Problem set #11 Cosmology and expanding confusion (here)

10.02.2023

Problem 1: *Recombination redshift*

When the universe expanded and cooled, the electrons ("re") combined with baryons and formed neutral atoms.

- Determine at which redshift z this recombination occurred in the early universe. For simplicity, assume the universe was pure hydrogen (no helium or metal nuclei) at that time. The average baryon density today is $n_0 = 2 \times 10{-}7$ cm^{-3}, the temperature of the microwave background is $T_0 = 2.73$ K.

(*Hint*: How do density and radiation temperature vary with redshift? What were the conditions in the universe before recombination?)

Problem 1: Solution

For temperature and number density, we note that

$$n = n_0 \left(1+z\right)^3 , \quad T = T_0 \left(1+z\right) . \tag{11.1.1}$$

Assuming local thermodynamical equilibrium, the fraction of ionisation x is given by the Saha equation:

$$x = \tag{11.1.2}$$

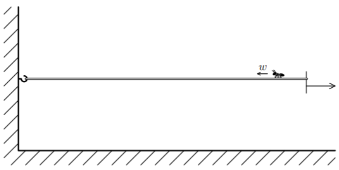
Solving for an ionisation fraction of $x = 0.5$, we find that

$$z_{\rm rec} = 1400 \tag{11.1.3}$$

See also **here****TO DO Actually calculate stuff. Sorry (!)**

Problem 2: *Ant on a rubber rope*

One end of an initially 1 m long infinitely stretchable rubber rope is anchored to a wall, while the other end is pulled away from the wall with a constant velocity of $u = 1$ cm/s. Starting out at the moving end of the rope is an ant that walks on the rope with a velocity of $w = 1$ mm/s (relative to the piece of rope it is currently on) towards the wall (see sketch).



- Derive an equation describing the motion of the ant as seen from a stationary observer, and compute the position of the ant as a function of time. Will the ant reach the wall? If yes, how long will it take? If no, how must the parameters of the system (length of the rope, velocities of the ant and the pull on the rope) be modified so that it can?

Note that an analytic solution exists for the above case of $u = const$. You can solve the equation numerically to explore different scenarios of $u = u(t)$ varying in time. What cosmological problem is this situation an analogy of?

Problem 2: Solution

To determine the time it takes for the ant to reach the end of the expanding rope, we consider the equation for its velocity $\dot{x}$.
Obviously, it has some constant component w in negative x-direction, but how do we factor in the stretch?
Let the total length of the rope be $X(t) = x_0 + ut$, where $x_0 = 1$ m is the initial length. Then, the velocity of the ant has an additional component that is determined by the expansion velocity u and the fraction of the rope the ant is currently located at.
Thus, $\dot{x}$ is given by

$$\dot{x} = -w + u\frac{x}{X} = -w + u\frac{x}{x_0 + ut}. \tag{11.2.4}$$

This is a linear ordinary differential equation (LODE) of the form $\dot{x} = p(t)x + q(t)$, which can be solved as these equations are usually solved:

- **Homogeneous solution**:
  First, we set $q(t) = -w$ to 0 and simply integrate the remaining equation, yielding

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{u}{x_0 + ut}x \tag{11.2.5}$$

$$\iff \int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}t}{t + x_0/u} \tag{11.2.6}$$

$$\iff \ln(x) = \ln(t + x_0/u) + c_0 \quad (c_0 \in \mathbb{R}) \tag{11.2.7}$$

$$\iff x(t) = c_1(t + x_0/u), \quad (c_1 \in \mathbb{R}) \tag{11.2.8}$$

  where c_0 and c_1 are constants.
- **Inhomogeneous solution** (variation of constants):
  For the inhomogeneous solution, we assume the 'constant' c_1 of eq. (11.2.8) to be time-dependent ($c_1(t)$), differentiate the solution, and plug in eq. (11.2.4):

$$\dot{x} = \dot{c}_1(t + x_0/u) + c_1 \tag{11.2.9}$$

$$\overset{(11.2.4)}{\iff} -w + u\overbrace{\frac{x}{x_0 + ut}}^{=\frac{c_1(t+x_0/u)}{t+x_0/u}} = \dot{c}_1(t + x_0/u) + c_1 \tag{11.2.10}$$

$$\overset{(11.2.8)}{\iff} \dot{c}_1 = -\frac{w}{t + x_0/u} + c_1 - c_1 \tag{11.2.11}$$

$$\implies c_1(t) = -w\ln(t + x_0/u) + c_2, \quad (c_2 \in \mathbb{R}), \tag{11.2.12}$$

  where we pick up c_2 as the independent integration constant.
- **Full general solution**:
  The full solution of the LODE is then simply given by

$$x(t) = (t + x_0/u)(c_2 - w\ln(t + x_0/u)). \tag{11.2.13}$$

- **Taking care of initial conditions**:
  In our case, we have the initial condition that $x(t=0) \stackrel{!}{=} x_0$, enabling us to determine c_2:

$$x_0 = \frac{x_0}{u}\left(c_2 - w\ln\left(0 + \frac{x_0}{u}\right)\right) \tag{11.2.14}$$

$$\iff c_2 = u + w\ln\left(\frac{x_0}{u}\right). \tag{11.2.15}$$

  Thus, the full solution, considering the initial conditions, is given by

$$x(t) = (ut + x_0)\left(1 - \frac{w}{u}\ln\left(\frac{ut}{x_0} + 1\right)\right). \tag{11.2.16}$$

- Sanity checks:
  This equation surprisingly even recovers the solution we'd expect for $\lim_{u\to 0}$:
  We need to know that we can expand

$$\ln(y+1) = \sum_{k=1} \frac{(-1)^{k+1} y^k}{k} = y - \frac{y^2}{2} + \frac{y^3}{3} + \ldots, \tag{11.2.17}$$

  so $\ln(ut/x_0 + 1) \approx ut/x_0$ for small u, so the solution becomes[a]

$$\lim_{u\to 0} x(t) \approx \lim_{u\to 0}(ut + x_0)\left(1 - w\frac{ut}{ux_0}\right) \tag{11.2.18}$$

$$= x_0 - wt. \tag{11.2.19}$$

Now that we have a nice solution for $x(t)$ describing the position of the ant on the rope at all times, we can solve for $x(t) = 0$ (i. e. the time when the ant would reach the start):

- We find

$$0 = \overbrace{(ut + x_0)}^{=0 \quad (i)} \overbrace{\left(1 - \frac{w}{u}\ln\left(\frac{ut}{x_0} + 1\right)\right)}^{=0 \quad (ii)} \tag{11.2.20}$$

$$(i) \qquad t_1 = -\frac{x_0}{u} \quad \text{(not physical as } t > 0\text{)} \tag{11.2.21}$$

$$(ii) \qquad \frac{u}{w} = \ln\left(\frac{ut_2}{x_0} + 1\right) \tag{11.2.22}$$

$$\iff t_2 = \frac{x_0}{u}\left(e^{\frac{u}{w}} - 1\right). \tag{11.2.23}$$

  The surprising result is therefore that no matter how great the difference of u and w, the ant will always reach its goal (although we might have to provide it with some snacks along the way).

- Sanity check:
  For $u \to 0$, we can expand $\exp(u/w) \approx 1 + u/w$, so

$$\lim_{u\to 0} t_2 \approx \frac{x_0}{u}(1 + u/w - 1) = \frac{x_0}{w}, \tag{11.2.24}$$

  so again we recovered the solution we'd expect.

- Numerical values:
  Plugging in the numbers, we obtain $x_0/u = 100$ s, $u/w = 10$, so

$$t_2 = 100\left(e^{10} - 1\right)\ \text{s} = 2.2 \times 10^6\ \text{s}\ = 25.5\ \text{days}. \tag{11.2.25}$$

  By that time, the rope will have grown to a whopping

$$X(t_2) = x_0 + ut_1 = 22\ \text{km}... \tag{11.2.26}$$

  That must be some sturdy material, where can I get it?

Alright, bad remarks aside, let's discuss the last part of the problem:
If $u = u(t)$ isn't constant but rather grows, the ant might not reach its goal at all.
This is reminiscent of the accelerated expansion of space, implying that we might never be able to see light sent out by stars too far away due to the constant speed the light can travel.

TO DO Numerical analysis and plotting (!)

[a] I know this isn't mathematically rigorous, but I just wanted to show that this ugly beast can be tamed.

Problem 3: *Expansion law for a flat universe*

The parameterised Friedmann equation is

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_\text{r}}{a^4} + \frac{\Omega_\text{m}}{a^3} + \frac{1 - \Omega_\text{m} - \Omega_\Lambda - \Omega_\text{r}}{a^2} + \Omega_\Lambda. \tag{11.3.27}$$

(a) Assume a flat universe without radiation. Rewrite the Friedmann equation for this case substituting $y = a^{\frac{3}{2}}$.

(b) Introduce the scaled time variable $x = t/t_\text{acc}$, where t_acc is some characteristic time, and replace the time derivative in the above equation with $\dfrac{\mathrm{d}}{\mathrm{d}x}$.
By appropriately choosing t_acc, write the equation in the form

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = C + y^2, \tag{11.3.28}$$

where C is a constant.

(c) Solve the differential equation for y (*Hint*: $\cosh^2 x - \sinh^2 x = 1$).
Derive the expansion law $a(t)$.

(d) Determine the asymptotic form of $a(t)$ for early and late times[a].

(e) Plot $a(t)$ for different cosmological parameter combinations.

[a] These limits should be consistent with a matter-dominated universe in the first case and a Λ-dominated one in the second.

Problem 3: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) For a flat universe without radiation, $\Omega_\mathrm{r} = 0$ and $\Omega_\mathrm{m} + \Omega_\Lambda = 1$.
Let us also already substitute $H = \frac{\dot{a}}{a}$, so the equation simplifies to

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\Omega_\mathrm{m} a^{-3} + \Omega_\Lambda\right). \tag{11.3.29}$$

Thus, substituting $y = a^{\frac{3}{2}}$ (or rather, $a = y^{\frac{2}{3}}$) and using the chain rule with $\dot{a} = \frac{2}{3} y^{-\frac{1}{3}} \dot{y}$, we obtain

$$\left(\frac{2}{3} \frac{\dot{y}}{y^{\frac{2}{3}} y^{\frac{1}{3}}}\right)^2 = H_0^2 \left(\Omega_\mathrm{m} y^{-2} + \Omega_\Lambda\right). \tag{11.3.30}$$

Solving for $\dot{y}^2$, we have

$$\dot{y}^2 = \frac{9}{4} H_0^2 \left(\Omega_\mathrm{m} + y^2 \Omega_\Lambda\right). \tag{11.3.31}$$

(b) Let us rewrite the equation once again to see that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t} \frac{2}{3H_0\sqrt{\Omega_\Lambda}}\right)^2 = \frac{\Omega_\mathrm{m}}{\Omega_\Lambda} + y^2. \tag{11.3.32}$$

Thus, we have found that $t_\mathrm{acc} = \frac{2}{3H_0\sqrt{\Omega_\Lambda}}$, and with $x = t/t_\mathrm{acc} \implies \mathrm{d}x = \mathrm{d}t/t_\mathrm{acc}$, we arrive at

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = y^2 + \frac{\Omega_\mathrm{m}}{\Omega_\Lambda}, \tag{11.3.33}$$

so the constant C in the above solution to derive is $C = \frac{\Omega_\mathrm{m}}{\Omega_\Lambda}$.

(c) To solve this equation, we simply integrate it after separation and substitute with a sinh to use $(*) : 1 + \sinh^2(x) = \cosh^2(x)$ as this alleviates the nasty term in the denominator:

$$\int \frac{\mathrm{d}y}{\sqrt{C + y^2}} = \int \mathrm{d}x \tag{11.3.34}$$

$$\iff \quad \frac{1}{\sqrt{C}} \int \frac{\mathrm{d}y}{\sqrt{1 + (y/\sqrt{C})^2}} = x - x_0 \tag{11.3.35}$$

$$\overset{y=\sqrt{C}\sinh(z)}{\iff} \quad \frac{1}{\sqrt{C}} \int \frac{\mathrm{d}z\sqrt{C}\cosh(z)}{\sqrt{1 + \sinh^2(z)}} = x - x_0 \tag{11.3.36}$$

$$\overset{(*),\,\cosh(z)>0\,\forall z}{\iff} \quad \int \mathrm{d}z = x - x_0 \tag{11.3.37}$$

$$\overset{z=\mathrm{arsinh}(y/\sqrt{C})}{\iff} \quad \mathrm{arsinh}(y/\sqrt{C}) = x - c_1 \quad (c_1 \in \mathbb{R}) \tag{11.3.38}$$

$$\iff \quad y(x) = \sqrt{C}\sinh(x - c_1), \tag{11.3.39}$$

where $c_1 = x_0 - \operatorname{arsinh}(y_0/\sqrt{C})$ is a constant depending on the initial conditions. To derive the expansion law, we have to resubstitute $y = a^{\frac{3}{2}}$ and $x = t/t_{\mathrm{acc}}$, finding

$$a(t) = \left(\frac{\Omega_{\mathrm{m}}}{\Omega_\Lambda}\right)^{\frac{1}{3}} \left(\sinh\left(\frac{t}{t_{\mathrm{acc}}} - c_1\right)\right)^{\frac{2}{3}} . \tag{11.3.40}$$

Kind of ugly, I'd say...

The initial conditions for our problem are $a(t \to 0) = 0$, implying that $c_1 \stackrel{!}{=} 0$ as $\operatorname{arsinh}(0) = 0$, so the final answer is

$$a(t) = \left(\frac{\Omega_{\mathrm{m}}}{\Omega_\Lambda}\right)^{\frac{1}{3}} \left(\sinh\left(\frac{t}{t_{\mathrm{acc}}}\right)\right)^{\frac{2}{3}} . \tag{11.3.41}$$

(d) To study the behaviour at early and late times, we note that

$$\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right) , \tag{11.3.42}$$

which has the following behaviour:

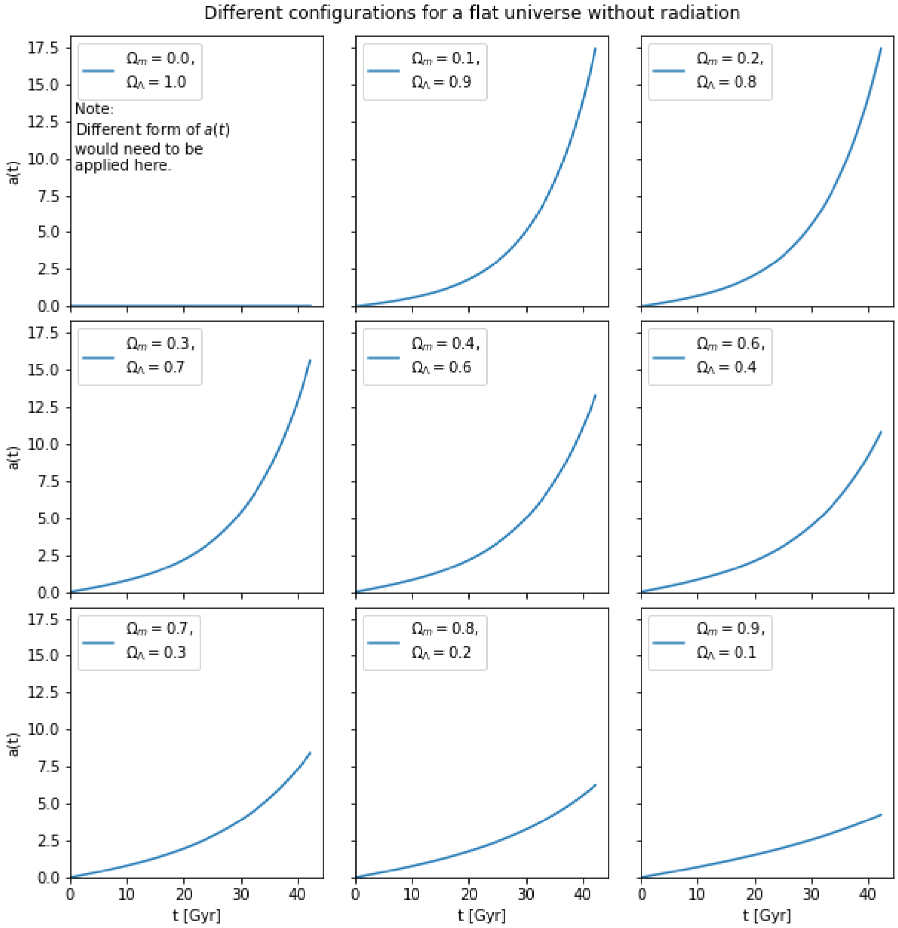
- **Early times**:
  For $x \to 0$, $\sinh(x) \approx \frac{1}{2}(1 + x - (1 - x)) = x$, so

$$t \to 0 \implies a(t) \approx \left(\frac{\Omega_{\mathrm{m}}}{\Omega_\Lambda t_{\mathrm{acc}}^2}\right)^{\frac{1}{3}} t^{\frac{2}{3}} = \left(\frac{3}{2} H_0 \sqrt{\Omega_{\mathrm{m}}} t\right)^{\frac{2}{3}} , \tag{11.3.43}$$

  which is independent of Ω_Λ (since $t_{\mathrm{acc}}^2 = \frac{2^2}{3^2 H_0^2 \Omega_\Lambda}$, it cancels out).
  This indeed reproduced the solution for the matter-dominated universe.
- **Late times**:
  For $x \to \infty$, the exponential term of the sinh dominates, so $\sinh(x) \approx \frac{1}{2}e^x$, leading to

$$t \to \infty \implies a(t) \approx \left(\frac{\Omega_{\mathrm{m}}}{8\Omega_\Lambda}\right)^{\frac{1}{3}} \exp\left(\frac{2}{3} t/t_{\mathrm{acc}}\right) = \left(\frac{\Omega_{\mathrm{m}}}{8\Omega_\Lambda}\right)^{\frac{1}{3}} \exp\left(H_0 \sqrt{\Omega_\Lambda} t\right) . \tag{11.3.44}$$

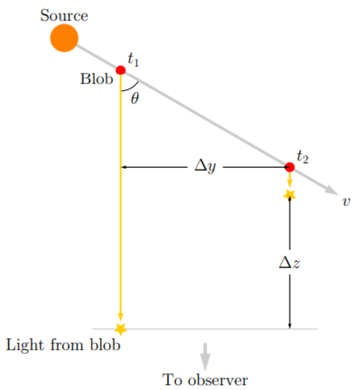
  This is the exponential expansion we'd indeed expect from a Λ-dominated universe (we note that for $\Omega_{\mathrm{m}} = 0$, the differential equation would already look a little different).
- For the timesteps in the plot, we use t in terms of $1/H_0$ from 0 to 3 Hubble times:



We see that for higher values of Ω_Λ, the exponential growth dominates earlier.

Problem 4: *Superluminal motion*

Jets from AGN sometimes show superluminal motion that appears to move on the sky faster than the speed of light.



A source ejects a blob in a direction tilted from the observer's line of sight by the angle θ. The blob moves with speed v.

(a) The light emitted by the blob at t_1 precedes the light emitted by the blob at t_2 by Δz. Show that

$$\Delta z = c(t_2 - t_1) - v(t_2 - t_1)\cos\theta, \tag{11.4.45}$$

and therefore

$$\Delta t = (t_2 - t_1) - \frac{v}{c}(t_2 - t_1)\cos\theta \tag{11.4.46}$$

in time.

(b) Show that the apparent speed of the blob that the observer sees is

$$v_y = \frac{\Delta y}{\Delta t} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta}. \tag{11.4.47}$$

(c) If you see superluminal motion at all, you can tell how much the minimum actual speed of the blob must be, regardless of your viewing angle. How much is it?

Problem 4: Solution

(a) From trigonometric considerations and by labelling the sides of the triangle as follows: $s := v(t_2 - t_1)$ as the distance between t_1 adn t_2, a as the vertical distance between the blob at t_1 and t_2, and l as the vertical distance between the Blob at t_1 and the observer.

In this case, we simply find that

$$\Delta z = l - a = c(t_2 - t_1) - \cos\theta v(t_2 - t_1). \tag{11.4.48}$$

Using $\Delta t = \frac{\Delta z}{c}$, we find the results given in the assignment as well:

$$\Delta t = (t_2 - t_1) - \frac{v}{c}(t_2 - t_1)\cos\theta. \tag{11.4.49}$$

(b) The apparent speed of the blob at the observer is simply

$$v_y = \frac{\Delta y}{\Delta t} = \frac{\sin\theta v(t_2 - t_1)}{(t_2 - t_1) - \frac{v}{c}(t_2 - t_1)\cos\theta} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta}. \tag{11.4.50}$$

(c) For us to be see any superluminal motion, the observed v_y has to be greater than the speed of light.

$$v_y > c \tag{11.4.51}$$

$$\frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta} > c \tag{11.4.52}$$

$$\frac{v}{c}\sin\theta > (1 - \frac{v}{c}\cos\theta) \tag{11.4.53}$$

$$\frac{v}{c}(\sin\theta + \cos\theta) > 1 \tag{11.4.54}$$

$$v > \frac{c}{\sin\theta + \cos\theta} \tag{11.4.55}$$

$$\implies v_{\min} = \frac{c}{\sqrt{2}}. \tag{11.4.56}$$

Problem 5: *Lorentz transformation and beaming*

K and K' are inertial frames; K' moves with a constant velocity v with respect to K along the x axis. The Lorentz transformation from K to K' is given by

$$x' = \gamma(x - vt) \tag{11.5.57}$$
$$y' = y \tag{11.5.58}$$
$$z' = z \tag{11.5.59}$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \tag{11.5.60}$$

where $\gamma := \frac{1}{\sqrt{1-v^2/c^2}}$.

(a) Find the forms for x, y, z, and t as functions of x', y', z', and t'.
(b) Find the differentials $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$, and $\mathrm{d}t$ as functions of $\mathrm{d}x', \mathrm{d}y', \mathrm{d}z'$, and $\mathrm{d}t'$.
(c) Find the velocities u_x, u_y, and u_z as functions of u'_x, u'_y, and u'_z.
(d) Generalise the velocities above to the velocities parallel and perpendicular to the motion of the frame K',

$$u_\parallel = \frac{u'_\parallel + v}{1 + vu'_\parallel/c^2}, \quad u_\perp = \frac{u'_\perp}{\gamma\left(1 + vu'_\parallel/c^2\right)}. \tag{11.5.61}$$

Define θ by $\tan\theta = \frac{u_\perp}{u_\parallel}$.
Draw a sketch of the geometry, indicating θ.
(e) Find $\tan\theta$ as a function of θ' and $u' := |\boldsymbol{u}'|$ (i. e., $u'_\perp = u'\sin\theta'$ and $u'_\parallel = u'\cos\theta'$).
(f) Find $\tan\theta$ and $\cos\theta$ where $u' = c$.
(g) Rewrite $\tan\theta$ for $\theta' = \frac{\pi}{2}$. Discuss qualitatively what this means.
(h) Find the form of θ when $\gamma \gg 1$.
(i) Explain qualitatively why we often miss counter jets in AGN.

Problem 5: Solution

TO DO Do the first parts of this exercise (!)

(a)
(b)
(c)
(d)
(e) Using $\theta' = \frac{u'_\perp}{u'_\parallel}$, we can show that

$$\tan\theta = \frac{u'_\perp}{\gamma(u'_\parallel + v)} = \frac{c\sin\theta'}{\gamma(c\cos\theta' + v}. \tag{11.5.62}$$

(f) For $\theta' = \frac{\pi}{2}$, we have $\tan\theta = \frac{c}{\gamma v}$.
(g) For low γ, it approaches $\frac{c}{v}$, but for high γ (where $v \to c$, $\tan\theta = \frac{\tan(\theta'/2)}{\gamma} \implies \theta \approx \frac{\tan(\theta'/2)}{\gamma}$, which means that the light would be in a very small cone in the direction of motion.

(h) Thus, the higher the velocity of the jets, the smaller the emission cone becomes, reducing the likelihood for us to be able to see it. ,