Competition and Innovation in Dynamic Oligopoly

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Papers

- Goettler, R. & B. R. Gordon (2011), “Does AMD Spur Intel to Innovate More?” *Journal of Political Economy*, 119(6), pp. 1141–1200.
- Goettler, R. & B. R. Gordon (2014), “Competition and Product Innovation in Dynamic Oligopoly,” *Quantitative Marketing and Economics*, 12, pp. 1–42.

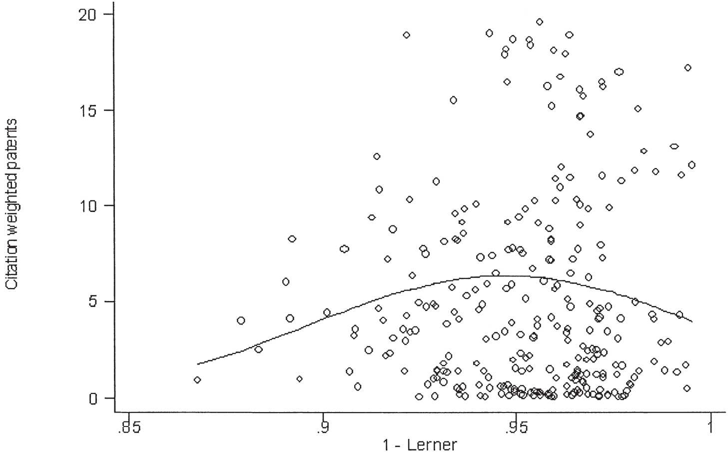
Market Structure and Innovation: Theory

Antitrust policymakers are increasingly concerned with innovation, but economic theory offers conflicting stories.

- Schumpeter (1934, 1942): competition **decreases** innovation
  - High market power $\longrightarrow$ extract gains from innovation
- Arrow (1962): competition **increases** innovation
  - Innovate to escape perfect competition
- Aghion et al. (2005): **inverted-U** shaped relationship
  - Competition increases incremental profit from innovating
  - But discourages laggard's innovation incentives
- Vives (2008): **depends** on measure of competition
  - More product substitutability $\longrightarrow$ higher cost-reducing R&D
  - More firms $\longrightarrow$ lower R&D

Market Structure and Innovation: Empirical Evidence

Aghion, Bloom, Blundell, Griffith, & Howitt (QJE 2005)



Note: points are industry-year observations

Market Structure and Innovation: Empirical Evidence

Empirical evidence is also inconclusive (Cohen & Levin, 1989; Gilbert, 2006)

- Difficult to measure innovation (patents, costs, productivity?)
- Difficult to measure competition (concentration, margins?)
- Cross-industry studies must control for myriad of factors

Our Approach

- Focus on the CPU industry because
  - Driver of economic growth (Jorgenson et al., 2010)
  - Easy to measure innovation
  - Duopoly
- We follow a structural approach:
  - Estimate consumer preferences
  - Estimate firms' costs and innovation processes
  - Solve for equilibrium under various market structures
- Our model permits either Arrow or Schumpeter to prevail

Model Motivation: Consumers

Empirical observations (sample from 1993 to 2004)

- Quality-adjusted prices have declined $\sim$30% per year
- Replacement drives demand (82% of purchases in 2004)

Model implications

- Consumers have expectations over future price and quality
- Time upgrade decisions depending on what they own
- Demand is dynamic

Model Motivation: Firms

Empirical observations

- Given lack of physical depreciation, firms innovate to rebuild replacement demand
- Frequent new product introductions, but variation in magnitude of improvements
- Cross-licensing and other arrangements produce "spillovers"

Model implications

- Firms compete with each other **and** the stock of used durables
- Products exist on a quality ladder
- Successful improvement is more likely for higher R&D
- Laggard firm benefits from innovation spillovers

Contributions

Methodological

- Model durable goods with endogenous durability/obsolescence in dynamic oligopoly
  - Modify Ericson & Pakes (1995) to incorporate durables
  - Demand is dynamic, driven by replacement purchases
    $\longrightarrow$ outside good depends on consumer's previous purchase
    $\longrightarrow$ the ownership distribution becomes a state variable
    $\longrightarrow$ buy today or wait for lower prices, higher quality?
  - Innovation, prices, and purchases are equilibrium outcomes based on dynamic trade-offs and strategic interaction
- Provide alternative approach to bounding the EP state space
  - Endogenizes long-run rate of innovation
  - Innovation therefore varies across market structures

Empirical

- Does AMD spur Intel to innovate more? No*

Related Literature

Empirical Applications:

- Ericson-Pakes type Dynamic Oligopoly
  - Gowrisankaran & Town (1997), Benkard (2004), Chan & Seetharaman (2004), Dube, Hitsch, & Manchanda (2005), Esteban & Shum (2006), Ryan (2007), Xu (2007), Dube, Hitsch, & Chintagunta (2007), Macieira (2007), Lee (2008)
- CPU Industry: Song (2003), Prince (2007), Gordon (2009)

Theoretical Literature:

- Optimal Durability by a Monopolist
  - Kleiman & Ophir (1966), Swan (1970, 1971), Sieper & Swan (1973), Rust (1986), Levinthal & Purohit (1989), Waldman (1996), Fudenberg & Tirole (1998), Hendel & Lizzeri (1999)
- Intertemporal Price Discrimination
  - Coase (1972), Stokey (1979, 1981), Bulow (1982), Bond & Samuelson (1984), Gul, Sonnenschein, & Wilson (1986), Nair (2007)
- Oligopoly: Sobel (1984), Gul (1987), Desai & Purohit (1999)
- Competition & Innovation: Aghion et al. (2001, 2005), Holmes, Levine, Schmitz (2008)

Model Overview

- J firms, one product per firm, no entry/exit
- Product log-quality denoted $q_{jt} \in \{0, \delta, 2\delta, \ldots\}$
  - innovations are proportional improvements in quality.
- Firms simultaneously choose price p_{jt} and investment x_{jt}
  - Lowering price today reduces future demand
  - Future quality is stochastically increasing in investment
  - Spillovers: innovation efficiency higher for laggards
- Consumers currently own a product with quality $\tilde{q}_t$
  - Decide whether to keep $\tilde{q}_t$ or buy some $q_{jt} \in q_t$
  - Endogenous value of the "no purchase" option
- No physical depreciation (trivial to add)
- No secondary market (nontrivial to add)

Consumers

- Utility for buying new product with log quality q_{jt}:

$$u_{jt} = \gamma q_{jt} - \alpha p_{jt} + \xi_j + \varepsilon_{jt}$$

- Utility for outside alternative (i.e., no purchase option):

$$u_{0t} = \gamma \tilde{q}_t + \varepsilon_{0t}$$

- Define a lower bound for $\tilde{q}_t$ as: $\underline{q}_t = \bar{q}_t - \bar{\delta}_c$
  where $\bar{q}_t = \max(q_{1t}, \ldots, q_{Jt})$ denotes frontier quality
  - We choose $\bar{\delta}_c$ high so that bound rarely hit before upgrade
    $\longrightarrow$ bound has minimal effect on equilibrium behavior
  - Consumers with $\tilde{q}_t = \underline{q}_t$ may be viewed as "non-owners"

Consumers

- Let $\Delta_{k,t}$ = share who own product with quality k at time t.

  Ownership distribution is $\Delta_t = (\Delta_{\underline{q}_t,t}, \ldots, \Delta_{k,t}, \ldots, \Delta_{\bar{q}_t,t})$

- Dynamic consumer problem:

$$V(q, \Delta, \tilde{q}, \varepsilon) = \max_{j \in (0,1,\ldots,J)} u_j \;\; + \\ \beta \sum_{\substack{q' \\ \Delta'}} \int_{\varepsilon} V(q', \Delta', \tilde{q}', \varepsilon')\, h_c(q'|q, \Delta)\, g_c(\Delta'|\Delta, q, q')\, F(d\varepsilon')$$

- Law of motion: $\quad \Delta'_k = s_{0|k}\Delta_k + \sum_j s_j \mathcal{I}(q_j = k)$

  where $s_{j|\tilde{q}} = \frac{\exp\{v_j(q,\Delta,\tilde{q}\}}{\sum_k \exp\{v_k(q,\Delta,\tilde{q})\}}$ and $s_j = \sum_{\tilde{q} \in \{\underline{q},\ldots,\bar{q}\}} s_{j|\tilde{q}}\, \Delta_{\tilde{q}}$

Firms

- Investment outcome $\tau_{jt} = q_{j,t+1} - q_{jt}$, with $\tau_{jt} \in \{0, \delta\}$
  - $f(\tau_{jt} | x_j, a_j(q_t))$ stochastically increasing in x_j
  - $a_j(q_t)$ increasing in $\bar{q}_t - q_{jt}$, captures spillover effects
- Period profits: $\pi_j(p_t, q_t, \Delta_t) = M s_{jt}(p_t, q_t, \Delta_t)(p_{jt} - mc_j(q_t))$
- Bellman equation for firm j:

$$W_j(q_j, q_{-j}, \Delta) = \max_{p_j, x_j} \quad \pi_j(p, q, \Delta) - x_j \ +$$

$$\beta \sum_{\tau_j,\, q'_{-j},\, \Delta'} W_j(q_j + \tau_j, q'_{-j}, \Delta')\ f(\tau_j | x_j, a_j(q))\ h_f(q'_{-j} | q, \Delta)\ g_f(\Delta' | \Delta, q, q', p)$$

Equilibrium

A Markov-Perfect Nash Equilibrium (MPNE) is the set

$$\left\{V^*, h_c^*, g_c^*, \left\{W_j^*, x_j^*, p_j^*, h_{f_j}^*, g_{f_j}^*\right\}_{j=1}^{J}\right\}$$

such that:

1. Firms' and consumers' strategies depend only on current state variables

2. Consumers have rational expectations about future qualities and Δ

   a. $h_c^*(q'|q,\Delta,\tilde{q}) = \prod_{j=1}^{J} f(\tau = q'_j - q_j \ |x_j^*, a_j(q))$

   b. g_c^* derived from Δ law of motion.

3. Each firm has rational expectations about future qualities and Δ

   a. $h_{f_j}^*(q'_{-j}|q,\Delta) = \prod_{j' \neq j}^{J} f(\tau = q'_{j'} - q_{j'} \ |x_{j'}^*, a_j(q))$

   b. g_f^* derived from Δ law of motion.

State Space Normalization

- State space is unbounded since qualities increase continually
- Normalize w.r.t. the frontier product's quality $\bar{q}_t$

$$\begin{aligned} W_j(q_{jt}, q_{-j,t}, \Delta_t) &= W_j(q_{jt} - \bar{q}_t, q_{-j,t} - \bar{q}_t, \Delta_t) \\ V(q_t, \Delta_t, \tilde{q}_t, \varepsilon_t) &= \tfrac{\gamma \bar{q}_t}{1-\beta} + V(q_t - \bar{q}_t, \Delta_t, \tilde{q}_t - \bar{q}_t, \varepsilon_t), \end{aligned}$$

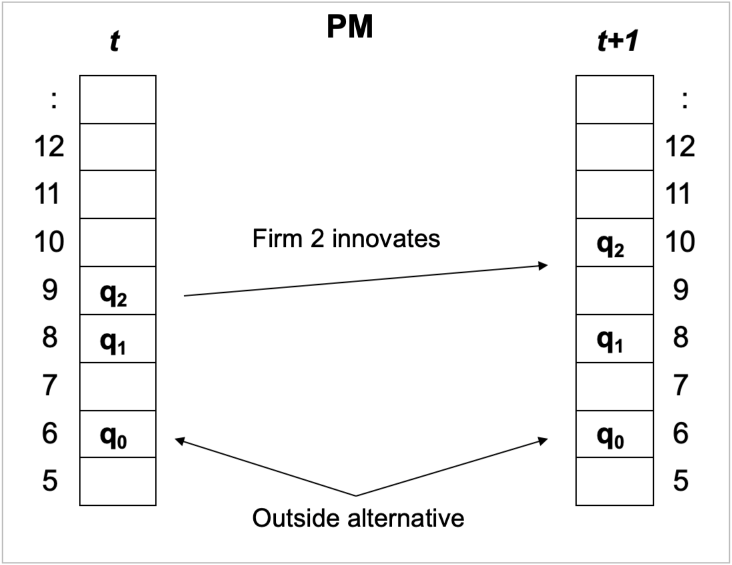
- Let $\omega_t = q_t - \bar{q}_t, \quad \tilde{\omega}_t = \tilde{q}_t - \bar{q}_t, \quad I_{\omega_t} = 1$ if $\bar{q}_{t+1} > \bar{q}_t$.

$$\begin{aligned} \hat{V}_j(\omega_t, \Delta_t, \tilde{\omega}_t) &= \gamma \omega_{jt} - \alpha p_{jt} + \beta \\ & \sum_{I_{\omega_t},\, \omega_{t+1},\, \Delta_{t+1}} \log \left(\sum_{j' \in \{0,\dots,J\}} \exp \left\{ \tfrac{\gamma \delta I_{\omega_t}}{1-\beta} + \hat{V}_{j'}(\omega_{t+1}, \Delta_{t+1}, \tilde{\omega}_{t+1}) \right\} \right) \\ & h_c(I_{\omega_t}, \omega_{t+1} | \omega_t, \Delta_t)\; g_c(\Delta_{t+1} | \Delta_t, \omega_t, p_t, I_{\omega_t}) \end{aligned}$$

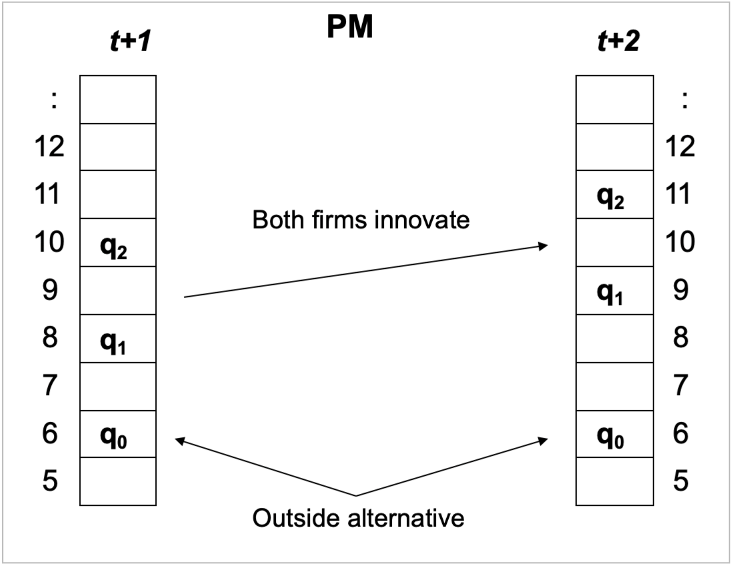
$$\begin{aligned} W_j(\omega_{jt}, \omega_{-j,t}, \Delta_t) &= \max_{p_{jt}, x_{jt}} \pi_j(p_t, \omega_t, \Delta_t) - c x_{jt} + \beta \\ & \sum_{\tau_{jt},\, \omega_{-j,t+1},\, I_{\omega_t},\, \Delta_{t+1}} W_j(\omega_{jt} + \tau_{jt} - I_{\omega_t}, \omega_{-j,t+1} - I_{\omega_t}, \Delta_{t+1}) \\ & h_f(I_{\omega_t}, \omega_{-j,t+1} | \omega_t, \Delta_t)\; g_f(\Delta_{t+1} | \Delta_t, \omega_t, p_t, I_{\omega_t})\; f(\tau_{jt} | x_{jt}) \end{aligned}$$

A digression to Pakes & McGuire (1994)...

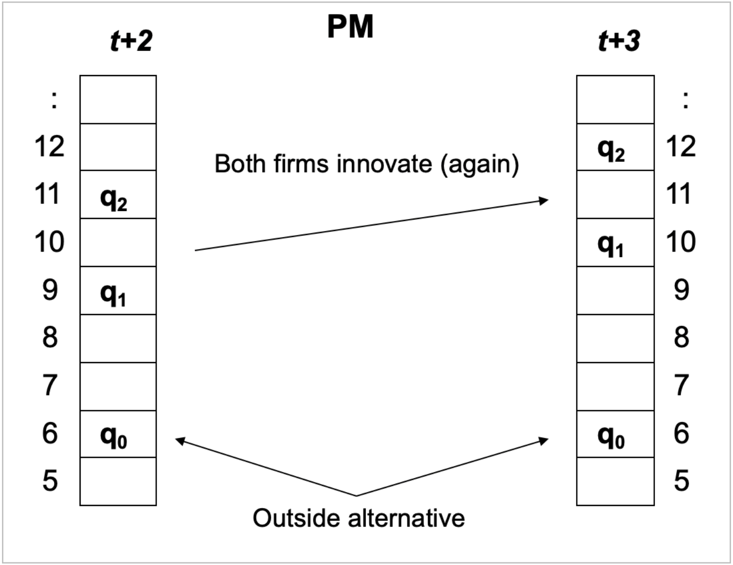
Innovation in PM



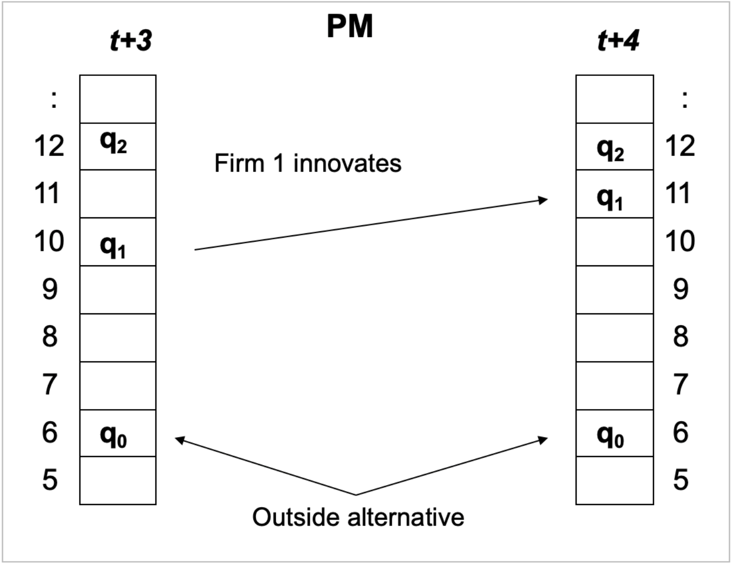
Innovation in PM



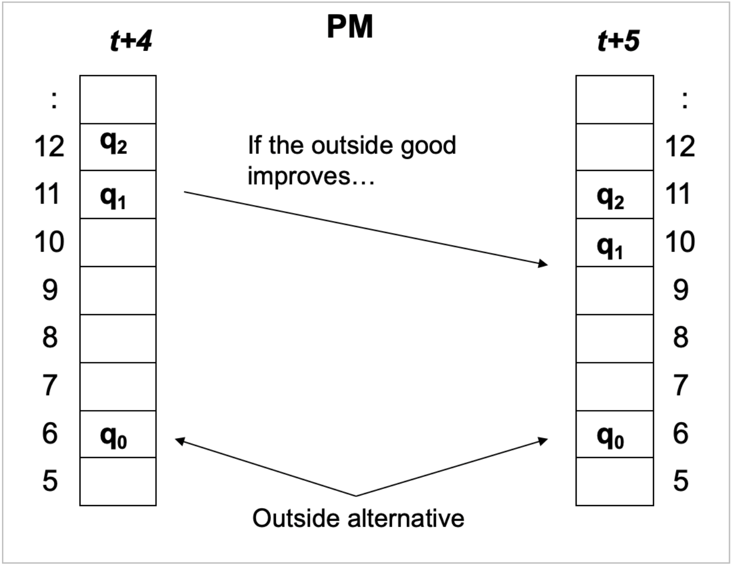
Innovation in PM



Innovation in PM



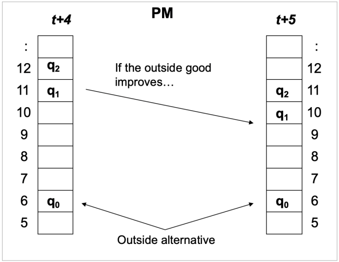
Innovation in PM



State Space Bound in Pakes & McGuire (1994)

- Numerical solution of a dynamic model of competition requires bounded state space
- PM creates lower bound with scrappage value for exit
- PM create upper bound by:
  - Defining firm's quality relative to outside good
  - Specifying consumer preferences such that benefit of higher quality $\to 0$, independent of competitors' qualities.

Why does this happen in PM?

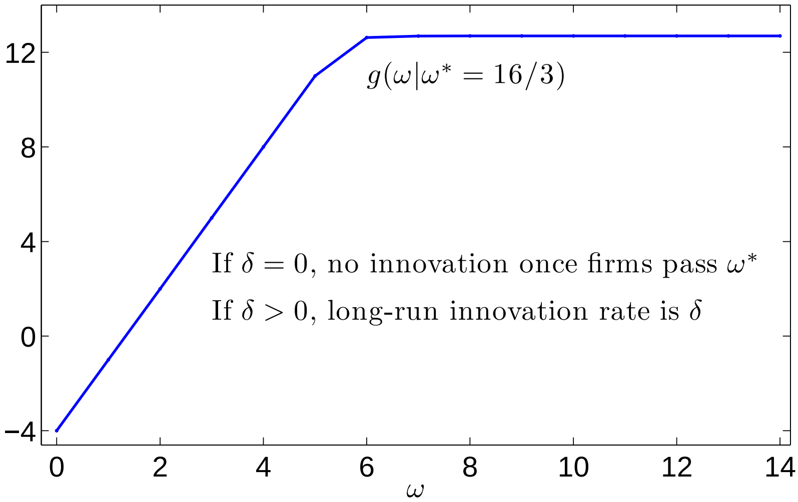


Pakes-McGuire: concave $g(\cdot)$	Standard Discrete Choice
$u_0 = \varepsilon_0$	$u_0 = \gamma q_0 + \varepsilon_0$
$u_1 = \gamma g(q_1 - q_0) - \alpha p_1 + \varepsilon_1$	$u_1 = \gamma q_1 - \alpha p_1 + \varepsilon_1$
$u_2 = \gamma g(q_2 - q_0) - \alpha p_2 + \varepsilon_2$	$u_2 = \gamma q_2 - \alpha p_2 + \varepsilon_2$

$$g(\omega) = \begin{cases} \lambda + \gamma\omega, & \text{if } \omega \leq \omega^* \\ \lambda + \gamma\omega^* + \ln(2 - \exp(\gamma(\omega^* - \omega))), & \text{otherwise.} \end{cases}$$

where $\omega_j = q_j - q_0$

$g(\omega)$

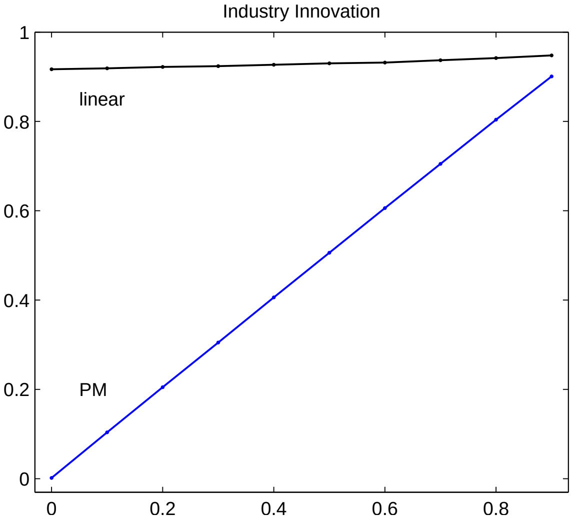


12
8
4
0
−4
0
2
4
6
8
10
12
14
ω
$g(\omega|\omega^* = 16/3)$
If $\delta = 0$, no innovation once firms pass ω^*
If $\delta > 0$, long-run innovation rate is δ

Implications

- Investment$\rightarrow 0$ as firm's quality improves, regardless of competitors' qualities.
  - Even neck-and-neck firms stop innovating
- When do firms start innovating again?
  - When the outside good improves enough
- Outside good's quality innovates with exogenous probability δ
- Long-run (steady-state) rate of innovation is determined by outside good's exogenous rate of improvement
  - Industry rate of innovation is independent of market structure
  - Let's look at a comparative static in δ...

Comparitive Static in δ: PM (1994) vs. GG (2014)



Simulations for $T = 100$ over 10,000 runs

This was all with non-durable goods

- Goettler & Gordon (2014), "Competition and Product Innovation in Dynamic Oligopoly," *Quantitative Marketing and Economics*
- Non-durable goods with entry/exit, spillovers
- Goal: understand how different competitive forces determine industry evolution, profits, and surplus
  - Entry barriers
  - Spillover level
  - Degree of product substitutability
- Matlab code posted on our websites

End of digression

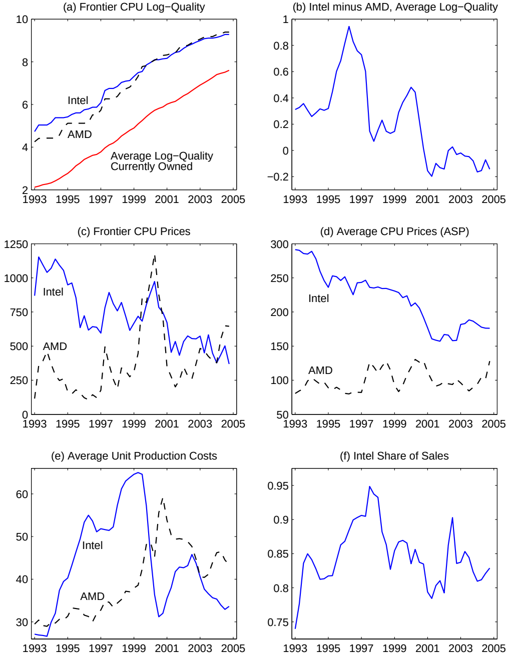
Implications of our bounds approach

- Uses standard discrete choice model
  - Normalization exactly represents the game in absolute qualities
  - Important for durable goods, since $\tilde{q}$ should yield $\gamma\tilde{q}$ utils next period too (if no depreciation)
- The bounds assumption (i.e., $\underline{q}_t = \bar{q}_t - \bar{\delta}_c$) has little effect on the behavior of lead firms, which generate majority of sales, profits, & surplus
- The long-run rate of innovation is endogenous, and is sensitive to industry structure and consumers' preferences.
- These points also apply to the nondurable dynamic oligopoly (see Goettler & Gordon, 2014)

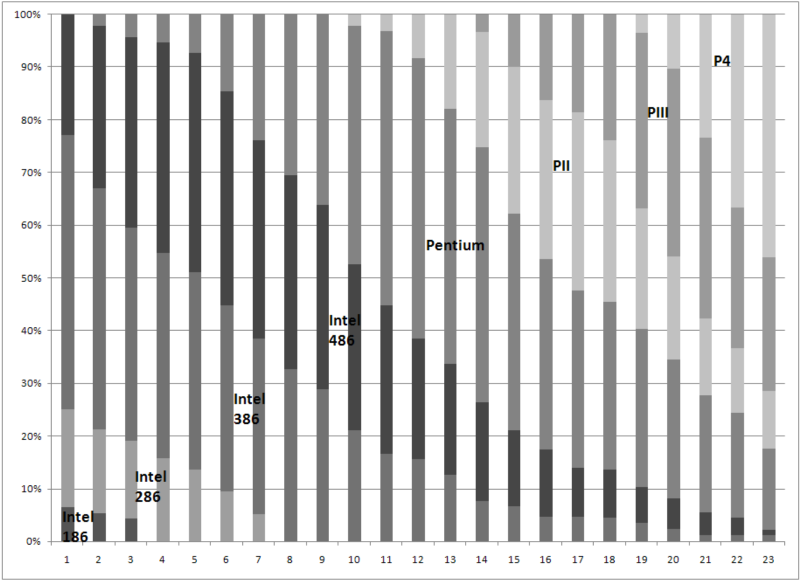
Data

- Desktop PC processors from 1993 to 2004
  - Intel and AMD control 95% of the market
  - AMD: $+1\%$ market share $\approx$ \$300 million in revenue
- Data components – assembled from many sources
  - Prices: complete series over life cycle for 217 CPU's
  - Quality: speed benchmark
  - Investment: quarterly R&D spending from annual reports
  - Sales: quarterly unit shipment information
  - Marginal Cost: blended unit production costs
  - Ownership Distribution: semi-annual consumer survey

Data: Quality, Prices, Shares, and Costs



Data: Ownership Distribution (semi-annual)



Estimation

- Use Simulated Method of Moments to match empirical moments to simulated counterparts

- Simulate model $NS = 10,000$ times over $T = 48$ periods

$$\hat{\theta}_T = \underset{\theta \in \Theta}{\operatorname{argmin}} \, (m_{S,T}(\theta) - m_T)' A_T (m_{S,T}(\theta) - m_T)$$

- Estimate A_T directly from m_T using bootstrap

- First estimate marginal costs: $mc_j = \mu_0 + \mu_1 \left(\bar{q}_t - q_{j,t}\right)$

- 7 remaining parameters, 15 moments

- See Andrews, Gentzkow, and Shapiro (2017)

Model Fit

Moment	Actual	Fitted	Actual SE
Intel price equation:			
average price	219.7		5.9
$q_{Intel,t} - q_{AMD,t}$	47.4		17.6
$q_{Intel,t} - \bar{\Delta}_t$	94.4		31.6
AMD price equation:			
average price	100.4		2.3
$q_{Intel,t} - q_{AMD,t}$	-8.7		11.5
$q_{AMD,t} - \bar{\Delta}_t$	16.6		15.4
Intel share equation:			
constant	0.834		0.007
$q_{Intel,t} - q_{AMD,t}$	0.055		0.013
Potential Upgrade Gains:			
Mean $(\bar{q}_t - \bar{\Delta}_t)$	1.146		0.056
Mean Innovation Rates:			
Intel	0.557		0.047
AMD	0.610		0.079
Relative Qualities:			
Mean $q_{Intel,t} - q_{AMD,t}$	1.257		0.239
Mean $\mathcal{I}(q_{Intel,t} \geq q_{AMD,t})$	0.833		0.054
Mean R&D / Revenue:			
Intel	0.114		0.004
AMD	0.203		0.009

Model Fit

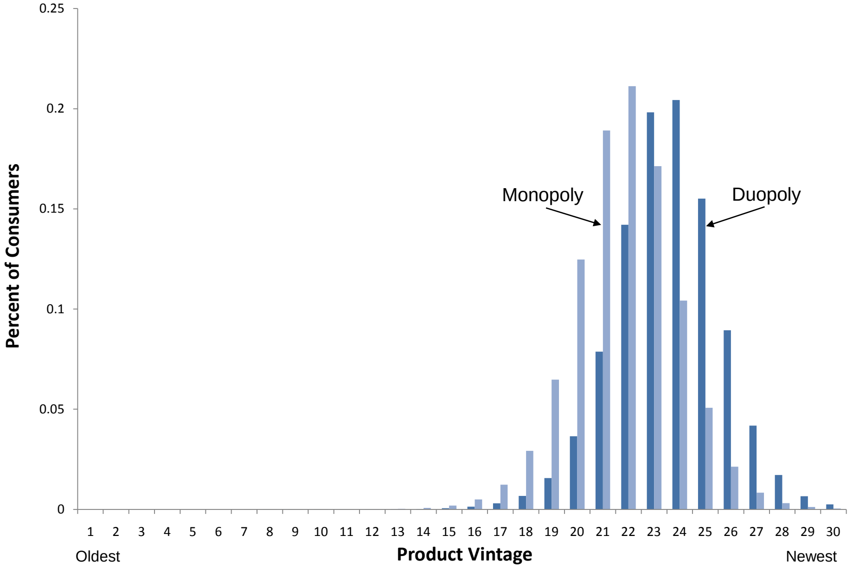
Moment	Actual	Fitted	Actual SE
Intel price equation:			
average price	219.7	206.2	5.9
$q_{Intel,t} - q_{AMD,t}$	47.4	27.3	17.6
$q_{Intel,t} - \bar{\Delta}_t$	94.4	43.0	31.6
AMD price equation:			
average price	100.4	122.9	2.3
$q_{Intel,t} - q_{AMD,t}$	-8.7	-22.3	11.5
$q_{AMD,t} - \bar{\Delta}_t$	16.6	5.9	15.4
Intel share equation:			
constant	0.834	0.846	0.007
$q_{Intel,t} - q_{AMD,t}$	0.055	0.092	0.013
Potential Upgrade Gains:			
Mean $(\bar{q}_t - \bar{\Delta}_t)$	1.146	1.100	0.056
Mean Innovation Rates:			
Intel	0.557	0.597	0.047
AMD	0.610	0.602	0.079
Relative Qualities:			
Mean $q_{Intel,t} - q_{AMD,t}$	1.257	1.352	0.239
Mean $\mathcal{I}(q_{Intel,t} \geq q_{AMD,t})$	0.833	0.929	0.054
Mean R&D / Revenue:			
Intel	0.114	0.101	0.004
AMD	0.203	0.223	0.009

Estimated Parameters

Parameter	Estimate	Std. Error
Price, α	0.0131	0.0017
Quality, γ	0.2764	0.0298
Intel Fixed Effect, ξ_{Intel}	-0.6281	0.0231
AMD Fixed Effect, ξ_{AMD}	-3.1700	0.0790
Intel Innov, $a_{0,Intel}$	0.0010	0.0002
AMD Innov, $a_{0,AMD}$	0.0019	0.0002
Spillover, a_1	3.9373	0.1453
Marginal Costs		
Constant, μ_0	44.5133	1.1113
slope, μ_1	-19.6669	4.1591

- Price elasticity: about -2
- WTP for 20% faster CPU to be used for 4 years: $51
- Intel premium: $\frac{\xi_{Intel} - \xi_{AMD}}{\alpha} = \$194$

Steady-State Ownership Distribution



Summary of Results

- Counterfactuals on competition and innovation
  - Compare outcomes from estimated duopoly, symmetric duopoly, monopoly, and social planner
  - Effect of market restriction
- Robustness checks: comparative statics in
  - Price and quality coefficients
  - Durability/depreciation rate
  - Market growth rate
  - Innovation spillover
  - Product Substitutability (var(ε))
  - Discount factor
- Other results
  - Optimal versus myopic pricing (ignoring durability)
  - Firms profits *increase* in consumer discount factor
  - Monte Carlo succeeds in recovering parameters

Summary: Other Results

Optimal vs. myopic pricing

- Duopoly profits 10% higher, margins 17% higher
- Monopoly profits 77% higher, margins 157% higher
- Implies static FOC may overestimate MC
- Effect of AMD on Intel's innovation is reversed

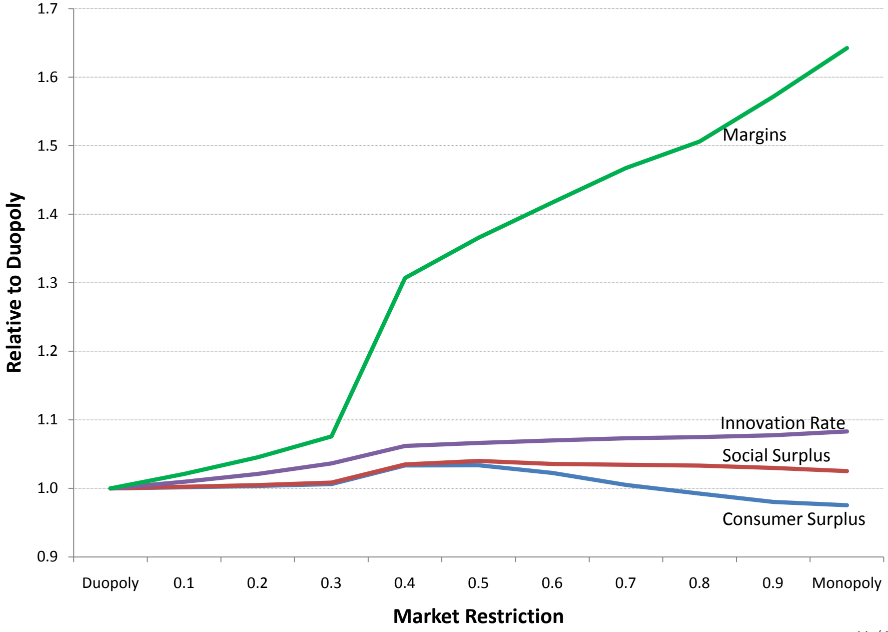
$\Rightarrow$ Importance of correctly accounting for durability when studying innovation and pricing

Industry Measures under Various Scenarios

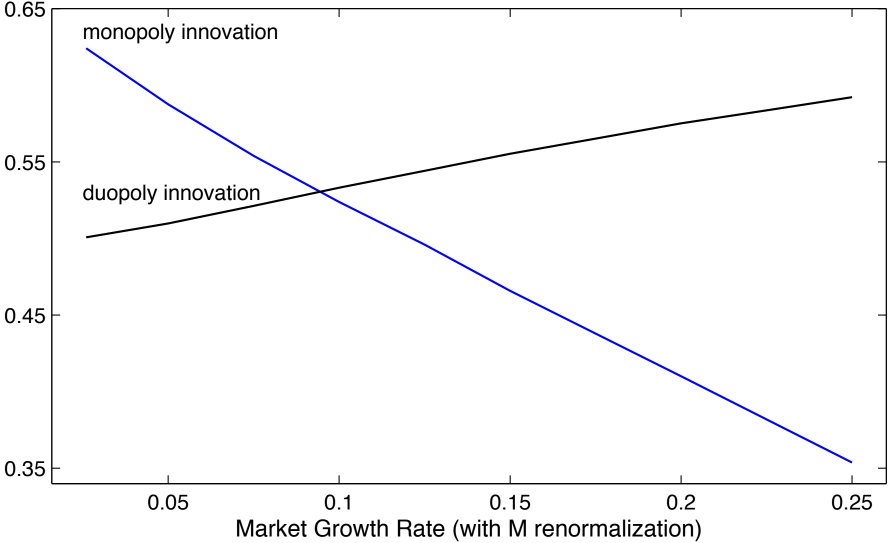
	Intel-AMD Duopoly	Symmetric Duopoly	Monopoly	Social Planner
Industry Profits ($billions)	408	400	567	-267
Consumer Surplus	2978	3012	2857	4032
Social Surplus	3386	3412	3424	3765
CS as share of Monopoly CS	1.042	1.054	1.000	1411
SS as share of Monopoly SS	0.989	0.997	1.000	1.100
Margins ($(p - mc)/mc$)	3.434	2.424	5.672	0.000
Frontier Innovation Rate	0.599	0.501	0.624	0.869
Mean Quality Upgrade %	261	148	410	97

- Effect of competition (Intel + AMD vs. Intel monopoly)
  - Competition increases CS by 4.2% ($12 billion per year)
  - ... but social surplus is 1.1% lower.
- Innovation is 4.2% higher in monopoly than duopoly.
- Stronger competition in symmetric duopoly lowers both margins and innovation

Counterfactual: Market Foreclosure



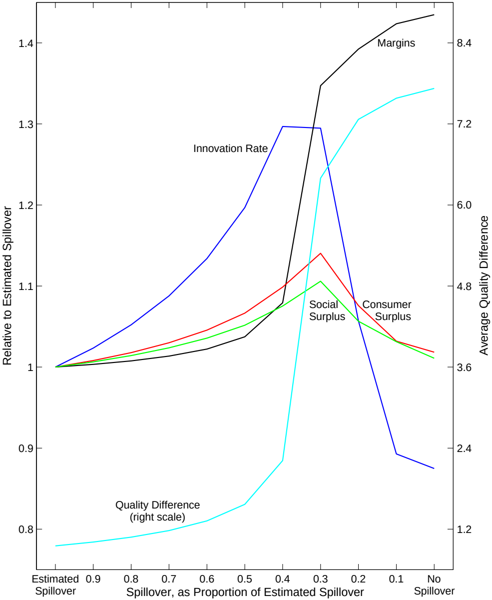
Comparative Static: Market Growth



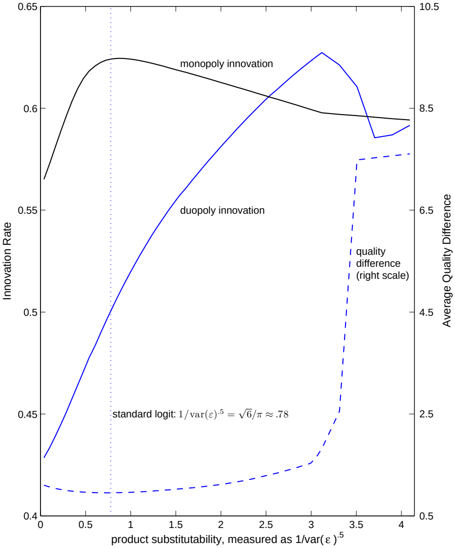
Conclusions and Future Research

- Consumers benefit from competition between Intel and AMD, but innovation would be higher with Intel as a monopolist (Schumpeter)
- Use structural econometric model to examine relationship between innovation and competition
- Extend EP-PM oligopoly model to durable goods
    - Endogenize industry innovation with new bounds approach
    - Accounting for durability raises margins & profits
- Future work
    - Multiproduct firms, heterogeneous consumers
    - sell vs. lease

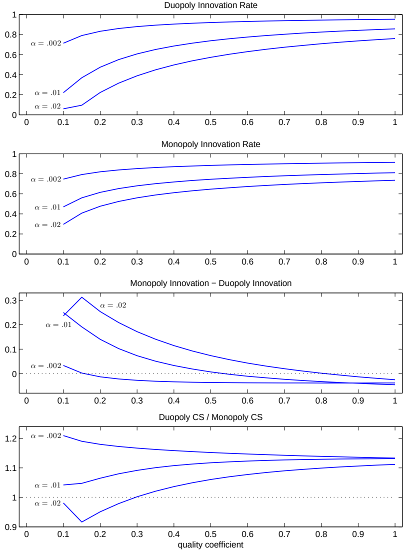
Comparative Static: Spillover



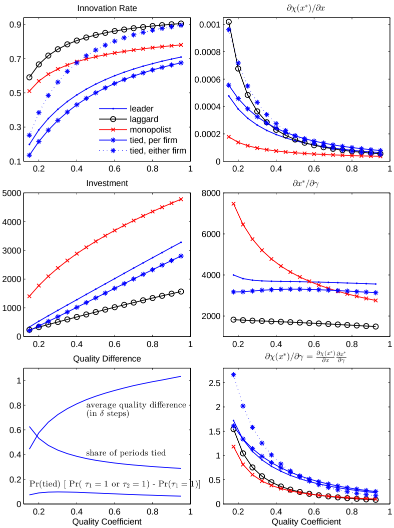
Comparative Static: Product Substitutability



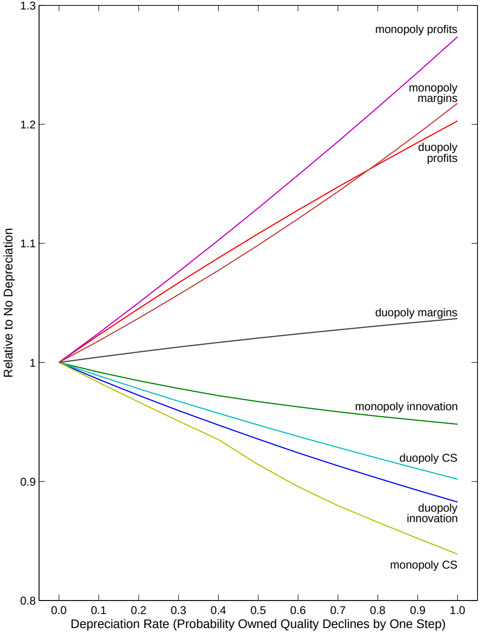
Comparative Static: α, γ



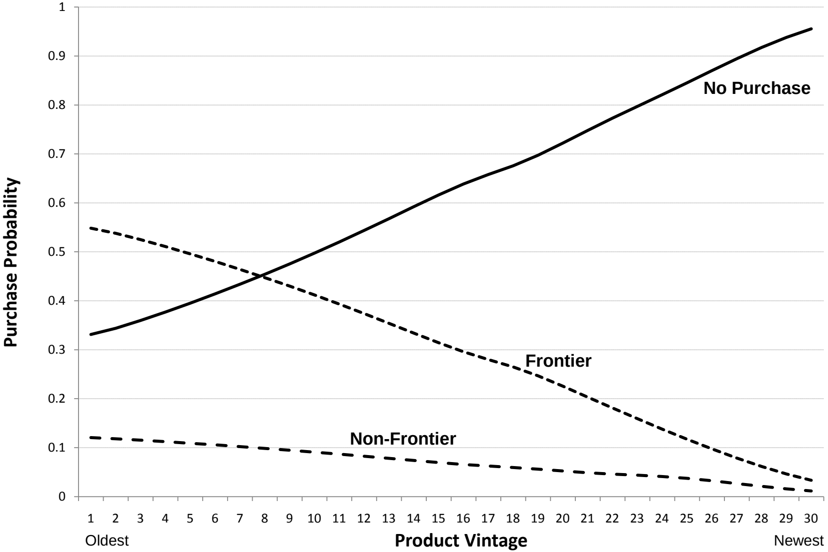
Comparative Static: decomposing effect of γ on innovation



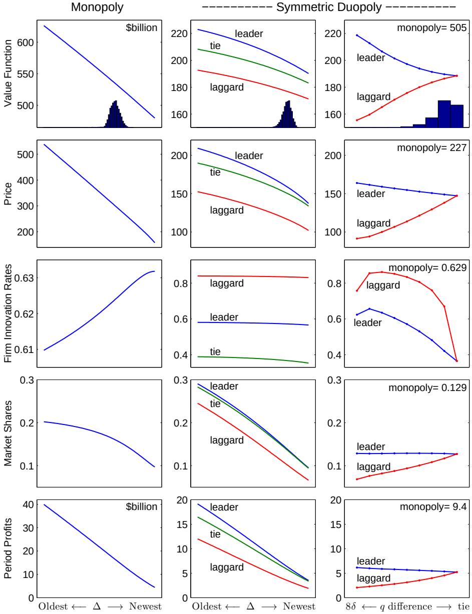
Comparative Static: Durability



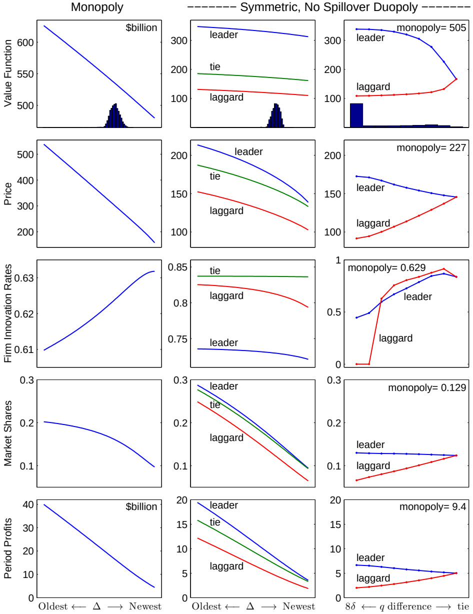
Steady-State Conditional Purchase Probabilities



Value and Policy Functions



Value and Policy Functions



Computing Welfare

- CS in period t given absolute $\tilde{q}_t$ and q_t is

$$\begin{aligned} CS(\tilde{q}_t, q_t) &= \frac{1}{\alpha}\mathsf{E}\left[u_{jt}\right] \\ &= \frac{1}{\alpha}\sum_{j\in\{0,\ldots,J\}} s_{jt|\tilde{q}}\left(\gamma q_{jt} - \alpha p_{jt} + \xi_j - \mathsf{E}\left[\varepsilon_{jt}|\text{choose } j\right]\right) \\ &= \frac{1}{\alpha}\sum_{j\in\{0,\ldots,J\}} s_{jt|\tilde{q}}\left(\gamma q_{jt} - \alpha p_{jt} + \xi_j - \log s_{jt|\tilde{q}}\right) \end{aligned}$$

- Integrating over Δ_t yields aggregate CS

$$CS = M\sum_{t=0}^{T}\beta^t\sum_{\tilde{q}=\underline{q}_t}^{\bar{q}_t} CS(\tilde{q}, q_t)\cdot\Delta_{\tilde{q},t}$$

- Equivalently $\quad CS = \frac{M}{\alpha}\sum_{\tilde{q}=\underline{q}_t}^{\bar{q}_t}\hat{V}(q_0, \Delta_0, \tilde{q})\cdot\Delta_{\tilde{q},0}$

- Assumes utility is always zero in the absence of inside firms. We could allow outside good's quality to improve exogenously over time, which would lower CS.

Computing Equilibrium: Backwards Recursion

Firms simultaneously choose p and x to satisfy the FOC

$$\frac{\partial W}{\partial p_j} = \frac{\partial \pi_j(p,q,\Delta)}{\partial p_j} + \beta \sum_{\tau_j, q'_{-j}, \Delta'} W_j(q_j+\tau_j, q'_{-j}, \Delta') h_f(q'_{-j}|q,\Delta) \frac{\partial g_f(\Delta'|\Delta,q,p)}{\partial p_j} f(\tau_j|x_j)$$

$$\frac{\partial W_j}{\partial x_j} = -1 + \beta \sum_{\tau_j, q'_{-j}, \Delta'} W_j(q_j+\tau_j, q'_{-j}, \Delta') h_f(q'_{-j}|q,\Delta) g_f(\Delta'|\Delta,q,p) f(\tau_j|x_j) \frac{\partial f(\tau_j|x_j)}{\partial x_j}$$

Given $f(\tau_j = 1|x) = a_j x/(1+a_j x)$ simplify $\frac{\partial W_j}{\partial x_j}$ to

$$x_j - \left(\frac{a_j}{1 - (\beta a_j (EW^+(p_j) - EW^-(p_j)))^{-1/2}} - a_j \right)^{-1} = 0$$

where

$$EW^+(p_j) = \sum_{q'_{-j}, \Delta'} W_j(q_j + \delta, q'_{-j}, \Delta')\; h_f(q'_{-j}|q,\Delta)\; g_f(\Delta'|\Delta,q,p)$$

$$EW^-(p_j) = \sum_{q'_{-j}, \Delta'} W_j(q_j + 0, q'_{-j}, \Delta')\; h_f(q'_{-j}|q,\Delta)\; g_f(\Delta'|\Delta,q,p)$$