customers. Since retail sales occur much more frequently than purchases on the wholesale market, retail price data $\{p_t^r\}$ can provide a key source of information for learning about $\{p_t\}$. However on the subset of days where both p_t and p_t^r are observed, we observe that markups $p_t^r - p_t$ are quite volatile, and vary by time, location, and type of the customer. In other words, there is considerable price discrimination in the retail market for steel. As a result the retail price of steel p_t^r is best regarded as a noisy and biased signal of the wholesale price p_t and therefore retail prices, while observed more frequently than wholesale prices, are not sufficient by itself for estimating the unknown parameters of the wholesale price process.[2]

The MSM estimation method requires nested numerical solution of a dynamic programming (DP) problem that determines the firm's optimal trading strategy, and hence the endogenous sampling rule. The DP problem is re-solved for each trial value for the unknown parameter vector θ, and as a result, the MSM estimator can be computationally intensive. However significant computational savings can be achieved by exploiting special features of the solutions to these dynamic programming problems. Extending a seminal result by Scarf (1959) for a simpler class of inventory investment problems, Hall and Rust (2007) showed that the optimal speculative investment strategy for a fairly general class of commodity price speculation problems takes the form of a *generalized* (S,s) *rule*. In a generalized (S,s) rule, S and s are functions of

---

[2]Our treatment of the wholesale price process $\{p_t\}$ as an exogenously specified "forcing process" that is known up to a finite number of parameters is admittedly only a first approximation to reality. The assumptions that $\{p_t\}$ is observed each day by the firm and evolves as an exogenous stochastic process (i.e. its realizations do not depend on actions of the firm) are particularly strong restrictions that we intend to relax in future work. As we noted above, prices in the steel market are determined via bilateral negotiations: there is no central market place where the lowest price can be easily observed. Instead, in order to get price quotes, purchasing agents within the firm must communicate with steel producers or other intermediaries via telephone, fax, telex, or the internet. Thus each price quote involves a small monetary and time cost. However this leads potential endogeneity problems, since the best price the firm is able to negotiate depends on the intensity of its search/bargaining process, and this intensity level could vary depending on the conditions it faces. We defer the difficult issues associated with potential endogeneity in $\{p_t\}$ to future research. However while a more realistic model of speculation would result in a more complicated dynamic programming problem, we believe the general approaches to estimation of the underlying price processes described in this paper will still apply. The main modification is that when there is no spot wholesale market and the "law of one price" does not hold, we would need to estimate a conditional probability distribution representing the firm's beliefs about the distribution of potential prices available at a given point in time.

3

the current wholesale price p and a vector of other state variables x such as interest rates, demand shifters, and other variables that affect the firm's beliefs about future prices and sales levels. The functions $S(p,x)$ and $s(p,x)$ satisfy $S(p,x) \geq s(p,x)$. The lower band $s(p,x)$ is the firm's *order threshold:* it is optimal for the firm to place an order whenever its current inventory level q falls below $s(p,x)$. The upper band $S(p,x)$ is the firm's *target inventory level:* whenever the firm places an order to replenish its inventory, it orders an amount sufficient to insure that inventory on hand (the sum of the current inventory plus new orders) equals $S(p,x)$.

The order threshold function $s(p,x)$ encodes the endogenous sampling of $\{p_t\}$ since the firm only records the wholesale price p_t on the days where purchases occur. Therefore the sampling is described by the following threshold rule

$$p_t \text{ is observed iff } q_t < s(p_t, x_t). \tag{1}$$

Conditional on a purchase occurring, we observe an order of size q_t^o given by

$$q_t^o = S(p_t, x_t) - q_t, \tag{2}$$

and $q_t^o = 0$ otherwise. Using the generalized (S, s) rule as our model of the endogenous determination of sampling dates, we propose estimators that are able to consistently estimate the unknown parameters of the $\{p_t\}$ process even though we only have incomplete information on $\{p_t\}$.

The endogenous sampling problem can potentially be solved via maximum likelihood, although at significant computational cost. The approach involves writing a full likelihood for the variables of the model including p_t, initially assuming we can observe p_t on all days t. We form a partial likelihood from the full likelihood by integrating out p_t on the days where it is unobserved, i.e. on the days the firm does not buy this product. However the dimension of the necessary integrations is very high. For the 3/4 inch plate product we analyze, we observe the firm's inventories over 1647 business days but observe purchases only on 230 of these days. Thus implementing a likelihood approach involves computing a 1417 dimensional integral, which is a daunting computational task even using the recursive likelihood function integration algorithm of Reich, 2018 which decomposes the overall multivariate integral into 1417 unidimensional integrals, each recursively computed as conditional expectations.

The MSM estimator builds on McFadden's (1989) insight that simulation of a structural model can be used to avoid the need to do numerical integration. The MSM estimator only requires the ability to simulate realizations of the optimal inventory model, resulting in a simulated time series of steel purchases, sales, and inventories. These simulations are then censored in exactly the same way as the observed

data are censored, similar to the way McFadden simulated choices from simulated random utilities. It is also similar in many respects to the strategy of "data augmentation" used in Bayesian inference of latent variable models. The idea behind the MSM estimator is to choose parameter values that result in simulated moments that match the observed moments as closely as possible, where both the real and simulated data are censored according to the same sampling rule; namely the one given in equation (1). Even though the moments relating to wholesale prices entering the MSM criterion (such as the mean price) are biased and inconsistent due to endogenous sampling, the fact that we can censor the data entering the simulated moments in the same way that the data we observed is censored can be used to establish the consistency of the MSM estimator. It should be apparent that although the details of how the time series data are censored are case-specific and our analysis of endogenous censoring is specific this particular steel example, it is straightforward to generalize the general approach to other types of endogenous sampling problems that arise in a variety of other contexts. In each case it is necessary to formulate a structural model that is rich enough to simulate data that *could potentially be observed in the absence of censoring* and then censor the simulated data in the same way that the actual data are censored.

Section 2 describes our data set and introduces the steel speculation and inventory problem that motivates this research. Section 3 discusses the challenges in trying to estimate this model using traditional approaches such as GMM or maximum likelihood and introduces several key assumptions that are necessary to establish the asymptotic properties of any estimator in a time series context where we have T time series observations on only a single firm. Section 4 introduces the MSM estimator and reviews its asymptotic properties. Section 5 present the results of an empirical application of the MSM estimator to two plate steel products for which wholesale prices are assumed to evolve according to a univariate truncated lognormal AR(1) process. We estimate the unknown parameters of the price process and the unknown parameters affecting the firm's cost of purchasing and holding inventory. We then evaluate how well our generalized (S,s) trading strategy fits these data, and use our results to infer the fraction of the firm's discounted profits are due to the markups it charges its retail customers, and the fraction that is due to pure commodity price speculation, i.e., its success in timing purchases of steel in order to profit from "buying low and selling high."

2 Description of the Data and the Model of Price Speculation

In this section we introduce the data and describe a generalized version of a model of commodity price speculation an inventory investment introduced by Hall and Rust (1999, 2007) that allows for additional covariates and unobserved state variables. This model provides the framework for inference and provides the key insights that enabled us to pose and solve the endogenous sampling problem.

2.1 The Data

Via a personal contact with an executive at a large U.S. steel wholesaler, we acquired a high frequency micro database that enables us to track its inventories and all sales and purchase transactions for all of its 8900+ individual steel products on a daily basis. The empirical results presented in section 4 are based on data on every transaction the firm made between July 1, 1997 to December 31, 2003 (1647 business days) for two of its highest volume steel products, 3/4 and 1 inch plate steel. For each transaction we observe the quantity (number of units and/or weight in pounds) of steel bought or sold, the sales price, the shipping costs, and the identity of the buyer or seller.

Although this is an exceptionally clean and rich dataset, we only observe prices on the days the firm actually made transactions: the firm does not record any price information on days that it does not transact (either as a buyer or seller of steel). This shortcoming of our dataset is much more important for steel purchases than steel sales, since the firm purchases new steel inventory in the wholesale market much less frequently than it sells steel to its retail customers. Indeed, even for its highest volume products, it makes purchases only about once every two weeks. The (S,s) theory we present below predicts that purchases are not made at random. Instead, the firm tends to make purchases when prices are low, so that the average price on the days the firm makes purchases will be lower than the average wholesale price on days the firm does not purchase. The exception to this general rule is that the firm may make purchases even when prices are relatively high if its inventories are low. Conversely, the firm may refrain from purchasing even if prices and inventories are low if it expects that the rate of retail sales will be depressed for a long period of time, say due to bad macroeconomic conditions. Thus, while the firm is attempting to "buy low and sell high", its purchase decisions involve a tradeoff among a number of different considerations.

We illustrate our data by plotting the time series of prices and inventories of 3/4 inch steel plate in figures 1 and 2. Steel plates are one of highest volume products sold by this firm. This type of plate is also a benchmark product within the industry since the prices of several other steel products are often
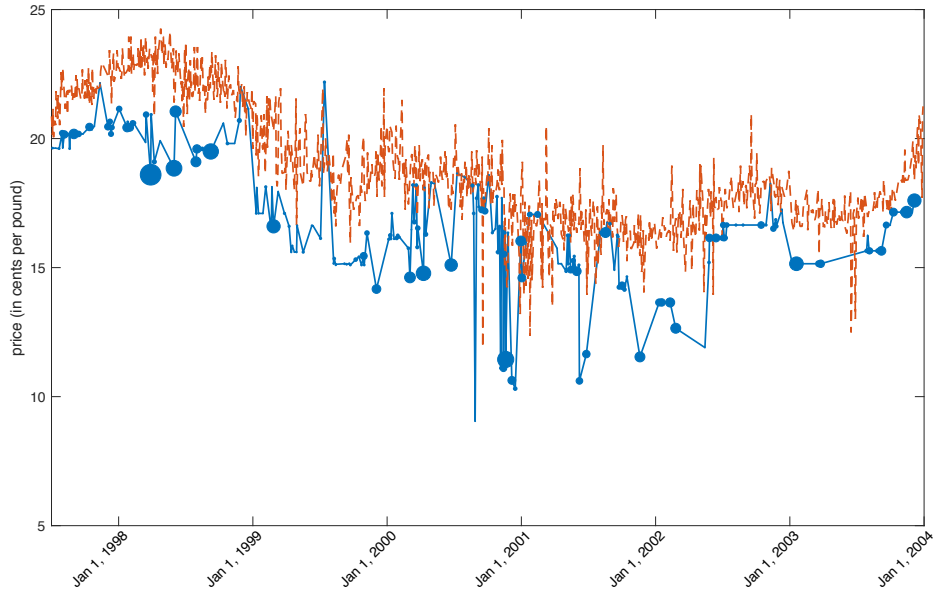
Figure 1: Purchase prices (solid line) and retail prices (dashed line) of 3/4 inch plate.

computed as a function of it's price. It is possible to get weekly and monthly survey data on prices for certain classes of steel products through trade publications such as *Purchasing Magazine* and *American Metal Market*. However, since there are no public exchanges for steel products, transactions in the steel market are carried out in private negotiations. Hence these price surveys rely on participants in the steel market to provide accurate reports of the prices they paid or received for various steel products. The firm we study often faces considerably different prices than those in the survey data.

As a result, in our plots of wholesale transaction prices in figure 1 (the lower curve with the large blue circles), we used straight line interpolations between observed purchase prices at successive purchase dates. The blue circle at each purchase date is proportional to the size of the firm's purchase in pounds. This gives us our first visual indication of the endogenous sampling problem. First, we see that even though we have 1647 observations on this firm, we observe purchases in the wholesale market on only 230 days. Second, the clustering of the blue dots at lower prices suggests that the firm is more likely to purchase large quantities of steel when wholesale prices are low, although other economic factors seem to be influencing the firm's purchase decisions as well. One key factor is the level of inventory: the firm tends to make large purchases when its inventory is low. We also see that even though wholesale prices continued to decline during 2000 and 2001, the firm's largest purchases of steel occurred during the "turning point" in prices in early 1998. The firm may have avoided making large purchases in late 2000 and 2001 due to economic
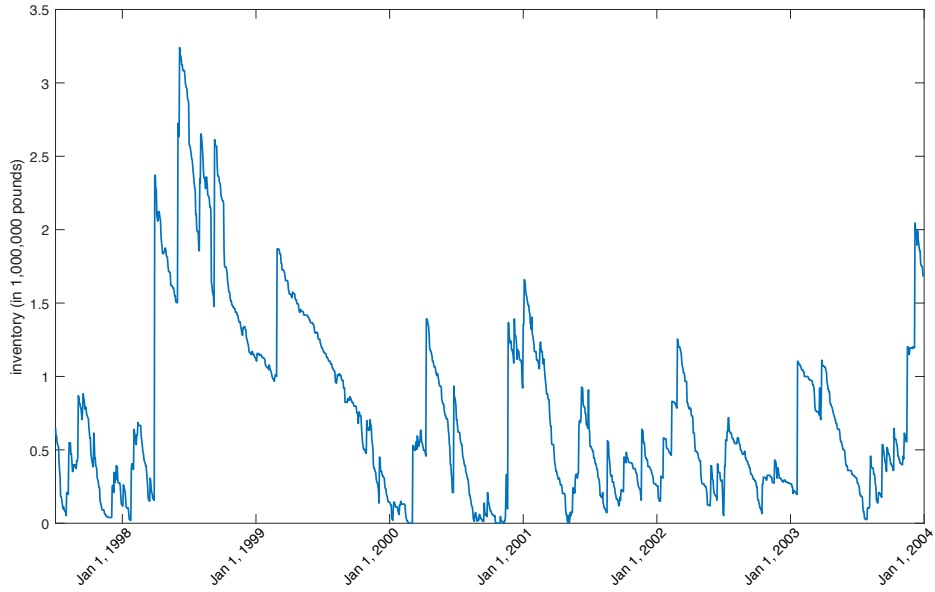
7

Figure 2: Times series plot of inventory of 3/4 inch plate

uncertainties resulting from the "dot com crash" and the economic uncertainties following the 9/11/2001 terrorist attack on the U.S.

Overall, our interpolated plot of steel wholesale prices in figure 2 suggests that we should be wary of using the relatively small number of irregularly spaced observations to make inferences about the underlying law of motion for $\{p_t\}$. The observed purchase prices are unlikely to be representative of the unconditional mean level of prices in the wholesale market (especially if the firm is attempting to "buying low and sell high"), and the estimated serial correlation coefficient for these irregularly spaced transactions is unlikely to be a good estimate of the serial correlation coefficient between daily wholesale prices (assuming we were able to observe them).

Figure 1 also plots the interpolated sequence of daily retail sales prices. Retail sales occur on about two out every three business days, so the amount of interpolation in the retail price series is modest. The wholesale and retail prices move in a roughly parallel way, although there appears to be considerable day-to-day variation in retail prices. Retail prices are quoted net of transportation costs, but still much of the the high frequency variation is due to observable factors. Athreya (2002) found that roughly 65% of the high frequency variation in retail prices can be explained by observable customer characteristics such as geographical location and past volume of purchases. The remaining 35% of the variation in retail prices appears to be due either to high frequency fluctuations in wholesale prices or to some sort of "informational

price discrimination" in the retail market. Using the limited number of days on which both wholesale and retail prices are available, Chan (2001) found that at most 50% of the variation in retail prices can be explained by variations in the wholesale price of steel. This conclusion is possible due to the fact that on many days there are multiple retail sales to different customers. These findings suggest that a large share of the high frequency variation in retail prices can be ascribed to price discrimination, i.e. the firm charges higher prices to more impatient or poorly informed retail customers. We conclude that even though retail sales occur much more frequently than wholesale purchases, the fact that retail prices involve a number of other different considerations (including price discrimination based on observable and unobservable characteristics of the customer) suggest that the retail price is at best a very noisy and (upward) biased signal of the underlying wholesale price.

Figure 2 plots the evolution of inventories over the same period. Purchases of steel are easily recognizable as the discontinuous upward jumps in the inventory trajectories. As is evident from the saw-tooth pattern of the inventory holdings, the firm purchases the product much less frequently than it sells it. The firm's opportunistic purchasing behavior is very clear for this product. As can be seen in figures 1 and 2, during the first ten months of the sample, from July, 1997 until March, 1998, the firm held relatively low levels of inventories at a time when the average price the firm paid for steel was about 20.5 cents per pound. However as the Asian financial crisis deepened, foreign steel producers began cutting their prices and aggressively increasing their exports. We see this clearly in our data, where in April 1998, wholesale prices dropped to 18.5 cents per pound. At that time the firm made a large purchase. As the price of steel continued to fall to historical lows during the remainder of 1998 the firm made a succession of large purchases that lead it to hold historically unprecedented high levels of inventories. We view this as clear evidence that the firm is attempting to profit from a "buy low, sell high" strategy.

2.2   The Model

Our model is an extension of previous work by Hall and Rust (1999), who showed that in a broad class of commodity price speculation problems, the optimal trading rule is a generalized version of the classic (S,s) rule from inventory theory. Their work can be viewed as linking contributions by Arrow *et. al.* (1951) and Scarf (1959) who first proved the optimality of (S,s) policies in inventory investment problems to more recent work by Williams and Wright (1991), Deaton and Laroque (1992) and Miranda and Rui (1997) on the rational expectations commodity storage model. The fixed (S,s) thresholds derived by Scarf under the assumption that the price (cost) of procuring (producing) inventories is constant are clearly suboptimal in

9

a speculative trading environment, since the stochastic fluctuations in the price of steel affects the firm's perception of the optimal level of inventory S, and the threshold for purchasing new inventory s. Hall and Rust (1999, 2007) showed that the firm's optimal speculative trading strategy is a *generalized the (S,s) rule* where S and s are functions of state variables that include the wholesale price of steel p.[3]

Before we describe how the generalized (S,s) rule allows us to formulate and solve the problem of endogenous sampling of steel wholesale prices, we describe the notation and key assumptions underlying Hall and Rust's model of commodity price speculation. Then we formally define the (S,s) trading strategy, and show how in a broad class of models of speculation, the (S,s) rule constitutes the optimal strategy for "buying low and selling high". We assume that a middleman (which we also refer to as the "firm") can purchase unlimited quantities of steel at a time-varying wholesale price p_t that evolves according to a Markov transition density to be specified below. We assume that the middleman subsequently sells this steel to retail customers at a retail price p_t^r that includes a randomly varying markup over the current wholesale price p_t (if we think of the firm as selling to different customers on different business days, this randomly varying markup is intended to be a "reduced-form" approach to capturing the pricing and price discrimination decisions by the firm).

On each business day t the following sequence of actions occurs:

1. At the start of day t the firm knows its inventory level q_t, the current wholesale price p_t, and the values of the other state variables x_t.

2. Given (q_t, p_t, x_t) the firm orders additional inventory q_t^o for immediate delivery.

3. Given (q_t, q_t^o, p_t, x_t) the firm sets a retail price p_t^r that is modeled as a random draw from a density $\gamma(p_t^r | q_t + q_t^o, p_t, x_t)$.

---

[3]This analysis extends previous results in the operations research literature such as Fabian *et. al.* (1959), Kingman (1969), Kalymon (1971), Golabi (1985), Song and Zipken (1993), Moinzadeh (1997), and Ozekici and Parlar (1999) that prove the optimality of generalized versions of the (S,s) rule when the cost (price) of producing (procuring) new inventory fluctuates stochastically. While Hall and Rust (2007) are not the first to prove the optimality of generalized versions of the (S,s) rule, they extend the OR literature by making the connection between models of optimal inventory policies and models of storage and commodity prices. Moreover in the current paper we computationally solve and estimate our model. Thus we can formally compare the model's optimal policies to the inventory policies we see in the data. Besides the work noted above, the most closely related recent work that we are aware of is the ambitious paper by Aguirregabiria (1999) that models price and inventory decisions by a supermarket chain. A supermarket is similar to our steel wholesaler in that both types of firms hold inventories of a substantial number of different products, purchasing them in the wholesale market and selling their inventories at a markup to retail customers. The key difference is that prices in supermarkets are almost always posted so there is no direct price discrimination and there is presumably a larger "menu cost" to changing prices on a day by day basis. Aguirregabiria also did not directly address the endogenous sampling issue, using monthly price averages as proxies for underlying daily prices. For this reason we are unable to directly employ his innovative and ambitious approach to estimation.

4. Given $(q_t, q_t^o, p_t, p_t^r, x_t)$ the firm observes a realized retail demand for its steel, q_t^r, modeled as a draw from a distribution $H(q_t^r | p_t, p_t^r, x_t)$ with a point mass at $q_t^r = 0$.

5. The firm cannot sell more steel than it has on hand, so the actual quantity sold satisfies

$$q_t^s = \min\left[q_t + q_t^o, q_t^r\right]. \tag{3}$$

6. Sales on day t determine the level of inventories on hand at the beginning of business day $t + 1$ via the standard inventory identity:

$$q_{t+1} = q_t + q_t^o - q_t^s. \tag{4}$$

7. New values of (p_{t+1}, x_{t+1}) are drawn from a Markov transition density $g(p_{t+1}, x_{t+1} | p_t, x_t)$.

Note that we abstract from delivery lags and assume that the firm cannot backlog unfilled orders. Thus, whenever demand exceeds quantity on hand, the residual unfilled demand is lost. Thus, in addition to the censoring of the purchase and retail prices (p_t, p_t^r), we only observe a truncated measure of the firm's retail demand, i.e., we only observe the *minimum* of q_t^r and $q_t + q_t^o$ as given in equation (3). Since the quantity demanded has support on the $[0, \infty)$ interval, equation (3) implies that there is always a positive probability of a *stockout* given by:

$$\delta(q, p, p^r, x) = 1 - H(q | p^r, p, x). \tag{5}$$

Since retail sales occur much more frequently than purchases of new inventory, the retail sales price p_t^r provides an important source of information about the wholesale price p_t. Presumably for most transactions we should have $p_t^r \geq p_t$, reflecting nonnegative markups over the current wholesale price of steel. However as noted above markups vary in an apparently random fashion from day to day, so at best p_t^r is a biased and noisy indicator of the wholesale price p_t. In this version of the paper we bypass some of the difficult issues associated with modeling endogenous price setting and price discrimination by adopting a "reduced-form" model of price setting. We model the daily average retail price as a draw from a conditional density $\gamma(p_t^r | q_t + q_t^o, p_t, x_t)$. This way of modeling prices is sufficiently flexible to be consistent with a variety of theories of bargaining and price discrimination by the firm.

The firm's expected sales revenue function, $ES(p, q, x)$ is the conditional expectation of realized sales revenue $p^r q^r$ given the current wholesale price p, quantity on hand q, and the observed information variables x. The firm's retail sales on date t is a random draw q_t^r from a conditional distribution $H(q_t^r | p_t^r, p_t, x_t)$

that depends on the retail price quote p_t^r, the current wholesale price p_t, and the values of the other observed state variables x_t. We assume that there is a positive probability $\eta(p^r,p,x) = H(0|p^r,p,x)$ that the firm will not make any retail sales on a particular day, so H can be represented by

$$H(q^r|p^r,p,x) = \eta(p^r,p,x) + [1 - \eta(p^r,p,x)] \int_0^{q^r} h(q|p^r,p,x)dq, \tag{6}$$

where h is a continuous strictly positive probability density function over the interval $[0,\infty)$. Given this stochastic "demand function", the firm's expected sales revenue $ES(p,q,x)$ is:

$$
\begin{aligned}
ES(p,q,x) &= E\{\tilde{p}^r \tilde{q}^s | p,q,x\} \\
&= E\left\{\tilde{p}^r E\{\min[q,\tilde{q}^r] | p^r,p,q,x\} | p,q,x\right\} \\
&= \int_0^\infty p^r[1 - \eta(p^r,p,x)]\left[\int_0^q q^r h(q^r|p^r,p,x)dq^r + \delta(q,p^r,p,x)q\right]\gamma(p^r|q,p,x)dp^r.
\end{aligned}
\tag{7}
$$

In order to state the per period profit function, we need to describe the costs that the firm incurs. The main cost is the cost of ordering new inventory, represented by the *order cost function* $c^o(q^o,p)$. We assume that the firm incurs a fixed cost $K \geq 0$ associated with placing new orders for inventory, which implies that $c^o(q^o,p)$ is given by

$$c^o(q^o,p) = \begin{cases} pq^o + K & \text{if} \quad q^o > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

The firm's remaining costs are summarized by the *holding cost function* $c^h(q,p,x)$. These costs include physical storage costs, and "goodwill costs" representing the present value of lost future business from customers whose orders cannot be filled due to a stockout. Goodwill costs can be viewed as the inverse of the "convenience yield" discussed in the commodity storage literature (Kaldor (1939), Williams and Wright (1991)). In this case a convenience yield emerges from a desire to hold a buffer stock or precautionary level of inventories in order to minimize goodwill costs from stockouts. This allows the model to capture other reasons besides pure price speculation for holding inventories.[4] The firm's single-period profits π equals its sales revenues, less the cost of new orders for inventory $c^o(q^o,p)$ and inventory holding costs $c^h(q+q^o,p,x)$:

$$\pi(p,p^r,q^s,q+q^o,x) = p^r q^s - c^o(q^o,p) - c^h(q+q^o,p,x). \tag{9}$$

---

[4]The firm obtains much of its steel from foreign sources. In the model orders occur instantaneously with certainty. In practice, however, delivery lags can be several months and the steel delivered can often be of lower quality than agreed on. The firm does have the option of refusing to take delivery if the steel is not of the quality promised. Having a buffer stock of inventories on hand reduces the cost to firm of exercising this option. Also foreign producers of steel do from time to time renege on previously negotiated deals, failing to deliver the amount of steel originally promised.

where $q^s = min[q^r, q + q^o]$. Each period the firm chooses investment q_t^o given $\{p_t, q_t, x_t\}$ to maximize the discounted present value of profits:

$$V(p,q,x) = \max_{\{q^o\}} E \left\{ \sum_{t=0}^{\infty} \beta^t \pi(p_t, p_t^r, q_t^r, q_t^o + q_t, x_t) \Big| p, q, x \right\}, \tag{10}$$

where $\beta = \exp\{-r/365\}$ and r is the firm's annual discount rate and $\{q^o\}$ denotes a sequence of steel orders given by state-dependent decision rules that described in more detail below. The exponential formula for β account for the fact that our model has decisions being made at daily frequency. The value function $V(p,q,x)$ is given by the unique solution to Bellman's equation:

$$V(p,q,x) = \max_{0 \le q^o \le \overline{q} - q} \left[W(p, q + q^o, x) - c^o(q^o, p) \right], \tag{11}$$

where $\overline{q}$ is the firm's maximum storage capacity and

$$W(p,q,x) \equiv \left[ES(p,q,x) - c^h(q,p,x) + \beta EV(p,q,x) \right], \tag{12}$$

and EV denotes the conditional expectation of V given by:

$$
\begin{aligned}
EV(p,q,x) &= E\{V(\tilde{p}, \max[0, q - \tilde{q}^r], \tilde{x}) | p, q, x) \tag{13}\\
&= \lambda_1(p,q,x) \int_{p'} \int_{x'} V(p', q, x') g(p', x' | p, x) dp' dx' \\
&+ \lambda_2(p,q,x) \int_{p'} \int_{x'} V(p', 0, x') g(p', x' | p, x) dp' dx' \\
&+ \lambda_3(p,q,x) \int_{p'} \int_{x'} \int_0^q V(p', q - q', x') h(q' | p, q, x) g(p', x' | p, x) dq' dp' dx',
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda_1(p,q,x) &= \int_{p^r} \eta(p^r, p, x) \gamma(p^r | p, q, x) dp^r \tag{14}\\
\lambda_2(p,q,x) &= \int_{p^r} [1 - \eta(p^r, p, x)] \delta(p^r, p, q, x) \gamma(p^r | p, q, x) dp^r \\
\lambda_3(p,q,x) &= \int_{p^r} [1 - \eta(p^r, p, x)] \gamma(p^r | p, q, x) dp^r \\
h(q' | p, q, x) &= \int_{p^r} h(q' | p^r, p, q, x) \gamma(p^r | p, q, x) dp^r.
\end{aligned}
$$

The optimal decision rule $q^o(p,q,x)$ is given by:

$$q^o(p,q,x) = \inf \operatorname*{argmax}_{0 \le q^o \le \overline{q} - q} \left[W(p, q + q^o, x) - c^o(q^o, p) \right]. \tag{15}$$

13

We invoke the inf operator in the definition of the optimal decision rule in equation (15) to handle the case where there are multiple maximizing values of q^o. We effectively break the tie in such cases by defining $q^o(p,q)$ as the *smallest* of the optimizing values of q^o.

In this model the variables q and q^o do not enter as separate arguments in the value function W given in (12): rather they enter as the sum $q+q^o$ as shown in equation (15). This symmetry property is a consequence of our timing assumptions: since new orders of steel arrive instantaneously, the firm's expected sales, inventory holding costs, and expected discounted profits only depend on the sum $q+q^o$, representing inventory on hand at the beginning of the period after new orders q^o have arrived. It follows that if the firm is holding less than its desired level of inventories $S(p_t,x_t)$ at the start of day t, it will only have to order the amount $q^o(p,q,x) = S(p,x) - q$ in order to achieve its target inventory level $S(p,x)$. Another way to see this is to note that when it is optimal for the firm to order, the optimal order level solves the first order condition:

$$\frac{\partial W}{\partial q^o}(p,q+q^o,x) = p. \tag{16}$$

If W were strictly concave in q, there would be a unique value of $q+q^o$ that solves equation (16) for any value of p. Call this solution $S(p,x)$:

$$\frac{\partial W}{\partial q^o}(p,S(p,x),x) = p. \tag{17}$$

Then we have $q+q^o = S(p,x)$, or $q^o(p,q,x) = S(p,x) - q$.

In turns out that if $K > 0$ the function $W(p,q,x)$ will not be strictly concave. However under fairly general conditions W is *K-concave* as a function of q for each fixed p.[5] Using the *K*-concavity property we can prove that whenever $q \geq s(p,x)$, it is not optimal to order: $q^o(p,q,x) = 0$. When $q < s(p,x)$ the symmetry property implies that $q^o(p,q,x) = S(p,x) - q$ as discussed above. In particular Hall and Rust (2007) proved:

Theorem 1: *Consider the function $W(p,q+q^o,x)$ defined in equation (12), where W is defined in terms of the unique solution V to Bellman's equation (11). Under appropriate regularity conditions given in Hall and Rust (2007), the optimal speculative trading strategy $q^o(p,q,x)$ takes the form of an (S,s) rule. That is, there exist a pair of functions (S,s) satisfying $S(p,x) \geq s(p,x)$ where $S(p,x)$ is the desired or target*

---

[5]A function $W(p,q) : [\underline{p},\overline{p}] \times [0,\overline{q}] \to R$ is *K-concave* in its second argument q if and only if $-W(p,q)$ is *K-convex* in its second argument. More directly, $W(p,q)$ is *K-concave* in q iff $\exists K \geq 0$ such that for every $p \in [\underline{p},\overline{p}]$, and for all $z \geq 0$ and $b \geq 0$ such that $q+z \leq \overline{q}$ and $q-b \geq 0$ we have $W(p,q+z) - K \leq W(p,q) + z[W(p,q) - W(p,q-b)]/b$.

inventory level and $s(p,x)$ is the inventory order threshold, i.e.

$$q^o(p,q,x) = \begin{cases} 0 & \text{if} \quad q \geq s(p,x) \\ S(p,x) - q & \text{otherwise} \end{cases} \tag{18}$$

where $S(p,x)$ is given by:

$$S(p,x) = argmax_{0 \leq q^o \leq \overline{q} - q} \left[W(p,q^o,x) - c^o(q^o,p) \right] \tag{19}$$

and the lower inventory order limit, $s(p,x)$ is the value of q that makes the firm indifferent between ordering and not ordering more inventory:

$$s(p,x) = \inf_{q \geq 0} \left\{ q | W(p,q,x) - pq \geq W(p,S(p,x),x) - pS(p,x) - K \right\}. \tag{20}$$

By a simple substitution of the generalized (S,s) rule in equation (18) into the definition of V in equation (11) we obtain the following corollaries:

Corollary 1: *The value function V is linear with slope p on the interval $[0, s(p,x)]$:*

$$V(p,q,x) = \begin{cases} W(p,S(p,x),x) - p[S(p,x) - q] - K & \text{if} \quad q \in [0, s(p,x)] \\ W(p,q,x) & \text{if} \quad q \in (s(p,x), \overline{q}]. \end{cases} \tag{21}$$

Corollary 2: *The $S(p,x)$ and $s(p,x)$ functions are non-increasing in p and are strictly decreasing in p in the set $\{p | 0 < S(p,x) < \overline{q}\}$.*

Corollary 3: *If fixed costs of ordering is zero, $K = 0$, then the minimum order size is 0 and*

$$S(p,x) = s(p,x). \tag{22}$$

3 Challenges in confronting theory to data

Our model of steel price speculation and inventory investment implies that $\{p_t, q_t, x_t\}$ evolves as a *controlled Markov process* where $\{p_t, x_t\}$ can be viewed as an "exogenous forcing process" whereas inventory $\{q_t\}$ is an endogenous variable whose law of motion is derived from the inventory accumulation identity (4). Initially we ignore the censoring of p_t due to the endogenous sampling problem described in the introduction, and focus on deriving a formula for the transition probability for the controlled Markov process that is the basis for inference by likelihood-based methods or simulation-based methods such as MSM or Markov Chain Monte Carlo (MCMC) Bayesian methods.

In addition to (p_t, x_t, q_t) we also observe (q_t^o, q_t^s, p_t^r) where q_t^o is the amount of steel ordered on day t, p_t^r is the retail price charged to customers on day t, and q_t^s is the amount sold to retail customers.[6] Thus, our interest is to model the evolution of the random vector $\{\xi_t\}$ given by

$$\xi_t \equiv (p_t, p_t^r, x_t, q_t^o, q_t^s, q_t) \tag{23}$$

and show that $\{\xi_t\}$ is a Markov process and provide sufficient conditions for it to have a unique invariant distribution. This invariant distribution plays a key role in the asymptotic properties of maximum likelihood or MSM esimators of the parameters θ of the model as the number of observations $T \to \infty$.

An immediate issue that we need to confront is the potential *statistical degeneracy* of the model: that is, unless there are unobservables included in the inventory speculation model, certain components of the vector ξ_t will be deterministic functions of the remaining components. For example, Theorem 1 implies that the optimal order quantity q_t^o is a deterministic of (p_t, q_t, x_t) given in equation (18). Further, q_{t+1} is a deterministic function of ξ_t given in the inventory law of motion (4). The fact that q_t^o can be perfectly predicted from knowledge of (p_t, q_t, x_t) creates problems for statistical inference of the inventory model if we assume all components of x_t can be observed by the econometrician, since it is unlikely that any econometric model could perfectly predict every steel order made by the firm.

Inference is also complicated by frequently binding inequality constraints on inventory investment, q_t^o. This implies that it is not possible to use standard Euler equation methods to estimate the unknown parameters of the model via generalized method of moments (GMM) (see, e.g. Hansen (1982)). Note that Theorem 1 does yield a first order condition that could possibly provide a basis for a GMM estimator of the unknown parameters of the model:

$$\frac{\partial W}{\partial q}(p, S(p, x), x) - p = 0. \tag{24}$$

If we assume that there is additive measurement error ε in the wholesale price p, or assume that ε represents other unobserved (per unit) components of the cost of ordering new inventory, then it is tempting to treat equation (24) as an "Euler equation" and use GMM to estimate parameters of the model. However there are several big obstacles to this approach. First, we do not have a convenient analytical formula for the partial derivative of the value function, $\partial W / \partial q$. Second, as we show in Theorem 2 below, even if the unconditional

---

[6]For simplicity, we assume that all sales on day t are done at a single posted price p_t^r. In actuality, the firm can sell to many different customers on a single day at individually negotiated prices. Thus the actual data we observe are more complex and a fully accurate treatment would necessitate modeling these individual negotiations and controlling for the individual customer characteristics and order sizes and delivery locations.

mean of ε is zero, the conditional mean of ε over those values of $(p,ε)$ for which it is optimal to purchase (i.e. for which $q < s(p,x)$), is generally nonzero. Finally, there is the issue of endogenous sampling, and the fact that we observe purchases only an a relatively small subset of business days in our overall sample.

These problems motivate a search for an alternative likelihood-based approach that is capable of incorporating other information such as retail sales prices in order to improve our ability to make inferences about the $\{p_t\}$ process. We briefly discuss how to derive a non-degenerate likelihood function via the inclusion of a single *IID* unobservable state variable $ε_t$ in the firm's optimization problem. The resulting conditional probability distribution function for q^o has a mass point at $q^o = 0$ that reflects the frequently binding constraint that inventory investment cannot be negative. This conditional distribution allows us to derive a full-information maximum likelihood estimator that provides a complete solution to the problem of endogenous sampling of the whole price process. It does this by integrating out the unobserved values of the wholesale prices in periods where they are unobserved. This likelihood is the analog of the Chapman-Kolmogorov equation for computing multi-step transition probabilities from a one-step transition probabilities. However, we have already discussed the drawbacks of maximum likelihood estimation in the introduction which motivates our use of the simpler and computationally tractable but less efficient MSM estimator in section 4. Thus, we will only derive the likelihood function but not go into detail about deriving the asymptotic distribution of the maximum likelihood estimator.

One way to avoid a "zero likelihood" problem for maximum likelihood estimation of θ using the full set of data $\{ξ_t\}$ is to allow for an unobserved component of the per unit cost of steel, denoted by $ε_t$. There is an economic rationale for including the unobservable $ε_t$ in the model since it captures transportation costs and quality variability in steel purchases that are not fully reflected in the observed purchase price p. We assume that the distribution of $ε_t$ has support on the entire real line and continuous, strictly positive density $φ(ε)$. Theorem 2 below derives the implied conditional distribution of q^o given (p,q,x) formed by integrating out ε from the deterministic decision rule $q^o(p,q,x,ε)$.

Theorem 2: *Let $ε_t$ be an (unobserved to the econometrician) component of the per unit cost of ordering new inventory. That is, $c^o(q^o,p,ε) = (p+ε)q^o + K$ if $q^o > 0$ and $c^o(q^o,p,ε) = 0$ if $q^o = 0$. Assume that $\{ε_t\}$ is an IID process whose density $φ$ is continuous and strictly positive over the entire real line. Then the optimal trading strategy is still a generalized (S,s) rule and the conditional distribution of the optimal*

order quantity q^o given (p,q,x) is given by

$$
\begin{aligned}
F(q^o|p,q,x) &= \Pr\{q^o(p,q,x,\varepsilon) \le q^o|p,q,x\} \\
&= \int_{-\infty}^{+\infty} I\{q^o(p,q,x,\varepsilon) \le q^o\}\phi(\varepsilon)d\varepsilon \\
&= \int_{s^{-1}(p,x,q)}^{\infty} \phi(\varepsilon)d\varepsilon \\
&+ I\{S(p,x,s^{-1}(p,x,q)) \le q^o + q \le \overline{q}\} \int_{S^{-1}(p,x,q+q^o)}^{s^{-1}(p,x,q)} \phi(\varepsilon)d\varepsilon \\
&+ I\{q^o + q > \overline{q}\} \int_{-\infty}^{S^{-1}(p,x,\overline{q})} \phi(\varepsilon)d\varepsilon,
\end{aligned}
\tag{25}
$$

where

$$
\begin{aligned}
S^{-1}(p,x,q) &= \inf\{\varepsilon|S(p,x,\varepsilon) = q\} \\
s^{-1}(p,x,q) &= \inf\{\varepsilon|s(p,x,\varepsilon) = q\}.
\end{aligned}
\tag{26}
$$

Let $f = dF$ denote the mixed discrete/continuous conditional density of q^o given (p,q,x). It is given by

$$
f(q^o|p,q,x) = \begin{cases} \int_{s^{-1}(p,x,q)}^{\infty} \phi(\varepsilon)d\varepsilon & \text{if } q^0 = 0 \\ \int_{-\infty}^{S^{-1}(p,x,\overline{q})} \phi(\varepsilon)d\varepsilon & \text{if } q^0 = \overline{q} - q \\ \dfrac{-\phi(S^{-1}(p,x,q+q^o))}{\frac{\partial^2}{\partial^2 q}W(p,x,q+q^o)} & \text{otherwise.} \end{cases}
\tag{27}
$$

The formula for the density of q^o in equation (27) can be derived by differentiating the conditional distribution in equation (25) with respect to q^o for q^o in the interval $[S(p,x,s^{-1}(p,x,q)) - q, \overline{q} - q]$ to obtain:

$$
dF(q^o|p,q,x) = -\phi(S^{-1}(p,x,q+q^o))\frac{\partial}{\partial q^o}S^{-1}(p,x,q+q^o).
\tag{28}
$$

Using the definition of $S(p,x,\varepsilon)$

$$
\frac{\partial}{\partial q}W(p,S(p,x,\varepsilon),x) = p + \varepsilon,
\tag{29}
$$

and the inverse and implicit function theorems we obtain:

$$
\frac{\partial}{\partial q^o}S^{-1}(p,x,q+q^o) = \frac{1}{\frac{\partial}{\partial \varepsilon}S(p,x,S^{-1}(p,x,q+q^o))} = \frac{1}{\frac{\partial^2}{\partial^2 q}W(p,q+q^o,x)}.
\tag{30}
$$

Note that Theorem 2 implies that the transition density for q^o is mixed discrete and continuous, with mass points at $q^o = 0$ and $q^o = \overline{q} - q$, and strictly positive density over the interval $[S(p,x,s^{-1}(p,x,q)) - q, \overline{q} - q]$. However there is a "gap" where there is zero density for q^o in the interval $[0, S(p,x,s^{-1}(p,x,q)) - q]$ since the quantity $S(p,x,s^{-1}(p,x,q)) - q$ represents the minimum order size implied by the (S,s) model in the

18

state (p, q, x). The gap is problematic for maximum likelihood estimation since a single observation with an order smaller than the predicted minimum order size would result in a zero value for the likelihood function. To obtain a fully nondegenerate likelihood function, we would have to introduce a second unobservable, such as an unobservable component υ of the fixed cost K of placing an order. If the distribution of this component is such that there is positive probability that the combined order cost $K + \upsilon$ is arbitrarily close to zero for sufficiently small realizations of υ, then consistent with Corollary 3 of section 2, the gap will be zero, thus fully eliminating the possibility of a "zero likelihood problem." In practice for the values of K we encountered in our estimation, the gap is sufficiently small that zero likelihood problems seem unlikely to arise. Therefore in order to simplify the model and the exposition we decided to omit the case where there are unobservable components of K as well as p.

Even after introducing an unobservable state ε_t there is still a statistical degeneracy present since the full vector ξ_t "overdetermines" the data we observe due to the inventory accumulation identity (4) which implies that knowledge of inventories at t, q_t, plus sales q_t^s and purchases q_t^o enables us to perfectly predict inventories at $t + 1$. If we drop sales q_t^s from the ξ_t vector we can still back out the value of sales q_t^s from knowledge of q_{t+1}, q_t and q_t^0, but the resulting model is no longer statistically degenerate, i.e. there are no current or future variables that can be perfectly predicted from the remaining variables. Hereafter, let ξ_t be the reduced vector of variables that excludes sales, q_t^s. Given this reduced vector ξ_t, q_{t+1} is no longer a deterministic function of its components: instead it is a random quantity that depends on the realization of retail sales $q_t^s = \min[q_t + q_t^o, \tilde{q}_t^r]$, where $\tilde{q}_t^r$ is a draw from the conditional distribution $H(q^r | p^r, p, x)$ given in equation (6). Given our timing convention, at the start of each day t the firm a) observes (q_t, p_t, x_t), b) orders q_t^o which we assume arrives immediately, c) announces the retail price p_t^r, and then d) given $(p_t, p_t^r, q_t, q_t^o, x_t)$ there will be some realized level of retail sales q_t^s during the remainder of the day. Once sales are realized, the firm knows the amount of inventory it will have at the beginning of day $t + 1$, q_{t+1}.

Let the conditional density of next period inventory q_{t+1} given $\xi_t = (p_t, p_t^r, x_t, q_t, q_t^o)$ be denoted by μ. From our discussion of the model in section 2, it is easy to see the μ is a mixed discrete/continuous density with three classes of outcomes for q_{t+1}: 1) with probability $\eta(p^r, p, x)$ the firm will not make any sales and $q_{t+1} = q_t + q_t^o$; 2) with probability $(1 - \eta(p_t^r, p_t, x_t))\delta(p_t^r, p_t, q_t + q_t^o, x_t)$ the firm will have a stockout and $q_{t+1} = 0$; 3) otherwise q_{t+1} is distributed continuously over the interval $(0, q_t + q_t^o)$ with density given by $(1 - \eta(p_t^r, p_t, x_t))h(q_t + q_t^o - q_{t+1} | p_t^r, p_t, x_t)$ where h is the density of retail sales and $q_t^r = q_t + q_t^o - q_{t+1}$ is the implied value of retail sales given (q_{t+1}, q_t, q_t^o). We summarize this as:

Theorem 3: *The density of next period inventory q' given (p, p^r, q, q^o, x) is given by:*

$$\mu(q'|p, p^r, q, q^o, x) = \begin{cases} (1 - \eta(p^r, p, x))\delta(p^r, p, q + q^o, x) & \text{if} \quad q' = 0 \\ \eta(p^r, p, x) & \text{if} \quad q' = q + q^o \\ (1 - \eta(p^r, p, x))h(q + q^o - q'|p^r, p, x) & \text{otherwise.} \end{cases} \quad (31)$$

Notice that μ is a mixed discrete/continuous distribution with mass points at 0 (stockout) and $q + q^o$ (no sales) and a continuous density in between (ignoring the minimum purchase size gap noted above).

Under our setup, we can show that the observables $\{p_t, p^r_t, q_t, q^o_t, x_t\}$ evolve as a joint Markov process which also has a discrete/continuous transition probability density λ. We state this as Theorem 4:

Theorem 4: *The joint process $\{\xi_t\} = \{p_t, p^r_t, q_t, q^o_t, x_t\}$ is Markov with (discrete/continuous) transition density λ given by:*

$$
\begin{aligned}
\lambda(p_{t+1}, p^r_{t+1}, q_{t+1}, q^o_{t+}, x_{t+1}|p_t, p^r_t, q_t, q^o_t, x_t) &= g(p_{t+1}, x_{t+1}|p_t, x_t) \\
&\times \mu(q_{t+1}|p_t, p^r_t, q_t, q^o_t, x_t) \\
&\times f(q^o_{t+1}|p_{t+1}, q_{t+1}, x_{t+1}) \\
&\times \gamma(p^r_{t+1}|p_{t+1}, q_{t+1} + q^o_{t+1}, x_{t+1}). \quad (32)
\end{aligned}
$$

Notice that even in cases where there is no unobservable state variable ε_t, the joint process $\{\xi_t\}$ will still be Markovian, however in that case there will be the statistical degeneracy that the q^o component is a deterministic function of (p, x, q). Now consider the full information case where *all* of the variables $\{p_t, p^r_t, q_t, q^o_t, x_t\}$ are observed over the entire sample period $t = 0, \ldots, T$.

Definition 1: *The full information maximum likelihood (FIML) estimator $\hat{\theta}^f_T$ is defined as:*

$$\hat{\theta}_f = \underset{\theta \in \Theta}{argmax} \ l_f(\{p_t, p^r_t, q_t, q^o_t, x_t\}^T_{t=1}|p_0, p^r_0, q_0, q^o_0, x_0, \theta), \quad (33)$$

where l_f is given by:

$$l_f(\{p_t, p^r_t, q_t, q^o_t, x_t\}^T_{t=1}|p_0, p^r_0, q_0, q^o_0, x_0, \theta) = \prod_{t=1}^{T} \lambda(p_t, p^r_t, q_t, q^o_t, x_t|p_{t-1}, p^r_{t-1}, q_{t-1}, q^o_{t-1}, x_{t-1}, \theta). \quad (34)$$

where $\theta \in R^D$ denotes a D-dimensional vector of unknown parameters of the densities $\{f, g, h, \eta, \mu, \gamma, \phi\}$ and the unknown parameters entering the firm's cost functions $\{c^o, c^h\}$ and the firm's discount factor β. Let $\Theta \subset R^D$ denote a compact parameter space. Now consider the partial information case where we only observe wholesale prices on the subset of n trading days, $T_n \equiv \{t_1, \ldots, t_n\}$ at which purchases occur. To simplify notation we assume (without loss of generality) that the data begin on the day of the first

20

observed purchase, so $t_1 = 0$, and end on the day of the last observed purchase, $t_n = T$. The relevant likelihood in this case is a marginal likelihood l_p formed by integrating the full likelihood function l_f in equation (34) over wholesale prices p_t for all time indices t in the complement of T_n. For simplicity, we will consider the case where retail sales are observed in every period. Otherwise, an additional set of integrations would need to be performed over the values of p_t^r for business days t where no retail sales occurred. As noted in the Introduction, it is notationally convenient to convert the endogenous sampling problem into a censored sampling problem by defining an observed censored price sequence $\{p_t\}$ in terms of the underlying uncensored price process $\{p_t^*\}$. Thus, the observed prices p_t are given by:

$$p_t = \begin{cases} p_t^* & \text{if} \quad q_t^o > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{35}$$

Definition 2: *The Partial Information Maximum Likelihood (PIML) estimator $\hat{\theta}_T^p$ is defined as:*

$$\hat{\theta}_p = \underset{\theta \in \Theta}{\mathrm{argmax}} \; l_p(\{p_t, p_t^r, q_t, q_t^o, x_t\}_{t=1}^T | p_0, p_0^r, q_0, q_0^o, x_0, \theta), \tag{36}$$

where l_p is given by:

$$l_p(\{p_t, p_t^r, q_t, q_t^o, x_t\}_{t=1}^T | p_0, p_0^r, q_0, q_0^o, x_0, \theta) = \int_{p_{T_n}} \int_{\varepsilon_{T_n}} \cdots \int_{p_{T_1}} \int_{\varepsilon_{T_1}} l_f \prod_{t \notin T_n} I\{q_t > s(p_t, x_t, \varepsilon_t)\} q(\varepsilon_t) dp_t d\varepsilon_t. \tag{37}$$

Thus, the PIML likelihood l_p is derived from the FIML likelihood l_f by integrating out the unobserved wholesale prices over the dates $t \notin T_n$ that purchases do not occur. The region of integration is limited to the region of the state space where making a purchase is not optimal. This is given by the indicator function $I\{1_t > s(p_t, x_t, \varepsilon_t)\}$. Notice that this region involves the unobserved state variable ε_t. Thus the integration must be done over both unobserved variables (p_t, ε_t) over all of the $T - n$ dates $t \notin T_n$.

In the next section we will present the alternative computationally simpler but less efficient MSM estimator that uses Monte Carlo simulations of the model in place of numerical integrations. However the asymptotic properties of both the PIML and MSM estimators depend on a key assumption of *ergodicity* of the inventory investment model, namely that there is a unique invariant distribution $\psi(\xi|\theta)$ for each $\theta \in \Theta$.

Assumption 1: *For any $\theta \in \Theta$ the corresponding controlled Markov process $\{\xi_t(\theta)\}$ from the solution to the inventory investment problem (11) is ergodic with unique invariant distribution $\psi(\xi|\theta)$ given by:*

$$\psi(\xi'|\theta) = \int \Lambda(\xi'|\xi, \theta) \psi(\xi|\theta) d\xi. \tag{38}$$

where Λ is the CDF corresponding to the transition density λ.

21

Assumption 1 can be established via lower level assumptions on the primitives of the model following the uniqueness and existence criterion in Futia (1982). Futia's uniqueness criterion states that a Markov process $\{\xi_t\}$ has a unique invariant distribution if and only if there exists a point ξ_0 with the following property: for any neighborhood U of ξ_0 and any point ξ there is an n such that $\Lambda^n(U|\xi,\theta) > 0$ where Λ^n is the n-step transition probability implied by the one step transition density λ given by

$$\Lambda^1(B|\xi,\theta) = \int_{\xi'} I\{\xi' \in B\}\lambda(\xi'|x,\theta)d\xi'$$

$$\Lambda^n(B|\xi,\theta) = \int_{\xi'} \Lambda^{n-1}(B|\xi',\theta)\lambda(\xi'|\xi,\theta)d\xi' \tag{39}$$

where B is a Borel set in R^J where J is the dimension of the random vector ξ. Futia's uniqueness assumption guarantees that there is at least one point ξ_0 in the state space that the process $\{\xi_t\}$ returns to a neighborhood of ξ_0 infinitely often as $T \to \infty$, where T is the horizon over which the process is simulated.[7]

However we need a stronger result than just existence of a unique invariant distribution $\psi(\xi|\theta)$ for each $\theta \in \Theta$. We also require this invariant distribution to be differentiable in its parameters and also that expectations of bounded, Borel-measurable functions $h : R^J \to R^M$, we have

$$\frac{\partial}{\partial\theta}E\{h|\theta\} = \frac{\partial}{\partial\theta}\int h(\xi')\psi(d\xi'|\theta) \tag{40}$$

exists and is a continuous function of $\theta \in \Theta$. Vázquez-Abad and Kushner (1992) provide sufficient conditions for differentiability to hold and the key condition, in addition to uniqueness of the invariant measure, is that the collection of invariant distributions $\{\psi(\xi'|\theta)|\theta \in \Theta\}$ is tight. Rather than become distracted by specifying even lower level assumptions necessary to establish tightness, we simply make

Assumption 2: *For any $\theta \in int(\Theta)$ and any bounded, Borel measurable function $h : R^J \to R^M$,*

$$E\{h|\theta\} = \int_{\xi'} h(\xi')\psi(d\xi'|\theta)$$

$$E\{h(\tilde{\xi})|\xi,\theta\} = \int_{\xi'} h(\xi')\lambda(\xi'|\xi,\theta)d\xi' \tag{41}$$

---

[7]From the decomposition of the transition density for λ in Theorem 4, we note that $\{p_t, x_t\}$ follows its own lower-dimensional exogenous Markovian "forcing process", which we assume satisfies Futia's uniqeness criterion so there is a point (p_0, x_0) that the process $\{p_t, x_t\}$ visits infinitely often as $T \to \infty$. Then we can see from figure 1 that a candidate for ξ_0 is any point of the form $\xi_0 = (p_0, p_0^r, q_0, q_0^o, x_0)$ where q_0 is any inventory level in the interval $(0, \overline{q})$, q_0^o is any point in the support of the conditional density $f(q^o|p_0, x_0, q_0)$ and p_0^r is any retail price in the support of the conditional density $\gamma(p^r|p_0, q_0 + q_0^o, x_0)$. Said more intuitively, the Markov process $\{\xi_t\}$ has a unique invariant distribution because the steady inventory decumulation from sales to customers and replenishments of inventories via new purchases combined with the ergodicity of the $\{p_t, x_t\}$ forcing process means that there are many points ξ_0 in the state space for which the process visits arbitrarily small neighboods of infinitely often, thereby satisfying Futia's uniqueness criterion.

are continuously differentiable in θ.

Assumptions 1 and 2, along with several additional assumptions to be introduced in the next section, are the key to establishing the asymptotic properties of the MSM as $T \to \infty$ where T is the number of time periods we observe the single steel firm that is the basis for or empirical analysis. In the next section we show how the MSM estimator is able to conveniently and easily handle the complexities of the endogenous sampling of the wholesale price process $\{p_t\}$.

4   Method of Simulated Moments Estimation

This section presents a method of simulated moments (MSM) estimator for the parameters of the AR(1) model of wholesale prices and the structural parameters of the steel inventory speculation model that we introduced in the previous section. As we noted in the introduction, the original idea for MSM is due to McFadden (1989) who realized that Monte Carlo simulation can be used to avoid numerical integrations to form moments necessary to implement standard method of moments estimators. McFadden's insight was that simulated moments could be constructed using *only a single Monte Carlo simulation per observation* since the Law of Large Numbers guarantees that asymptotically the "Monte Carlo noise" will average out along with the usual random sampling noise.

In the context of our steel example, these same favorable properties of the MSM estimator carry over, except that a Law of Large Numbers and Central Limit Theorem for serially correlated data need to be used, rather than ones for *IID* data that were appropriate in McFadden's analysis. Implementation of the MSM estimator is straightforward. First we calculate sample moments using the censored observations in the data, i.e. with $p_t = 0$ when $q_t^o = 0$. Then we generate one or more simulated realizations of the (S, s) model for a given trial value θ of the unknown parameter vector. We define $\hat{\theta}_T$ as the value of θ that minimizes a quadratic form in the difference between the sample moments for the actual data and the sample moments of the simulated data, where the simulated data has been censored in exactly the same fashion as the actual data, i.e. we set $p_t = 0$ whenever the simulated value of $q_t^o = 0$. Thus even though various moments based on censored data may be biased and inconsistent estimators of the corresponding moments of the ergodic process in the absence of censoring, this does not prevent us from deriving a consistent MSM estimator for θ^*.

Similar to McFadden's (1989) analysis of the cross sectional case, the asymptotic variance of the MSM estimator is multiplied by a factor $(1 + 1/n)$ where *n* is the number of independently drawn time series

simulations of the model. The asymptotic properties of the MSM are indexed by T, the number of time-series observations. The MSM estimator is consistent even if we estimate it using only a single simulated realization of the (S, s) model. Similar to McFadden's (1989) analysis, the cost of using only a single realization, $n = 1$, compared to an "exact" method of moments estimator (equivalent to setting $n = \infty$) is a doubling of the asymptotic variance of the MSM parameter estimates. The increase in asymptotic variance of the MSM estimator due to using relatively small numbers of simulations n seems small in comparison to the substantial reduction in computational burden from from using many simulations. Though MSM estimation requires a nested fixed point algorithm to solve for the optimal (S, s) policy and repeated re-simulations of the model using a fixed set of random shocks (see below) each time the parameter θ is updated, we find that the computer time required to simulate the model exceeds the computer time to solve the model when n is set to even moderate sizes such as $n = 50$ or $n = 100$. However n smoothes the MSM criterion and the added smoothness helps the numerical minimization algorithm to find the estimator $\hat{\theta}_T$, the global minimizer of the MSM objective function, in a smaller number of function evaluations. Thus, there are important tradeoffs to be considered in setting the number of simulations n.

MSM requires the analyst to determine an appropriate set of moments to represent the relevant metric for assessing the distance between the predictions of the model and the data. In principle an infinite number of different moment conditions could be specified, but only a finite number can be used in practice. Identification of the model parameters θ^* can often depend on the types of moments the analyst chooses, but to our knowledge there is no formal theory that specifies which moments and how many moments must be chosen in order to secure identification and relatively efficient estimation of the parameters. Moments that approximate efficient score (i.e. the gradient of the likelihood with respect to the parameters of the model) result in more efficient MSM estimators but these are often difficult to construct and result in intractable high dimensional integrations in order to compute the efficient score, which is the key reason that lead us to prefer the MSM estimator over maximum likelihood estimation.

The MSM estimator is easiest to implement computationally in cases where it is possible to construct *smooth simulators* i.e. where the the simulated observations can be recursively constructed to be continuously differentiable functions of the structural parameters θ of the model. However while it is easy to construct a smooth simulator for the wholesale price process $\{p_t\}$ itself, the MSM estimator requires us to simulate a vector of endogenous variables, including orders of steel q_t^o and inventories q_t, and the nature of the (S, s) policy generates inherent discontinuities in the quantity variables since it implies that $q_t^o = 0$ when $s(p_t, x_t, \theta) \leq q_t$ but q_t^o discontinuously jumps to $q_t^o = S(p_t, x_t, \theta) - q_t$ when $s(p_t, x_t, \theta) > q_t$.

However the asymptotics of the MSM estimator are robust to the presence of discontinuous simulations of the model for the same reason that McFadden's original MSM estimator is also robust to potential discontinuities arising from small changes in the parameter values causing simulated discrete decisions to discontinuously jump from one value to another. The main complexity caused by discontinuities is not in the asymptotics (which are still $\sqrt{T}$ consistent and asymptotically normal in the presence of discontinuities in the simulated realizations of $\{p_t, x_t\}$) but rather in terms of computation of the MSM estimator, since the discontinuities in the simulations can lead to an estimation criterion that is locally flat and thus difficult to optimize. However by employing the ideas of Frazier and Zhu (2017) including the use of automatic differentiation, the practical difficulties caused by discontinuities in the MSM objective function can be significantly ameliorated.

Let $\{\xi_t\}$ denote the censored process introduced in section 3 (i.e. with $p_t = 0$ when $q_t^o = 0$), and let θ denote the $D \times 1$ vector of parameters to be estimated. The MSM estimator is based on finding a parameter value that best fits a $M \times 1$ vector of moments of the observed process:

$$h_T \equiv \frac{1}{T} \sum_{t=1}^{T} h(\xi_t, \xi_{t-1}), \tag{42}$$

where $M \geq D$ and h is a known function of (ξ_t, ξ_{t-1}) that determines the moments we wish to match. We include ξ_t and its lag ξ_{t-1} as arguments of h in order to handle situations where we are trying to fit moments such as means and covariances of the components of ξ_t. It is straightforward to allow moments that involve more than one lag: we only include a single lagged value of ξ_t in our presentation below for notational simplicity.

By Assumption 1, the process $\{\xi_t\}$ is ergodic so that, with probability 1, h_T converges to a limit $E\{h(\xi', \xi)\}$ where the expectation is taken with respect to the ergodic distribution of (ξ', ξ) (i.e. the limiting distribution of (ξ_{t+1}, ξ_t) as $t \to \infty$). Under suitable additional regularity conditions, a central limit theorem will hold for h_T, i.e. we have

$$\sqrt{T}\left[h_T - E\{h\}\right] \Longrightarrow N(0, \Omega(h)), \tag{43}$$

where

$$\Omega(h) = \sum_{j=-\infty}^{\infty} E\left\{(h(\xi_{t+1}, \xi_t) - E\{h\})(h(\xi_{t+j+1}, \xi_{t+j}) - E\{h\})'\right\}, \tag{44}$$

where the expectations in (44) are taken with respect to the ergodic distribution of (ξ', ξ). Formula (44) is the standard formula for the variance of a sum for a serially correlated process when the number of elements in the sum tend to infinity, and is the same formula given in Assumption 3 of Duffie and Singleton

(1993). To operationalize the infinite sum of covariances at all leads and lages in our empirical work, we employ the Newey-West (1987) estimator of $\Omega(h)$ in our empirical work, since as Duffie and Singleton note, the assumptions that insure the geometric ergodicity of the process $\{\xi_t\}$ also imply the α-mixing property necessary to insure the consistency of the Newey-West estimator.

Now assume it is possible to generate simulated realizations of the $\{\xi_t\}$ process for any candidate value of θ, and that this process is censored in exactly same way as the observed $\{\xi_t\}$ process is censored, i.e., with $p_t = 0$ when $q_t^o = 0$. These simulations depend on a $T \times 1$ vector, u, of *IID* $U(0,1)$ random variables that are drawn once at the start of the estimation process and held fixed thereafter in order for the estimator to satisfy a stochastic equicontinuity condition necessary to establish the consistency and asymptotic normality of the MSM estimator. We will consider simulated processes of the form

$$\{\xi_t(\{u_s\}_{s\leq t}, \theta, \xi_0)\}, \quad t = 2, \ldots, T \tag{45}$$

where for each $t > 1$, $\xi_t(\{u_s\}_{s\leq t}, \theta, \xi_0)$ is recursively defined to depend on previous simulated values for periods $s \leq t$. The notation $\{u_s\}_{s\leq t}$ reflects the fact that the simulated process is *adapted* to the realization of the $\{u_t\}$ process, i.e. the first t realized values of $\{\xi_t(\{u_s\}_{s\leq t}, \theta)\}$ depend only on the first t realized values of $\{u_s\}$ and not on subsequent realized values of u_s for $s > t$. Note that we allow the simulated process to depend on the first value ξ_0 of the observed data as an initial condition.

To illustrate the recursive procedures for constructing simulated realizations, consider first the simulation of the (uncensored) wholesale price process $\{p_t\}$ itself. Let $\lambda(p_{t+1}|p_t, \theta)$ denote the Markov transition density for $\{p_t\}$ and $P(p_{t+1}|p_t, \theta)$ be the corresponding conditional CDF. The first value of the simulated process is simply set to the observed value p_0. Using the probability integral transform, we can define $p_1(u_1, \theta, p_0)$ by:

$$p_1(u_1, \theta, p_0) = P^{-1}(u_1|p_0, \theta). \tag{46}$$

Clearly $p_1(u_1, \theta, p_0)$ will be a continuously differentiable function of θ if $P^{-1}(u_1|p_0, \theta)$ is a continuously differentiable function of θ. Now define recursively for $t = 2, 3, 4, \ldots$

$$p_t(\{u_s\}_{s\leq t}, \theta, p_0) = P^{-1}(u_t|p_{t-1}(\{u_s\}_{s\leq t-1}, \theta, p_0), \theta). \tag{47}$$

We can see via a simple recursive argument that $p_t(\{u_s\}_{s\leq t}, \theta, p_0)$ will be a continuously differentiable function of θ provided that $P^{-1}(u|p, \theta)$ is a continuously differentiable function of p and θ.

We can simulate the full multivariate process $\{\xi_t\}$ where $\xi_t = (p_t, p_t^r, q_t, q_t^o, x_t)$, using a similar recursively defined simulation procedure as in the univariate case described above, but using a factorization of

26

the transition density of $\{x_t\}$ a product of univariate conditional densities. For example, consider a case where ξ_t has two components, $\xi_t = (\xi_{1,t}, \xi_{2,t})$, and suppose that its transition density λ can be factored as

$$\lambda(\xi_{t+1}|\xi_t, \theta) = \lambda_2(\xi_{2,t+1}|\xi_{1,t+1}, \xi_t, \theta)\lambda_1(\xi_{1,t+1}|\xi_t, \theta), \tag{48}$$

with corresponding conditional CDFs denoted by P_1 and P_2. Now we can generate simulations of $\{\xi_t\}$ that will be smooth function of θ just as in the univariate case, except that in the two-dimensional case we need to generate two random $U(0,1)$ variables $u_t = (u_{1,t}, u_{2,t})$ for each time period simulated. For example to generate a simulated value of $\xi_1 = (\xi_{1,1}, \xi_{2,1})$ we compute

$$
\begin{aligned}
\xi_{1,1} &= P_1^{-1}(u_{1,1}|\xi_0, \theta) \\
\xi_{2,1} &= P_2^{-1}(u_{2,1}|\xi_{1,2}, \xi_0, \theta).
\end{aligned} \tag{49}
$$

Clearly the resulting realization for ξ_1 is of the form $\xi_1(u_1, \xi_0, \theta)$ and will be a smooth function of θ provided that P_1 and P_2 are smooth functions of (ξ, θ). Continuing recursively we have:

$$
\begin{aligned}
\xi_{1,t+1} &= P_1^{-1}(u_{1,t+1}|\xi_t, \theta) \\
\xi_{2,t+1} &= P_2^{-1}(u_{2,t+1}|\xi_{1,t+1}, \xi_t, \theta).
\end{aligned} \tag{50}
$$

The resulting simulations take the form $\{\xi_t(\{u_s\}_{s \leq t}, \theta, \xi_0)\}$ and will be smooth functions of θ provided that P_1 and P_2 are smooth functions of their conditioning arguments (ξ, θ).

However, it is not possible to construct a smooth simulator for the full vector $\xi_t = (p_t, p_t^r, q_t, q_t^o, x_t)$ due to the inherent discontinuity in q_t^o created by the fact that the optimal ordering strategy follows an (S, s) rule, by Theorem 1 in section 2. If q_t is sufficiently close to the optimal order threshold $s(p_t, x_t, \theta)$, small changes in θ can result in perturbations of the simulated values of p_t, x_t and q_t that push q_t above or below $s(p_t, x_t, \theta)$ resulting in discontinuous jumps in q_t^o from $q_t^0 = 0$ to $q_t^o = S(p_t, x_t, \theta) - q_t$ as we discussed in section 2. However the discontinuities along the lower $s(p, x)$ band are of measure zero within the full state space, and the Continuous Mapping Theorem implies that for any continuous, bounded mapping $h : R^J \to R^M$ we have $E\{h(\tilde{\xi})|\xi, \theta\} = \int_{\xi'} h(\xi')\lambda(\xi'|\xi, \theta)d\xi'$ will be a continuous function of θ provided the transition probability λ is continuous in θ.

We have actually imposed a stronger assumption, Assumption 2 in the previous section, that not only guarantees that this conditional expectation is a continuous function of θ but also that it is a continuously differentiable function of θ. This assumption is necessary to establish the asymptotic normality of the

27

MSM estimator, whereas continuity alone is all that is required (along with additional assumptions guaranteeing geometric ergodicity of the $\{\xi_t(\theta)\}$ process uniformly for $\theta \in \Theta$ given in Duffie and Singleton (1993)) to establish the consistency of the MSM estimator.[8]

Finally, as part of the recursive simulation of $\{\xi_t(\{u_s\}_{s \leq t}, \theta, \xi_0)\}$ we censor the p_t component in the same way it is censored in the data we observe. Let p_t denote the uncensored simulated wholesale price series (from the smooth univariate simulator described in equation (47) above), and let p_t^c be the corresponding censored wholesale price given in equation (35) of section 3. Using the DP solution, we calculate the (S, s) bands for each θ and set $q_t^o = 0$ if $q_t \geq s(p_t, x_t, \theta)$, and set $q_t^o = S(p_t, x_t, \theta) - q_t$ otherwise, and then based on the value of q_t^o we censor $\{p_t\}$ according to equation (35).

Now consider using a single simulated realization of $\{\xi_t(\{u_s\}_{s \leq t}, \theta, \xi_0)\}$ to form a simulated sample moment $h_T(\{u_s\}_{s \leq T}, \xi_0, \theta)$ given by

$$h_T(\{u_s\}_{s \leq T}, \xi_0, \theta) = \frac{1}{T} \sum_{t=1}^{T} h\left(\xi_t(\{u_s\}_{s \leq t}, \theta, \xi_0), \xi_{t-1}(\{u_s\}_{s \leq t-1}, \theta, \xi_0)\right). \tag{51}$$

Let $(\{u_s^1\}_{s \leq T}, \ldots, \{u_s^n\}_{s \leq T})$ denote n IID $T \times 1$ sequences of $U(0,1)$ random vectors used to generate the n independent realizations of the endogenously sampled process $\{\xi_t(\{u_s^i\}_{s \leq t}, \theta, \xi_0)\}$, $i = 1, \ldots, n$. Define $h_{n,T}(\theta)$ as the average of r independent time averages $h_T(\{u_s^i\}_{s \leq T}, \xi_0, \theta)$

$$h_{n,T}(\theta) = \frac{1}{n} \sum_{i=1}^{n} h_T(\{u_s^i\}_{s \leq T}, \xi_0, \theta). \tag{52}$$

Definition 6: *The* **method of simulated moments (MSM) estimator** $\hat{\theta}_T$ *is defined by:*

$$\hat{\theta}_T = \underset{\theta \in \Theta}{argmin}(h_{n,T}(\theta) - h_T)' W_T (h_{n,T}(\theta) - h_T), \tag{53}$$

where W_T is an $M \times M$ positive definite weighting matrix.

Note that in the definition of the MSM it is crucial for the *IID* $U(0,1)$ random vectors $\{u_s^i\}$ for $s = 1, \ldots, T$ and $i = 1, \ldots, n$ to be drawn *only once* at the start of the estimation process and *remain fixed* as we search over θ for a global $\hat{\theta}_T$ of the MSM criterion (53). This is needed to guarantee the stochastic equicontinuity of the MSM criterion as a function of θ which is the key to establishing the consistency and asymptotic normality of the MSM estimator $\hat{\theta}_T$.

---

[8]Similar issues arose in McFadden's (1989) original analysis and application to discrete choice models: the simulated choices are not everywhere continuous in θ and can jump from one choice to another at points where simulated utilities of two different choices are both close to optimal. However as long as the additive random component of utility is continuously distributed, the probability of a ties is zero, so the choice probabilities (equal to the expectation of the discontinuous indicator for the utility maximizing choice over the continuous multivariate distribution of utility shocks) are continuous in θ and in fact continuously differentiable in $\theta \in int(\Theta)$.

In order to simplify the asymptotic analysis, we initially assume that we have a correct parametric specification of the endogenous sampling problem. That is we make

Assumption 3: *The parametric model introduced in section 2 is correctly specified, i.e., there is a $\theta^* \in \Theta$ such that:*

$$\{\xi_t(\{u_s\}_{s \leq t}, \theta^*, \xi_0)\} \sim \{\xi_t | \xi_0\} \tag{54}$$

that is, when $\theta = \theta^*$, the simulated sequence initialized from the observed value ξ_0 has the same conditional probability distribution as the observed sequence $\{\xi_t\}$ given ξ_0.

We believe that it is possible to relax assumption 3 to allow the parametric model to be misspecified, following an analysis similar to that of Hall and Inoue (2003) who characterized the asymptotic properties of the GMM estimator in the misspecified case. We conjecture that their analysis will also apply to the case of MSM estimation and that the asymptotic properties of the MSM estimator that we derive for the correctly specified case will still hold, except that now θ^* is interpreted as the value of θ the minimizes the distance between the moments of the true data generating process and the parametric simulated process, where the expectation is taken in the limit as both $S \to \infty$ and $T \to \infty$.[9]

We now sketch the derivation of the asymptotic distribution of the MSM estimator, listing the key assumptions and showing how its asymptotic variance depends on the number of simulations n. By Assumption 2, we have that for any $\theta \in \Theta$ the simulated process $\{\xi_t(\{u_s\}_{s \leq t}, \theta, \xi_0)\}$ is ergodic with unique invariant probability $\psi(\xi|\theta)$. Define the functions $E\{h|\theta\}$, $\nabla E\{h|\theta\}$, and $\nabla h_{S,T}(\theta)$ by:

$$
\begin{aligned}
E\{h|\theta\} &= \int h(\xi', \xi) \lambda(\xi'|\xi, \theta) \psi(d\xi|\theta) d\xi' \\
\nabla E\{h|\theta\} &= \frac{\partial}{\partial \theta} E\{h|\theta\} \\
\nabla h_{S,T}(\theta) &= \frac{\partial}{\partial \theta} h_{S,T}(\theta).
\end{aligned}
\tag{55}
$$

By Assumption 2 these gradients exist and are continuous in θ for $\theta \in \text{int}(\Theta)$.

Assumption 4: θ^* *is* **identified***; that is, if $\theta \neq \theta^*$, then $E\{h|\theta\} \neq E\{h|\theta^*\} = E\{h\}$. Furthermore, $rank(\nabla E\{h|\theta\}) = D \leq M$ and $\lim_{T \to \infty} W_T = W$ with probability 1 where W is a $M \times M$ positive definite matrix.*

---

[9] When there is misspecification, the standard formula for the asymptotic covariance matrix when the model is correctly specified will generally not be consistent when the model is misspecified. However similar to the case of maximum likelihood estimation of misspecified models (White, 1982), there are alternative estimators of the asymptotic covariance matrix which are consistent when the model is misspecified and when the model is correctly specified.

The consistency of the MSM estimator can be established by providing appropriate regularity conditions under which the simulated process is uniformly ergodic, i.e., under which with probability 1 we have

$$\lim_{T \to \infty} \sup_{\theta \in \Theta} \left| (h_{n,T}(\theta) - h_T)' W_T (h_{n,T}(\theta) - h_T) - (E\{h|\theta\} - E\{h|\theta^*\})' W (E\{h|\theta\} - E\{h|\theta^*\}) \right| = 0. \quad (56)$$

Assumption 3 guarantees that the unique minimizer of $(E\{h|\theta\} - E\{h|\theta^*\})' W (E\{h|\theta\} - E\{h|\theta^*\})$ is θ^*, and this combined with the uniform consistency result implies the consistency of $\hat{\theta}_T$. The asymptotic normality of $\hat{\theta}_T$ can be established by a Taylor series expansion of the first order condition

$$(h_{n,T}(\hat{\theta}_T) - h_T)' W_T \nabla h_{n,T}(\hat{\theta}_T) = 0. \quad (57)$$

Expanding $h_{n,T}(\hat{\theta}_T)$ about $\theta = \theta^*$ we have

$$h_{n,T}(\hat{\theta}_T) = h_{n,T}(\theta^*) + \nabla h_{n,T}(\tilde{\theta}_T)(\hat{\theta}_T - \theta^*), \quad (58)$$

where $\tilde{\theta}_T$ denotes a vector that is (elementwise) on the line segment between $\hat{\theta}_T$ and θ^*. Substituting (58) into the first order condition for $\hat{\theta}_T$ in equation (57) and solving for $(\hat{\theta}_T - \theta^*)$ we obtain

$$(\hat{\theta}_T - \theta^*) = - \left[\nabla h_{n,T}(\tilde{\theta}_T)' W_T \nabla h_{n,T}(\hat{\theta}_T) \right]^{-1} \nabla h_{n,T}(\hat{\theta}_T)' W_T \left[h_{n,T}(\theta^*) - h_T \right], \quad (59)$$

where we assume that $\left[\nabla h_{n,T}(\tilde{\theta}_T)' W_T \nabla h_{n,T}(\hat{\theta}_T) \right]$ is invertible, which will be the case with probability 1 for sufficiently large T due to assumptions 3 and 4. Now multiply both sides of equation (59) by $\sqrt{T}$ and apply the Central Limit Theorem for geometrically ergodic Markov prcoesses to the difference $\sqrt{T}[h_{n,T}(\theta^*) - h_T]$ to obtain

$$\sqrt{T} \left[h_{n,T}(\theta^*) - h_T \right] \Longrightarrow N \left(0, (1 + 1/n)\Omega(h, \theta^*) \right). \quad (60)$$

To understand this result, note that $h_{n,T}(\theta^*)$ is an average of n independent realizations of $\{\xi_t(\{u_s^i\}_{s \leq t}, \theta, \xi_0)\}$, $i = 1, \ldots, n$ which by assumption 3 has the same distribution as $\{\xi_t\}$. As a result each of the terms entering $h_{n,T}(\theta^*)$, $h_T(\{u^i\}, \theta^*)$, has the same probability distribution as h_T and are distributed independently of h_T. The Central Limit Theorem applied to h_T yields

$$\sqrt{T} \left[h_T - E\{h|\theta^*\} \right] \Longrightarrow N \left(0, \Omega(h, \theta^*) \right). \quad (61)$$

Similarly, for each $i = 1, \ldots, n$ we have

$$\sqrt{T} \left[h_T(\{u_s^i\}_{s \leq T}, \theta^*) - E\{h|\theta^*\} \right] \Longrightarrow N \left(0, \Omega(h, \theta^*) \right). \quad (62)$$

Note that

$$[h_{n,T}(\theta^*) - h_T] = \left[\frac{1}{n} \sum_{i=1}^{n} [h_T(\{u_s^i\}_{s \leq T}, \theta^*) - E\{h|\theta^*\}] + E\{h|\theta^*\} - h_T \right], \tag{63}$$

so that we have

$$\sqrt{T}[h_{n,T}(\theta^*) - h_T] \Longrightarrow \left[\frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i + \tilde{X}_0 \right], \tag{64}$$

where $(\tilde{X}_0, \tilde{X}_1, \ldots, X_n)$ are *IID* $N(0, \Omega(h, \theta^*))$ random vectors. It follows immediately that the asymptotic distribution of $\sqrt{T}[h_{n,T}(\theta^*) - h_T]$ is $N(0, (1 + 1/n)\Omega(h, \theta^*))$. Using this result and equation (59) we have

$$\sqrt{T}\left[\hat{\theta}_T - \theta^*\right] \Longrightarrow N(0, (1 + 1/n)\Lambda_1^{-1}\Lambda_2\Lambda_1^{-1}), \tag{65}$$

where

$$\begin{aligned} \Lambda_1 &= [\nabla E\{h|\theta^*\}]' W \nabla [E\{h|\theta^*\}] \\ \Lambda_2 &= [\nabla E\{h|\theta^*\}]' W \Omega(h, \theta^*) W [\nabla E\{h|\theta^*\}]. \end{aligned} \tag{66}$$

Borrowing from the literature on generalized method of moments estimation, the optimal weight matrix $W = [\Omega(h, \theta^*)]^{-1}$ results in an MSM estimator with minimal variance. In this case the asymptotic distribution of $\hat{\theta}_T$ simplifies to:

Theorem 5: *Consider the MSM estimator $\hat{\theta}_T$ formed using a weighting matrix W_T equal to the inverse of any consistent estimator of $\Omega(h, \theta^*) = \Omega(h)$ such as the Newey-West estimator. Then we have:*

$$\sqrt{T}[\hat{\theta}_T - \theta^*] \Longrightarrow N(0, (1 + 1/n)\Lambda^{-1}) \tag{67}$$

where:

$$\Lambda = \left[\nabla E\{h|\theta^*\}' [\Omega(h, \theta^*)]^{-1} \nabla E\{h|\theta^*\} \right]. \tag{68}$$

The most important point to note about this result is that the penalty to forming an MSM estimator using only a single realization $n = 1$ of the endogenously sampled process $\{\xi_t(\{u_s\}_{s \leq t}, \theta, \xi_0)\}$ is fairly small. The variance of the resulting estimator is only twice as large as an estimator that computes the expectation of $h_T(\{u\}, \theta)$ exactly, such as would be done via Monte Carlo integration when $n \to \infty$.

The MSM estimator can be implemented in practice by solving

$$\hat{\theta}_T = \underset{\theta \in \Theta}{argmin}(h_{S,T}(\theta) - h_T)' \left[\hat{\Omega}(h) \right]^{-1} (h_{S,T}(\theta) - h_T), \tag{69}$$

where $\hat{\Omega}(h)$ is the Newey-West (1987) estimator of $\Omega(h)$ given in equation (44). Thus, it is not necessary to recompute the optimal weighting matrix $\Omega(h, \theta)$ each time the parameter θ is updated, but instead we

can pre-compute the empirical covariance matrix of the moments $\hat{\Omega}(h)$ using the Newey-West estimator at the start of the estimation process and the inverse of this matrix is the required consistent estimator of the optimal weighting matrix that can be used for the entire estimation process.

Finally, we note that it is appears that it is possible to relax assumption 3 that the parametric model is correctly specified. As long as assumption 2 holds, there will still exist well defined limiting moments for the simulated process, $E\{h|\theta\}$, for each $\theta \in \Theta$. Define θ^* as the "psuedo-true" parameter value that minimizes the distance between the simulated model and the true data generating process:

$$\theta^* = \underset{\theta \in \Theta}{argmin}\left[E\{h|\theta\} - E\{h\}\right]' W\left[E\{h|\theta\} - E\{h\}\right], \tag{70}$$

where $E\{h\}$ denotes the limit of h_T as $T \to \infty$ for the true data generating process. We believe that the results of Hall and Inoue (2003, 2007) on GMM estimation of misspecified models can be adapted to establish the asymptotic distribution of the MSM estimator in the misspecified case. However given the space constraints we leave this topic, together with Monte Carlo tests and an empirical application of the MSM estimator for a misspecified model, as a topic for subsequent research.

5   Empirical Application

To illustrate the MSM estimator, we consider a special case of the model in which there are no additional state variables, x. In this case, the (S,s) bands are only functions of the current wholesale price, $S(p)$ and $s(p)$. We estimate the model twice using actual data for two products, both large construction grade plates, from the steel service center. Using these two estimated models, we decompose the firm's profits by product into four components. We use this decomposition to infer the share of the firm's profits that are due to markups paid by retail customers and the share due to price speculation. We also use this decomposition to compare the general manager's purchasing decisions to the model's trading rules.

5.1   A special case of the model

Consider a version of the model in which the firm's general manager solves the following problem:

$$\max_{\{q_t^o\}} E \sum_{t=0}^{\infty} \beta^t \left\{ p_t^r q_t^s - c^o(q_t^o, p_t) - c^h(q_t + q_t^o, p_t) \right\} \tag{71}$$

subject to (3) and (4), and where

$$c^o(q_t^o, p_t) = \begin{cases} p_t q_t^o + K & \text{if} \quad q_t^o > 0 \\ 0 & \text{otherwise,} \end{cases} \text{ and }$$

$$c^h(q_t + q_t^o, p_t) = \phi_1(q_t + q_t^o) + \phi_2(q_t + q_t^o)^2.$$

As before, the manager takes the wholesale price p_t and quantity demanded q_t^r as given. The manager knows p_t before deciding q_t^o. The manager then draws q_t^r. The order cost function, $c^o(\cdot, \cdot)$ and holding cost function, $c^h(\cdot)$, are described in section 2. The holding cost function is quadratic so the marginal holding cost is increasing in the level of inventories.

We assume the wholesale price evolves according to a truncated lognormal $AR(1)$ process:

$$log(p_{t+1}) = \mu_p + \lambda_p log(p_t) + w_t^p \tag{72}$$

where w_t^p is an IID $N(0, \sigma_p^2)$ sequence. The firm sets the retail price by using a fixed linear markup rule over the current wholesale price:

$$p_t^r = \alpha_0 + \alpha_1 p_t. \tag{73}$$

The firm draws a quantity demanded q_t^r each period from a mixed truncated lognormal distribution conditional on p_t That is, with probability η, $q_t^d = 0$, and with probability $1 - \eta$, q_t^d is drawn from a truncated normal distribution with location parameter $\mu_q(p) = \mu_p - \varsigma log(p_t)$. Both ς, the price elasticity of demand, and η are fixed, time-invariant constants. Let θ denote the $(D \times 1)$ parameter vector to be estimated: $\theta = \{r, 1 - \eta, \alpha_0, \alpha_1, \mu_q, \sigma_q, \varsigma, K, \phi_1, \phi_2, \lambda_p, \mu_p, \sigma_p\}$.

5.2 Computation

The MSM estimation procedure requires us to solve for the optimal inventory investment rule each time we evaluate the criterion for a new parameter vector. We solve the model by the method of discrete policy iteration (DPI). The DPI algorithm is equivalent to solving for the fixed point of Bellman's equation, which can be stated abstractly as

$$V = \Gamma(V) \tag{74}$$

by the Newton-Kantorovich (NK) algorithm which converts the fixed point into an equivalent zero of the nonlinear operator $F(V) = (I - \Gamma)(V) = 0$ resulting in iterations of the form

$$V_{t+1} = V_t - [I - \Gamma'(V_t)]^{-1}[V_t - \Gamma(V_t)]. \tag{75}$$

where $\Gamma'(V_t)$ is the Gateaux derivative of the nonlinear operator Γ. An alternative name for the NK iterations is *policy iteration*.[10]

---

[10]See Rust (1996) for further discussion of the equivalence between the policy iteration method introduced by Howard (1960) and NK iterations.

Though the NK iterations are well defined even when V is an infinite-dimensional object (i.e. a function in a Banach space of continuous, bounded functions of (p,q,x) under the supremum norm), to compute the NK iterations in practice we rely on discretizing the (p,q,x) state space into grids involving a finite number of points n. Then $V_t \in R^n$ and we can use multidimensional interpolation to approximate $V_t(p,q,x)$ at any point (p,q,x) that is not on the pre-defined grid. Under this discretization the linear operator $[I - \Gamma'(V)]$ is equivalent to an $n \times n$ matrix and the linear operator $\Gamma'(V)$ is given by

$$\Gamma'(V) = \beta P(V) \tag{76}$$

where $P(V)$ is an $n \times n$ transition probability matrix that implements that conditional expectation operator for the controlled process $\{p_t, q_t, x_t\}$ at each of the n grid points. Typically the NK/policy iteration algorithm converges in a relatively small number of iterations: in our case we found it nearly always converges to very close to the true solution in 15 or fewer iterations.

The DPI algorithm can be further speeded up by employing the special structure of the (S,s) policy. In the "policy improvement step" of policy iteration, we need to compute the optimal order quantity $q^o(p,q,x)$ for each (p,q,x) coordinate in the pre-defined grid of n points over the state space. Since this is a continuous optimization problem, this continuous optimization can be slow, especially when n is large since the optimization must be repeated at each of the n (p,q,x) grid points. However because we know the optimal inventory policy is of the (S,s) form, we only have to search for the optimal order quantity $S(p,x)$ at a subset of grid points of the form $(p,0,x)$, i.e. where the firm has zero inventory. Then using this solution $S(p,x) = q^o(p,0,x)$, we calculate the $s(p,x)$ threshold numerically using Newton's method, using equation (20). Once we know the two numbers $S(p,x)$ and $s(p,x)$ we can calculate $q^o(p,q,x)$ for all other grid points (p,q,x) using equation (18) of Theorem 1.

In the results we present below we did not include any other state variables x so our state space consists only of the two continuous variables (p,q). We used a discrete, uniformly spaced grid of 40 points for the p state variable, and 25 non-uniformly spaced grid points for the q state variable (i.e. with the grid points spaced more closely together when q is closer to 0 and more widely spaced when q is close to the upper bound $\overline{q} = 55000$). The resulting grid has $n = 25 * 40 = 1000$ points, so each NK/Policy iteration step requires the solution of a 1000×1000 system of linear equations, but only 40 continuous optimizations to find $S(p)$ are required at each policy improvement step, so the NK/Policy iteration runs quite fast, taking less than 0.7 seconds to solve the Bellman equation (11) on a Macbook Pro computer. It actually takes longer for the computer to do $n = 100$ simulations of the inventory investment model than to solve it.

34

5.3 Estimation

We have considerable freedom in our choice of moments functions, the h vector, to use in the criterion. We match the means, variances and histograms (four of the five quintile binds) of the p, p^r, q^o, q^s, and q processes, average markups conditional on the sale size, the fraction of days a purchase is made, the fraction of days a sale is made, and the mean and variance of a 4-day moving average of purchase prices for a total of 42 moment conditions. We set the number of simulations, S, to 100. The Newey-West estimator of $\Omega(h)$ in equation (44) requires a bandwidth parameter (number of lags l to estimate autocovariances) to form an consistent estimator $\hat{\Omega}(h)$ of $\Omega(h)$ as $T \to \infty$. Consistent estimation requires the number of lags to be a function of T, $l(T)$, and satisfy $l(T) \to \infty$ as $T \to \infty$ and

$$\lim_{T \to \infty} \frac{l(T)}{T^{1/4}} = 0. \tag{77}$$

We used the bandwidth $l(T) = T^{1/5}$ and for our sample size of $T = 1647$ this implies that $l(T)$ should be either 4 or 5. We experimented with these values and several larger and smaller values. The auto-covariances of the MSM residuals die out rather quickly so the results with $l = 5$ or more lags are very similar to results with $l = 4$. However reducing l below 4 affects the estimated standard errors of the MSM significantly and with $l = 0$ (i.e. when $\hat{\Omega}(h)$ is the standard variance covariance matrix estimator that is consistent for *IID* data) the MSM standard errors using formula (68) appeared to be significantly underestimated. Thus, the results we present below are for $l(T) = 4$.

Computing histogram bins requires the use of indicator functions. To ensure these indicator functions do not add discontinuities into the criterion function, we use the exponential transformation of the indicator functions. In general, the criterion is (at least visually) a smooth function of the parameters. However we did find regions in the parameter space in which concentrated "slices" of the criterion function had "steps" and "cliffs." Therefore to verify that MATLAB's constrained minimization routine `fmincon.m` found a global minimum in each case, we visually inspected concentrated slices of the criterion function after each estimation.

We now estimate the model for two products independently. In table 1 we report the point estimates and standard errors for the parameters of the model for two large construction grade steel plates: one is 3/4 inch thick, the other is 1 inch thick. Although we estimated the parameters for each of these products independently, it is reassuring that several of the point estimates are similar across the two products. It is reasonable to expect that the parameters, r, η, K, α_0, α_1, λ_p, ς, ϕ_1 and ϕ_2 would be quite similar, if not

	3/4 inch plate		1 inch plate	
parameter	point estimate	standard error	point estimate	standard error
r	0.0794	0.0070	0.0746	0.0120
$1-\eta$	0.581	0.002	0.571	0.003
α_0	0.553	0.070	0.550	0.045
α_1	1.041	0.005	1.014	0.002
μ_q	6.011	0.024	6.225	0.031
σ_q	1.152	0.011	1.322	0.021
ς	0.560	0.010	0.525	0.011
K	3.05	0.33	3.36	0.94
ϕ_1	6.56×10^{-3}	1.71×10^{-4}	6.01×10^{-3}	7.11×10^{-4}
ϕ_2	1.12×10^{-8}	1.83×10^{-7}	1.06×10^{-8}	1.03×10^{-7}
λ_p	0.978	4.37×10^{-5}	0.979	4.48×10^{-5}
μ_p	0.0613	5.68×10^{-5}	0.0595	5.68×10^{-5}
σ_p	0.0192	2.97×10^{-4}	0.0204	3.64×10^{-4}

Table 1: Estimation Results Using Data for 3/4 and 1 inch Plates

identical, across products.[11] In general this is case.

Our point estimates of the annual interest rate r, 7.9% for the 3/4 inch plate and 7.5% for the 1 inch plate, align with our expectations. The general manager would not provide us with specific data on the firm's borrowing and lending (many sales involve trade credit), but told us that one and three-quarter points over a short-term LIBOR rate was a good estimate of the interest rate they faced. The average 3-month LIBOR rate over the period studied is about 5.75%, which implies an average annual borrowing rate for the firm of about 7.5%.

The demand side parameters are pinned down by the frequency and distribution of sales. The point estimates of the customer arrival probability, $1-\eta$, of roughly 0.6 matches the share of days that a sale occurs. The estimates of ς of little over 1/2 imply that demand is relatively inelastic, but that the quantity demanded is sensitive to the price.

Our estimate of the fixed cost, K, is just a little over \$3 per order. We asked the general manager for his estimate of K. He stated that the cost of placing an order was the value of the time it took for the general manager and his administrative assistant to complete the paperwork; he pointed out that the firm was paying for their time whether an order was made or not.

---

[11]We could have estimated the model jointly across the two products, constraining these value to be equal across products.
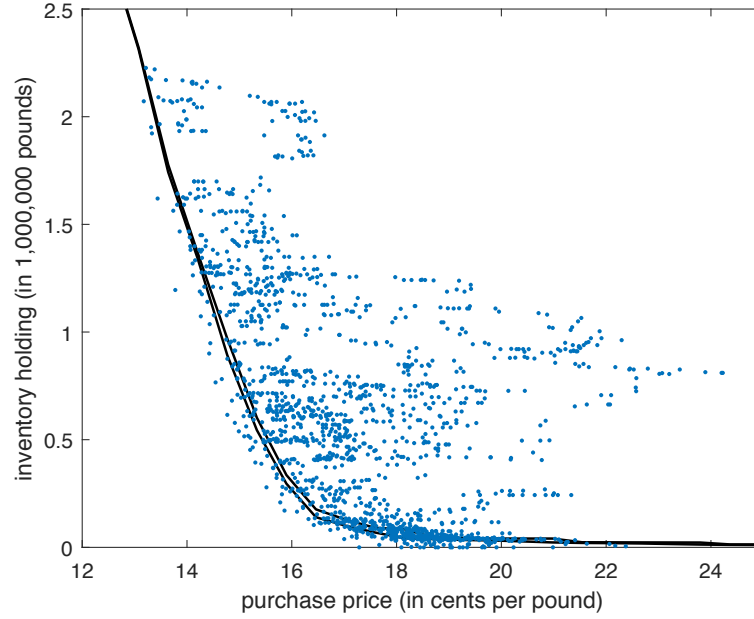
Figure 3: Scatterplot of Purchase Price and Inventory Holding Pairs from a Simulation for 3/4 inch Plate. The solid lines are the $S(p)$ and $s(p)$ bands from the model.

The endogenous sampling problem is illustrated in figures 3, 4, and 5. In figure 3 we plot we the $S(p)$ and $s(p)$ bands derived from the optimal decision rules for the manager's problem using the estimated parameter vector for 3/4 inch plate. Due to the fixed costs of ordering, the $S(p)$ band is strictly above the $s(p)$ band although the difference between the two bands decreases as the price increases. In other words, the minimum order size is a decreasing function of the price. In figure 3 we also scatterplot a set of simulated state space pairs (p_t, q_t). According to the firm's optimal trading rule, the firm only makes purchases when the (p_t, q_t) pair is below the $s(p)$ band (in the southwest corner of the graph). In the simulation presented, this occurs less than 15 percent of time.

In table 2 we report a subset of the 42 moments we used in the SMD estimator along with the average moments from 100 simulations of the model. For both types of the plates, the means of the buy prices and sell prices are within a standard deviation of their counterpart in the data. However, note that the model underestimates the buy price and overestimates the sell price; hence the model underestimates the markups. Further the model fails to capture the negative relationship between the markup and sales quantity. In the data, the larger the sale, the smaller the markup. This pattern holds in the model though the size of these "quantity discounts" is very small.
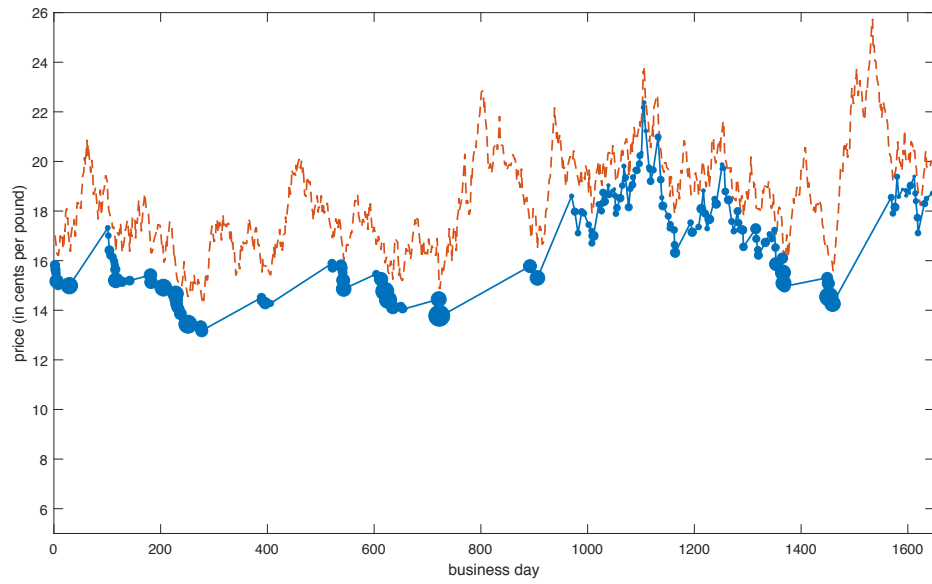
Figure 4: Censored (solid line) and uncensored (dotted line) purchase prices, p_t from a simulation for 3/4 inch plate.
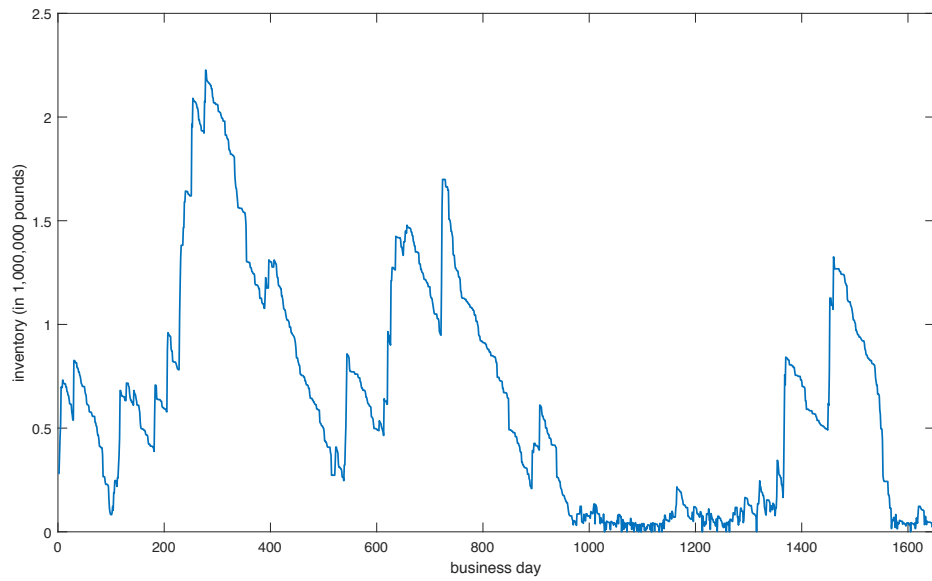

Figure 5: Simulated inventory data from the estimated model for 3/4 inch plate.

38

moment	3/4 inch plate		1 inch plate	
	model simulation	data	model simulation	data
mean(buy price)	17.61	16.78	17.55	16.95
var(buy price)	3.41	4.37	3.88	5.58
mean(sell price)	18.83	19.09	18.36	18.96
var(sell price)	2.73	5.37	3.01	5.58
mean(purchase size)	674	1138	1066	1393
var(purchases)	50	370	155	629
mean(sale size)	156	260	244	313
var(sales)	5	5	21	12
mean(inventory)	4383	5247	7947	7561
mean(markup)—small sale	1.27	2.14	0.80	2.02
mean(markup)—medium sale	1.27	1.71	0.80	1.79
mean(markup)—large sale	1.26	1.63	0.80	1.59
share of days with buys	0.142	0.144	0.140	0.140
share of days with sales	0.582	0.605	0.572	0.594
$\chi^2(28)$	1623		1076	

Table 2: Comparison of Selected Moments from 100 Simulations and Data for 3/4 in Plate and 1 inch Plate

Prices are in cents per pound. Quantities are in hundredweight (i.e. 100s of pounds).

In table 2 we also report the minimized MSM estimation criterion. Although both models are formally rejected, the models at the estimated parameter values capture several of the salient features of the inventory and price data. The model correctly replicates the fact that the average purchase is four times the size of the average sale and days with sales are four times as frequent as days with purchases. Figures 3, 4, and 5 highlight some of these strengths of the model. First, in the data purchases are made infrequently. Figure 4 presents the censored and uncensored purchase price series, p_t. The solid line is the analogue of what we observe in the data: we linearly interpolated between the prices at which transactions took place; the dotted line includes the unobserved prices at which no transactions occurred. During periods of low prices (e.g. days 189-302, 613-755 and 1359-1473) the firm aggressively made purchases to build up large levels of inventories. The large levels of inventories were slowly drawn down as prices inevitably rose; for example, there were only nine purchases made between business days 302 and 413. Thus after exploiting a low price opportunity, the firm may subsequently make no new purchases for many days. Second, we observe both small and large purchases in the data. Again this can been in both graphs. In figure 3 when the (p_t, q_t) pair (dot) is below the $s(p)$ band, the size of the order is the vertical distance between the $S(p)$

band and the (p_t, q_t) pair (dot). When the purchase price is less than 16 cents per pound, we observe both large and small orders. When the purchase price is above 18 cents per pound we only observe small orders. In figure 4, the size of the marker is proportional to the size of the purchase. Again one can see that the model predicts relatively large purchases when the price is low and relatively small purchases when the price is high. Third, in the data we observe periods with high levels of inventories and periods with low levels of inventories. From the scatterplot in figure 3 and the time path of inventories plotted in figure 5 we can see that the model predicts that inventory levels will vary over the sample between almost zero and 2.0 million pounds.

The main shortcoming of the estimation is our inability to match the downward trend of the price process that we see in almost all of the firm's products. As illustrated in figure 1 the wholesale price for the 3/4 inch plate fell from 20 cents per pound in 1997 to about 12 cents per pound in 2002. No such trend is evident in simulations such as the one presented in figures 4 and 5. In our model, prices are stationary though highly persistent. Consequently, as can been seen in the (S, s) bands plotted in figure 3 the optimal decision rules imply counterfactually that the firm should make only small purchases and hold low levels of inventories whenever the procurement price is above 17 cents per pound. From figures 1 and 2 we see that, for the 3/4 inch plate, the firm made large purchases around 18.5 cents per pound in April 1998, and around 15 dollars per hundred-weight in the later part of the sample.

An often suggested solution to this trend problem is that we assume prices follow a random walk. However if we assume the price process follows a (or a very nearly) random walk, the optimal decision rules imply frequent small- to medium-size orders such that the inventory level fluctuates closely around a fixed target level. A version of the model which assumes p_t follows a random walk will not imply the large variation in inventory holdings that we see in the data. A second potential solution is to detrend the data. However when we first started working on this project, no one we talked to expected steel prices to decline 40% in four years. To some extent we are just working with too short a sample period. A third candidate solution is to add an additional macroeconomic state variable. Such a variable could allow for "high price" regimes and "low price" regimes. As we discuss below, we view this third solution as the most promising.

5.4 A profit decomposition exercise

Finally, we use simulations of the estimated model to deduce the relative importance of capital gains versus markups for the overall profitability of the firm. By substituting the law of motion for inventories (4) into

the firm's objective function, (71), the discounted present value of the firm's profits can be expressed by

$$\sum_{t=1}^{T} \beta^t \pi(p_t, p_t^r, q_t^r, q_t + q_t^o) = \sum_{t=1}^{T} \beta^t (p_t^r - p_t) q_t^s + q_1 p_1 + \sum_{t=2}^{T} \beta^t (p_t - (1+r) p_{t-1}) q_t -$$
$$\sum_{t=1}^{T} \beta^t I(q_t^o) K - \sum_{t=1}^{T} \beta^t c^h(q_t + q_t^o, p_t). \tag{78}$$

The first term on the right hand side of equation (78) can be interpreted as the discounted present value of the markup paid by the firm's retail customers over the current wholesale price while the third term can be interpreted as the discounted present value of the capital gains or losses from holding the steel from period $t-1$ into period t. The fourth and fifth terms are the discounted present values of the order costs and the holding costs incurred by the firm over the sample period.

Since this decomposition depends on the wholesale price path between purchases, we simulate between purchase dates via importance sampling. That is, for each interval between successive purchase dates, we simulate wholesale price paths that are consistent with the estimated law of motion (72) and the observed purchase prices at the beginning and end of the interval. Since our theory implies that the firm places an order anytime the quantity falls below the order threshold, $s(p)$, we truncate the simulated price process by rejecting any paths such that $q_t < s(p_t)$ for any draw within the simulated realizations. We discuss our simulation method in more detail in the appendix.

We first employ this decomposition to evaluate the general manager's actual performance over the six-and-a-half year sample period for the two plates. For a given interpolated price series, we decomposed the firm's profits using the actual data for q_t, q_t^s, and q_t^o and our point estimates for r, K, ϕ_1, and ϕ_2. In table 3 we report the average decomposition from 100 simulated wholesale price paths. As discussed in the introduction, the price of steel fell steadily over the sample period. Never the less, by our accounting, the firm made \$396,000 (3/4 inch plate) and \$459,000 (1 inch plate) from the markup and capital gains net holding and order costs on each of these two products over the six-and-a-half year period.[12] Ignoring the fixed order and holding costs, about 90 percent (3/4 inch plate) and 86 percent (1 inch plate) of these profits came from the markup, while the remaining 10 and 14 percent came from capital gains. We find it remarkable and evidence of the general manager's acumen in steel trading that the firm made positive capital gains over this period despite the price of steel falling about 40 percent. While the firm's success in price speculating is good for its profits, it increases the potential biases from failing to account for the endogeneity of the sampling process.

---

[12]Profits are discounted back to July 1, 1997.
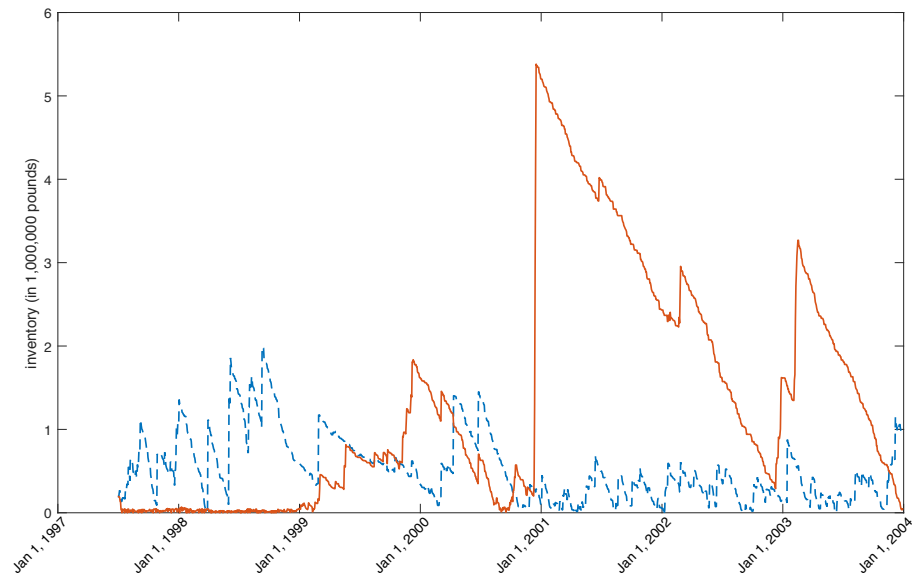
Figure 6: Actual (dashed line) and counter-factual (solid line) inventory holdings for 3/4 inch plate.
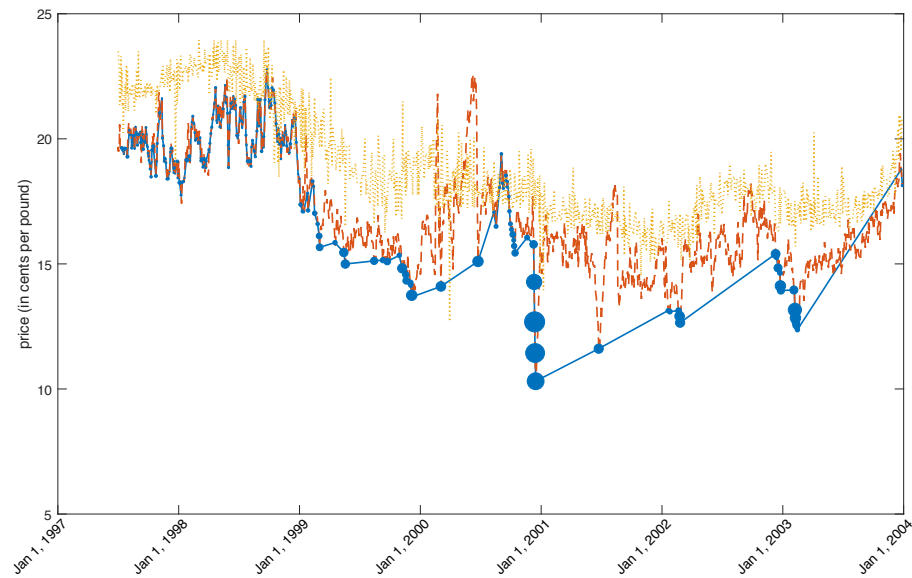

Figure 7: Counter-factual uncensored purchase prices (dotted line), censored purchase prices (solid line), and retail prices (dashed line) for 3/4 inch plate. For the censored purchase price series, the size of the marker is proportional to the size of the purchase.

42

3/4 inch plate	G.M.'s actual Performance			Model's Policy Prescription		
markup	$401,240	(15,034)	90%	$330,244	(15,134)	60%
capital gain	45,682	(16,160)	10%	223,415	(9,462)	40%
holding cost	-50,214	(0)		-122,592	(3,359)	
order costs	-606	(0)		-743	(27)	
total profits	396,102	(1,739)		430,324	(15,861)	

1 inch plate	G.M.'s actual Performance			Model's Policy Prescription		
markup	$450,540	(15,684)	86%	$376,997	(13,769)	51%
capital gain	76,162	(16,739)	14%	365,363	(14,252)	49%
holding cost	-66,776	(0)		-197,358	(3,976)	
order costs	-665	(0)		-749	(25)	
total profits	459,261	(1,863)		544,252	(12,690)	

Table 3: Profit Decomposition for 3/4 inch Plate and 1 inch Plate Using Equation (78)
Both the actual and the counter-factual profits cover the 1647 days studied and are discounted back to the start of the sample period, July 1, 1997. The profit numbers reported are the average across 100 simulations. The numbers in parentheses are the standard deviations from the 100 simulations. Total profits are the sum of the first four rows.

As a diagnostic of our model, we compare the general manager's performance to the model's predictions. In this exercise we take as given the 100 interpolated wholesale price series, the firm's quantity demanded series, and the firm's initial level of inventories for each product. But in this case, we let the model's optimal decision rule dictate when and how much to order. Inventories follow the accumulation identity given by equation (4). As reported in table 3, had the general manager counter-factually followed the optimal order strategy implied by our model, his discounted profits from the markup would have been considerably smaller: $71,000 less for 3/4 inch plate; $73,000 less for 1 inch plate. However, his capital gains would have been considerable larger: $177,000 more for 3/4 inch plate; $289,000 more for 1 inch plate.

The model implies that the firm should aggressively price speculate. In figures 6 and 7 we plot the prices and inventory holdings for one simulation of the model. In figure 6 we plot both the actual inventory holdings along with the implied holdings under the model's decision rules. In figure 7 we plot the corresponding retail and wholesale price paths. The model's counter-factual inventory path differs considerably from the firm's actual inventory path. In the beginning of the sample, years 1997 and 1998, when

prices were high, the model implies the firm should have made frequent small purchases and held relatively low levels of inventories. As was discussed in the introduction, in April 1998 when the wholesale price of steel dropped from 20 cents per pound to 18.5 cents per pound, the firm built up its inventory of 3/4 inch plate substantially. In contrast the model does not view 18.5 cents as a particularly good price; as can be seen in the (S, s) bands plotted in figure 3, the target inventory level at 18.5 cents is around 100,000 pounds. In April 1998, the firm's inventory of 3/4 inch plate was roughly 2,000,000 pounds.

It is not until December 1999 when prices fell below 13 cents a pound that the model recommends holding more than 1,000,000 pounds of inventory. However during January and February 2000, the general manager let his inventory of 3/4 inch plate fall to almost zero. The sharp contrast between model's counter-factual inventory policy and the firm's behavior is also evident during the second half of the sample. In mid-December 2000, the general manager had an opportunity to buy steel for a little over 10 cent per pound Our model dictates that he should have purchased very large quantities at these prices.[13] In years 2001 and 2002, the firm held relatively low levels of inventories, whereas the model's recommended inventory level was often in excess of 2,000,000 pounds. Basically, the model recommends the firm's purchasing strategy should have been the opposite of what it did: the firm should have held low inventory levels in 1997, 1998 and 1999, and high inventory levels from 2001 to the start of 2004.

This counter-factual exercise is "rigged" in the model's favor in one dimension and "rigged" against the model in another. Since we used the entire sample period to both estimate the model and evaluate the model's performance, the model "knows" the mean and the standard deviations of prices and quantity demanded for the entire period. The model knows, whereas the general manager did not know, that a price of 18.5 cents per pound in the Spring of 1998 was an above-average price for the 1997-2003 period. In this way the model has an advantage over the manager. However the model is constrained to sell at most the quantity of steel that the general manager actually sold. The model does not get the opportunity make any sales the general manager might have had the option to make but decided to turn down.

While we do not report an out-of-sample comparison between our model and the general manager, if we had estimated the model through the Fall of 2001, and then used our model to dictate purchases for the firm for the Winter and Spring of 2002, our model would have outperformed the general manager. In the Fall of 2001, the firm was purchasing steel for around 10 to 12 cents per pound. We told the general

---

[13]In fairness to the general manager, he bought as much steel as he could at these prices. If we constrained the model to purchase no more steel than he actually did on those particular days, figure 6 would not be quite as dramatic; but the model would still imply that inventories should considerable higher in years 2001 and 2002 than the levels we observed in the company's records.

manager at that time that our model recommended building up inventories at these prices. He did not follow this advice since he anticipated further price declines. He argued (and to be honest, we did not disagree) that our model did not take into account the potential slowdown in the economy in the wake of the terrorist attack of September 11, 2001 and possible resulting reduction in the demand for steel. He also expected new production capacity from the Nucor Corporation to put additional downward pressure on prices. However, with the bankruptcy of Bethlehem Steel in October 2001 as well as both the anticipation of an increase and the actual increase in steel tariffs imposed by President Bush in March 2002, steel prices increased about 20 percent in the Spring of 2002 to the 12 to 14 cent range. In the Spring of 2002, we reminded the general manager that in the fall our model recommended he build up inventories. He sighed, "I wish I had."

In this case, our model "got it right" but perhaps not for the right reasons. Our model was predicting an increase in prices since our model always expects prices to return to the sample mean. Our model had no way of predicting where the economy was going. It just expected demand to be stationary; moreover, our model could provide no help in forecasting the President's actions regarding steel tariffs. In future work, we hope to add a macroeconomic state variable to the analysis. Thus we could use our model jointly with a macroeconomic forecasting model to provide conditional inventory level recommendations to the firm such as "If you expect the economy to remain strong, the model recommends holding inventories in a range from X to Y; if you expect the economy to weaken, ..."

6 Conclusion

A huge and highly influential econometric literature has been developed to enable us to do valid inference in the presence of various types of censoring, or more generally *endogenous sampling* including the hugely influential work of Heckman on "sample selection bias" (Heckman, 1979) and the work on "choice based sampling" (Heckman, Manski and McFadden, 1981). Most of the early work has been done in cross sectional contexts, but there is also a growing time series literature that recognizes and deals with the econometric problems arising from irregular sampling of financial time series (Engle and Russell, 1998). This paper contributes to this literature by focusing on the problem of both irregular and endogenous sampling of time series data on prices, in the context of commodity price speculation in the steel market. The commodity speculator we study obtains profits in part from attempting to "buy low and sell high." The speculator only records the wholesale prices of steel on the days he buys, but we have no data on

these prices on the vast majority of days where he does not buy steel. This implies that the prices that we observe are not only irregularly sampled, but they are also endogenously sampled and thus do not constitute a random sample from the time series of all wholesale prices. Without correcting for this endogeneity, our inferences about the time series properties of wholesale prices will be clearly invalid. In particular, the average price of steel on the days the speculator buys steel will be a downward biased estimate of the ergodic or long run average wholesale price of steel.

In this paper we present two econometric procedures for estimating an endogenously-sampled Markov process. We discuss a parametric partial information maximum likelihood (PIML) estimator that solves the endogenous sampling problem. While the PIML estimator efficiently estimates the unknown parameters of a Markov transition probability, it requires repeatedly computing numerical approximations to high dimensional integrals. Therefore, inspired by the work of McFadden (1989), we proposed an alternative consistent, less efficient, method of simulate moments (MSM) estimator. This estimation method is computationally simpler than the PIML estimator, but it still requires solving the dynamic programming problem at each trial value of the unknown parameter vector for the endogenous sampling rule. Using this sampling rule, the MSM estimator is able to consistently estimate the unknown parameters of the Markov process even though the econometrician has incomplete information on the process.

We showed that the MSM estimator can successfully identify the parameters of the underlying endogenously sampled time series despite a high degree of censoring: we observe purchases and wholesale prices in fewer than 15% of the business days over our sample period. We estimated a truncated lognormal AR(1) specification for the wholesale price series at daily frequencies and find strong mean reversion in these commodity prices: the coefficient estimate of the log of lagged wholesale price is precisely estimated and significantly below 1 so we can strongly reject the hypothesis that it has a unit root.

However we are not just interested in making inferences about the $\{p_t\}$ process, we are also interested in the behavior of the steel company and the extent to which its behavior conforms to the prediction of our theory. We found that the MSM parameters for the firm's retail demand for steel, storage and ordering costs and other parameters are both plausible and precisely estimated. In particular, MSM delivers precise estimates of the firm's discount factor, a surprising finding in view of the difficulty of identifying the discount factor in other contexts such as dynamic discrete choice models (see Rust, (1987) and Abbring and Dalfjord (2018) for further discussion).

Despite the plausible MSM parameter estimates, we conclude that our theory of commodity price speculation and inventory investment is unable to provide a sufficiently good approximation to the actual

behavior of the firm we analyzed. Although there are many strong parametric assumptions we imposed in order to estimate the the model, we think the strongest assumption is the hypothesis of discounted expected profit maximization and we provided calculations that suggest that the behavior of the firm we analyzed may not be optimal. We are attempting to see if the firm is open to field experiments that might enable us to test the profit maximization hypothesis more directly, by using our model and (S, s) inventory policy to manage inventories of several "treatment products" and compare their profitability to a set of corresponding "control products" that are managed by the firm's own inventory policies.

We are also pursuing the question of whether the endogenous sampling problem can be solved without resorting to as many strong assumptions as we have imposed. It is an open question, for example, whether it is possible to relax the assumption of expected profit maximization and somehow non-parametrically uncover the firm's wholesale ordering policy $q^o(p, x)$ (which may not necessarily be of the (S, s) form that is appropriate if the firm is an expected discount profit maximizer) given the high degree of censoring and serial correlation in the data. If it is possible to identify the nature of the endogenous sampling under weak assumptions with a non-parametrically estimated "first stage" it may be possible to estimate the parameters of the $\{p_t\}$ process using the MSM estimator in the second stage. If this sort of two step estimation is possible, it would enable us to relax the strong assumption of optimal behavior by the firm which we relied on to characterize the nature of the endogenous sampling rule via the generalized (S, s) strategy. Misspecification of the sampling rule could potentially lead to inconsistent estimates of the underlying endogenously sampled process $\{p_t\}$.

While this research was motivated by a new dataset from a single steel wholesaler, most datasets in which agents have the choice of whether and when to participate in a market activity will be endogenously sampled. In most markets, the only prices recorded are the transaction prices – econometricians almost never get to observe prices offered but not transacted on. For example, econometricians rarely get to observe the wages unemployed job seekers are offered but refuse. It should be straightforward to apply the MSM estimator to other types of endogenous sampling problems that arise in time series contexts.

References

[1] Abbring, J. and O. Daljord (2018) "Identifying the Discount Factor in Dynamic Discrete Choice Models" manuscript, Tilburg University.

[2] Aït-Sahalia, Y. and P. A. Mykland (2001) "The Effects of Random and Discrete Sampling When Estimating Continuous-Time Diffusions" working paper, Princeton University.

[3] Aguirregabiria, V. (1999) "The Dynamics of Markups and Inventories in Retailing Firms" *Review of Economic Studies* 66, 275–308.

[4] Allenby, G., McCulloch, R. and P. Rossi (1996) "On the Value of Household Information in Target Marketing" *Marketing Science* 15, 321–340.

[5] Arrow, K.J., Harris, T. and J. Marschak (1951) "Optimal Inventory Policy" *Econometrica* 19-3, 250–272.

[6] Athreya, R. (2002) "Price Dispersion in the Wholesale Market for Steel" manuscript, Yale University.

[7] Billingsley, P. (1961) *Statistical Inference for Markov Processes* University of Chicago Press.

[8] Chan, H. M. (2001) "Analysis of Variance in Steel Sales Price" manuscript, Yale University.

[9] Cong, Y., B. Chen and M. Zhou (2004) "Fast Simulation of Hyperplane-Truncated Multivariate Normal Distributions" *Bayesian Analysis* **1-1** 1–28.

[10] Deaton, A. and G. Laroque (1992) "On the Behavior of Commodity Prices" *Review of Economic Studies* 59, 1–23.

[11] Duffie, D. and K.J. Singleton (1993) "Simulated Moments Estimation of Markov Models of Asset Prices" *Econometrica* 61-4, 929–952.

[12] Duffie, D. and P. Glynn (2004) "Estimation of Continuous-Time Markov Processes Sampled at Random Time Intervals" *Econometrica* **72-6** 1773–1808.

[13] Engle, R. and J. Russell (1998) "Autoregressive Conditional Duration: A New Model for Irregularly Spaced Data" *Econometrica* **66-5** 1127–1162.

[14] Erdem, T. and M. Keane (1996) "Decision-making Under Uncertainty: Capturing Dynamic Brand Choice Processes in Turbulent Consumer Goods Markets" *Marketing Science* 15–1, 1–20.

[15] Fabian, T., J.L. Fisher, M.W. Sasieni, and A. Yardeni (1959) "Purchasing Raw Material on a Fluctuating Market" *Operations Research* 7, 107-122.

[16] Frazier, D. and D. Zhu (2017) "Derivative-Based Optimization with a Non-Smooth Simulated Criterion" manuscript, Department of Econometrics, Monash University.

[17] Futia, C. (1982) "Invariant Distributions and the Limiting Behavior of Markovian Economic Models" *Econometrica* **50-2** 377–408.

[18] Golabi, K. (1985) "Optimal Inventory Policies when Ordering Prices are Random" *Operations Research* 33–3, 575-588.

[19] Hall, A. and A. Inoue (2003) "The Large Sample Behavior of the Generalized Method of Moments Estimator in Misspecified Models" *Journal of Econometrics* **114** 361–394.

[20] Hall, A. and A. Inoue (2007) "Corrigendum to: The Large Sample Behavior of the Generalized Method of Moments Estimator in Misspecified Models" *Journal of Econometrics* **141** 1418.

[21] Hall, G. and J. Rust (1999) "An Empirical Model of Inventory Investment by Durable Commodity Intermediaries" *Carnegie-Rochester Conference Series on Public Policy*, 52.

[22] Hall, G. and J. Rust (2007) "The (S,s) Rule is an Optimal Trading Strategy in a Class of Commodity Price Speculation Problems" *Economic Theory* **30** 515–538.

[23] Hansen, L. P. (1982) "Large Sample Properties of Generalized Method of Moments Estimators" *Econometrica* 50, 1929-1954.

[24] Heckman, J. (1979) "Sample Selection Bias as Specification Error" *Econometrica* **47-1** 153–161.

[25] Heckman, J.(1981) "Statistical Models for Discrete Panel Data" in C. Manski and D. McFadden (eds.) *Structural Analysis of Discrete Data* MIT Press.

[26] Howard, R. (1960) *Dynamic Programming and Markov Processes* New York, Wiley.

[27] Kaldor, N. (1939) "Speculation and Economic Stability" *Review of Economic Studies* 7, 1-27.

[28] Kalymon, B. (1971) "Stochastic Prices in a Single-Item Inventory Purchasing Problem" *Operation Research* 19, 1434-1458.

[29] Kingman, B.G. (1969) "Commodity Purchasing" *Operational Research Quarterly* 20-1, 59-79.

[30] Lee, B. and B.F. Ingram (1991) "Simulation Estimation of Time-Series Models" *Journal of Econometrics* 47, 197–205.

[31] Manski, C. and D. McFadden (1981) "Alternative Estimators and Sample Designs for Discrete Choice Analysis" in C. Manski and D. McFadden (eds.) *Structural Analysis of Discrete Data* MIT Press.

[32] McFadden, D. (1997) "On the Analysis of Endogenously Recruited Panels" manuscript, University of California at Berkeley.

[33] McFadden, D. (1989) "A Method of Simulated Moments for Estimation of Discrete Models Without Numerical Integration" *Econometrica* 57–5, 995–1026.

[34] Miranda, M. and Rui, X. (1997) "An Empirical Reassessment of the Nonlinear Rational Expectations Commodity Storage Model" manuscript, Ohio State University, forthcoming, *Review of Economic Studies*

[35] Moinzadeh, K. (1997) "Replenishment and Stocking Policies for Inventory Systems with Random Deal Offerings" *Management Science* 43–3, 334-342.

[36] Newey, W. and K. West (1987) "A Simple, Positive Definite, Heteroskedasticity and Autocorrelation Consistent Covariance Matrix" *Econometrica* **55** 703–708.

[37] Reich, G. (2018) "Divide and Conquer: Recursive Likelihood Function Integration for Hidden Markov Models with Continuous Latent Variables" forthcoming, *Operations Research*

[38] Resnick, S. (1992) *Adventures in Stochastic Processes* Birkhäuser, Boston.

[39] Rust, J. (1987) "Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher" *Econometrica* **55-5** 999-1033.

[40] Rust, J. (1996) "Numerical Dynamic Programming in Economics" *Handbook of Computational Economics, Volume 1* Hans Amman, David Kendrik and John Rust (eds.) Elsevier, North Holland.

[41] Rust, J. and G. Hall (2003) "Middlemen versus Market Makers: A Theory of Competitive Exchange" *Journal of Political Economy* **111-2** 353–403.

[42] Rust, J. Traub, J. and H. Woźniakowski (2002) "Is There a Curse of Dimensionality for Contraction Fixed Points in the Worst Case?" *Econometrica* 70-1, 285–329.

[43] Scarf, H. (1959) "The Optimality of (S,s) Policies in the Dynamic Inventory Problem" In *Mathematical Methods in the Social Sciences* K. Arrow, S. Karlin and P. Suppes (ed.), Stanford, CA: Stanford University Press.

[44] Song, J. and P. Zipkin (1993) "Inventory Control in a Fluctuating Demand Environment" *Operations Research* 41-2, 351-370.

[45] Vázquez-Abad, F.J. and H.J. Kushner (1992) "Estimation of the Derivative of a Stationary Measure with Respect to a Control Parameter" *Journal of Applied Probability* **29-2** 343–352.

[46] White, H. (1982) "Maximum Likelihood Estimation of Misspecified Models" *Econometrica* 50–1, 1–26.

[47] Williams, J.C. and B. Wright (1991) *Storage and Commodity Markets* Cambridge University Press, New York.

Appendix: Simulating Price Paths with Fixed Starting and Ending Points

When the spot procurement price follows the AR(1) process given in equation (72) of the paper, as is well known, we can rewrite p_T, the logarithmically transformed spot price of steel in an equivalent moving average (MA) representation. For notational simplicity, we rewrite the AR(1) representation in equation (72) as

$$p_{t+1} = \alpha + \beta p_t + \varepsilon_{t+1} \tag{79}$$

where we assume that $\beta \in (0,1)$ and $\{\varepsilon_t\}$ is an *IID* $N(0,\sigma^2)$ noise process. Then the MA representation for this process is

$$p_T = \alpha \sum_{t=0}^{T-1} \beta^t + \sum_{t=1}^{T} \varepsilon_t \beta^{T-t} + \beta^T p_0. \tag{80}$$

Suppose we want to simulate values for the intervening logarithmic prices $(p_1, p_2, \ldots, p_{T_1})$ conditional on known prices p_0 and p_T. We can rewrite the MA representation of the price process (80) that imposes the constraints implied by the known beginning and ending prices as a linear constraint on the *IID* shocks $\vec{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_T)$ as follows

$$L\vec{\varepsilon} = r \tag{81}$$

where L is a $T \times 1$ vector $L' = (\beta^{T-1}, \beta^{T-2}, \ldots, \beta, 1)$ and r is a scalar given by

$$r = p_T - \alpha \sum_{t=0}^{T-1} \beta^t - \beta^T p_0. \tag{82}$$

To generate a simulated sequence $(p_1, p_2, \ldots, p_T)$ subject to a known initial condition p_0 and terminal condition p_T, we simulate a vector $\vec{\varepsilon}$ of T *IID* $N(0, \sigma^2)$ random variables. We define a new $T \times 1$ vector $\vec{\eta}$ given by

$$\vec{\eta} = \vec{\varepsilon} + L'(r - L\vec{\varepsilon})/L'L. \tag{83}$$

Cong, Chen and Zhou (2004) proved that $\vec{\eta}$ is a realization from the conditional distribution of a multivariate normal distribution for $\vec{\varepsilon}$ given the linear restriction $L\vec{\varepsilon} = r$. Using the resulting $T \times 1$ vector $\vec{\eta}$, we can construct simulated draws for $(p_1, p_2, \ldots, p_{T-1}, p_T)$ as follows

$$
\begin{aligned}
p_1 &= \alpha + \beta p_0 + \eta_1 \\
p_2 &= \alpha + \beta p_1 + \eta_2 \\
&\cdots \\
p_{T-1} &= \alpha + \beta pT - 2 + \eta_{T-1} \\
p_T &= \alpha + \beta pT - 1 + \eta_T
\end{aligned}
\tag{84}
$$

Note that $\vec{\eta}$ is a multivariate normal distribution implied by the *IID* shocks $\vec{\varepsilon}$ but conditional on the linear restriction $L\vec{\varepsilon} = r$, which implies that regardless of the realized values of $\vec{\eta}$ the terminal constraint that p_T equals a known, fixed value is satisfied with probability 1. It follows that the simulated prices $(p_1, p_2, \ldots, p_T)$ from (84) are a draw from the conditional distribution of $(p_1, p_2, \ldots, p_{T_1})$ conditional on known values (p_0, p_T).