Problem set #2 Radiation and the ISM (here) 17.11.2022

Problem 1: *Radiation through an isothermal layer*

Electromagnetic radiation can be described by the Planck law and its relatives (the Wien displacement law and the Stefan–Boltzmann law). These equations hold strictly in TE ("thermodynamic equilibrium") and reasonably well in most stellar photospheres.
The Planck function specifies the radiation intensity emitted by a gas or a body in TE (a black body, cf. Figure 1).

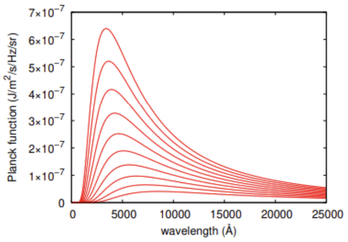


Figure 1: Planck function B_ν for temperatures from 6000 K to 15000 K.

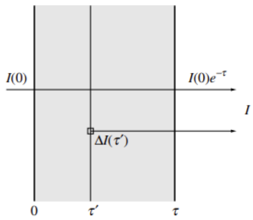


Figure 2: Emergent radiation from a layer consists of transmitted radiation and radiation produced in the layer itself.

Another quantity next to the radiation produced by a gas of temperature T is the amount of absorption. Take the situation sketched in Figure 2.
A beam of radiation with intensity $I(0)$ passes through a layer in which it is attenuated. The weakened intensity I that emerges on the right is given by

$$I = I(0)e^{-\tau} \tag{2.1.1}$$

in which the decay parameter τ specifies the attenuation by absorption in the layer. It is a dimensionless measure of the opaqueness of the layer that is usually called the "optical thickness" because it measures how thick the layer is, not in cm but in terms of its effect on the passing radiation.
Nothing comes through if $\tau \gg 1$ and (almost) everything comes through if $\tau \ll 1$. The next step in computing the total emergent radiation is to add the radiation that originates within the layer itself. The amount ΔI of radiation that is generated locally at position τ' within the layer is equal to $\underline{\quad\quad}_1$ (assume the layer has a fixed temperature $T(\tau')$).
This radiation is subsequently attenuated by the remainder of the layer to the right, so that its contribution to the emergent beam is given by $\underline{\quad\quad}_2$. The total emergent intensity (containing all contributions) is therefore $\underline{\quad\quad}_3$, which for an isothermal layer (one in which T and thus also $B(T)$ is independent of τ') simplifies to $\underline{\quad\quad}_4$.

(a) Derive the 4 equations which are marked by "$\underline{\quad\quad}$".

(b) Make plots of the emergent intensity I for given values B and $I(0)$ against the total layer thickness τ. Use $B = 2, I(0) = 0, 1, 2, 3, 4$ and $\tau = 0...10$.

(c) How does I depend on τ for $\tau \ll 1$ when $I(0) = 0$?
And when $I(0) > B$? Such a layer is called *optically thin*. Why?

(d) A layer is called *optically thick* when it has $\tau \gg 1$. Why?
The emergent intensity becomes independent of τ for large τ. Explain why this is so in physical terms.

Problem 1: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) Let's consider the "_____" one by one:

- $\_\_\__1$: The amount of radiation generated in one layer is temperature-dependent and given by the Stefan-Boltzmann law:

$$\Delta I = B(\tau')\mathrm{d}\tau'. \tag{2.1.2}$$

- $\_\_\__2$: Therefore, the amount of radiation from this layer reaching the observer (and therefore emerging through a layer of thickness τ') after attenuation is given by

$$I'(\tau') = \Delta I e^{-\tau'} = B(\tau')e^{-\tau'}\mathrm{d}\tau'. \tag{2.1.3}$$

- $\_\_\__3$: To calculate the total emergent intensity I_{tot}, we need to consider all intensity that has been picked up, which we obtain by integration:

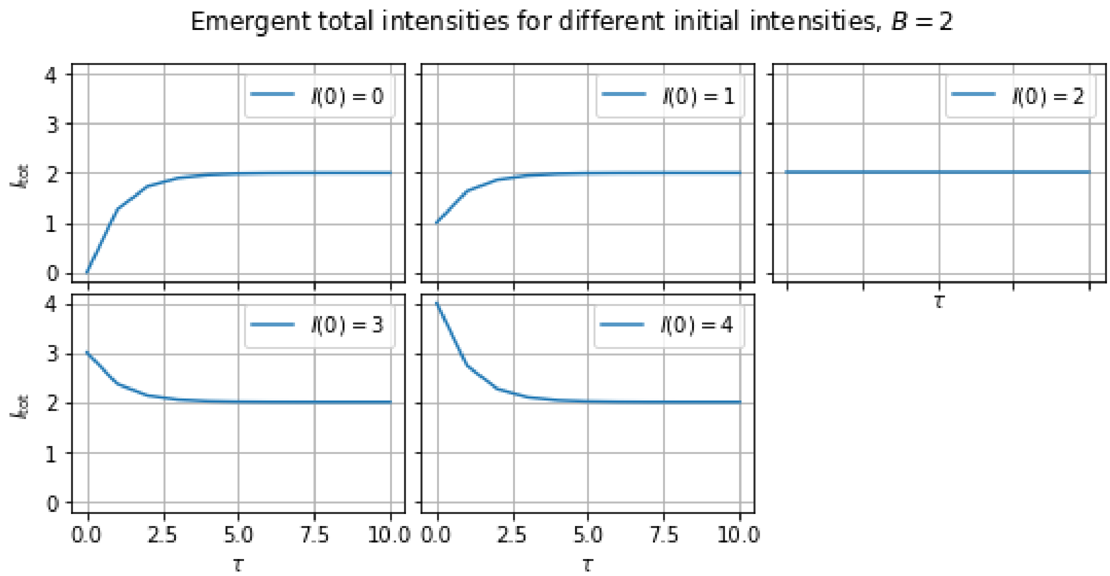
$$I_{\text{tot}} = I(0)e^{-\tau} + \int_0^\tau B(\tau')e^{-(\tau-\tau')}\mathrm{d}\tau'. \tag{2.1.4}$$

Here, we considered the fact that $\tau = 0$ is defined on the left side.

- $\_\_\__4$: For an isothermal layer, $B(\tau') = B$, so the integral can be calculated:

$$I_{\text{tot}} = I(0)e^{-\tau} + \left[-Be^{-\tau'}\right]_0^\tau = \underline{\underline{I(0)e^{-\tau} + (1-e^{-\tau})B}}. \tag{2.1.5}$$

(b) As instructed, we plot the τ-dependence for different values of $I(0)$, keeping $B = 2$ fixed.



As we'll explore in the next two parts of this exercise, we see that for large values of τ, the intensity always approaches B.

(c) For $\tau \ll 1$, we have

$$I_{\text{tot}} = (I(0)-B)e^{-\tau} + B \overset{I(0)=0}{=} (1-e^{-\tau})B \overset{e^x\approx 1+x}{\approx} \tau B. \tag{2.1.6}$$

For $I(0) > B$, we have

$$I_{\text{tot}} \approx \tau B + I(0)(1+\tau) = I(0) + \tau(B - I(0)). \tag{2.1.7}$$

For $\tau \gg 1$, we have $e^{-\tau} \approx 0$, so we arrive at

$$I_{\text{tot}} \approx B. \tag{2.1.8}$$

This makes sense, as the initial intensity is completely attenuated, and we can basically only see the outer layer of the isothermal sheet.

Problem 2: *Flux from the sun in a clever way*

As observations show, on Oct. 1 the sun subtends an angular diameter of 32 arcmin.

(a) Calculate the solid angle $\Omega_\odot$ subtended by the sun, in steradians.

(b) Show that the flux (in W m-2s^{-1}) or its cgs equivalent) of solar radiation on Earth is $F = I(T_\odot)\Omega_\odot$ with $T_\odot = 5777$ K, and calculate this value numerically. *Note that to calculate the flux from a blackbody of known temperature (or other source of known specific intensity), you do not need to know the distance or luminosity, but only the temperature and angle subtended! Both of these are direct observables, unlike distance and luminosity.*

(c) Show that the answer to the previous part is the same as you would get by the more obvious but unnecessarily complicated method $F = \frac{L_\odot}{4\pi a^2}$, with $a = 1$ au and $L_\odot = 4\pi R_\odot^2 \sigma T_\odot^4$.

Problem 2: Solution

(a) If we consider a circular area on a sphere, its size (given an angle θ) can be integrated as follows:

$$S = \int_0^\theta 2\pi R^2 \sin\theta' \mathrm{d}\theta' = 2\pi R^2(1-\cos\theta). \tag{2.2.9}$$

The solid angle is then

$$\Omega = \frac{S}{R^2}(1-\cos\theta). \tag{2.2.10}$$

Applying $\theta = 16$ arcmin (half of the diameter) as the aperture angle, we have

$$\Omega_\odot = 2\pi\left(1-\cos(\theta)\right)\ \mathrm{sr} = 6.8\times10^{-5}\ \mathrm{sr}. \tag{2.2.11}$$

Approximating small angles θ, we have $\cos\theta \approx 1-\theta^2/2$ (taylor), so the equation simplified to one of a circle:

$$\Omega_\odot \approx 2\pi(1-(1-\theta^2/2) = \pi\theta^2 = \pi\left(\frac{32/2}{60}\right)^2 \mathrm{deg}^2 = \pi\left(\frac{4\pi}{15\cdot180}\right)^2 \mathrm{sr} = \underline{\underline{6.8\times10^{-5}\ \mathrm{sr}}}, \tag{2.2.12}$$

where we again used $\theta = 16$ arcmin as the angular radius and converted it to radians applying 360 deg $= 2\pi$.

(b) In the lecture, we saw that the forward-flux term is $F^+ = \pi I$, and $\Omega = \frac{A}{d^2} = \frac{\pi R_\odot^2}{d^2}$, so we can derive the flux as follows:

$$F_\oplus = \frac{L_\odot}{4\pi d^2} = \frac{F_\odot^+ A}{4\pi d^2} = \frac{F_\odot^+ 4\pi R_\odot}{4\pi d^2} = \frac{\pi I R_\odot}{d^2} = I(T_\odot)\Omega_\odot. \tag{2.2.13}$$

Applying the Stefan-Boltzmann-Law for the flux density ($I = \frac{\sigma_\mathrm{B} T^4}{\pi}$), we have

$$F = \frac{\sigma_\mathrm{B} T_\odot^4}{\pi}\Omega_\odot = \underline{\underline{1360\ \mathrm{Wm}^{-2}}}. \tag{2.2.14}$$

(c) We find that

$$F = \frac{L_\odot}{4\pi a^2} = \frac{4\pi R_\odot \sigma_\mathrm{B} T_\odot^4}{4\pi a^2} = \underline{\underline{\sigma_\mathrm{B} T_\odot^4 \left(\frac{R_\odot}{a}\right)^2}}. \tag{2.2.15}$$

Problem 3: *Mean molecular weight*

The equation of state often depends on the particle density n (e .g., for the ideal gas the pressure is $P = nk_\mathrm{B}T$). One can relate the particle density to the mass density ρ and the mass m_H of the hydrogen atom by introducing the mean molecular weight μ,

$$\mu = \frac{\rho}{n}\frac{1}{m_\mathrm{H}}, \tag{2.3.16}$$

i.e., the average mass of a particle (ρ/n) in units of the hydrogen atom mass, so that

$$n = \frac{\rho}{\mu m_\mathrm{H}}. \tag{2.3.17}$$

- What is the mean molecular weight of a pure hydrogen plasma? Assume that the gas is fully ionised.
- What is the mean molecular weight of a pure helium plasma? Again assume full ionisation.
- What is the analogous expression for heavier elements? Give a simplified approximation. (Hint: how many electrons do you get from carbon? from nitrogen? from oxygen? given full ionisation.)
- Now combine the above and derive an expression for n for a plasma consisting of hydrogen, helium, and heavier elements in mass fractions X, Y , and Z (i.e., in one gram of material, X grams are in the form of (ionised) hydrogen atoms, Y grams are in the form of (ionised) helium atoms, and Z grams are in the form of (ionised) heavier atoms). Express Z in terms of X and Y and derive an equation for μ that depends only on X and Y.

Problem 3: Solution

- We can summarise the results (assuming the easiest isotopes):

Atom	Nucleus (corresponding to mass)	# Electrons	# free particles	μ
H	1 p	1	2	$\frac{1}{2}$
He	2 p + 2 n	2	3	$\frac{4}{3}$
C	6 p + 6 n	6	7	$\frac{12}{7}$
N	7 p + 7 n	7	8	$\frac{14}{8}$
O	8 p + 8 n	8	9	$\frac{16}{9}$

  This table includes $\mu_\mathrm{H} = \frac{1}{2}$ and $\mu_\mathrm{He} = \frac{4}{3}$.
  Generalised, for fully ionised plasma, we have

$$\mu = \frac{A}{Z+1}, \tag{2.3.18}$$

  where A is the atomic mass number and Z is the atomic number, denoting the amount of electrons.
- We see that for heavier elements, μ approaches 2, although we note that the heavier the element, the less likely it is that it is fully ionised.
- To obtain the combined number density n for a plasma of H, He, and heavier elements with mass fractions X, Y, and Z, we first note that the sum of the mass fractions has to be 1:

$$X + Y + Z = 1. \tag{2.3.19}$$

  We can then sum up the individual number densities of the different species:

$$\begin{aligned} n = \sum_i n_i &\approx 2\frac{\rho}{m_\mathrm{H}}X + \frac{3}{4}\frac{\rho}{m_\mathrm{H}}Y + \frac{1}{\mu_Z}\frac{\rho}{m_\mathrm{H}}Z & (2.3.20)\\ &\approx \left(2X + \frac{3}{4}Y + \frac{1}{2}\right)\frac{\rho}{m_\mathrm{H}} & (2.3.21)\\ &= \frac{8X+3Y+2Z}{4}\frac{\rho}{m_\mathrm{H}} & (2.3.22)\\ &\overset{(2.3.19)}{=} \underline{\underline{\underbrace{\frac{6X+Y+2}{4}}_{=1/\mu_\mathrm{tot}}\frac{\rho}{m_\mathrm{H}}}}. & (2.3.23) \end{aligned}$$

  Here, we have found $\mu_\mathrm{tot} = \frac{4}{6X+Y+2}$ as the effective μ of the combined plasma, while assuming that $\mu \approx 2$ for the heavy elements.

Problem 4: *Pressures and energies in the ISM*

The interstellar medium consists of five main phases. Recall the typical densities and temperatures of cold molecular gas, cold and warm neutral (atomic) gas, and warm and hot ionised gas. Where do we find the different phases in the Milky Way and other galaxies?
Calculate the gas pressure and the mean energy per particle for each of the five phases.

(Hint: Assume ideal gas in equilibrium at the corresponding temperature and consider the number of degrees of freedom available.)

Problem 4: Solution

Note: The code used for parts of this exercise is available in **this repository**.
The different phases can be found in almost all regions of the galaxy, although they are more prevalent near the spiral arms.
The gas pressure P and the mean energy per particle $\langle E \rangle$ (assuming an ideal gas in equilibrium) can be calculated as follows:

$$P = nk_\mathrm{B}T, \quad \langle E \rangle = \frac{f}{2}k_\mathrm{B}T, \tag{2.4.24}$$

where f denotes the degrees of freedom.
All of the monoatomic species only have the three standard degrees of freedom.
For the molecular gas, in principle there would be two additional rotational and two vibrational degrees of freedom possible, but they only become available for temperatures $T \gtrsim 100$ K (rotational) and $T \gtrsim 1000$ K (vibrational), so we have $f = 3$ for all phases in the interstellar medium.

Phase	**Where to find them**	n [cm^{-3}]	T [K]	P [K k_B^{-1}cm^{-3}]	$\langle E \rangle$ [eV]
Molecular	Star formation sites	300.000	10	3000	0.0013
Cold atomic	Sheets and filaments	50.000	80	4000	0.0103
Warm atomic	Photodissociation regions	0.500	8000	4000	1.0341
Warm ionised	Near O and B stars	0.300	8000	2400	1.0341
Hot ionised	Near SF regions and SNe	0.003	500000	1500	64.6300

Problem 5: *Sound speeds in the ISM*

What are the typical sound speeds in the five phases of the ISM? Be aware that the average particle mass as well as the adiabatic index γ may differ from one phase to the other. For simplicity you can assume that the ISM consists only of hydrogen.
(Hint: What is the connection between the equation of state and the propagation of small disturbances?)

Problem 5: Solution

Note: The code used for parts of this exercise is available in **this repository**.
The sound speed can be obtained via

$$c_\mathrm{s} = \sqrt{\gamma\frac{P}{\rho}}, \quad \text{where } \gamma = 1 + \frac{2}{f}, \tag{2.5.25}$$

so $\gamma = 5/3$ for the monoatomic and the cold (< 100 K) molecular gas.

Therefore, we have (applying the equation in exs. 3)

$$c_\mathrm{s} = \sqrt{\frac{\gamma k_\mathrm{B} T}{\mu m_\mathrm{H}}}. \tag{2.5.26}$$

Note that $\mu = 1$ for neutral and $\mu = \frac{1}{2}$ for ionised hydrogen.

Phase	T [K]	μ	c_s [km/s]
Molecular	10	1.0	0.3709
Cold atomic	80	1.0	1.0491
Warm atomic	8000	1.0	10.4909
Warm ionised	8000	0.5	14.8364
Hot ionised	500000	0.5	117.2917

Problem 6: *Virial theorem and Jeans mass*

The Jeans criterion for the collapse of gas clouds is an important concept for interpreting the behaviour of interstellar matter. The corresponding Jeans mass can (among other derivations) be obtained by an application of the virial theorem.

(a) Show that for a self-gravitating spherically-symmetric ideal-gas cloud in (hydrostatic) pressure equilibrium,

$$E_\mathrm{kin} = -\frac{1}{2} E_\mathrm{pot} \tag{2.6.27}$$

(Hint: all the descriptive words above are relevant.

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{GM(r)\rho(r)}{r^2} \tag{2.6.28}$$

Don't compute any numbers here, just show that the corresponding integrals are equal.)

(b) For a cloud to be able to contract, what can we then say about the energies involved? Use simple approximations to evaluate the integrals appearing in (a) above. What minimum mass must a cloud of given mean density and mean temperature therefore at least have in order to contract?
Compare your result to the one from the lectures and discuss possible differences.

Problem 6: Solution

(a) We start with the expression given in eq. (2.6.28), which we can assume if we consider hydrostatic equilibrium, in which the pressure gradient should depend on the gravity pulling everything inwards. $M(r)$ is the enclosed mass inside of radius r.

The potential energy can then be obtained as

$$\begin{aligned}
E_{\text{pot}} &= \int_0^M -\frac{Gm(r)}{r}\mathrm{d}m \\
&\overset{\mathrm{d}m=\rho(r)4\pi r^2\mathrm{d}r}{=} \int_0^R -\frac{GM(r)\rho(r)}{r^2}4\pi r^3\mathrm{d}r \\
&= 4\pi\int_0^R \frac{\mathrm{d}p}{\mathrm{d}r}r^3\mathrm{d}r \\
&\overset{\text{int. by parts}}{=} 4\pi \underbrace{r^3P(r)\Big|_0^R}_{=0\text{, see below}} -4\pi\int_0^R P(r)3r^2\mathrm{d}r \\
&= -3\int_0^R 4\pi r^2 P(r)\frac{V}{V}\mathrm{d}r \\
&= -3V\langle P\rangle, \qquad (2.6.29)
\end{aligned}$$

where $\langle P\rangle$ is the volume-averaged pressure.
The term $r^3P(r)$ vanishes at $r=0$ because the pressure has a finite value, and vanishes at $r=R$ because we can assume $P(R)=0$.
Applying the ideal gas law, for the kinetic energy we find

$$E_{\text{kin}} = \frac{3}{2}Nk_{\mathrm{B}}\langle T\rangle = \frac{3}{2}\langle P\rangle V. \qquad (2.6.30)$$

$$\Longrightarrow E_{\text{pot}} = -2E_{\text{kin}}. \qquad (2.6.31)$$

(b) The assumption is that the potential energy is approximately $E_{\text{grav}} \sim \frac{GM^2}{R}$, so the criterion for instability becomes

$$\frac{GM^2}{R} > \frac{3}{2}Nk_{\mathrm{B}}T = \frac{3}{2}\frac{M}{\bar{m}}k_{\mathrm{B}}T \qquad (2.6.32)$$

$$\Longleftrightarrow M > \frac{3Rk_{\mathrm{B}}T}{2G\bar{m}}. \qquad (2.6.33)$$

This deviates from the known Jeans criterion since the potential depends on the density profile that we apply.

Note: The code used for parts of this exercise is available in **this repository**.