Problem set #9 Orbital mechanics and more (here)

26.01.2023

Problem 1: *Jaffe density distribution*

The spherically symmetric density distribution of the singular isothermal sphere,

$$\rho_{\rm sis}(r) = \frac{\sigma^2}{2\pi G r^2} \tag{9.1.1}$$

is only of limited use for describing astronomical objects such as galaxies or globular because the volume integral (the total mass) diverges. Jaffe (1983) derived a modified density distribution which is more suitable for many analytic studies:

$$\rho_{\rm J}(r) = \left(\frac{M_{\rm J}}{4\pi r_{\rm J}^3}\right)\frac{r_{\rm J}^4}{r^2(r+r_{\rm J})^2}. \tag{9.1.2}$$

(a) Prove the divergence of the total mass for the isothermal sphere.
(b) How large is the total mass of the Jaffe density distribution? Which radius contains half of the total mass? Give an expression for the circular velocity as a function of r for this density distribution. Compare this with the circular velocity profile of a point mass.
(c) Calculate the potential that corresponds to the Jaffe density distribution.

Problem 1: Solution

(a) Since the density of the isothermal sphere profile goes with $\rho \propto r^{-2}$, the mass contained in it diverges if we integrate over it (due to the $4\pi r^2$ factor in the integral):

$$M_{\rm sis} = \int_0^\infty 4\pi^2 r^2 \frac{\sigma^2}{2\pi G r^2}\mathrm{d}r = \frac{2\sigma^2}{G} r\Big|_0^\infty \to \infty. \tag{9.1.3}$$

(b) For the Jaffe density distribution, the included mass is

$$M_{\rm J} = \int_0^\infty \frac{4\pi r^2 M_{\rm J}}{4\pi r_{\rm J}^3}\frac{r_{\rm J}^4}{r^2(r+r_{\rm J})^2}\mathrm{d}r \tag{9.1.4}$$

$$= M_{\rm J} r_{\rm J}\int_0^\infty \frac{1}{(r+r_{\rm J})^2}\mathrm{d}r \tag{9.1.5}$$

$$= -M_{\rm J} r_{\rm J}\frac{1}{r+r_{\rm J}}\Big|_0^\infty \tag{9.1.6}$$

$$= M_{\rm J}\frac{r_{\rm J}}{r_{\rm J}} = M_{\rm J}. \tag{9.1.7}$$

To find the radius R where the enclosed mass is $M_{\rm J}/2$, we solve the equation:

$$\int_0^R 4\pi r^2 \rho_{\rm J}(r) = \frac{M_{\rm J}}{2} \tag{9.1.8}$$

$$\iff M_{\rm J} r_{\rm J}\left(\frac{1}{r_{\rm J}} - \frac{1}{R+r_{\rm J}}\right) = \frac{M_{\rm J}}{2} \tag{9.1.9}$$

$$\iff \frac{R}{R + r_J} = \frac{1}{2} \tag{9.1.10}$$

$$\iff R = r_J, \tag{9.1.11}$$

so conveniently, the Jaffe density profile already contains the half-mass radius and the total enclosed mass.
To find the circular velocity profile, we use the well-known relation where the centripetal acceleration is given by the gravitational acceleration:

$$\frac{GM(r)}{r^2} = \frac{v_{\text{rot}}^2}{r} \tag{9.1.12}$$

$$\iff v_{\text{rot}} = \sqrt{\frac{GM_J r_J}{r}\left(1 - \frac{r_J}{r + r_J}\right)} = \sqrt{\frac{GM_J}{r + r_J}}. \tag{9.1.13}$$

If we set $r_J = 0$, we reproduce the circular velocity profile for a point mass.
It makes sense that compared to the circular velocity profile of a point mass, the circular velocity in a Jaffe distribution of the same mass is always lower, which makes sense as the mass is more distributed and the enclosed mass is always lower. For $r \gg r_J$, the velocity profile approaches the one of a point mass and tends to 0.

(c) Due to spherical symmetry, we know that the following equation holds:

$$\frac{GM(R)}{R^2} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}\Big|_R \tag{9.1.14}$$

$$\implies \Phi(R) = -\int_R^\infty \frac{GM_J}{r(r + r_J)}\mathrm{d}r \tag{9.1.15}$$

$$\stackrel{6}{=} -GM_J \int_R^\infty \frac{1}{r_J x r_J (1 + x)}\mathrm{d}x r_J \tag{9.1.16}$$

$$= -\frac{GM_J}{r_J}\int_{R/r_J}^\infty \left(\frac{1}{x} - \frac{1}{1 + x}\right)\mathrm{d}x \tag{9.1.17}$$

$$= -\frac{GM_J}{R_J}\ln\left(\frac{x}{1 + x}\right)\Big|_{R/r_J}^\infty \tag{9.1.18}$$

$$= \frac{GM_J}{r_J}\ln\left(\frac{R}{R + R_J}\right). \tag{9.1.19}$$

[6] Substituting $x = \frac{r}{r_J}$, so $\mathrm{d}x = \frac{\mathrm{d}r}{r_J}$

Problem 2: *Fundamental orbit equation*

The fundamental orbit equation (for the unit mass $m = 1$) is given by

$$\frac{l^2}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right) - \frac{l^2}{r^3} = f(r). \tag{9.2.20}$$

(a) Why is it called and "orbit equation"?

(b) For which force fields $f(r)$ is the circular orbit a solution of the fundamental orbit equation?
(c) Show that Kepler ellipses are a solution of the fundamental orbit equation.
(d) Derive the force law for Kepler ellipses.
(e) Show that the total energy $E = T + \Phi$, with the kinetic energy per unit mass T and the potential Φ, is an integral of the fundamental orbit equation for any force law $f(r)$.

Problem 2: Solution

(a) This equation is an orbital equation because it is a time-independent differential equation for r which is only dependent on φ.
This can be shown using Lagrangian mechanics as an ansatz.

(b) For a circular orbit, the solution should be independent of φ ($r(\varphi) = c$, so we require

$$f(r) = -\frac{l^2}{r^3}. \tag{9.2.21}$$

This leads to

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{l^2}{r^3} \implies \varphi = l^2 \int_R^\infty \frac{1}{r^3} = -\frac{l^2}{2R^2}, \tag{9.2.22}$$

so $\varphi_{\text{eff}} = \varphi + \frac{l^2}{2R^2}$.

(c) A Kepler ellipse is given by

$$r = \frac{p}{1 + \epsilon \cos\varphi}, \tag{9.2.23}$$

where p is a constant (some scale factor) and ϵ is the eccentricity.
The derivative is given by

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} \frac{p\epsilon \sin\varphi}{(1 + \epsilon\cos\varphi)^2}, \tag{9.2.24}$$

so multiplying this with $1/r^2 = \frac{(1+\epsilon\cos\varphi)^2}{p^2}$ and taking the derivative again, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{p\epsilon\sin\varphi}{r^2(1 + \epsilon\cos\varphi)^2}\right) = p^3\epsilon\cos\varphi, \tag{9.2.25}$$

so this plugging this into the fundamental orbit equation, we arrive at

$$\frac{l^2(1+\epsilon\cos\varphi)^2}{p^2} p^3\epsilon\cos\varphi - \frac{l^2(1+\epsilon\cos\varphi)^3}{p^3} = f(r) \tag{9.2.26}$$

$$\iff \frac{c}{r^2} = f(r), \tag{9.2.27}$$

which is indeed the proportionality we know for the Kepler potential.

(d) Starting with $f(r) = -\frac{\mathrm{d}\varphi}{\mathrm{d}r}$ and the following equation (which comes out of the Lagrangian equations of motion)

$$\ddot{r} - \frac{l^2}{r^3} = f(r), \tag{9.2.28}$$

we can multiply it by $\dot{r}$ and integrate to have the following

$$\frac{\dot{r}^2}{2} + \frac{l^2}{r^2} = \int f(r)\dot{r}\mathrm{d}t \tag{9.2.29}$$

$$= \int -\frac{\mathrm{d}\Phi}{\mathrm{d}r}\dot{r}\mathrm{d}\Phi = -\Phi. \tag{9.2.30}$$

Thus, we have the kinetic energy and potential in polar coordinates:

$$\frac{\dot{r}^2}{2} + \frac{l^2}{r^2} + \Phi = const = \frac{T+\Phi}{m} \tag{9.2.31}$$

Problem 3: *Timescales and collisionless systems*

A globular cluster contains about 10^6 stars that move with a typical velocity of 10 km/s. It has a half-light diameter of approximately 10 pc, and thus has a mean density of about $n = 10$ stars/pc^3.

(a) Calculate the timescale τ_c for direct collisions of stars in the globular cluster ($\tau_c = 1/(nv\pi d^2)$) for stars of the diameter of our sun ($R_\odot$) and for red giant stars ($100R_\odot$).

(b) In the centre of a globular cluster, the stellar density can be as high as 500 stars/pc^3. How does this change the collision timescales?

(c) The dynamical timescale of a systems gives the time it takes to go around one orbit in the system. Calculate the dynamical timescale for the cluster.

(d) The relaxation timescale measures the time it takes for one object in a system to be significantly perturbed by the other objects in the system. It is given by

$$T_{\text{relax}} = \frac{v^3}{8\pi G^2 m^2 n \ln\Lambda}, \tag{9.3.32}$$

where $\ln\Lambda$ is called the Coulomb logarithm, and is usually in the range $14...22$. Here we will assume $\ln\Lambda = 15$. Calculate the relaxation time for the globular cluster, and compare it to the dynamical time.

(e) What does this mean for the dynamical state of the cluster?

Problem 3: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) Using $d_{\text{sun-like}} = R_\odot$ and $d_{\text{rg-like}} = 100R_\odot$ (and $v = 10$ km/s), $n_{\text{stars}} = 10$ stars/pc^3, we have

$$\tau_{c,\text{ sun-like}} = 1.53\times10^9 \text{ Gyr} \tag{9.3.33}$$

$$\tau_{c,\text{ sun-like}} = 1.53\times10^5 \text{ Gyr} \tag{9.3.34}$$

for the collisional timescales, which is longer than the age of the universe ($T_{\text{universe}} \approx 13$ Gyr), so collisions are near impossible.

(b) Changing $n_{\text{stars}} = 500$ stars/pc^3, we have

$$\tau_{c,\text{ sun-like}} = 3.06 \times 10^7 \text{ Gyr} \tag{9.3.35}$$
$$\tau_{c,\text{ sun-like}} = 3.06 \times 10^3 \text{ Gyr.} \tag{9.3.36}$$

Still longer than the age of the universe.

(c) The dynamical timescale of the cluster can be calculated by using the distance where one typical orbit takes place (e. g. the half-mass radius $r_{1/2} \approx 5$ pc), dividing it by the typical speed:

$$\tau_{\text{dyn}} = \frac{2\pi r_{1/2}}{v} = 3 \times 10^6 \text{ years.} \tag{9.3.37}$$

This is less than the age of the universe (yay!).

(d) Using $v = 10$ km/s as the average velocity again, $m \approx 5M_\odot$ (the average mass of a star in the cluster, which we had derived on an earlier problem sheet), $n = 10$ stars/pc^3, $\ln\Lambda = 15$, we arrive at

$$T_{\text{relax}} = 1.4 \times 10^{10} \text{ yr} \tag{9.3.38}$$

Problem 4: *The black-hole-mass-velocity dispersion relation*

There is observational evidence that most elliptical galaxies harbour supermassive black holes at their centres. The observations indicate that the black hole masses correlate with global properties of the galaxies. In the table below you will find measurements of the black hole masses M_{BH} and the velocity dispersions σ of the stellar component for a number of elliptical galaxies. Plot the black hole mass against the velocity dispersion and verify that a correlation exists.

The data can be found in the this repository (note: not by the lecturers, see the data folder) along with some code.

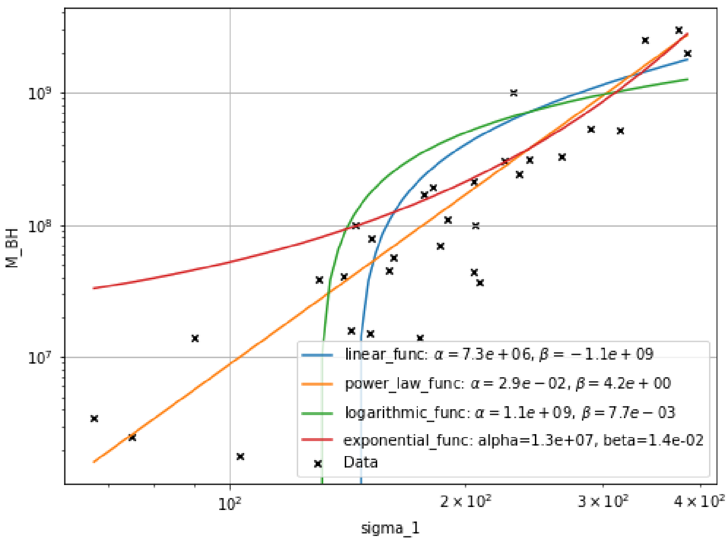
Which of the following functions represents the data best?

- $M_{\text{BH}} = \alpha\sigma + \beta$ (linear)
- $M_{\text{BH}} = \alpha\exp(\beta\sigma)$ (exponential)
- $M_{\text{BH}} = \alpha\sigma^\beta$ (power law)
- $M_{\text{BH}} = \alpha\sigma\log(\beta\sigma)$ (logarithmic)

Determine the parameters α and β of that function.

Problem 4: Solution

Note: The code used for parts of this exercise is available in **this repository**.



Both power law fits the data best, which can be seen from the log-log-plot above. Obviously, it has been fitted perfectly.
Here are the fit parameters:

- Power law:

$$M_{\mathrm{BH}} = \alpha\sigma^{\beta} : \tag{9.4.39}$$
$$\alpha = 7.33 \times 10^{6} \quad \beta = -1.07 \times 10^{9} \tag{9.4.40}$$

- Exponential:

$$M_{\mathrm{BH}} = \alpha \exp(\beta\sigma) : \tag{9.4.41}$$
$$\alpha = 1.3 \times 10^{7} \quad \beta = 0.0139 \tag{9.4.42}$$

- Power law:

$$M_{\mathrm{BH}} = \alpha\sigma^{\beta} \tag{9.4.43}$$
$$\alpha = 0.0288 \quad \beta = 4.24 \tag{9.4.44}$$

- Logarithmic:

$$M_{\mathrm{BH}} = \alpha\sigma \log(\beta\sigma) \tag{9.4.45}$$
$$\alpha = 1.15 \times 10^{9} \quad \beta = 0.00769 \tag{9.4.46}$$

Problem 5: *Major mergers of elliptical galaxies*

Elliptical galaxies can theoretically be described as collisionless stellar systems. Explain what we mean by this. Observations and evolutionary models for elliptical galaxies indicate they can grow by mergers with other small and big ellipticals. In this exercise we will use the virial theorem to estimate how the size, the velocity dispersion, and the mean density of an elliptical galaxy will change if it accretes stars.

- Assuming the systems have isotropic velocity dispersions and no rotation, give expressions for the total kinetic and gravitational energy (and the total energy) of a galaxy as function of Mass M, stellar mean square velocity $\langle v^2 \rangle$, and effective gravitational radius r_{g}.
  (*Hint:* How could an effective radius be suitably defined to describe a galaxy's binding energy?)

Let $\eta = M_{\mathrm{a}}/M_{\mathrm{i}}$ be the ratio of the accreted mass M_{a} to the initial mass M_{i} (so that the final galaxy has a mass $M_{\mathrm{f}} = M_{\mathrm{i}} + M_{\mathrm{a}} = (1+\eta)M_{\mathrm{i}}$), and let $\epsilon = \langle v_{\mathrm{a}}^2 \rangle / \langle v_{\mathrm{i}}^2 \rangle$ be the ratio of the initial average square speeds to the average square speeds of the accreted material.

- What is the energy of the merged galaxy?
- How do the velocities, sizes, and densities change when merging two systems? Compute the final-to-initial ratios of these quantities. Give the general expressions as functions of η and ϵ and numerical values for two identical systems.
- What happens if you have accretion of smaller systems with $\langle v_{\mathrm{a}}^2 \rangle \ll \langle v_{\mathrm{i}}^2 \rangle$? Can the first galaxies have formed through the mergers of single stars?

Problem 5: Solution

For this exercise, we only need the virial theorem and the conservation of energy.
Starting with an initial galaxy with parameters $r_{\mathrm{i}}, \langle v_{\mathrm{i}}^2 \rangle, M_{\mathrm{i}}$ and an accreted galaxy with parameters $r_{\mathrm{a}}, \langle v_{\mathrm{a}}^2 \rangle, M_{\mathrm{a}}$, they end up in the final galaxy with subscript f.
We assume that initially, their respective and rotational velocities are 0.
We use the ratios of mass and velocities, again given by

$$\eta = M_{\mathrm{a}}/M_{\mathrm{i}}, \quad \epsilon = \langle v_{\mathrm{a}}^2 \rangle / \langle v_{\mathrm{i}}^2 \rangle, \tag{9.5.47}$$

and also $M_{\mathrm{f}} = (1+\eta)M_{\mathrm{i}}$.
The virial theorem states that

$$\frac{E_{\mathrm{pot}}}{2} = -E_{\mathrm{kin}} \implies -\frac{GM^2}{2r} = -\frac{M\langle v^2 \rangle}{2} \implies \langle v^2 \rangle = \frac{GM}{r}. \tag{9.5.48}$$

Energy conservation also states that the total energy is

$$E_{\mathrm{tot}} = E_{\mathrm{kin}} + E_{\mathrm{pot}} = -\frac{1}{2}M\langle v^2 \rangle = -\frac{1}{2}\frac{GM^2}{r}. \tag{9.5.49}$$

We know that the total energy of the final galaxy has to be the cumulative energy of the initial and the accreted galaxy, so we have

$$E_{\mathrm{tot,\,f}} = E_{\mathrm{tot,\,i}} + E_{\mathrm{tot,\,a}} = -\frac{1}{2}M_{\mathrm{i}}\langle v_{\mathrm{i}}^2 \rangle - \frac{1}{2}M_{\mathrm{a}}\langle v_{\mathrm{a}}^2 \rangle = -\frac{1}{2}M_{\mathrm{i}}\langle v_{\mathrm{i}}^2 \rangle(1+\eta\epsilon). \tag{9.5.50}$$

This has to be equal to the total energy of the final galaxy, so we find

$$-\frac{1}{2}M_\mathrm{i}\langle v_\mathrm{i}^2\rangle(1+\eta\epsilon) \stackrel{!}{=} -\frac{1}{2}M_\mathrm{f}\langle v_\mathrm{f}^2\rangle \iff \frac{\langle v_\mathrm{f}^2\rangle}{\langle v_\mathrm{i}^2\rangle} = \frac{1+\eta\epsilon}{1+\eta}. \tag{9.5.51}$$

On the other hand, we know that

$$E_{\mathrm{tot,\,f}} == -\frac{1}{2}M_\mathrm{i}\langle v_\mathrm{i}^2\rangle(1+\eta\epsilon) = -\frac{1}{2}\frac{GM_\mathrm{f}^2}{r_\mathrm{f}} = -\frac{1}{2}G\frac{M_\mathrm{i}^2(1+\eta)^2}{r_\mathrm{f}}, \tag{9.5.52}$$

so using $\langle v_\mathrm{i}^2\rangle = \frac{GM_\mathrm{i}}{r_\mathrm{i}}$ from the virial theorem, we have

$$\frac{r_\mathrm{f}}{r_\mathrm{i}} = \frac{(1+\eta)^2}{1+\eta\epsilon}. \tag{9.5.53}$$

Assuming spherical distributions, the ratio of densities (using $\rho \propto \frac{M}{r^3}$) is

$$\frac{\rho_\mathrm{f}}{\rho_\mathrm{i}} = \frac{M_\mathrm{f}}{M_\mathrm{i}}\frac{r_\mathrm{i}^3}{r_\mathrm{f}^3} = \frac{(1+\eta)(1+\eta\epsilon)^3}{(1+\eta)^6} = \frac{(1+\eta\epsilon)^3}{(1+\eta)^5}. \tag{9.5.54}$$

We thus have derived three relations for the ratios of velocities, radii and densities.

- If two identical galaxies merge, we have $M_\mathrm{i} = M_\mathrm{a}$, $\langle v_\mathrm{i}^2\rangle = \langle v_\mathrm{a}^2\rangle$, so $\eta = 1$ and $\epsilon = 1$. Therefore, the velocity dispersion of the final galaxy is the same as the one of the initial galaxy since
$$\frac{\langle v_\mathrm{f}^2\rangle}{\langle v\rangle_\mathrm{i}^2} = \frac{1+1}{1+1} = 1. \tag{9.5.55}$$
For the radius, we find that
$$r_\mathrm{f} = \frac{(1+1)^2}{(1+1)}r_\mathrm{i} = 2r_\mathrm{i}. \tag{9.5.56}$$
Finally, for the density, we find that
$$\rho_\mathrm{f} = \frac{2^3}{2^5}\rho_\mathrm{i} = \frac{1}{4}\rho_\mathrm{i}, \tag{9.5.57}$$
so the density is lower by a factor of four.
- If $\langle v_\mathrm{a}^2\rangle \ll \langle v_\mathrm{i}^2\rangle$ with a mass ratio of $\eta = 1$, $\epsilon \approx 0$, leading to
$$\frac{\langle v_\mathrm{f}^2\rangle}{\langle v\rangle_\mathrm{i}^2} = \frac{1}{1+1} = \frac{1}{2} \tag{9.5.58}$$
for the velocity,
$$r_\mathrm{f} = \frac{(1+1)^2}{(1+0)}r_\mathrm{i} = 4r_\mathrm{i} \tag{9.5.59}$$
for the radius and
$$\rho_\mathrm{f} = \frac{1^3}{2^5}\rho_\mathrm{i} = \frac{1}{32}\rho_\mathrm{i} \tag{9.5.60}$$
for the density.