6 Electromagnetic Fields and Waves

James Clerk Maxwell's unification of electromagnetic phenomena, published in 1865, is perhaps the best example of a successful modern scientific theory [Maxwell, 1998]. In just a few simple equations he was able to show that the apparently distinct phenomena of electricity and magnetism were actually intimately related through a common theoretical framework that contained unexpected predictions, such as electromagnetic waves, and led to significant succeeding discoveries including the theory of special relativity. This chapter studies these *Maxwell's equations* for static and time-varying electric and magnetic fields. This theory is called *electrodynamics* because it describes the time variation of electromagnetic phenomena, and it will be the foundation of much the rest of the book.

6.1 VECTOR CALCULUS

Working with Maxwell's equations will require differentiating and integrating field vectors, and so our first step will be a review of the necessary vector calculus.

6.1.1 Vectors

Let $\vec{x} \equiv (x, y, z) \equiv (x_1, x_2, x_3)$ be the coordinates of a point expressed as a vector in rectangular coordinates (Figure 6.1). The *magnitude* of a vector is

$$|\vec{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} \quad . \tag{6.1}$$

For two vectors $\vec{A}$ and $\vec{B}$ with an angle θ between them, the *dot product* measures their overlap:

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= |A||B| \cos(\theta) \\
&= A_1 B_1 + A_2 B_2 + A_3 B_3 \\
&= \sum_{i=1}^{3} A_i B_i \\
&\equiv A_i B_i \quad .
\end{aligned} \tag{6.2}$$

Because such sums recur frequently in manipulating vectors, the last line introduces the *Einstein summation convention* of summing over repeated indices.

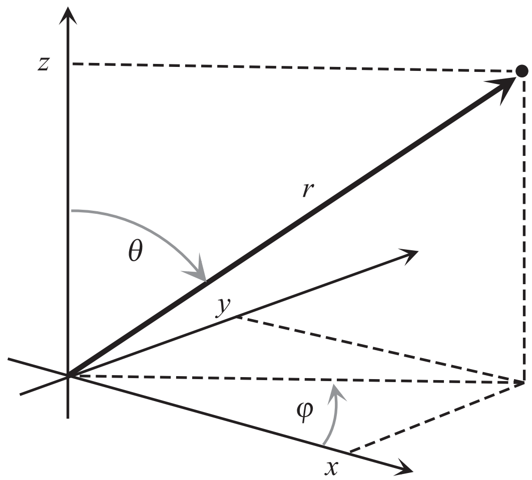

Figure 6.1. Rectangular, cylindrical, and spherical coordinate systems.

The *cross product* of these vectors is

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$
$$= (A_2B_3 - A_3B_2)\hat{x}_1 + (A_3B_1 - A_1B_3)\hat{x}_2 + (A_1B_2 - A_2B_1)\hat{x}_3 \quad , \quad (6.3)$$

where $\hat{x}_1$ (pronounced "x hat") is a unit vector in the x_1 direction. The magnitude of the cross product is equal to the product of the lengths of the vectors times the sine of the angle between them, and its direction is perpendicular to the plane containing the vectors. The orientation can be remembered by the *right hand rule*: if the fingers of your right hand curl from $\vec{A}$ towards $\vec{B}$, your thumb points in the direction of their cross product. The ith term of the cross product can be written in terms of the summation convention as

$$(\vec{A} \times \vec{B})_i = \epsilon_{ijk} A_j B_k \quad (6.4)$$

by using the *antisymmetric tensor*

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } (ijk) = (123),\ (231),\ \text{or } (312)\ \text{(cyclic permutation)} \\ -1 & \text{if } (ijk) = (132),\ (321),\ \text{or } (213)\ \text{(anticyclic permutation)} \\ 0 & \text{otherwise} \end{cases} \quad . \quad (6.5)$$

Interchanging indices shows that the cross product is *anticommutative*:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \quad . \quad (6.6)$$

A useful expansion for the product of antisymmetric tensors is

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \quad , \quad (6.7)$$

where δ_{ij} is the *Kroenecker delta*

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad . \quad (6.8)$$

6.1.2 Differential Operators

Now let $\varphi(\vec{x})$ be a scalar function of $\vec{x}$. The *gradient* of φ is defined to be

$$\nabla\varphi(\vec{x}) = \frac{\partial\varphi(\vec{x})}{\partial x_1}\,\hat{x}_1 + \frac{\partial\varphi(\vec{x})}{\partial x_2}\,\hat{x}_2 + \frac{\partial\varphi(\vec{x})}{\partial x_3}\,\hat{x}_3 \quad . \tag{6.9}$$

The gradient is a vector that points in the direction of the fastest change of φ, and its magnitude is equal to the rate of change in that direction. If $\varphi(\vec{x})$ is the height of a hill, then a ball released at $\vec{x}$ would roll down the hill in the direction $-\nabla\varphi(\vec{x})$. The vector operator ∇ is called "del."

For a vector-valued function $\vec{A}(\vec{x}) = (A_1(\vec{x}), A_2(\vec{x}), A_3(\vec{x}))$, the *divergence* is

$$\begin{aligned}\nabla\cdot\vec{A}(\vec{x}) &= \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} \\ &= \sum_i \frac{\partial A_i}{\partial x_i} \\ &\equiv \sum_i \partial_i A_i \\ &= \partial_i A_i \quad .\end{aligned} \tag{6.10}$$

The divergence is a number that measures the rate at which the vector field is locally expanding or contracting.

The *curl* of a vector field in three dimensions is defined to be

$$\nabla\times\vec{A} = (\partial_2 A_3 - \partial_3 A_2)\hat{x}_1 + (\partial_3 A_1 - \partial_1 A_3)\hat{x}_2 + (\partial_1 A_2 - \partial_2 A_1)\hat{x}_3 \quad . \tag{6.11}$$

The curl points in the direction of circulation of the vector field. Written in the summation convention,

$$(\nabla\times\vec{A})_i = \epsilon_{ijk}\partial_j A_k \quad . \tag{6.12}$$

Plugging in the definitions shows that the curl of a gradient vanishes,

$$(\nabla\times\nabla\varphi)_i = \epsilon_{ijk}\partial_j\partial_k\varphi = 0 \quad , \tag{6.13}$$

as does the divergence of a curl,

$$\nabla\cdot\nabla\times\vec{A} = \epsilon_{ijk}\partial_i\partial_j A_k = 0 \quad . \tag{6.14}$$

The *Laplacian* of a scalar quantity is

$$\begin{aligned}\nabla^2\varphi &= \nabla\cdot\nabla\varphi \\ &= \frac{\partial^2\varphi}{\partial x_1^2} + \frac{\partial^2\varphi}{\partial x_2^2} + \frac{\partial^2\varphi}{\partial x_3^2} \\ &= \partial_i\partial_i\varphi \quad .\end{aligned} \tag{6.15}$$

It measures the curvature at a point. The Laplacian of a vector is a vector-valued quantity defined to be the Laplacian of each of the components of the vector

$$(\nabla^2\vec{A})_j = \partial_i\partial_i A_j \quad . \tag{6.16}$$

In addition to rectangular coordinates, there are two other common coordinate systems, *cylindrical* (r, φ, z) and *spherical* (r, θ, φ), also shown in Figure 6.1. When the coordinate system reflects the symmetries of a problem the math is much simpler. Inserting the trigonometric relationships among the varaibles into the definitions of the differential operators in rectangular coordinates, and taking the appropriate partial derivatives, shows that in cylindrical and spherical coordinates a differential volume element is

$$\begin{aligned} dV &= dx\ dy\ dz \\ &= r\ dr\ d\theta\ dz \\ &= r^2 \sin\theta\ dr\ d\theta\ d\varphi \quad , \end{aligned} \tag{6.17}$$

the gradient is

$$\begin{aligned} \nabla\Phi &= \frac{\partial\Phi}{\partial x}\hat{x} + \frac{\partial\Phi}{\partial y}\hat{y} + \frac{\partial\Phi}{\partial z}\hat{z} \\ &= \frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\varphi}\hat{\varphi} + \frac{\partial\Phi}{\partial z}\hat{z} \\ &= \frac{\partial\Phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\varphi}\hat{\varphi} \quad , \end{aligned} \tag{6.18}$$

the divergence is

$$\begin{aligned} \nabla\cdot\vec{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial\varphi} + \frac{\partial A_z}{\partial z} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial\varphi} \quad , \end{aligned} \tag{6.19}$$

the Laplacian is

$$\begin{aligned} \nabla^2\Phi &= \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} \\ &= \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right)}_{\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2} \quad , \end{aligned} \tag{6.20}$$

and the curl is

$$\begin{aligned} \nabla\times\vec{A} &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \\ &= \left(\frac{1}{r}\frac{\partial A_z}{\partial\varphi} - \frac{\partial A_\varphi}{\partial z}\right)\hat{r} + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)\hat{\varphi} + \frac{1}{r}\left(\frac{\partial(rA_\varphi)}{\partial r} - \frac{\partial A_r}{\partial\varphi}\right)\hat{z} \\ &= \frac{1}{r\sin\theta}\left(\frac{\partial(A_\varphi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\varphi}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial(rA_\varphi)}{\partial r}\right)\hat{\theta} \\ &\quad + \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\varphi} \quad . \end{aligned} \tag{6.21}$$

Finally, Table 6.1 lists some vector and trigonometric relationships that will be needed later.

Table 6.1. *Vector and trigonometric identities.*

$\nabla \times \nabla\varphi = 0$
$\nabla \cdot (\nabla \times \vec{A}) = 0$
$\nabla \cdot (\varphi\vec{A}) = \vec{A} \cdot \nabla\varphi + \varphi\nabla \cdot \vec{A}$
$\nabla \times (\varphi\vec{A}) = \nabla\varphi \times \vec{A} + \varphi\nabla \times \vec{A}$
$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$
$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$
$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
$\cos(2A) = \cos^2 A - \sin^2 A$
$\sin(2A) = 2 \sin A \cos A$
$\sin^2 A + \cos^2 A = 1$

6.1.3 Integral Relationships

The differential operators introduced in the last section measure the local properties of scalar and vector fields. Not surprisingly, there are intimate relationships between these local properties and the global properties of the fields. These will be very useful for relating fields to their sources. Without proof, here are two important cases of general theorems relating local and global properties:

- *Divergence Theorem* (or *Gauss' Theorem*)

$$\int_V \nabla \cdot \vec{E}\, dV = \int_S \vec{E} \cdot d\vec{A} \quad . \tag{6.22}$$

  The volume integral of the divergence is equal to the surface integral of the normal component of the field. V is an arbitrary volume, S is its surface, dV is a volume element, and $d\vec{A}$ is a surface area element. $d\vec{A}$ is the same as $\hat{n}\, dA$, where $\hat{n}$ is an outward-pointing unit vector that is perpendicular to the patch dA. Adding up the net flux into or out of the volume is equivalent to adding up all the local sources and sinks.
- *Stokes' Theorem*

$$\int_S \nabla \times \vec{E} \cdot d\vec{A} = \oint_l \vec{E} \cdot d\vec{l} \quad . \tag{6.23}$$

  The line integral around a closed path is equal to the surface integral of the curl over any arbitrary surface bounded by the path (Figure 6.2). If your fingers curl in the direction of the line integral, your thumb points in the direction of the surface normal. The total circulation around the path is equal to the sum of all the local circulations.

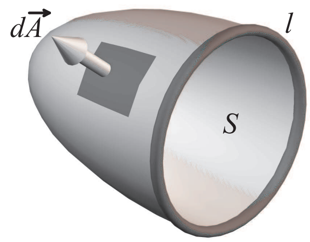

Figure 6.2. Definition of Stokes' Theorem.

6.2 STATICS

This section reviews the governing equations for time-independent electromagnetic phenomena; the following one turns on time dependence to arrive at Maxwell's equations. Together these will serve as the foundation for much of the rest of the book. As surprising as the remarkable phenomenology contained in these apparently simple relationships is the sophistication of the techniques needed to reveal it.

6.2.1 Electrostatics

The force in a vacuum between two charges is given by *Coulomb's Law*:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r} \ \ (\mathrm{N}) \quad . \tag{6.24}$$

$\epsilon_0 = 10^7/(4\pi c^2) = 8.854\times10^{-12}$ F/m is a constant called the *permittivity of free space*, q_1 and q_2 are the size of the charges in coulombs, r^2 is the distance between them in meters, and $\hat{r}$ is a unit vector pointing between them. This relationship was determined experimentally by Charles Augustin Coulomb in 1785.

The force on one charge due to an applied electric field is

$$\vec{F} = q\vec{E} \tag{6.25}$$

and so the electric field due to a single charge is

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r} \ \ \left(\frac{\mathrm{V}}{\mathrm{m}}\right) \quad . \tag{6.26}$$

With this definition the electric field points from positive charge towards negative charge. The field diverges as you approach the charge; very close to a charge the expression is no longer valid and quantum electrodynamics is needed to describe the field.

We will shortly see that the curl of the electric field vanishes if there are no time-varying magnetic fields, which according to equation (6.13) means that the electric field can be written as the gradient of a *potential* Φ

$$\vec{E} = -\nabla\Phi \quad . \tag{6.27}$$

Given this definition, the potential from a point charge is

$$\Phi = \frac{q}{4\pi\epsilon_0 r} \ \ (\mathrm{V}) \quad . \tag{6.28}$$

Since the electric field is the gradient of the potential, the potential difference between two points $\vec{x}$ and $\vec{y}$ can be found by integrating the electric field along an arbitrary path between the points

$$\begin{aligned} V(\vec{x},\vec{y}) &= \Phi(\vec{y}) - \Phi(\vec{x}) \\ &= -\int_{\vec{x}}^{\vec{y}} \vec{E}\cdot d\vec{l} \quad \text{(V)} \quad . \end{aligned} \tag{6.29}$$

It is convenient to view the electric field in terms of fictitious *field lines*, which are perpendicular to the lines of constant potential and have an areal density proportional to the field strength.

An electric field inside a material is modified by the response of the material to the field. In a *dielectric*, the charge is bound so that it is not free to move, but an applied electric field will polarize the bound charge. Because of this polarization, the strength of the field generated by a free charge in the material will be reduced by a factor called the *permittivity* ϵ

$$\vec{E} = \frac{q}{4\pi\epsilon r^2}\hat{r} \quad . \tag{6.30}$$

The permittivity ϵ equals the *relative permittivity* of the material ϵ_r times the permittivity of free space

$$\epsilon = \epsilon_0\epsilon_r \quad . \tag{6.31}$$

ϵ_r, also called the *dielectric constant*, is 1 in a vacuum, between 2 and 5 for typical plastics, and can be over 100 in a material such as $SrTiO_3$. Depending on the symmetry of the material the permittivity can be a tensor that depends on direction, and for strong fields (such as those generated by a laser, Problem 6.5) it will depend nonlinearly on the field. This latter property is very useful for mixing and generating harmonics of incident beams of light.

Let's take the divergence of the field due to a charge and integrate over an infinitesimal spherical volume of radius r around the charge. According to Gauss' Theorem,

$$\begin{aligned} \int_V \nabla\cdot\vec{E}\, dV &= \int_S \vec{E}\cdot d\vec{A} \\ &= \int_S \frac{q}{4\pi\epsilon r^2}\hat{r}\cdot\hat{r}\, dA \\ &= \frac{q}{4\pi\epsilon r^2}4\pi r^2 \\ &= \frac{q}{\epsilon} \\ &= \int_V \frac{\rho}{\epsilon}\, dV \quad . \end{aligned} \tag{6.32}$$

In the last line we've introduced the *charge density* ρ, which for this point charge is just a delta function. Since the left side must equal the right side independent of the volume, the integrands must be equal, and we see that

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon} = \frac{\rho}{\epsilon_0\epsilon_r} \quad . \tag{6.33}$$

If we define $\vec{D} = \epsilon\vec{E}$, and if ϵ is a constant, this reads

$$\nabla \cdot \vec{D} = \rho \quad . \tag{6.34}$$

$\vec{D}$ is called the *displacement* field, and this is the differential form of *Gauss' Law*. $\vec{E}$ is the real physical field that exerts a force on charges; $\vec{D}$ is the effective field that results from source charges.

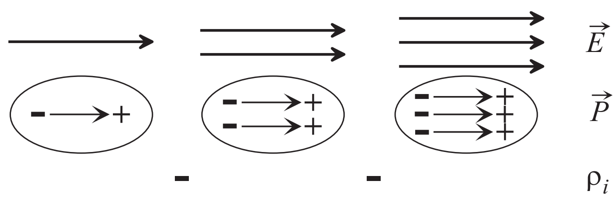

Figure 6.3. Relationship between an electric field with a gradient, the resulting spatial variation of the polarization, and the net charge induced by the local charge imbalance.

If the $\vec{E}$ field in the material is uniform, the induced polarization will be constant. However, if the $\vec{E}$ field varies in space (Figure 6.3) then there will be a spatially varying induced polarization, leading to an average induced charge density. To understand this, let's return to equation (6.33). The electric field in the material can be viewed as being the sum of the field due to any free charge ρ_{free} that would be there if the material was not present ($\epsilon_r = 1$), and the field due to the induced charge ρ_{induced}. The induced charge is conventionally defined in terms of a *polarization vector* $\vec{P}$

$$\rho_{\text{induced}} \equiv -\nabla \cdot \vec{P} \quad , \tag{6.35}$$

and so

$$\begin{aligned} \nabla \cdot \epsilon_0\vec{E} &= \rho_{\text{free}} + \rho_{\text{induced}} \\ &= \rho_{\text{free}} - \nabla \cdot \vec{P} \end{aligned} \tag{6.36}$$

or

$$\nabla \cdot \underbrace{(\epsilon_0\vec{E} + \vec{P})}_{\equiv \vec{D}} = \rho_{\text{free}} \quad . \tag{6.37}$$

If the field is not too strong then $\vec{P}$ will be linearly related to $\vec{E}$, and this relationship defines the *electric susceptibility* χ_e

$$\vec{D} \equiv \epsilon_0\vec{E} + \vec{P} = \epsilon_0(1 + \chi_e)\vec{E} = \epsilon_0\epsilon_r\vec{E} = \epsilon\vec{E} \quad . \tag{6.38}$$

The *dipole moment* of a charge distribution is defined as the integral of the charge times the position $\int \vec{x}\rho(\vec{x})\, dV$. To relate this to $\vec{P}$, first note that by differentiating $x\vec{P}$ and writing out the terms,

$$\nabla \cdot (x\vec{P}) = x\nabla \cdot \vec{P} + P_x \quad . \tag{6.39}$$

Therefore,

$$\int x\rho_{\text{induced}}\, dV = -\int x\nabla \cdot \vec{P}\, dV$$

$$
\begin{aligned}
&= \int P_x \, dV - \int \nabla \cdot (x\vec{P}) \, dV \\
&= \int P_x \, dV - \int x\vec{P} \cdot d\vec{A} \quad .
\end{aligned} \tag{6.40}
$$

In the limit that the volume goes to zero, $\vec{P}$ will be uniform and so the second term will vanish. Dropping it and repeating the calculation for the y and z components gives

$$
\int \vec{x}\rho_{\text{induced}} \, dV = \int \vec{P} \, dV \quad . \tag{6.41}
$$

Since this must be true for any volume, we see that the polarization vector is equal to the local density of the dipole moment. Note that unlike the $\vec{E}$ field, the dipole moment is defined to point from negative charge to positive charge.

Substituting the definition of the potential into Gauss' Law and assuming a homogeneous polarizibility gives *Poisson's equation*

$$
\nabla^2\Phi(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon} \tag{6.42}
$$

For a point charge located at $\vec{x}_0$, $\rho(\vec{x}) = q \; \delta(|\vec{x} - \vec{x}_0|)$, and the potential is given by equation (6.28). Plugging these into Poisson's equation,

$$
\begin{aligned}
\nabla^2\Phi(\vec{x}) &= -\frac{\rho(\vec{x})}{\epsilon} \\
\nabla^2 \frac{q}{4\pi\epsilon|\vec{x} - \vec{x}_0|} &= -\frac{q}{\epsilon} \; \delta(|\vec{x} - \vec{x}_0|) \\
\nabla^2 \frac{1}{|\vec{x} - \vec{x}_0|} &= -4\pi \; \delta(|\vec{x} - \vec{x}_0|) \quad .
\end{aligned} \tag{6.43}
$$

This relationship can then be used to show that

$$
\Phi(\vec{x}) = \frac{1}{4\pi\epsilon} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \; d^3x' \tag{6.44}
$$

solves Poisson's equation. $1/|\vec{x} - \vec{x}'|$ is a *Green's function* for this problem, relating the field to an integral over its source. A similar solution can be found for $\vec{E}$ by using

$$
\begin{aligned}
\nabla \frac{1}{|\vec{x} - \vec{x}'|} &= -\frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \\
\Rightarrow \; \vec{E}(\vec{x}) &= -\nabla\Phi(\vec{x}) \\
&= \frac{1}{4\pi\epsilon} \int \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} \; d^3x' \quad .
\end{aligned} \tag{6.45}
$$

In free space, Poisson's equation reduces to

$$
\nabla^2\Phi(\vec{x}) = 0 \quad . \tag{6.46}
$$

This is called *Laplace's equation*, and governs many other phenomena including the profile of a membrane such as a drumhead stretched around a boundary. One of the properties of its solution is that it can take on an extremum only on the boundary, and so electromagnetic particle traps require time-varying fields. The solution of Laplace's equation requires the specification these boundary conditions, usually given by either the distribution of the potential on the boundary (*Dirichlet* boundary conditions) or its normal derivative (*Neumann* boundary conditions).

Note that we found these equations by integrating Coulomb's Law, which involved canceling the r^2 dependence of a three-dimensional surface area with the r^{-2} dependence of the field. In anything other than three dimensions this would not work. Laplace's equation is routinely solved numerically, and to make it tractable it is frequently solved in two dimensions, but it is very important to remember that solving Laplace's equation in two dimensions corresponds to taking Coulomb's law to have a r^{-1} form because the length of a two-dimensional perimeter is proportional to r. This may effectively be correct if the problem has two-dimensional symmetry so that each point corresponds to a line of charge, but it will be incorrect for an arbitrary two-dimensional slice of a three-dimensional problem.

The *capacitance* between two electrodes relates the charge Q on each of them to the potential difference V between them:

$$C = \frac{Q}{V} \quad . \tag{6.47}$$

The relationship to current is found by differentiating:

$$C\frac{dV}{dt} = \frac{dQ}{dt} = I \quad . \tag{6.48}$$

6.2.2 Magnetostatics

Electric fields are produced by stationary charges; magnetic fields are produced by moving charges. The strength of the *magnetic field* $\vec{H}$ due to an infinitesimal section of a wire carrying a current (Figure 6.4) was found experimentally in 1820 to be governed by the *Biot–Savart Law*

$$d\vec{H} = \frac{I\ d\vec{l} \times \hat{r}}{4\pi r^2}, \tag{6.49}$$

or integrated over space

$$\begin{aligned}\vec{H} &= \int \frac{I\ d\vec{l} \times \hat{r}}{4\pi r^2} \\ &= \frac{1}{4\pi}\int \vec{J}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\ d^3x' \quad (\mathrm{A/m}) \quad ,\end{aligned} \tag{6.50}$$

where $\vec{J}$ is the current density. Using the right hand rule, if your thumb points in the direction of current flow then you fingers will curl in the direction of the field.

The relationship between $\vec{H}$ and $\vec{J}$ can be written more simply by taking the curl,

$$\begin{aligned}\vec{H}(\vec{x}) &= \frac{1}{4\pi}\int \vec{J}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3}\ d^3x' \\ &= \frac{1}{4\pi}\int \vec{J}(\vec{x}') \times \nabla_{\vec{x}} \frac{1}{|\vec{x} - \vec{x}'|}\ d^3x' \\ \nabla_{\vec{x}} \times \vec{H}(\vec{x}) &= \frac{1}{4\pi}\int \nabla_{\vec{x}} \times \vec{J}(\vec{x}') \times \nabla_{\vec{x}} \frac{1}{|\vec{x} - \vec{x}'|}\ d^3x'\end{aligned}$$

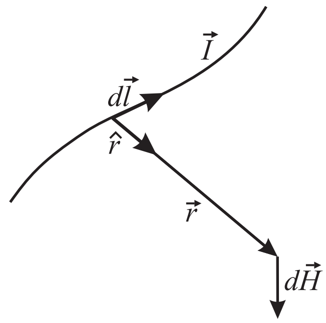

Figure 6.4. The magnetic field due to a differential current element.

$$= \frac{1}{4\pi} \int \vec{J}(\vec{x}') \underbrace{\nabla^2_{\vec{x}} \frac{1}{|\vec{x}-\vec{x}'|}}_{\delta(\vec{x}-\vec{x}')} - \nabla_{\vec{x}} \frac{1}{|\vec{x}-\vec{x}'|} \underbrace{\nabla_{\vec{x}} \cdot \vec{J}(\vec{x}')}_{0} \, d^3x'$$

$$\nabla \times \vec{H} = \vec{J} \tag{6.51}$$

(using the BAC–CAB rule $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$, Problem 6.1). We will soon see that a term must be added to this equation if the fields are time-varying.

The force on an infinitesimal current element is given in terms of the *magnetic flux density* $\vec{B}$ by

$$d\vec{F} = I d\vec{l} \times \vec{B} \quad , \tag{6.52}$$

or for a single moving charge

$$\vec{F} = q\vec{v} \times \vec{B} \quad . \tag{6.53}$$

The continuum version is

$$\vec{F} = \int \vec{J} \times \vec{B} \, dV \quad . \tag{6.54}$$

Analogous to the relationship between $\vec{D}$ and $\vec{E}$, in a material $\vec{H}$ and $\vec{B}$ are related by the *permeability* μ:

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1 + \chi_m)\vec{H} = \mu_0\mu_r\vec{H} = \mu\vec{H} \quad (\mathrm{T}) \quad . \tag{6.55}$$

$\mu_0 = 4\pi \times 10^{-7}$ H/m is the *permeability of free space*, $\vec{M}$ is the *magnetization*, the *relative permeability* $\mu_r = 1$ for a vacuum, and χ_m is the *magnetic susceptibility*. $\vec{H}$ is the effective field that results from source currents, and $\vec{B}$ is the physical field that exerts a force on charges. As with $\vec{D} = \epsilon\vec{E}$, the linear relationship $\vec{B} = \mu\vec{H}$ applies only to weak fields; magnetic recording depends on the nonlinear hysteresis in μ. Chapter 13 will explain the origin of this, as well as the reason why $\chi_m < 0$ for diamagnetic materials but $\chi_m > 0$ for paramagnetic, ferromagnetic, and ferrimagnetic materials. For iron $\chi_m \sim 10^3$; for non-magnetic materials $\chi_m \sim 10^{-5}$ and so $\mu_r \sim 1$ for them. In a high-permeability alloy such as *mumetal* ($Fe_{18}Ni_{75}Cu_5Cr_2$) the relative permeability is $\sim 10^5$.

Taking the curl of $\vec{H}$,

$$\nabla \times \vec{H} = \vec{J}$$

$$
\begin{aligned}
&= \nabla \times \frac{1}{\mu_0}\vec{B} - \nabla \times \vec{M} \\
&= J_{\text{free}} - J_{\text{induced}} \quad .
\end{aligned} \tag{6.56}
$$

As with the electrostatic case, the effective current in the material can be decomposed into free and induced currents. The induced current can be understood in terms of the *magnetic moment* of a current distribution, defined to be

$$
\vec{m} = \frac{1}{2}\int \vec{x} \times \vec{J}(\vec{x})\, dV \quad . \tag{6.57}
$$

For example, for a circular current loop for radius r,

$$
\begin{aligned}
|\vec{m}| &= \frac{1}{2}\, r\ I\ 2\pi r \\
&= I\pi r^2 \\
&= I \cdot \text{area} \quad .
\end{aligned} \tag{6.58}
$$

The magnetic moment associated with the induced current is equal to

$$
\begin{aligned}
\vec{m}_{\text{induced}} &= \frac{1}{2}\int \vec{x} \times \vec{J}_{\text{induced}}(\vec{x})\, dV \\
&= \frac{1}{2}\int \vec{x} \times (\nabla \times \vec{M})\, dV \quad ,
\end{aligned} \tag{6.59}
$$

or for the ith component

$$
m_{i,\text{induced}} = \frac{1}{2}\int \epsilon_{ijk} x_j \epsilon_{klm} \partial_l M_m\ dV \quad . \tag{6.60}
$$

Since

$$
\partial_l (x_j M_m) = x_j \partial_l M_m + M_m \underbrace{\partial_l x_j}_{\delta_{jl}} \quad , \tag{6.61}
$$

this can be rewritten as

$$
m_{i,\text{induced}} = \frac{1}{2}\int \epsilon_{ijk}\epsilon_{klm}\left[\partial_l (x_j M_m) - M_m \delta_{jl}\right] dV \quad . \tag{6.62}
$$

The integral over all space of the first term in the brackets will vanish. This is because the integral over all space of the magnetization M is bounded, therefore asymptotically M must fall off faster than $1/x$. Hence $\lim_{x\to\infty} xM(x) = 0$, so the integral of the derivative of this quantity is just its value at $\pm\infty$, which is 0. That leaves the remaining term

$$
\begin{aligned}
m_{i,\text{induced}} &= -\frac{1}{2}\int \epsilon_{ijk}\epsilon_{klm} M_m \delta_{jl}\ dV \\
&= -\frac{1}{2}\int \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) M_m \delta_{jl}\ dV \\
&= -\frac{1}{2}\int \delta_{il}\delta_{jm}\delta_{jl} M_m - \delta_{im}\delta_{jl}\delta_{jl} M_m\ dV \\
&= -\frac{1}{2}\int \delta_{il}\delta_{ml} M_m - \delta_{im} 3 M_m\ dV
\end{aligned}
$$

$$= -\frac{1}{2}\int M_i - 3M_i \; dV$$
$$= \int M_i \; dV \quad , \tag{6.63}$$

or for all components

$$\vec{m}_{\text{induced}} = \int \vec{M} \; dV \quad . \tag{6.64}$$

The magnetic dipole moment is equal to the integral of the magnetization, therefore the magnetization is the local density of the dipole moment.

As far as we know there is no such thing as a magnetic "charge", called a magnetic monopole, so $\nabla \cdot \vec{B} = 0$. This means that $\vec{B}$ can be written as the curl of a vector field $\vec{A}$ (equation 6.14):

$$\vec{B} = \nabla \times \vec{A} \quad , \tag{6.65}$$

called the *vector potential*. $\vec{A}$ is related to source currents by

$$\vec{A}(\vec{x}) = \frac{\mu}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \; d^3x' \quad , \tag{6.66}$$

verified by taking the curl to obtain equation (6.50). This relation holds for static current distributions; Section 8.1 will extend the scalar and vector potentials to time-dependent sources.

In quantum mechanics the vector potential takes on a deep significance beyond this formal definition. The *Aharonov–Bohm effect* considers a particle moving outside an infinite solenoid; the magnetic field vanishes there but the vector potential does not, and this leads to observable quantum interference effects [Sakurai, 1967]. This effect demonstrates the physical reality of the vector potential.

6.2.3 Multipoles

The theory of *multipoles* provides a systematic way to approximate the fields produced by more complex charge and current distributions. One way to understand it is by expanding the inverse distance

$$\frac{1}{|\vec{r} - \vec{x}|} = \frac{1}{|\vec{r}|} + \frac{\vec{r}\cdot\vec{x}}{|\vec{r}|^3} + \cdots$$
$$= \frac{1}{r} + \frac{\vec{r}\cdot\vec{x}}{r^3} + \cdots \quad , \tag{6.67}$$

where $\vec{r}$ is the distance from the source to where the field is being evaluated, $\vec{x}$ is the location within the source relative to its origin, and $\vec{x}$ is assumed to be much smaller than $\vec{r}$. Substituting this series into the potential rewrites it as

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon}\int \frac{\rho(\vec{x})}{|\vec{r} - \vec{x}|} \; dV$$
$$= \frac{1}{4\pi\epsilon}\left(\frac{1}{r}\int \rho(\vec{x}) \; dV + \frac{\vec{r}}{r^3}\cdot\int \rho(\vec{x})\vec{x} \; dV + \cdots\right)$$
$$\equiv \frac{1}{4\pi\epsilon}\left(\frac{q}{r} + \frac{\vec{p}\cdot\vec{r}}{r^3} + \cdots\right) \quad . \tag{6.68}$$

q is the *monopole* term, $\vec{p}$ is the *dipole moment*, and the next term would be the *quadrupole moment*. The corresponding electric field can be found by taking the gradient

$$\begin{aligned}\vec{E} &= -\nabla\Phi \\ &= \frac{q\hat{r}}{4\pi\epsilon r^2} + \frac{3\hat{r}(\vec{p}\cdot\hat{r}) - \vec{p}}{4\pi\epsilon r^3} + \cdots \\ &= \frac{q}{4\pi\epsilon r^2}\hat{r} + \frac{2p\cos\theta}{4\pi\epsilon r^3}\hat{r} + \frac{p\sin\theta}{4\pi\epsilon r^3}\hat{\theta} + \cdots \quad ,\end{aligned} \tag{6.69}$$

and the energy associated with the charge distribution is

$$\begin{aligned}U &= \int \rho(\vec{x})\Phi(\vec{x})\, dV \\ &= \int \rho(\vec{x})\left[\Phi(0) + \vec{x}\cdot\nabla\Phi(\vec{x})\big|_{\vec{x}=0} + \cdots\right]\, dV \\ &= q\Phi(0) - \vec{p}\cdot\vec{E}(0) + \cdots \quad .\end{aligned} \tag{6.70}$$

The same expansion can be used with the vector potential,

$$\begin{aligned}A_i(\vec{r}) &= \frac{\mu}{4\pi}\int \frac{J_i(\vec{x})}{|\vec{r}-\vec{x}|}\, dV \\ &= \frac{\mu}{4\pi}\left(\frac{1}{r}\int J_i(\vec{x})\, dV + \frac{\vec{r}}{r^3}\cdot\int J_i(\vec{x})\vec{x}\, dV + \cdots\right) \quad .\end{aligned} \tag{6.71}$$

Because of the vectorial character, finding this field is a bit more tricky. To start, notice that for arbitrary functions f and g, integration by parts shows that

$$\begin{aligned}\int f\vec{J}\cdot\nabla g\, dV &= 0 - \int g\nabla\cdot(f\vec{J})\, dV \\ &= -\int g\vec{J}\cdot\nabla f\, dV - \int fg\nabla\cdot\vec{J}\, dV \quad ,\end{aligned} \tag{6.72}$$

where the integrals are over all space, and the only assumption that's been made is that J vanishes at infinity. Rearranging terms,

$$\int\left(f\vec{J}\cdot\nabla g + g\vec{J}\cdot\nabla f + fg\nabla\cdot\vec{J}\right) dV = 0 \quad . \tag{6.73}$$

If we plug in $\nabla\cdot\vec{J} = 0$, and take $f = 1, g = x_i$, then

$$\int J_i\, dV = 0 \tag{6.74}$$

and so the monopole vector potential term vanishes (there are no free magnetic charges). Now taking $f = x_i$ and $g = x_j$,

$$\int\left(x_iJ_j + x_jJ_i\right) dV = 0 \tag{6.75}$$

or

$$\frac{1}{2}\int\left(x_iJ_j - x_jJ_i\right) dV = \int x_iJ_j\, dV \quad . \tag{6.76}$$

This can be substituted into the dipole term to relate it to the vector potential:

$$\begin{aligned}
A_i(\vec{r}) &= \frac{\mu}{4\pi}\frac{\vec{r}}{r^3}\cdot\int \vec{x}J_i(\vec{x})\,dV \\
&= \frac{\mu}{4\pi}\frac{1}{r^3}\int r_j x_j J_i\,dV \\
&= -\frac{1}{2}\frac{\mu}{4\pi}\frac{1}{r^3}\int (r_j x_i J_j - r_j x_j J_i)\,dV \\
\vec{A}(\vec{r}) &= -\frac{1}{2}\frac{\mu}{4\pi}\frac{\vec{r}}{r^3}\times\int \vec{x}\times\vec{J}(\vec{x})\,dV \\
&= \frac{\mu}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} \quad .
\end{aligned} \tag{6.77}$$

The magnetic field is found from the curl

$$\begin{aligned}
\vec{B} &= \nabla\times\vec{A} \\
&= \frac{\mu}{4\pi}\nabla\times\frac{\vec{m}\times\vec{r}}{r^3} \\
B_i &= \frac{\mu}{4\pi}\epsilon_{ijk}\partial_j\frac{1}{r^3}\epsilon_{klm}m_l r_m \\
&= \frac{\mu}{4\pi}\epsilon_{ijk}\epsilon_{klm}m_l\partial_j\frac{r_m}{r^3} \\
&= \frac{\mu}{4\pi}\left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)m_l\left(\frac{\delta_{jm}}{r^3}-\frac{3r_j r_m}{r^5}\right) \\
&= \frac{\mu}{4\pi}\left(\frac{3m_i}{r^3}-\frac{m_i}{r^3}-\frac{3m_i}{r^3}+\frac{3r_i m_j r_j}{r^5}\right) \\
\vec{B} &= \frac{\mu}{4\pi}\frac{3\hat{r}(\vec{m}\cdot\hat{r})-\vec{m}}{r^3} \quad .
\end{aligned} \tag{6.78}$$

This is exactly the same as the electrostatic dipole field (equation 6.69).

The force on a magnetic dipole can be derived by applying the substitution used to find the vector potential:

$$\begin{aligned}
\vec{F} &= \int \vec{J}\times\vec{B}\,dV \\
F_i &= \int (\vec{J}\times\vec{B})_i\,dV \\
&= \epsilon_{ijk}\int J_j B_k\,dV \\
&= \epsilon_{ijk}\int J_j[B_k(0)+\vec{x}\cdot\nabla B_k(\vec{x})|_{\vec{x}=0}]\,dV \\
&= 0+\epsilon_{ijk}\int J_j\vec{x}\cdot\nabla B_k\,dV \\
&= \epsilon_{ijk}\nabla B_k\cdot\int J_j\vec{x}\,dV \\
&= \epsilon_{ijk}\left(\vec{m}\times\nabla B_k\right)_j \\
&= \epsilon_{ijk}\epsilon_{jlm}m_l\partial_m B_k \\
&= \left(\delta_{im}\delta_{kl}-\delta_{il}\delta_{km}\right)m_l\partial_m B_k
\end{aligned}$$

$$
\begin{aligned}
&= m_k \partial_i B_k - m_i \partial_k B_k \\
\vec{F} &= \nabla(\vec{m} \cdot \vec{B}) - \vec{m} \underbrace{(\nabla \cdot \vec{B})}_{0} \quad .
\end{aligned}
\tag{6.79}
$$

Since a conservative force is the gradient of the potential energy, the energy of a magnetic dipole in a field is

$$
\begin{aligned}
U &= -\vec{m} \cdot \vec{B} \\
&= -mB \cos\theta \quad ,
\end{aligned}
\tag{6.80}
$$

where θ is the angle between the dipole and the local field. There is an angular dependence to this that will seek to align the dipole with the field,

$$
\frac{\partial U}{\partial \theta} = mB \sin\theta \quad ,
\tag{6.81}
$$

i.e., there will be a torque about the axis perpendicular to them of

$$
\vec{\tau} = \vec{m} \times \vec{B} \quad .
\tag{6.82}
$$

Note that all of these calculations have assumed that the distance to the point where the field is being evaluated is large compared to the special extent of this source. If the fields are needed closer to the source it's necessary to either use the full distribution or carry the multipole approximation out to a high order.

6.3 DYNAMICS

6.3.1 Maxwell's Equations

We're now ready for Maxwell's contribution. The statics equations tell us that the divergence of $\vec{B}$ and the curl of $\vec{E}$ vanish, and relate the divergence of $\vec{D}$ and the curl of $\vec{H}$ to their sources. Faraday had found that a varying magnetic field induces a current in a wire, and Ampère that a current produces a magnetic field; Maxwell realized that for these equations to be consistent (Problem 6.2) there must also be a time derivative of $\vec{D}$:

$$
\begin{aligned}
\nabla \cdot \vec{D} &= \rho(\vec{x}) \\
\nabla \cdot \vec{B} &= 0 \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\
\nabla \times \vec{H} &= \vec{J}(\vec{x}) + \frac{\partial \vec{D}}{\partial t} \quad .
\end{aligned}
\tag{6.83}
$$

These are now called *Maxwell's equations*; they show that electric and magnetic fields are connected through a more general theory of electromagnetic phenomena.

Material properties appear in Maxwell's equations through $\vec{D} = \epsilon\vec{E}$ and $\vec{B} = \mu\vec{H}$. In addition, the current $\vec{J}$ is related to the electric field by $\vec{J} = \sigma\vec{E}$. The coefficient σ is the *conductivity*, equal to the inverse of the *resistivity* ρ. In real materials ϵ, μ, and σ can become tensors that depend on direction, and can be complex quantities because of loss mechanisms.

We've already seen the first of Maxwell's equations; integrating it over a volume gives the integral form of Gauss' Law

$$\int_S \vec{D} \cdot d\vec{A} = \int_V \rho \, dV = Q \quad . \tag{6.84}$$

The surface integral of the normal component of $\vec{D}$ is equal to the charge Q enclosed. The second equation lacks a source term and the third one lacks a current term because there are no (known) magnetic monopoles. Integrating the last equation over a surface gives *Stokes' Law*

$$\oint \vec{H} \cdot d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{A} \quad . \tag{6.85}$$

The line integral of the magnetic field around a path is equal to the current crossing an arbitrary surface bounded by the path. The first term on the right hand side of equation (6.85) is the conventional current, and the second term $\partial \vec{D}/\partial t$ is called the *displacement current*. This acts just like a real current, but instead of charge moving it is associated with an electric field changing. The current that flows into and out of a capacitor appears to travel through the space between the capacitor plates; Problem 6.2 shows that this is accounted for by the displacement current. Anyone who has been shocked by a charged capacitor can attest to the reality of this current.

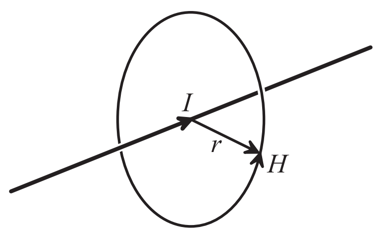

Figure 6.5. The magnetic field around a current-carrying wire.

It can be possible to use Stokes' Law to find magnetic fields without direct integration. For example, if a wire is carrying a current I, by symmetry according to the Biot–Savart Law the magnetic field must be directed circumferentially around the wire (Figure 6.5). This means that the line integral is just the field strength times the circumference, and the surface integral of the current density is equal to the current flowing through the wire, therefore

$$2\pi r H = I \Rightarrow H = \frac{I}{2\pi r} \quad . \tag{6.86}$$

6.3.2 Boundary Conditions

The integral forms of Maxwell's equations can be used to find the conditions that the fields must satisfy at interfaces between materials, as specified by the dielectric constant ϵ, the conductivity σ, and the permeability μ. Start by integrating Gauss' Law over the volume V in Figure 6.6:

$$\int \nabla \cdot \vec{D} \, dV = \int \vec{D} \cdot d\vec{A} = \int \rho \, dV \quad . \tag{6.87}$$

In the limit that the height of the box $h \to 0$, the only contributions to the surface integral will come from the top and bottom. If the box if infinitesimal then the fields can be taken to be constant, and so the surface integral is just the normal component of the field times the area:

$$(\vec{D}_1 - \vec{D}_2) \cdot \hat{n} A = \int_V \rho \, dV \quad , \tag{6.88}$$

where $\hat{n}$ is a unit vector normal to the interface. There is a sign change between the integrals over the top and the bottom because the surface normal changes directions. If there is charge at the interface with an areal density ρ_s, then

$$\int_V \rho \, d\vec{V} = \rho_s A \quad . \tag{6.89}$$

Therefore,

$$(\vec{D}_1 - \vec{D}_2) \cdot \hat{n} = \rho_s \quad . \tag{6.90}$$

The change in the normal component of $\vec{D}$ across the interface is equal to the charge density at the interface. An applied field will create such surface charge to match the boundary conditions.

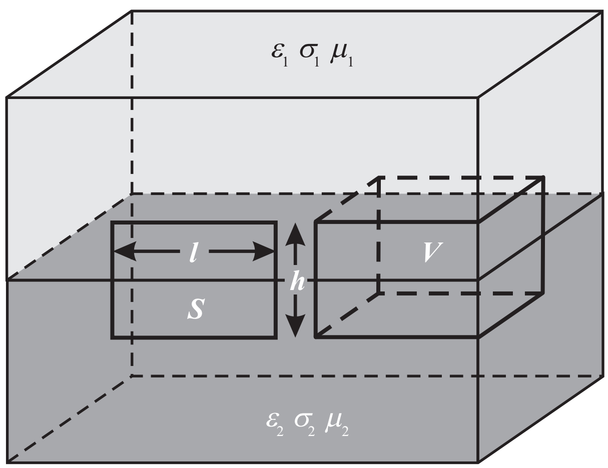

Figure 6.6. Loop and volume used for evaluating boundary conditions.

Next, integrate the curl of $\vec{E}$ over the surface S in Figure 6.6:

$$\int \nabla \times \vec{E} \cdot d\vec{A} = \oint \vec{E} \cdot d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} \quad . \tag{6.91}$$

As before, if the height h of the loop goes to zero then the only contribution to the line integral comes from the top and bottom, and the integral over the surface on the right hand side vanishes. Therefore

$$(\vec{E}_1 - \vec{E}_2) \times \hat{n} \; l = 0 \tag{6.92}$$

or

$$(\vec{E}_1 - \vec{E}_2) \times \hat{n} = 0 \quad . \tag{6.93}$$

The tangential component of $\vec{E}$ is continuous across the interface.

Since $\vec{J} = \sigma\vec{E}$, and in a perfect conductor $\sigma = \infty$, there can be no $\vec{E}$ field in an ideal conductor otherwise there would be an infinite current. According to equation (6.93), this means that there can be no tangential component on either side of the interface. Equation (6.90) does permit a normal component outside the interface, but it must be screened by a surface charge $\rho_s = D_1$. We see that the electric field must be perpendicular to the surface of a perfect conductor, and that a surface charge is induced to screen the interior from the field.

Integrating the divergence of $\vec{B}$ over the volume as we did for the divergence of $\vec{D}$ shows that

$$(\vec{B}_1 - \vec{B}_2) \cdot \hat{n} = 0 \quad . \tag{6.94}$$

The normal component of the $\vec{B}$ field is continuous across the interface. Similarly, integrating the curl of $\vec{H}$ over the surface instead of the curl of $\vec{E}$ gives

$$(\vec{H}_1 - \vec{H}_2) \times \hat{n} = \vec{J}_s \quad , \tag{6.95}$$

where $\vec{J}_s$ is the density of current at the surface. The tangential component of $\vec{H}$ changes by the surface current density across the interface.

The solution to Laplace's equation is unique. This means that *any* solution that satisfies the boundary conditions, no matter how it is found, is *the* solution. This observation leads to the useful *method of images*. Consider a point charge about an infinite ground plane, shown in Figure 6.7. A fictitious *image charge* is shown equidistant below the plane. By symmetry, the electric field lines are perdendicular to the surface. This is exactly the boundary condition that must be satisfied for a perfect conductor, hence to find the field above the plane the continuous surface charge distribution can be replaced by the single image charge. There are some geometries like this for which it's possible to use an image charge to easily solve for the fields; in more complex geometries image charges are still useful as an expansion technique based on iterating the induced image charge produced by the source image charge.

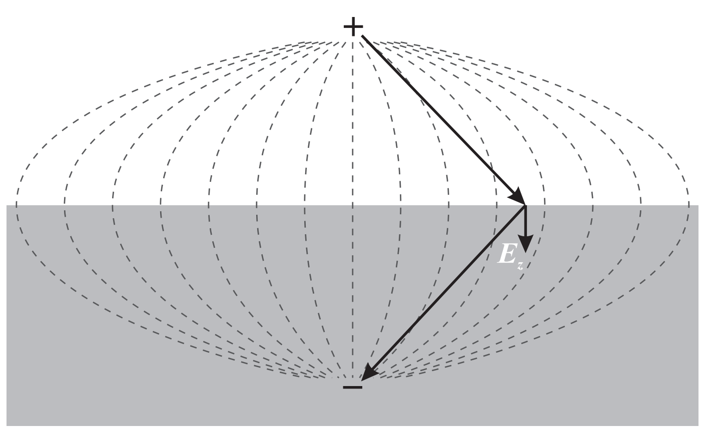

Figure 6.7. Image charge solution for the field of a charge above a ground plane.

6.3.3 Electromagnetic Units

There are two common systems of electromagnetic units: MKS that we've used here, and *Gaussian* or CGS. MKS uses familiar macroscopic quantities (volts, amps, ohms) and hence is most suitable for macroscopic phenomena and is commonly used in engineering; CGS is better matched to microscopic phenomena and is commonly used in physics. Table 6.2 gives the governing equations in these two systems, and in Table 6.3 the (non-obvious) conversion factors are summarized. In addition to these conventional definitions there are many other possible systems offering endless opportunities to go astray, ranging from minor variations of these to the theorist's favorite system in which all fundamental constants are set equal to 1, with units being put back in at the end of a calculation based on dimensional grounds. [Jackson, 1999] has a thorough discussion of the logic of, and the relationships among, the systems.

Table 6.2. *MKS and CGS governing equations.*

MKS	CGS
$\vec{D} = \epsilon_0\vec{E} + \vec{P}$	$\vec{D} = \vec{E} + 4\pi\vec{P}$
$\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$	$\vec{H} = \vec{B} - 4\pi\vec{M}$
$\nabla\cdot\vec{D} = \rho$	$\nabla\cdot\vec{D} = 4\pi\rho$
$\nabla\cdot\vec{B} = 0$	$\nabla\cdot\vec{B} = 0$
$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$	$\nabla\times\vec{E} = -\frac{1}{c}\frac{\partial\vec{B}}{\partial t}$
$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$	$\nabla\times\vec{H} = \frac{4\pi}{c}\vec{J} + \frac{1}{c}\frac{\partial\vec{D}}{\partial t}$
$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$	$\vec{F} = q\vec{E} + q\frac{\vec{v}}{c}\times\vec{B}$

6.4 RADIATION AND ENERGY

6.4.1 Waves

Perhaps the single most remarkable feature of Maxwell's equations is that they contain a wave solution. To see this, start with the equations for free space

$$
\begin{aligned}
\nabla\cdot\vec{D} &= 0 \\
\nabla\cdot\vec{B} &= 0 \\
\nabla\times\vec{E} &= -\mu_0\frac{\partial\vec{H}}{\partial t} \\
\nabla\times\vec{H} &= \epsilon_0\frac{\partial\vec{E}}{\partial t} \quad .
\end{aligned} \tag{6.96}
$$

Table 6.3. *Conversion between MKS and CGS units. All prefactors of 3 are actually 2.99792..., from the speed of light.*

Quantity	Symbol	MKS	CGS
charge	q	1 coulomb	3×10^9 statcoulombs
current	I	1 ampere	3×10^9 statamps
potential	V	1 volt	1/300 statvolt
polarization	P	1 coulomb/m^2	3×10^5 dipole moment/cm^3
electric field	E	1 volt/m	$\frac{1}{3}\times10^{-4}$ statvolt/cm
displacement	D	1 coulomb/m^2	$4\pi\times3\times10^5$ statvolt/cm
resistance	R	1 ohm	$\frac{1}{3^2}\times10^{-11}$ s/cm
capacitance	C	1 farad	$3^2\times10^{11}$ cm
magnetic flux	φ	1 weber	10^8 gauss·cm^2 (maxwells)
magnetic induction	B	1 tesla	10^4 gauss
magnetic field	H	1 ampere/m	$4\pi\times10^{-3}$ oersted
magnetization	M	1 ampere/m	10^{-3} magnetic moment/cm^3
inductance	L	1 henry	$\frac{1}{3^2}\times10^{-11}$ stathenry

Take the curl of the last two equations:

$$\nabla\times\nabla\times\vec{E} = -\mu_0\frac{\partial}{\partial t}\nabla\times\vec{H}$$
$$\nabla\times\nabla\times\vec{H} = \epsilon_0\frac{\partial}{\partial t}\nabla\times\vec{E} \quad . \tag{6.97}$$

These can be simplified with the identity (Problem 6.1)

$$\nabla\times\nabla\times\vec{E} = \nabla(\nabla\cdot\vec{E}) - \nabla^2\vec{E} \tag{6.98}$$

to give

$$-\nabla^2\vec{E} = -\mu_0\frac{\partial}{\partial t}\nabla\times\vec{H}$$
$$-\nabla^2\vec{H} = \epsilon_0\frac{\partial}{\partial t}\nabla\times\vec{E} \quad , \tag{6.99}$$

noting that the divergences will be 0 for $\vec{E}$ and $\vec{B}$. Substituting in the curls from equation (6.96),

$$\nabla^2\vec{E} = \mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$
$$\nabla^2\vec{H} = \mu_0\epsilon_0\frac{\partial^2\vec{H}}{\partial t^2} \quad . \tag{6.100}$$

These are wave equations, solved by a plane wave for the electric field

$$\vec{E}(\vec{x},t) = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)} \quad , \tag{6.101}$$

with the wave vector $\vec{k}$ pointing in the direction of propagation with a magnitude $|k| = 2\pi/\lambda$, and the velocity $c = \omega/|k| = (\mu_0\epsilon_0)^{-1/2}$. Let there be light.

If we take $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$ and $\vec{H} = \vec{H}_0 e^{i(\vec{k}\cdot\vec{x}-\omega t)}$,

$$\begin{aligned}
\nabla \times \vec{E} &= -\mu_0 \frac{\partial \vec{H}}{\partial t} \\
i\vec{k} \times \vec{E} &= i\omega\mu_0 \vec{H} \\
\frac{k}{\omega\mu_0}\hat{k} \times \vec{E} &= \vec{H} \\
\sqrt{\frac{\epsilon_0}{\mu_0}}\hat{k} \times \vec{E} &= \vec{H} \quad .
\end{aligned} \tag{6.102}$$

Similarly,

$$-\sqrt{\frac{\mu_0}{\epsilon_0}}\hat{k} \times \vec{H} = \vec{E} \quad . \tag{6.103}$$

The electric and magnetic fields of a plane electromagnetic wave are perpendicular to each other and to the direction of travel $\hat{k}$. Their ratio

$$\frac{|\vec{E}|}{|\vec{H}|} = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\ \Omega \tag{6.104}$$

has the units of resistance and defines the *impedance of free space*. It will return in Chapter 8 in the effective impedance of antennas.

Since it's reasonable to assume that a wave travels in a medium, the recognition of the wave solution to Maxwell's equations led to a search for the "ether" that supports the wave. The failure of the Michelson–Morely experiment in 1887 to detect the motion of the Earth through the ether helped plant the seeds for the discovery of quantum mechanics, and special relativity. The resolution of the paradox is that electromagnetic waves are carried by photons, which are particles that travel in free space but which also act like waves. One way to understand electromagnetic propagation is to remember that information cannot travel faster than the speed of light. If a charge is moved instantaneously, there is a "kink" in its electric field that travels out at the speed of light: that is an electromagnetic wave packet. Moving the charge periodically creates a wave.

6.4.2 Electromagnetic Energy

If electromagnetic waves can propagate, and if an electromagnetic field can accelerate a charge that is initially at rest, then it must be possible to store and transmit energy in the fields. In this section we will calculate that energy.

A charge in an electric field feels a force $\vec{F} = q\vec{E}$. If the charge moves a distance $d\vec{x}$, work $dW = q\vec{E} \cdot d\vec{x}$ is done against this force. If the charge is moving at a velocity $\vec{v}$ then the rate at which work is being done, or power is being consumed, is $W = q\vec{E} \cdot \vec{v}$. If there is a continuous current density $\vec{J}$, the total rate of work is this quantity integrated over space

$$W = \int_V \vec{E} \cdot \vec{J}\, dV \quad . \tag{6.105}$$

There is no work done by a magnetic field alone on a charge, because the magnetic force is perpendicular to the velocity:

$$\vec{F} \cdot \vec{v} = (q\vec{v} \times \vec{B}) \cdot \vec{v} = 0 \quad . \tag{6.106}$$

Since

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \;\Rightarrow\; \vec{J} = \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \quad , \tag{6.107}$$

equation (6.105) can be rewritten as

$$W = \int_V \left[\vec{E} \cdot (\nabla \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] \, dV \quad . \tag{6.108}$$

This in turn can be rewritten by using the vector identity

$$\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H}) \tag{6.109}$$

as

$$W = \int_V \left[\vec{H} \cdot (\nabla \times \vec{E}) - \nabla \cdot (\vec{E} \times \vec{H}) - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \right] \, dV \quad . \tag{6.110}$$

Plugging in $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$,

$$W = -\int_V \left[\nabla \cdot (\vec{E} \times \vec{H}) + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right] \, dV \quad . \tag{6.111}$$

Since $\vec{D} = \epsilon \vec{E}$,

$$\begin{aligned} \frac{\partial}{\partial t}(\vec{E} \cdot \vec{D}) &= \frac{\partial \vec{E}}{\partial t} \cdot \vec{D} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \\ &= 2\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \end{aligned} \tag{6.112}$$

and similarly

$$\frac{\partial}{\partial t}(\vec{B} \cdot \vec{H}) = 2\vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \quad . \tag{6.113}$$

Therefore, if we define

$$U = \frac{1}{2}(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}) \quad \left(\frac{\mathrm{J}}{\mathrm{m}^3} \right) \tag{6.114}$$

then equation (6.111) becomes

$$W = -\int_V \left[\nabla \cdot (\vec{E} \times \vec{H}) + \frac{\partial U}{\partial t} \right] \, dV \tag{6.115}$$

with

$$\frac{\partial U}{\partial t} = \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \quad . \tag{6.116}$$

Further defining

$$\vec{P} = \vec{E} \times \vec{H} \quad \left(\frac{\mathrm{J}}{\mathrm{m}^2 \cdot \mathrm{s}} \right) \quad , \tag{6.117}$$

the first term can be turned into a surface integral:

$$W = -\int_S \vec{P} \cdot d\vec{A} - \int_V \frac{\partial U}{\partial t} \, dV \quad . \tag{6.118}$$

This has a very natural interpretation. The first term represents an energy flux transported across the boundary of the integration volume by the field, and the second term represents the change in the energy stored in the volume by the field. $\vec{P}$ is called the *Poynting vector*, and U is the energy density. Note that the Poynting vector $\vec{P}$ has nothing to do with the polarization vector $\vec{P}$, they just use the same symbol by convention. Integrating P over an area gives the energy being carried by an electromagnetic wave; integrating U over a volume gives the energy stored in a static field.

Since the energy stored in an electric or magnetic field is equal to the volume integral of the square of the field strength, field lines behave like "*furry rubber bands*" in finding the lowest-energy configuration. They want to be as short as possible to minimize the volume of the integral, and they want to be as far apart from each other as possible to minimize the field density and hence the quadratic energy density.

6.5 SELECTED REFERENCES

[Jackson, 1999] Jackson, John David. (1999). *Classical Electrodynamics*. 3rd edn. New York: Wiley.

The definitive electrodynamics reference.

[Heald & Marion, 1995] Heald, Mark A., & Marion, Jerry B. (1995). *Classical Electromagnetic Radiation*. 3rd edn. Fort Worth: Saunders.

Less depth than Jackson, but a more accessible introduction to electrodynamics.

6.6 Problems

(6.1) Prove the BAC–CAB rule

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \tag{6.119}$$

by writing it out in the summations convention, and use it to show that

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} \quad . \tag{6.120}$$

(6.2) (*a*) Use Gauss' Law to find the capacitance between two parallel plates of area A at a potential difference V and with a spacing d. Neglect the fringing fields by assuming that this is a section of an infinite capacitor.

(*b*) Show that when a current flows through the capacitor, the integral over the internal displacement current is equal to the external electrical current.

(*c*) Integrate the energy density to find the stored energy at a fixed potential. The answer should be expressed in terms of the capacitance.

(*d*) Batteries are rated by amp-hours, the current they can supply at the design voltage for an hour. Consider a 10 V laptop battery that provides 10 A · h. Assuming a plate spacing of 10^{-6} m $\equiv$ 1 μm and a vacuum dielectric, what area would a capacitor need to be able to store this amount of energy? If such plates were 10 cm on a side and stacked vertically, how tall would the stack have to be to provide this total area?

(6.3) (*a*) Use Stokes' Law to find the magnetic field of an infinite solenoid carrying a current I with n turns/meter.

(*b*) Integrate the energy density to find the energy stored in a solenoid of radius r and length l, once again neglecting fringing fields.

(*c*) Consider a 10 T MRI magnet (Section 10.4) with a bore diameter of 1 m and a length of 2 m. What is the outward force on the magnet? Remember – force is the gradient of potential for a conservative force.

(6.4) Calculate the force per meter between two parallel wires one meter apart, each carrying a current of one ampere (this is the geometry used to define the ampere).

(6.5) (*a*) Assume that sunlight has a power energy density of 1 kW/m^2 (this is a peak number; the typical average value in the continental USA is $\sim$ 200 W/m^2). Estimate the electric field strength associated with this radiation.

(*b*) If 1 W of power is focused in a laser beam to a square millimeter, what is the field strength? What about if it is focused to the diffraction limit of $\sim$ 1 μm^2?