Conditional Independence and the Inversion Theorem

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- Each period $t \in \{1, 2, \ldots, T\}$ for $T \leq \infty$, an individual chooses among J mutually exclusive actions.
- Let d_{jt} equal one if action $j \in \{1, \ldots, J\}$ is taken at time t and zero otherwise:

$$d_{jt} \in \{0, 1\}$$

$$\sum_{j=1}^{J} d_{jt} = 1$$

- Suppose that actions taken at time t can potentially depend on the state $z_t \in Z$.
- The current period payoff at time t from taking action j is $u_{jt}^{*}(z_t)$.
- Given choices $(d_{1t}, \ldots, d_{Jt})$ in each period $t \in \{1, 2, \ldots, T\}$ the individual's expected utility is:

$$E\left\{\sum_{t=1}^{T}\sum_{j=1}^{J}\beta^{t-1}d_{jt}u_{jt}^{*}(z_t)\,|z_1\right\}$$

- Write the optimal decision rule as $d_t^o(z_t) \equiv (d_{1t}^o(z_t), \ldots, d_{Jt}^o(z_t))$.
- Denote the value function by $V_t^*(z_t)$:

$$\begin{aligned} V_t^*(z_t) &\equiv E\left\{\sum_{s=t}^{T}\sum_{j=1}^{J}\beta^{t-1} d_{js}^o(z_s)\, u_{js}^*(z_s) \,|z_t\right\} \\ &= \sum_{j=1}^{J} d_{jt}^o\left[u_{jt}^*(z_t) + \beta \int_{z_{t+1}} V_{t+1}^*(z_{t+1}) dF_{jt}(z_{t+1}\,|z_t)\right] \end{aligned}$$

- Let $v_{jt}^*(z_t)$ denote the flow payoff of action j plus the expected future utility of behaving optimally from period $t+1$ on:

$$v_{jt}^*(z_t) \equiv u_{jt}^*(z_t) + \beta \sum_{z_{t+1}=1}^{Z} V_{t+1}^*(z_{t+1}) dF_{jt}(z_{t+1}\,|z_t)$$

- Bellman's principle implies:

$$d_{jt}^o(z_t) \equiv \prod\nolimits_{k=1}^{K} I\left\{v_{jt}^*(z_t) \geq v_{kt}^*(z_t)\right\}$$

- Partition the states $z_t \equiv (x_t, \epsilon_t)$ into:
  - those which are observed, x_t
  - and those that are unobserved, ϵ_t.
- Without loss of generality we can express $u^*_{jt}(z_t)$ as the sum of its conditional expectation on the observed variables plus a residual:

$$u^*_{jt}(x_t, \epsilon_t) \equiv E\left[u^*_{jt}(x_t, \epsilon_t) \,|x_t\right] + \epsilon_{jt} \equiv u_{jt}(x_t) + \epsilon_{jt}$$

- For identification and estimation purposes we typically treat β, $u_{jt}(z_t)$, $dF_{jt}\left(z_{t+1} | z_t\right)$ and $dG\left(\epsilon_1 \,| x_1\right)$, the density/probability for ϵ_1, as the primititves to our model.
- We often index the family of models we are considering (and limiting our search to), by say Θ.

- The maximum likelihood (ML) estimator, $\theta_{ML} \in \Theta$ selects θ to maximize the joint probability (density) of the observed occurrences:

$$\prod_{n=1}^{N} \int_{\epsilon_T} \cdots \int_{\epsilon_1} \left[\begin{array}{l} \sum_{j=1}^{J} I\left\{d_{njT} = 1\right\} d_{jT}^{o}\left(x_{nT}, \epsilon_T\right) \times \\ \prod_{t=1}^{T-1} H_{nt}\left(x_{n,t+1}, \epsilon_{t+1} \left| x_{nt}, \epsilon_t\right.\right) dG\left(\epsilon_1 \left| x_{n1}\right.\right) \end{array} \right]$$

where:

$$\begin{array}{l} H_{nt}\left(x_{n,t+1}, \epsilon_{t+1} \left| x_{nt}, \epsilon_t\right.\right) \equiv \\ \sum_{j=1}^{J} I\left\{d_{njt} = 1\right\} d_{jt}^{o}\left(x_{nt}, \epsilon_t\right) dF_{jt}\left(x_{n,t+1}, \epsilon_{t+1} \left| x_{nt}, \epsilon_t\right.\right) \end{array}$$

is the probability (density) of the pair $\left(x_{n,t+1}, \epsilon_{t+1}\right)$ conditional on $\left(x_{nt}, \epsilon_t\right)$ when the observed choices are optimal for $\theta \in \Theta$.

- What are the computational challenges to large state space?
  1. Computing the value function;
  2. Solving for equilibrium in a multiplayer setting;
  3. Integrating over unobserved heterogeneity.
- These challenges suggest on several dimensions:
  1. Keep the dimension of the state space small;
  2. Assume all choices and outcomes are observed;
  3. Model unobserved states as a matter of computational convenience;
  4. Consider only one side of market to finesse equilibrium issues;
  5. Adopt parameterizations based on convenient functional forms.

- Suppose the transition of the observed variables does not depend on the unobserved variables for all (j, t, x_t, ϵ_t):

$$F_{jt}(x_{t+1} | x_t, \epsilon_t) = F_{jt}(x_{t+1} | x_t)$$

- Assuming x_{t+1} conveys all the information of x_t for the purposes of forming probability distributions at $t+1$:

$$\begin{aligned} F_{jt}(x_{t+1}, \epsilon_{t+1} | x_t, \epsilon_t) &\equiv G_{j,t+1}(\epsilon_{t+1} | x_{t+1}, x_t, \epsilon_t) F_{jt}(x_{t+1} | x_t, \epsilon_t) \\ &\equiv G_{j,t+1}(\epsilon_{t+1} | x_{t+1}, \epsilon_t) F_{jt}(x_{t+1} | x_t) \end{aligned}$$

- The ML estimator maximizes the same criterion function but $H_{nt}(x_{n,t+1}, \epsilon_{t+1} | x_{nt}, \epsilon_t)$ simplifies to:

$$H_{nt}(x_{n,t+1}, \epsilon_{t+1} | x_{nt}, \epsilon_t) =$$
$$\sum_{j=1}^{J} I\{d_{njt} = 1\} d_{jt}^{o}(x_{nt}, \epsilon_t) dG_{j,t+1}(\epsilon_{t+1} | x_{n,t+1}, \epsilon_t) dF_{jt}(x_{n,t+1} | x_{nt}$$

- Instead of estimating all the parameters at once, we could use a two stage estimator to reduce computation costs:
  1. Estimate $F_{jt}\left(x_{t+1}\left|x_t\right.\right)$ with a cell estimator (for x finite), a nonparametric estimator, or a parametric function;
  2. Define:
$$\widehat{H}_{nt}\left(x_{n,t+1},\epsilon_{t+1}\left|x_{nt},\epsilon_t\right.\right)\equiv \sum_{j=1}^{J}\left[\begin{array}{l} I\left\{d_{njt}=1\right\}d_{jt}^{o}\left(x_{nt},\epsilon_t\right) \\ \times dG_{j,t+1}\left(\epsilon_{t+1}\left|x_{n,t+1},\epsilon_t;\theta\right.\right)d\widehat{F}_{jt}\left(x_{n,t+1}\left|x_{nt}\right.\right)\end{array}\right]$$
  3. Select the remaining (preference) parameters to maximize:
$$\prod_{n=1}^{N}\int_{\epsilon_T}\cdots\int_{\epsilon_1}\left[\begin{array}{l}\sum_{j=1}^{J}I\left\{d_{njT}=1\right\}d_{jT}^{o}\left(x_{nT},\epsilon_T\right)\times \\ \prod_{t=1}^{T-1}\widehat{H}_{nt}\left(x_{n,t+1},\epsilon_{t+1}\left|x_{nt},\epsilon_t\right.\right)dG_1\left(\epsilon_1\left|x_{n1}\right.\right)\end{array}\right]$$
  4. Correct standard errors from the first stage estimator to account for the loss in asymptotic efficiency.

Conditional Independence

Conditional independence defined

- Separable transitions do not, however, free us from:
  1. the curse of multiple integration;
  2. numerical optimization to obtain the value function.
- Suppose in addition, that conditional on x_{t+1}, the unobserved variable ϵ_{t+1} is independent of (x_t, ϵ_t, d_t).
- Conditional independence embodies both assumptions:

$$
\begin{aligned}
dF_{jt}\left(x_{t+1} \left| x_t, \epsilon_t \right.\right) &= dF_{jt}\left(x_{t+1} \left| x_t \right.\right) \\
dG_{j,t+1}\left(\epsilon_{t+1} \left| x_{t+1}, x_t, \epsilon_t \right.\right) &= dG_{t+1}\left(\epsilon_{t+1} \left| x_{t+1} \right.\right)
\end{aligned}
$$

- It implies:

$$
dF_{jt}\left(x_{t+1}, \epsilon_{t+1} \left| x_t, \epsilon_t \right.\right) = dF_{jt}\left(x_{t+1} \left| x_t \right.\right) dG_{t+1}\left(\epsilon_{t+1} \left| x_{t+1} \right.\right)
$$

Conditional Independence

Exante value functions and conditional value functions defined

- Given conditional independence, define the exante valuation function as:

$$V_t(x_t) \equiv E\left[V_t^*(x_t, \epsilon_t) \left| x_t \right.\right]$$

and the conditional valuation function as:

$$v_{jt}(x_t) \equiv u_{jt}(x_t) + \beta \int_{x_{t+1}} V_{t+1}(x_{t+1}) dF_{jt}\left(x_{t+1} \left| x_t \right.\right)$$

- Optimal behavior implies that $d_{jt}^o(x_t, \epsilon) = 1$ if and only if:

$$\epsilon_{kt} - \epsilon_{jt} \leq v_{jt}(x_t) - v_{kt}(x_t)$$

for all $k \in \{1, \ldots, J\}$.

- Under conditional independence, the conditional choice probability (CCP) for action j is defined for each (t, x_t, j) as the probability of observing the j^{th} choice conditional on the values of the observed variables when behavior is optimal:

$$p_{jt}(x_t) \equiv E\left[d_{jt}^{o}(x_t, \epsilon_t)\,|x_t\right] = \int_{\epsilon_t} d_{jt}^{o}(x_{nt}, \epsilon_t)\, g_t(\epsilon_t\,|x_{nt})\, d\epsilon_t$$

  where we now assume (following the literature) that $G_t(\epsilon_t\,|x_{nt})$ has probability density function $g_t(\epsilon_t\,|x_{nt})$.

- The previous slide now implies:

$$p_{jt}(x_t) = \int_{\epsilon_t} \prod_{k=1}^{J} I\{\epsilon_{kt} - \epsilon_{jt} \leq v_{jt}(x_{nt}) - v_{kt}(x_{nt})\}\, g_t(\epsilon_t\,|x_t)\, d\epsilon_t$$

Conditional Independence

Simplifying expressions within the likelihood

- Conditional independence simplifies $H_{nt}\left(x_{n,t+1},\epsilon_{t+1}\left|x_{nt},\epsilon_t\right.\right)$ to:

$$H_{nt}\left(x_{n,t+1},\epsilon_{t+1}\left|x_{nt},\epsilon_t\right.\right) =$$
$$\sum_{j=1}^{J} I\left\{d_{njt}=1\right\} d_{jt}^{o}\left(x_{nt},\epsilon_t\right) g_{t+1}\left(\epsilon_{t+1}\left|x_{n,t+1}\right.\right) dF_{jt}\left(x_{n,t+1}\left|x_{nt}\right.\right)$$

- Also note that:

$$\begin{aligned} & \prod\nolimits_{t=1}^{T}\left\{\sum\nolimits_{j=1}^{J} I\left\{d_{njt}=1\right\} d_{jt}^{o}\left(x_{nt},\epsilon_t\right) dF_{jt}\left(x_{n,t+1}\left|x_{nt}\right.\right)\right\} \\ = & \prod\nolimits_{t=1}^{T}\left\{\sum\nolimits_{j=1}^{J} I\left\{d_{njt}=1\right\} dF_{jt}\left(x_{n,t+1}\left|x_{nt}\right.\right)\right\} \\ & \times\prod\nolimits_{t=1}^{T}\left\{\sum\nolimits_{j=1}^{J} I\left\{d_{njt}=1\right\} d_{jt}^{o}\left(x_{nt},\epsilon_t\right)\right\} \end{aligned}$$

- Hence the contribution of $n \in \{1, \ldots, N\}$ to the likelihood is the product of:

$$\prod_{t=1}^{T-1} \sum_{j=1}^{J} I\{d_{njt} = 1\} \, dF_{jt}(x_{n,t+1} | x_{nt})$$

and:

$$\int_{\epsilon_T} \cdots \int_{\epsilon_1} \prod_{t=1}^{T-1} \sum_{j=1}^{J} \left[\begin{array}{l} I\{d_{njt} = 1\} \, d_{jt}^{o}(x_{nt}, \epsilon_t) \\ \times g_{t+1}(\epsilon_{t+1} | x_{n,t+1}) \, g_1(\epsilon_1 | x_{n1}) \, d\epsilon_1 \ldots d\epsilon_T \end{array} \right]$$

$$= \prod_{t=1}^{T} \left[\sum_{j=1}^{J} I\{d_{njt} = 1\} \int_{\epsilon_t} d_{jt}^{o}(x_{nt}, \epsilon_t) \, g_t(\epsilon_t | x_{nt}) \, d\epsilon_t \right]$$

Conditional Independence

A compact expression for the ML criterion function

- Since:

$$p_{jt}\left(x_t\right) \equiv \int_{\epsilon_t} d_{jt}^o\left(x_{nt}, \epsilon_t\right) g_t\left(\epsilon_t \left| x_{nt}\right.\right) d\epsilon_t = E\left[d_{jt}^o\left(x_t, \epsilon_t\right) \left| x_t\right.\right]$$

the log likelihood can now be compactly expressed as:

$$\sum_{n=1}^{N}\sum_{t=1}^{T-1}\sum_{j=1}^{J} I\left\{d_{njt}=1\right\} \ln\left[dF_{jt}\left(x_{n,t+1}\left| x_{nt}\right.\right)\right] + \sum_{n=1}^{N}\sum_{t=1}^{T}\sum_{j=1}^{J} I\left\{d_{njt}=1\right\} \ln p_{jt}\left(x_t\right)$$

- Suppose we only had data on the last period T, and wished to estimate the preferences determining choices in T.
- By definition this is a static problem in which $v_{jT}(x_T) \equiv u_{jT}(x_T)$.
- For example to the probability of observing the J^{th} choice is:

$$p_{JT}(x_T) \equiv \int_{-\infty}^{\epsilon_{JT}+u_{JT}(x_T)-u_{1T}(x_T)} \cdots \int_{-\infty}^{\epsilon_{JT}+u_{JT}(x_T)-u_{J-1,T}(x_T)} \int_{-\infty}^{\infty} g_T(\epsilon_T | x_T)\, d\epsilon_T$$

- The only essential difference between a estimating a static discrete choice model using ML and a estimating a dynamic model satisfying conditional independence using ML is that parametrizations of $v_{jt}(x_t)$ based on $u_{jt}(x_t)$ do not have a closed form, but must be computed numerically.

- The starting point for our analysis is to define differences in the conditional valuation functions as:

$$\Delta v_{jkt}\left(x\right) \equiv v_{jt}\left(x\right) - v_{kt}\left(x\right)$$

- Although there are $J\left(J-1\right)$ differences all but $\left(J-1\right)$ are linear combinations of the $\left(J-1\right)$ basis functions.
- For example setting the basis functions as:

$$\Delta v_{jt}\left(x\right) \equiv v_{jt}\left(x\right) - v_{Jt}\left(x\right)$$

  then clearly:

$$\Delta v_{jkt}\left(x\right) = \Delta v_{jt}\left(x\right) - \Delta v_{kt}\left(x\right)$$

- Without loss of generality we focus on this particular basis function.

Inversion

Each CCP is a mapping of differences in the conditional valuation functions

- Using the definition of $\Delta v_{jt}(x)$:

$$
\begin{aligned}
p_{jt}(x) &\equiv \int d_{jt}^{o}(x,\epsilon)\, g_t(\epsilon\,|x)\, d\epsilon \\
&= \int I\left\{\epsilon_k \le \epsilon_j + \Delta v_{jt}(x) - \Delta v_{kt}(x) \forall\ k \neq j\right\} g_t(\epsilon\,|x)\, d\epsilon \\
&= \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)-\Delta v_{1t}(x)} \ldots \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)-\Delta v_{J-1,t}(x)} \int_{-\infty}^{\epsilon_j+\Delta v_{jt}(x)} g_t(\epsilon\,|x)\, d\epsilon
\end{aligned}
$$

- Noting $g_t(\epsilon\,|x) \equiv \partial^J G_t(\epsilon\,|x)/\partial\epsilon_1,\ldots,\partial\epsilon_J$, integrate over $(\epsilon_1,\ldots,,\epsilon_{j-1},\epsilon_{j+1}\ldots,\epsilon_J)$.
- Denoting $G_{jt}(\epsilon\,|x) \equiv \partial G_t(\epsilon\,|x)/\partial\epsilon_j$, yields:

$$
p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt}\left(\begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \ldots \\ \ldots, \epsilon_j, \ldots, \epsilon_j + \Delta v_{jt}(x) \end{array} |x \right) d\epsilon_j
$$

Inversion

There are as many CCPs as there are conditional valuation functions

- For any vector $J-1$ dimensional vector $\delta \equiv (\delta_1, \ldots, \delta_{J-1})$ define:

$$Q_{jt}(\delta, x) \equiv \int_{-\infty}^{\infty} G_{jt}\left(\epsilon_j + \delta_j - \delta_1, \ldots, \epsilon_j, \ldots, \epsilon_j + \delta_j \,|x\right) d\epsilon_j$$

- We interpret $Q_{jt}(\delta, x)$ as the probability taking action j in a static random utility model (RUM) where the payoffs are $\delta_j + \epsilon_j$ and the probability distribution of disturbances is given by $G_t(\epsilon \,|x)$.
- It follows from the definition of $Q_{jt}(\delta, x)$ that:

$$0 \leq Q_{jt}(\delta, x) \leq 1 \text{ for all } (j, t, \delta, x) \text{ and } \sum_{j=1}^{J-1} Q_{jt}(\delta, x) \leq 1$$

- In particular the previous slide implies that for any given (j, t, x):

$$p_{jt}(x) = \int_{-\infty}^{\infty} G_{jt}\left(\begin{array}{c} \epsilon_j + \Delta v_{jt}(x) - \Delta v_{1t}(x), \\ \ldots, \epsilon_j, \ldots, \epsilon_j + \Delta v_{jt}(x) \end{array} \,|x\right) d\epsilon_j \equiv Q_{jt}(\Delta v_t(x), x)$$

Theorem (Inversion)

For each (t, δ, x) *define:*

$$Q_t(\delta, x) \equiv (Q_{1t}(\delta, x), \ldots Q_{J-1,t}(\delta, x))'$$

Then the vector function $Q_t(\delta, x)$ *is invertible in* δ *for each* (t, x).

- Note that $p_{Jt}(x) = Q_{Jt}(\Delta v_t, x)$ is a linear combination of the other equations in the system because $\sum_{k=1}^{J} p_k = 1$.
- Let $p \equiv (p_1, \ldots, p_{J-1})$ where $0 \leq p_j \leq 1$ for all $j \in \{1, \ldots, J-1\}$ and $\sum_{j=1}^{J-1} p_j \leq 1$. Denote the inverse of $Q_{jt}(\Delta v_t, x)$ by $Q_{jt}^{-1}(p, x)$.
- The inversion theorem implies:

$$\begin{bmatrix} \Delta v_{1t}(x) \\ \vdots \\ \Delta v_{J-1,t}(x) \end{bmatrix} = \begin{bmatrix} Q_{1t}^{-1}[p_t(x), x] \\ \vdots \\ Q_{J-1,t}^{-1}[p_t(x), x] \end{bmatrix}$$

- In what sense does the inversion theorem help us to finesse optimization and integration by exploiting conditional independence?
- We use the Inversion Theorem to:
    1. provide empirically tractable representations of the conditional value functions.
    2. analyze identification in dynamic discrete choice models.
    3. provide convenient parametric forms for the density of ϵ_t that generalize the Type 1 Extreme Value distribution.
    4. provide cheap estimators for dynamic discrete choice models and dynamic discrete choice games of incomplete information.
    5. introduce new methods for incorporating unobserved state variables.

- From the definition of the optimal decision rule, and then appealing to the inversion theorem:

$$
\begin{aligned}
d_{jt}^{o}(x_t, \epsilon_t) &= \prod_{k=1}^{J} 1\{\epsilon_{kt} - \epsilon_{jt} \leq v_{jt}(x) - v_{kt}(x)\} \\
&= \prod_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq \begin{array}{l} v_{jt}(x) - v_{Jt}(x_t) \\ - [v_{kt}(x) - v_{Jt}(x_t)] \end{array}\right\} \\
&= \prod_{k=1}^{J} 1\{\epsilon_{kt} - \epsilon_{jt} \leq \Delta v_{jt}(x) - \Delta v_{kt}(x)\} \\
&= \prod_{k=1}^{J} 1\left\{\epsilon_{kt} - \epsilon_{jt} \leq Q_{jt}^{-1}[p_t(x), x] - Q_{kt}^{-1}[p_t(x), x]\right\}
\end{aligned}
$$

- If $G_t(\epsilon | x)$ is known and the data generating process (DGP$)$ is (x_t, d_t), then $p_t(x)$ and hence $d_t^o(x_t, \epsilon_t)$ are identified.

Corollaries of the Inversion Theorem

Definition of the conditional value function correction

- Define the conditional value function correction as:

$$\psi_{jt}(x) \equiv V_t(x) - v_{jt}(x)$$

- In stationary settings, we drop the t subscript and write:

$$\psi_j(x) \equiv V(x) - v_j(x)$$

- Suppose that instead of taking the optimal action she committed to taking action j instead. Then the expected lifetime utility would be:

$$v_{jt}(x_t) + E_t[\epsilon_{jt} | x_t]$$

so committing to j before ϵ_t is revealed entails a loss of:

$$V_t(x_t) - v_{jt}(x_t) - E_t[\epsilon_{jt} | x_t] = \psi_{jt}(x) - E_t[\epsilon_{jt} | x_t]$$

- For example if $E_t[\epsilon_t | x_t] = 0$, the loss simplifies to $\psi_{jt}(x)$.

Corollaries of the Inversion Theorem

Identifying the conditional value function correction

- From their respective definitions:

$$
\begin{aligned}
& V_t(x) - v_{it}(x) \\
= \; & \sum_{j=1}^{J} \left\{ p_{jt}(x) \left[v_{jt}(x) - v_{it}(x) \right] + \int \epsilon_{jt} d_{jt}^{o} \left(x_t, \epsilon_t \right) g_t \left(\epsilon_t \left| x \right. \right) d\epsilon_t \right\}
\end{aligned}
$$

- But:

$$
v_{jt}(x) - v_{it}(x) = Q_{jt}^{-1} \left[p_t(x), x \right] - Q_{it}^{-1} \left[p_t(x), x \right]
$$

and

$$
\begin{aligned}
& \int \epsilon_{jt} d_{jt}^{o} \left(x, \epsilon_t \right) g \left(\epsilon_t \left| x \right. \right) d\epsilon_t \\
= \; & \int \prod_{k=1}^{J} 1 \left\{ \begin{array}{l} \epsilon_{kt} - \epsilon_{jt} \\ \leq Q_{jt}^{-1} \left[p_t(x), x \right] - Q_{kt}^{-1} \left[p_t(x), x \right] \end{array} \right\} \epsilon_{jt} g_t \left(\epsilon_t \left| x \right. \right) d\epsilon_t
\end{aligned}
$$

- Therefore $\psi_{it}(x) \equiv V_t(x) - v_{it}(x)$ is identified if $G_t \left(\epsilon \left| x \right. \right)$ is known.

Conditional Valuation Function Representation

Telescoping one period forward

- From its definition:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^{X} V_{t+1}(x) f_{jt}\left(x_{t+1} | x_t\right)$$

- Substituting for $V_{t+1}(x_{t+1})$ using conditional value function correction we obtain for any k:

$$v_{jt}(x_t) = u_{jt}(x_t) + \beta \sum_{x=1}^{X} \left[v_{k,t+1}(x) + \psi_{k,t+1}(x) \right] f_{jt}\left(x | x_t\right)$$

- We could repeat this procedure ad infinitum, substituting in for $v_{k,t+1}(x)$ by using the definition for $\psi_{kt}(x)$.

Conditional Valuation Function Representation

Recursively defining the distribution of future state variables

- To formalize this idea, consider a random sequence of weights from t to T which begins with $\omega_{jt}(x_t, j) = 1$.
- For periods $\tau \in \{t+1, \ldots, T\}$, the choice sequence maps x_τ and the initial choice j into

$$\omega_\tau(x_\tau, j) \equiv \{\omega_{1\tau}(x_\tau, j), \ldots, \omega_{J\tau}(x_\tau, j)\}$$

where $\omega_{k\tau}(x_\tau, j)$ may be negative or exceed one but:

$$\sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j) = 1$$

.

- The weight of state $x_{\tau+1}$ conditional on following the choices in the sequence is recursively defined by $\kappa_t(x_{t+1}|x_t, j) \equiv f_{jt}(x_{t+1}|x_t)$ and for $\tau = t+1, \ldots, T$:

$$\kappa_\tau(x_{\tau+1}|x_t, j) \equiv \sum_{x_\tau=1}^{X} \sum_{k=1}^{J} \omega_{k\tau}(x_\tau, j)\, f_{k\tau}(x_{\tau+1}|x_\tau)\kappa_{\tau-1}(x_\tau|x_t, j)$$

Theorem (Representation)

For any state $x_t \in \{1, \ldots, X\}$, choice $j \in \{1, \ldots, J\}$ and weights $\omega_\tau(x_\tau, j)$ defined for periods $\tau \in \{t, \ldots, T\}$:

$$\begin{aligned} v_{jt}(x_t) &= u_{jt}(x_t) \\ &+ \sum_{\tau=t+1}^{T} \sum_{k=1}^{J} \sum_{x=1}^{X} \beta^{\tau-t} \left[u_{k\tau}(x) + \psi_k[p_\tau(x)] \right] \omega_{k\tau}(x, j) \kappa_{\tau-1}(x|x_t, j \end{aligned}$$

- The theorem yields an alternative expression for $v_{jt}(x_t)$ that dispenses with recursive maximization.
- Intuitively, the individuals have already solved their optimization problem, so their decisions, as reflected in their CCPs, are informative of their value functions.

Generalized Extreme Values

Definition

- Can we exploit this representation in identification and estimation?
- To make the approach operational requres us to compute $\psi_k(p)$ for at least some k.
- Suppose ϵ is drawn from the GEV distribution function:

$$G(\epsilon_1, \epsilon_2, \ldots, \epsilon_J) \equiv \exp\left[-\mathcal{H}\left(\exp[-\epsilon_1], \exp[-\epsilon_2], \ldots, \exp[-\epsilon_J]\right)\right]$$

where $\mathcal{H}(Y_1, Y_2, \ldots, Y_J)$ satisfies the following properties:

1. $\mathcal{H}(Y_1, Y_2, \ldots, Y_J)$ is nonnegative, real valued, and homogeneous of degree one;
2. $\lim \mathcal{H}(Y_1, Y_2, \ldots, Y_J) \to \infty$ as $Y_j \to \infty$ for all $j \in \{1, \ldots, J\}$;
3. for any distinct $(i_1, i_2, \ldots, i_r)$ the cross derivative $\partial\mathcal{H}(Y_1, Y_2, \ldots, Y_J) / \partial Y_{i_1}, Y_{i_2}, \ldots, Y_{i_r}$ is nonnegative for r odd and nonpositive for r even.

- Suppose $G(\epsilon)$ factors into two independent distributions, one a nested logit, and the other any GEV distribution.
- Let $\mathcal{J}$ denote the set of choices in the nest and denote the other distribution by $G_0(Y_1, Y_2, \ldots, Y_K)$ let K denote the number of choices that are outside the nest.
- Then:

$$G(\epsilon) \equiv G_0(\epsilon_1, \ldots, \epsilon_K) \exp\left[-\left(\sum_{j\in\mathcal{J}} \exp\left[-\epsilon_j/\sigma\right]\right)^{\sigma}\right]$$

- The correlation of the errors within the nest is given by $\sigma \in [0, 1]$ and errors within the nest are uncorrelated with errors outside the nest. When $\sigma = 1$, the errors are uncorrelated within the nest, and when $\sigma = 0$ they are perfectly correlated.

Generalized Extreme Values

Lemma 2 of Arcidiacono and Miller (2011)

- Define $\phi_j(Y)$ as a mapping into the unit interval where

$$\phi_j(Y) = Y_j \mathcal{H}_j \left(Y_1, \ldots, Y_J\right) / \mathcal{H} \left(Y_1, \ldots, Y_J\right)$$

- Since $\mathcal{H}_j \left(Y_1, \ldots, Y_J\right)$ and $\mathcal{H} \left(Y_1, \ldots, Y_J\right)$ are homogeneous of degree zero and one respectively, $\phi_j(Y)$ is a probability, because $\phi_j(Y) \geq 0$ and $\sum_{j=1}^{J} \phi_j(Y) = 1$.

Lemma (GEV correction factor)

When ϵ_t is drawn from a GEV distribution, the inverse function of $\phi(Y) \equiv (\phi_2(Y), \ldots \phi_J(Y))$ exists, which we now denote by $\phi^{-1}(p)$, and:

$$\psi_j(p) = \ln \mathcal{H} \left[1, \phi_2^{-1}(p), \ldots, \phi_J^{-1}(p)\right] - \ln \phi_j^{-1}(p) + \gamma$$

Generalized Extreme Values

Correction factor for extended nested logit

Lemma

For the nested logit $G(\epsilon_t)$ defined above:

$$\psi_j(p) = \gamma - \sigma \ln(p_j) - (1-\sigma)\ln\left(\sum_{k\in\mathcal{J}} p_k\right)$$

- Note that $\psi_j(p)$ only depends on the conditional choice probabilities for choices that are in the nest: the expression is the same no matter how many choices are outside the nest or how those choices are correlated.
- Hence, $\psi_j(p)$ will only depend on $p_{j'}$ if ϵ_{jt} and $\epsilon_{j't}$ are correlated. When $\sigma = 1$, ϵ_{jt} is independent of all other errors and $\psi_j(p)$ only depends on p_j.

Adapting Dynamic Games to the CCP Framework

Players and choices

- This framework naturally lends itself to studying equilibrium in games of incompete information.
- For example consider a dynamic infinite horizon game for finite I players.
- Thus $T = \infty$ and $I < \infty$.
- Each player $i \in I$ makes a choice $d_t^{(i)} \equiv \left(d_{1t}^{(i)}, \ldots, d_{Jt}^{(i)}\right)$ in period t.
- Denote the choices of all the players in period t by:

$$d_t \equiv \left(d_t^{(1)}, \ldots, d_t^{(I)}\right)$$

and denote by:

$$d_t^{(-i)} \equiv \left(d_t^{(1)}, \ldots, d_t^{(i-1)}, d_t^{(i+1)}, \ldots, d_t^{(I)}\right)$$

the choices of $\{1, \ldots, i-1, i+1, \ldots, I\}$ in period t, that is all the players apart from i.

- Denote by x_t the state variables of the game that are not *iid*.
- For example x_t includes the capital of every firm. Then:
  - firms would have the same state variables.
  - x_t would affect rivals in very different ways.
- We assume all the players observe x_t.
- Denote by $F\left(x_{t+1}\left|x_t, d_t\right.\right)$ the probability of x_{t+1} occurs when the state variables are x_t and the players collectively choose d_t.
- Similarly let:

$$F_j\left(x_{t+1}\left|x_t, d_t^{(-i)}\right.\right) \equiv F\left(x_{t+1}\left|x_t, d_t^{(-i)}, d_{jt}^{(i)} = 1\right.\right)$$

denote the probability distribution determining x_{t+1} given x_t when $\{1, \dots, i-1, i+1, \dots, I\}$ choose $d_t^{(-i)}$ in t and i makes choice j.

- Suppose $\epsilon_t^{(i)} \equiv \left(\epsilon_{1t}^{(i)}, \ldots, \epsilon_{Jt}^{(i)}\right)$, identically and independently distributed with density $g\left(\epsilon_t^{(i)}\right)$, affects the payoffs of i in t.
- Also let $\epsilon_t^{(-i)} \equiv \left(\epsilon_t^{(1)}, \ldots, \epsilon_t^{(i-1)}, \epsilon_t^{(i+1)}, \ldots, \epsilon_t^{(I)}\right)$.
- The systematic component of current utility or payoff to player i in period t form taking choice j when everybody else chooses $d_t^{(-i)}$ and the state variables are z_t is denoted by $U_j^{(i)}\left(x_t, d_t^{(-i)}\right)$.
- Denoting by $\beta \in (0, 1)$ the discount factor, the summed discounted payoff to player i throughout the course of the game is:

$$\sum_{t=1}^{T} \sum_{j=1}^{J} \beta^{t-1} d_{jt}^{(i)} \left[U_j^{(i)}\left(x_t, d_t^{(-i)}\right) + \epsilon_{jt}^{(i)}\right]$$

- Players noncooperatively maximize their expected utilities, moving simultaneously each period. Thus i does not condition on $d_t^{(-i)}$ when making his choice at date t, but only sees $\left(x_t, \epsilon_t^{(i)}\right)$.

- This is a stationary environment and we focus on Markov decision rules, which can be expressed $d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)$.
- Let $d^{(-i)}\left(x_t, \epsilon_t^{(-i)}\right)$ denote the strategy of every player but i:

$$\left(\begin{array}{r} d^{(1)}\left(x_t, \epsilon_t^{(1)}\right), \ldots, d^{(i-1)}\left(x_t, \epsilon_t^{(i-1)}\right), d^{(i+1)}\left(x_t, \epsilon_t^{(i+1)}\right), \\ d^{(i+2)}\left(x_t, \epsilon_t^{(i+2)}\right) \ldots, d^{(I)}\left(x_t, \epsilon_t^{(I)}\right) \end{array} \right)$$

- Then the expected value of the game to i from playing $d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)$ when everyone else plays $d\left(x_t, \epsilon_t^{(-i)}\right)$ is:

$$V^{(i)}\left(x_1\right) \equiv$$
$$E\left\{\sum_{t=1}^{\infty} \sum_{j=1}^{J} \beta^{t-1} d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right)\left[U_j^{(i)}\left(z_t, d\left(x_t, \epsilon_t^{(-i)}\right)\right) + \epsilon_{jt}^{(i)}\right] \middle| x_1\right\}$$

- Integrating over $\epsilon_t^{(i)}$ we obtain the j^{th} conditional choice probability for the i^{th} player at t as $p_j^{(i)}(x_t)$:

$$p_j^{(i)}(x_t) = \int d_j^{(i)}\left(x_t, \epsilon_t^{(i)}\right) g\left(\epsilon_t^{(i)}\right) d\epsilon_t^{(i)}$$

- Let $P\left(d_t^{(-i)} | x_t\right)$ denote the joint probability firm i's competitors choose $d_t^{(-i)}$ conditional on the state variables z_t.
- Since $\epsilon_t^{(i)}$ is distributed independently across $i \in \{1, \ldots, I\}$:

$$P\left(d_t^{(-i)} | x_t\right) = \prod_{\substack{i'=1 \\ i' \neq i}}^{I} \left(\sum_{j=1}^{J} d_{jt}^{(i')} p_j^{(i')}(x_t)\right)$$

- The strategy $\left\{ d^{(i)} \left(x_t, \epsilon_t^{(i)} \right) \right\}_{i=1}^{I}$ is a Markov perfect equilibrium if, for all $\left(i, x_t, \epsilon_t^{(i)} \right)$, the best response of i to $d^{(-i)} \left(x_t, \epsilon_t^{(-i)} \right)$ is $d^{(i)} \left(x_t, \epsilon_t^{(i)} \right)$ when everybody uses the same strategy thereafter.
- That is, suppose the other players collectively use $d^{(-i)} \left(x_t, \epsilon_t^{(-i)} \right)$ in period t, and $V^{(i)} (x_{t+1})$ is formed from $\left\{ d^{(i)} \left(x_t, \epsilon_t^{(i)} \right) \right\}_{i=1}^{I}$.
- Then $d^{(i)} \left(x_t, \epsilon_t^{(i)} \right)$ solves for i choosing j to maximize:

$$\sum_{d_t^{(-i)}} P \left(d_t^{(-i)} \middle| x_t \right) \left\{ \begin{array}{l} U_j^{(i)} \left(x_t, d_t^{(-i)} \right) \\ +\beta \sum_{z=1}^{X} V^{(i)} (x) F_j \left(x \middle| x_t, d_t^{(-i)} \right) \end{array} \right\} + \epsilon_{jt}^{(i)}$$

- In equilibrium, the systematic component of the current utility of player i in period t, as a function of x_t, the state variables for game, and his own decision j, is:

$$u_j^{(i)}(x_t) = \sum_{d_t^{(-i)}} P\left(d_t^{(-i)} \left| x_t\right.\right) U_j^{(i)}\left(x_t, d_t^{(-i)}\right)$$

- Similarly the probability transition from x_t to x_{t+1} given action j by firm i is given by:

$$f_j^{(i)}\left(x_{t+1}\left|x_t^{(i)}\right.\right) = \sum_{d_t^{(-i)}} P\left(d_t^{(-i)}\left|x_t^{(i)}\right.\right) F_j\left(x_{t+1}\left|x_t, d_t^{(-i)}\right.\right)$$

- The setup for player i is now identical to the optimization problem described in the second lecture for a stationary environment.

- Both theorems apply to this multiagent setting with two critical differences, and both are relevant for studying identification:
  1. $u_{jt}(x_t)$ is a primitive in single agent optimization problems, but $u_{jt}^{(i)}(x_t)$ is a reduced form parameter found by integrating $U_{jt}^{(i)}\left(x_t, d_t^{(\sim i)}\right)$ over the joint probability distribution $P_t\left(d_t^{(\sim i)} \left| x_t \right.\right)$.
  2. $f_{jt}\left(x_{t+1} \left| x_t \right.\right)$ is a primitive in single agent optimization problems, but $f_{jt}^{(i)}\left(x_{t+1} \left| x_t \right.\right)$ depends on CCPs of the other players, $P_t\left(d_t^{(\sim i)} \left| x_t \right.\right)$, as well as the primitive $F_{jt}\left(x_{t+1} \left| x_t, d_t^{(\sim i)} \right.\right)$. It is easy to interpret restrictions placed directly on $f_{jt}\left(x_{t+1} \left| x_t \right.\right)$ but placing restrictions on $F_{jt}\left(x_{t+1} \left| x_t, d_t^{(\sim i)} \right.\right)$ complicates matters in dynamic games because of the endogenous effects arising from $P_t\left(d_t^{(\sim i)} \left| x_t \right.\right)$ on $f_{jt}^{(i)}\left(x_{t+1} \left| x_t \right.\right)$.