Problem set #7 Special Holiday Edition (here)

12.01.2023

Because who needs holidays anyways...

Problem 1: *Critical density for forbidden lines*

The spectra of gaseous nebulae show prominent emission from forbidden transitions, whereas these forbidden lines are absent from the spectra of stars.

- What is the critical density (order of magnitude) above which these forbidden lines will not appear?

Make a rough approximation regarding the velocity of the electrons and the interaction cross-section between electrons and ions. The Einstein "A" coefficient of the forbidden O[III] transition at $4959, 5007$ Å is $A = 2.6 \times 10^{-2}$/s. Assume the nebular kinetic temperature to be $T = 8000$ K.

Problem 1: Solution

We assume the Maxwell-Boltzmann distribution, so the average velocity of the electrons is

$$\langle v \rangle = \sqrt{\frac{2k_\mathrm{B}T}{m_e}} = 2.4 \times 10^4 \text{ m/s} \tag{7.1.1}$$

We use a geometric approach to estimate the cross-section of the ions and electrons, where we can use only the radius of the ion and neglect the extent of the electron, so we have

$$\sigma = \pi d^2 \approx \pi a_0^2 \tag{7.1.2}$$

where d denotes the distance between the 'centres' of electron and ion, and a_0 is the Bohr radius which we can assume as the typical radius of the ions.
For the gas to be able to emit the 'forbidden emission', the radiation rate $R_\mathrm{rad} = A$ (which in this case corresponds to the spontaneous emission Einstein coefficient A) needs to be higher than the average collision rate R_coll, which we can estimate to be

$$R_\mathrm{coll} = \sigma \langle v \rangle n_\mathrm{e}. \tag{7.1.3}$$

Applying the criterion $R_\mathrm{coll} < R_\mathrm{rad}$ and solving for the number density, we find

$$n_\mathrm{e,\ crit} < \frac{A}{\pi a_0^2}\sqrt{\frac{m_e}{2k_\mathrm{B}T}} \approx \underline{\underline{10^6}}\frac{1}{\mathrm{cm}^3} \tag{7.1.4}$$

to still be able to see the emission lines.

Problem 2: *H II regions around hot stars*

Hot stars emit enough UV photons to ionise the surrounding interstellar medium, forming a sharply-bounded "Strömgren sphere", inside of which the material is practically fully ionised, and outside of which it is essentially neutral.

- How large is this H[II] region for a typical 40000 K main-sequence star, if the interstellar medium is pure hydrogen with a number density of $n_{\mathrm{H}} = 102/\mathrm{cm}^3$? What is its mass? (What is the star's mass?)

Approximate the star by a blackbody with a radius of $10R_\odot$, and assume that the ionised region has a temperature of $T = 8000$ K. At this temperature, the recombination rate coefficient for hydrogen is about $\alpha_{\mathrm{B}} = 3.5 \times 10^{-13}$ cm^3/s.

(Hint: In equilibrium, the number of ionising photons emitted by the star per second must equal the number of recombinations per second in the H[II] region. [Why?] The number of Photons in the range $[\nu, \nu + \mathrm{d}\nu]$ emitted per second per unit area by a blackbody is $\frac{\pi B_\nu}{h\nu}$. What is the ionisation energy of hydrogen? Is the Wien approximation applicable?)

Problem 2: Solution

If the gas is ionised inside of the region and not ionised outside of it, there has to be photoionisation equilibrium in the sphere for it to be able to be ionised.

Therefore, $P_\gamma \overset{!}{=} R_{\mathrm{rec}}$, the photon production rate from the star has to equal the recombination rate.

Since $P_\gamma = \int_{\nu_0}^{\infty} \frac{\pi B_\nu}{h\nu} \mathrm{d}\nu$ (where ν_0 corresponds to $E = 13.6$ eV as the frequency needed at minimum to ionise the photons, so we have

$$P_\gamma = \int_{\nu_0}^{\infty} \frac{2\pi\nu^2}{c^2} \exp\left(\frac{-h\nu}{k_{\mathrm{B}}T}\right) \mathrm{d}\nu = 1.2 \times 10^{49} \frac{1}{\mathrm{s}}, \tag{7.2.5}$$

while the recombination rate gives us an angle at the stromgren radius:

$$R_{\mathrm{rec}} = n^2 \alpha_{\mathrm{B}} \frac{4\pi}{3} R_{\mathrm{S}}^3 \iff R_{\mathrm{S}} = \sqrt[3]{\frac{3P_\gamma}{n^2 \alpha_{\mathrm{B}} 4\pi}} = 3\,\mathrm{pc}. \tag{7.2.6}$$

Via the density of the HII region we were given, we can determine its mass to be

$$M_{\mathrm{S}} = \frac{\bar{\rho}}{\frac{4}{3}\pi R_{\mathrm{S}}^3} = \frac{n_{\mathrm{H}} m_{\mathrm{H}}}{\frac{4}{3}\pi R_{\mathrm{S}}^3} \approx 290 M_\odot. \tag{7.2.7}$$

Using $T_{\mathrm{eff}} = 40000$ K **TO DO Once we talk about the solution (we never discussed in the end, sorry!) (!)**

Problem 3: *Type Ia supernova brightness*

Type Ia supernovae (SN Ia) are stellar explosions with no signs of hydrogen and helium in their spectra, but showing lines of intermediate mass elements such as Si, S, Ca, and Mg near the maximum of their light curves, and many Fe lines at later times. In contrast to massive stars which are the progenitors of type II supernovae (SN II), SN Ia progenitors are thought to be white dwarfs (WDs) expected to consist mainly of carbon and oxygen. In the canonical model the white dwarf accretes mass from a companion star. When the white dwarf approaches the Chandrasekhar mass, thermonuclear burning ignites in its core, and the white dwarf is disrupted by an explosion, leaving no compact remnant.

The luminosity of the ejecta is powered by the radioactive decay of ^{56}Ni. This is observationally supported by the exponential luminosity decline, suggesting that unstable ^{56}Ni decays into ^{56}Co and then into ^{56}Fe. The initial decay of ^{56}Ni into ^{56}Co emits gamma-rays with an average energy of 1.71 MeV and a half-life of 6.1 days.
Supernova 1991T in NGC 4527 showed an absolute brightness of -19.55 mag in the visual band. The total mass of ^{56}Ni produced by thermonuclear burning is estimated to be $1.1M_\odot$.

- How many ^{56}Ni atoms can decay?
- What percentage of ^{56}Ni has decayed after 20 days? What is the average luminosity of the supernova during these 20 days, in solar luminosities? (Assume that all produced energy is converted to light and radiated away.)
- The absolute magnitude of the sun is $M_{V,\odot} = 4.83$ mag. Calculate the brightness of SN 1991T based on the decay of ^{56}Ni above, and compare your result to the observed value. Discuss possible reasons for discrepancies.
- How does the luminosity of SN 1991T compare with the luminosity of the Milky Way?

Problem 3: Solution

- How many ^{56}Ni atoms can decay:
  If we assume the thermonuclear ^{56}Ni mass to be $M_0 = 1.1M_\odot$, and a mass[a] of $m_{\text{Ni-56}} = 55.942128u$, we have[b]

$$N_{\text{Ni-56}} = \frac{M_0}{m_{\text{Ni-56}}} = 2.35 \times 10^{55} \tag{7.3.8}$$

  ^{56}Ni atoms that could possibly decay.
- What percentage of ^{56}Ni has decayed after 20 days?
  Using the formula

$$N(t) = N_0 0.5^{\frac{t}{t_{\text{half}}}}, \tag{7.3.9}$$

  we find (with $t_{\text{half}} = 6.1$ days) that

$$\frac{N(20\text{ d})}{N_0} = 0.5^{\frac{20}{6.1}} = 0.103043, \tag{7.3.10}$$

  so $\eta_{20} = 89.69\,\%$ of the initial ^{56}Ni atoms have decayed after 20 days.
  What is the average luminosity of the supernova during these 20 days, in solar luminosities?
  Assume that all produced energy is converted to light and radiated away, we can just calculate[c] the average luminosity as follows:

$$\langle L\rangle_{20} = \frac{\Delta E}{\Delta t} = \frac{\Delta N_{\text{Ni-56}} E_{\text{dec}}}{20\text{ d}} = \frac{\eta_{20} N_{\text{Ni-56}} E_{\text{dec}}}{20\text{ d}} = 8.685 \times 10^9 L_\odot, \tag{7.3.11}$$

  where we used $E_{\text{dec}} = 1.71$ MeV as the energy per decay, and assumed that the fraction η_{20} of ^{56}Ni that we derived above decayed.
- The absolute magnitude of the sun is $M_{V,\odot} = 4.83$ mag. Calculate the brightness of SN 1991T based on the decay of ^{56}Ni above, and compare your result to the

observed value:
To convert the given luminosity to an absolute magnitude, we can compare it to the sun's absolute magnitude, so we have[d]

$$M_{\mathrm{abs}} = M_{\mathrm{abs},\odot} - 2.5\log\left(\frac{\langle L\rangle_{20}}{L_\odot}\right) = 4.83 - \frac{2.5\ln(8.685)}{\ln(10)} - 2.5\cdot 9 = -20.017 \quad (7.3.12)$$

This is brighter than the absolute brightness $M_{V,\,\mathrm{abs}} = -19.55$ mag that was reported above for the visual band.
The reason for this discrepancy is primarily that we weren't careful about the band that $\langle L\rangle_{20}$ was emitted in - gamma-rays are usually not covered in the V band, therefore we'd have to estimate the amount of flux only in that band (or, more precisely, the $V - \gamma_{\mathrm{ray}}$ colour).

- How does the luminosity of SN 1991T compare with the luminosity of the Milky Way?
  The Milky Way luminosity is[e] around $M_V = -20.74$, so the supernova is almost as bright as the whole Milky Way over the course of a few days. Wow.

[a] according to **this website**
[b] see **here** for my WolframAlpha calculation
[c] see **here** for my WolframAlpha calculation
[d] see **here** for my WolframAlpha calculation
[e] according to **this** paper by Licquia et al, table 3

Problem 4: *How old is Ann?*

Mary is 24 years old. She is twice as old as Ann was when Mary was as old as Ann is now. How old is Ann?

Problem 4: Solution

Let's rewrite this in terms of equations.
Let $A_{\mathrm{M}} = 24$ be Mary's age right now, and A_{A} be Ann's age now.
We know that at some time in the past Δt, Mary was as old as Ann is now, so

$$A_{\mathrm{M}} - \Delta t = A_{\mathrm{A}}. \quad (7.4.13)$$

Also, we know that at that point of time in the past, Ann *was* half as old as Mary is now, so

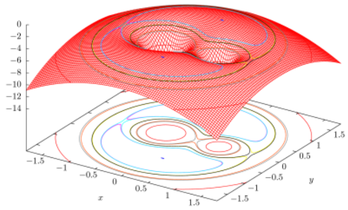
$$A_{\mathrm{M}} = 2(A_{\mathrm{A}} - \Delta t). \quad (7.4.14)$$

Solving these equations yields

$$\begin{aligned}
&A_{\mathrm{A}} + \Delta t = 24, && (7.4.15)\\
&A_{\mathrm{A}} + \Delta t = 2(A_{\mathrm{A}} - \Delta t) \iff 3\Delta t = A_{\mathrm{A}} && (7.4.16)\\
\iff\quad &4\Delta t = 24 \implies \Delta t = 6 && (7.4.17)\\
&\implies A_{\mathrm{A}} = 3\Delta t = 18. && (7.4.18)
\end{aligned}$$

Therefore, Ann is now 18 years and was 12 (half the age of Mary now) when Mary was 18 and therefore her current age. Checks out, but can cause some headache.

Problem 5: *Roche potential*



Use a computer program (for example, gnuplot) to draw a surface plot of the Roche potential, as in the figure on the right. Experiment with the parameters. For the final plot use masses of $M_1 = 2$ and $M_2 = 1$ and a distance of $d = 1$ between them. Use a gravitational constant of $G = 1$.

- Why are we allowed to rescale the units this way? How many independent dimensional physical parameters does the problem have? Why does the shape of the potential depend only on the mass ratio, not the absolute masses?
- Numerically calculate the potential at the Lagrange points and draw contour lines at (or a bit above and below) these values.
- Why are the Lagrange points L4 and L5 usually considered as stable in the literature, even though they correspond to local maxima of the potential?
  Extra credit: Calculate why this is so.
  (Hint: the potential actually depends on four coordinates. Linearise the equation of motion around the Lagrange points and compute the eigenvalues.)

Problem 5: Solution

Thanks and credits to Iliya Tikhonenko!

The usual way to start solving an average classical mechanics problem is to write down the Lagrangian. Let's assume that $\Phi(x)$ is the gravitational potential in the rest frame and then derive equations of motions in the rotating coordinate frame.

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\left|\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r}\right|^2 - \Phi(\boldsymbol{r}) \\ &= \tfrac{1}{2}\left|\dot{\boldsymbol{r}}\right|^2 + \dot{\boldsymbol{r}}\cdot(\boldsymbol{\Omega}\times\boldsymbol{r}) - \underbrace{\left(\Phi(\boldsymbol{r}) - \tfrac{1}{2}\left|\boldsymbol{\Omega}\times\boldsymbol{r}\right|^2\right)}_{\Phi_{\text{eff}}}\end{aligned} \tag{7.5.19}$$

Thus, Euler-Lagrange equations for the system would look like:

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{r}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{r}}} = 0 \quad\Longrightarrow\quad \ddot{\boldsymbol{r}} = -\nabla\Phi_{\text{eff}} - 2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}}. \tag{7.5.20}$$

If $\boldsymbol{\Omega} = \boldsymbol{k}\Omega$ the vector product from the effective potential definition can be simplified to $\Omega^2 R^2$, where $R = \sqrt{x^2+y^2}$. When all motion is constrained in xy plane, r would coincide with R. Thus, one can write the effective potential for a test mass in the gravitational field of two bodies as

$$\Phi_{\text{eff}} = \Phi(r) - \tfrac{1}{2}\Omega^2 r^2 = -\sum_{i=1}^{2}\frac{GM_i}{r_i} - \tfrac{1}{2}\Omega^2 r^2 \tag{7.5.21}$$

Using 3rd Kepler's law we can get rid of Ω:

$$\frac{G(M_1+M_2)}{4\pi^2} = \frac{a^3}{T^2} \implies \Omega^2 = \frac{G(M_1+M_2)}{a^3}, \tag{7.5.22}$$

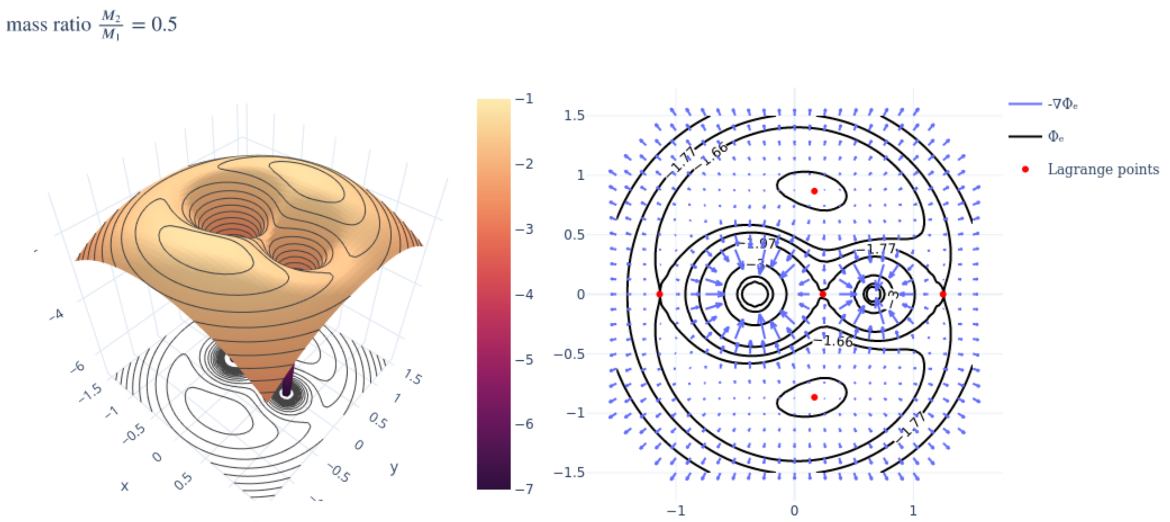
so, plugging this expression inside eq. (7.5.21) we get

$$\Phi_{\text{eff}} = -GM_1\left(\frac{1}{r_1}+\frac{\mu}{r_2}+\frac{1}{2}\frac{(1+\mu)r^2}{a^3}\right), \tag{7.5.23}$$

where a is the distance between two bodies with masses M_1 and M_2, r_1 and r_2 are the distance from the test particle to body #1 and #2 respectively, and $\mu = \frac{M_2}{M_1}$.
Assuming that both bodies lie on x axis at $x_1 < 0 < x_2$, we get

$$r_1 = \sqrt{(x-x_1)^2+y^2}, \quad x_1 = -\frac{a\mu}{1+\mu} \tag{7.5.24}$$

$$r_2 = \sqrt{(x-x_2)^2+y^2}, \quad x_2 = \frac{a}{1+\mu}. \tag{7.5.25}$$



We can also factor a out (effectively expressing all lengths in units of a), so

$$\Phi_{\text{eff}} = -\frac{GM_1}{a}\left(\frac{1}{r_1/a}+\frac{\mu}{r_2/a}+\frac{(1+\mu)\,(r/a)^2}{2}\right), \tag{7.5.26}$$

- In this form, it is immediately visible that we are allowed to rescale the units freely: the actual shape of the potential depends only on μ. You can see it in action in the **interactive version** of the plot above.
- Lagrange points are solutions of $-\nabla\Phi_{\text{eff}} = 0$, which in our case means that

$$\frac{\boldsymbol{r}_1}{r_1^3}+\frac{\mu\boldsymbol{r}_2}{r_2^3}+\frac{(1+\mu)\boldsymbol{r}}{a^3} = 0. \tag{7.5.27}$$

This equation has 5 roots in the complex plane, but we can almost immediately find two of them(L_4 and L_5) by assuming $r_1 = a = r_2$:

$$\frac{\boldsymbol{r}_1}{a^3}+\frac{\mu\boldsymbol{r}_2}{a^3}+\frac{(1+\mu)\boldsymbol{r}}{a^3} = 0, \tag{7.5.28}$$

which holds because in our coordinate frame the center of mass of the system coincides with 0:

$$\boldsymbol{r} = \frac{\boldsymbol{r}_1 + \mu \boldsymbol{r}_1}{1+\mu} = \frac{M_1 \boldsymbol{r}_1 + M_2 \boldsymbol{r}_2}{M_1 + M_2}. \tag{7.5.29}$$

The other 3 solutions ($L1$, L_2, and L_3) lie on x and cannot be expressed in an analytical form in the general case (see the right side of the plot above).

Assuming $G = 1$, $a = 1$, $M_1 + M_2 = 1$, and $\mu = \frac{1}{2}$, we can compute the values of the effective potential at Lagrange points:

point	x	y	Φ_{eff}
L_1	0.237	0	−1.973
L_2	1.249	0	−1.774
L_3	−1.136	0	−1.661
L_4	1/6	$\sqrt{3}/2$	−1.389
L_5	1/6	$-\sqrt{3}/2$	−1.389

- Considering the motion around L4 and L5, we need to take into account the second term in eq. (7.5.20): the Coriolis force. If a test particle tries to move away from one of the stable points, the Coriolis force will act in the direction orthogonal to the particle velocity, effectively diverging it from the initial trajectory.
  A more rigorous proof can be done in the following way.
  First we linearise Φ_{eff} near (say) L_4. All first order terms would be zero, due to L_4 being an equilibrium point, thus

$$\Phi_{\text{eff}}(x,y) = \Phi_0 + \underbrace{\frac{\partial^2 \Phi}{\partial x^2}}_{\Phi_{xx}} (\underbrace{x - x_{\text{L4}}}_{\xi})^2 + \underbrace{\frac{\partial^2 \Phi}{\partial y^2}}_{\Phi_{yy}} (\underbrace{y - y_{\text{L4}}}_{\eta})^2, \tag{7.5.30}$$

so we have

$$\nabla \Phi_{\text{eff}} = 0 + 2\Phi_{xx} \xi \, \boldsymbol{i} + 2\Phi_{yy} \eta \, \boldsymbol{j}. \tag{7.5.31}$$

On the other hand, we have

$$\Omega \times \dot{\boldsymbol{r}} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \Omega \\ \dot{x} & \dot{y} & \dot{z} \end{pmatrix} = \boldsymbol{i}(-\Omega \dot{y}) + \boldsymbol{j}(\Omega \dot{x}) = \boldsymbol{i}(-\Omega \dot{\eta}) + \boldsymbol{j}(\Omega \dot{\xi}), \tag{7.5.32}$$

so the EOM simplify to

$$\ddot{\xi} = -2\left(\Phi_{xx}\xi - \Omega \dot{\eta}\right) \tag{7.5.33}$$

$$\ddot{\eta} = -2\left(\Phi_{yy}\eta - \Omega \dot{\xi}\right), \tag{7.5.34}$$

which are solved with the ansatz $\xi = Ae^{\lambda t}$, $\eta = Be^{\lambda t}$.

TO DO expand.. :((!) We arrive at an equation for λ^2, requiring it to be negative, and have the following boundary conditions:

$$\Phi_{xx}\Phi_{yy} > 0 \tag{7.5.35}$$

$$\Phi_{xx} + \Phi_{yy} + 4\Omega^2 > 0 \tag{7.5.36}$$

$$\Phi_{xx} + \Phi_{yy} + 4\Omega^2 > 4\Phi_{xx}\Phi_{yy}. \tag{7.5.37}$$

The derivation can also be found in *Galactic dynamics*, page 182.

Problem 6: *Pulsar periods*

(a) Some of the fastest pulsars have pulse frequencies of around $f = 700$ Hz. Assuming that the pulses are correlated with a "hot spot" on a rigidly spinning sphere, what is the maximum size of that sphere? If the equator spins at $v = 0.2c$, what mass must the sphere at least have so it does not break up? (Make the crude assumption that the sphere isn't deformed by centrifugal forces before it breaks up.) Derive an expression for the density of the sphere. What parameters does it depend on? Compare your calculated density with that of nuclear material. (A uranium nucleus has a radius of about 10 fm and consists of 238 nucleons.)

(b) A certain pulsar has a frequency of 6.6 Hz. Assuming a homogeneous rotating sphere of mass $M = 1.4M_\odot$ and radius $R = 10$ km, what is the rotational kinetic energy of the sphere? The pulsar has a period rate of change of $\dot{P} = 1.54 \times 10^{-12}$ s/s. Compute the energy loss per unit time and compare it to the solar luminosity. What is the current best explanation for this spindown? Discuss the mechanism.

Problem 6: Solution

(a) We can just assume for a point at the surface of the sphere having to be slower than the speed of light, so the maximum size of such a sphere is

$$\omega R < c \iff R < \frac{c}{2\pi f} = 68 \text{ km}. \tag{7.6.38}$$

If the equator spins at $0.2c = \omega R$, we find $R = \frac{0.2c}{2\pi f} = 14$ km.
The mass then can be derived from an equilibrium of centrifugal force and gravity (assuming classical mechanics (which arguably might not be appropriate in this case)):

$$\omega^2 R \leq G\frac{M}{R^2} \tag{7.6.39}$$

which we can rewrite to

$$R \leq \left(\frac{G}{(2\pi f)^2}M\right)^{\frac{1}{3}} \lesssim 20 \text{ km}\left(\frac{M}{M_\odot}\right)^{\frac{1}{3}}, \tag{7.6.40}$$

so we find that

$$\frac{M}{M_\odot} \geq \left(\frac{14 \text{ km}}{20 \text{ km}}\right)^3 = 0.4. \tag{7.6.41}$$

Alternatively, we find $\frac{M}{R^3} \geq \frac{\omega^2}{G}$ Therefore, the mean density required is

$$\bar{\rho} = \frac{M}{V} \geq \frac{3\omega^2}{4\pi G} \approx 10^{14}\frac{\text{g}}{\text{cm}^3}. \tag{7.6.42}$$

Comparing this to the approximate density of nuclear material, we find

$$\rho_{\text{nuc}} = \frac{238 m_{\text{p}}}{r_{\text{nuc}}^3 4\pi/3} \approx 10^{14} \frac{\text{g}}{\text{cm}^3}, \tag{7.6.43}$$

so we see that the required density of the neutron star is of the same order of magnitude (we used $r_{\text{nuc}} = 10^{-12}$ m).

(b) For this pulsar, we have $f = 6.6$ Hz, $M = 1.4 M_\odot$, $R = 10$ km and $\dot{P} = 1.54 \times 10^{12}$ s/s.
To find the energy loss per unit time, we have to take a look at the rotational energy possessed by the neutron star, which is

$$E_{\text{rot}} = \frac{1}{2}\omega = \frac{1}{5} M R^2 \left(\frac{2\pi}{P}\right)^2 = 9.6 \times 10^{40} \text{ J}, \tag{7.6.44}$$

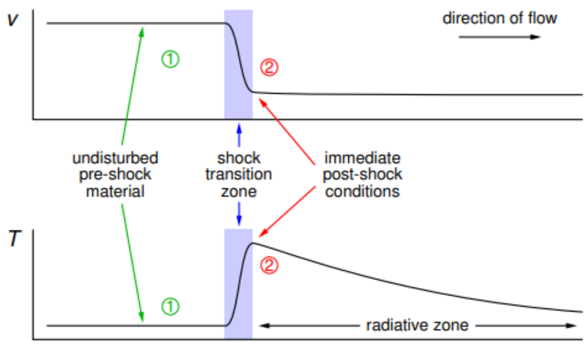
where we assumed the neutron star to be as sphere, having a moment of inertia of $I = \frac{2}{5} M R^2$.
Thus,

$$\dot{E_{\text{rot}}} = \frac{(2\pi)^2}{5} M R^2 \left(-\frac{2}{P^3}\dot{P}\right) = E_{\text{rot}} \left(-2\frac{\dot{P}}{P}\right) = 2 \times 10^{30} \text{ W} \approx 10^4 L_\odot. \tag{7.6.45}$$

Due to the strong magnetic field of the pulsar, currents inside it are induced, which leads to energy loss (magnetorotational breaking), and some energy is lost due to particles being accelerated by the magnetic fields.

Problem 7: *Simple one-dimensional shock model*

Shock waves play an important role in astrophysics. They occur on all scales in the ISM, for example in supernova blast waves, cloud-cloud collisions, H[II] regions, interstellar bubbles (very fast winds), accretion and outflow phenomena (jets), spiral shocks in the galactic disk, galaxy mergers, and even at the edges of galaxy clusters.
From a kinetic viewpoint, shocks convert much of the ordered motion of the pre-shock gas into random, thermal motion. If the shock speed is sufficiently high, the hot post-shock gas radiates and heats more distant gas both upstream and downstream from the front itself.



The conservation of mass flow, momentum flow, and energy flow through the shock zone are described by

$$\rho_1 v_1 = \rho_2 v_2 \tag{7.7.46}$$
$$P_1 + \rho_1 v_1^2 = P_2 = \rho_2 v_2^2 \tag{7.7.47}$$
$$v_1 \left(\frac{\rho_1 v_1^2}{2} + \rho_1 U_1 + P_1 \right) = v_2 \left(\frac{\rho_2 v_2^2}{2} + \rho_2 U_2 + P_2 \right), \tag{7.7.48}$$

where ρ, v, and P are the mass density, speed, and pressure, and the subscripts 1 and 2 refer to the pre- and post-shock gas, respectively.
The internal energy per unit mass is

$$U = \frac{P}{(\gamma - 1)\rho} \tag{7.7.49}$$

where γ is the adiabatic index of the gas.
The preceding set of equations can be rewritten as the *Rankine-Hugoniot jump conditions*:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1)M_1^2}{2 + (\gamma - 1)M_1^2} \tag{7.7.50}$$
$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - \gamma + 1}{\gamma + 1} \tag{7.7.51}$$
$$\frac{T_2}{T_1} = \frac{P_2}{P_1} \frac{v_2}{v_1} \tag{7.7.52}$$

with $M_1 = \frac{v_1}{c_1}$ the *Mach number* and $c_1 = \sqrt{\frac{\gamma P_1}{\rho_1}}$ the sound speed in the pre-shock region. (You can derive these equations yourself. Only basic algebra is involved, but finding the correct order of operations is not entirely trivial.)

(a) Why are there only *compression shocks* [in which fast, thin gas runs into the shock and slower, denser gas leaves it] and no *expansion shocks* [in which slow, dense gas enters the shock and fast, thin gas runs out of it], subject to those same conservation laws?

(b) For fast shocks ($M_1 \gg 1$), how much denser is the post-shock gas than the pre-shock material? Assume a non-relativistic, monatomic gas ($\gamma = \frac{5}{3}$).

(c) For a shock with $v_1 = 1000$ km/s (a typical value for the speed of the blast wave around a supernova remnant) and an ambient ISM temperature of 1000 K, what is the Mach number? What is the temperature of the shocked material? (You can simplify the Rankine–Hugoniot conditions when $M_1 \gg 1$.) What is the Mach number in the post-shock gas?

Problem 7: Solution

TO DO Once we get the solution (we never discussed it in the end... sorrey) (!)