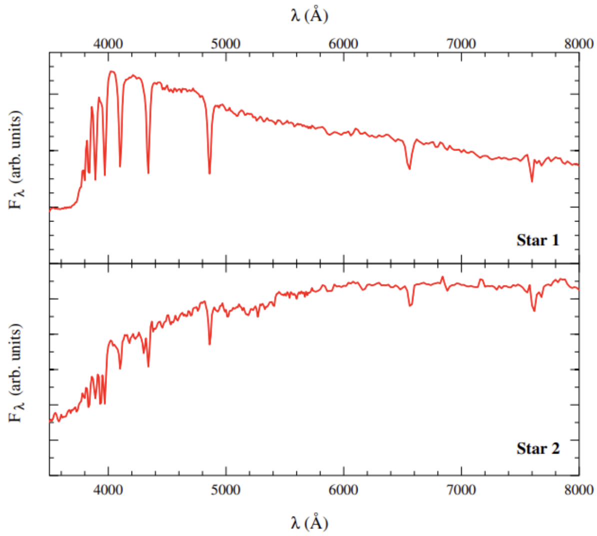
Problem set #5 I am a star! ⋆ (here)

08.12.2022

Problem 1: *Flux and colour with filters*

Remember that $F_\lambda(\lambda)$ is defined such that $F_\lambda(\lambda)\Delta\lambda$ is equal to the flux (rate that energy is collected by a 1 m^2 telescope) arriving at wavelengths between λ and $\lambda + \Delta\lambda$. Consider the two stars whose spectra are plotted below.



(a) Suppose you can make a filter that is exactly 1 Å wide, that is, it will only transmit colours between λ_0 and $\lambda_0 + 1$ Å. If you were designing this filter to transmit the maximum amount of energy possible from Star 1, at what wavelength would you choose to make the λ_0 of this filter?

(b) Consider the B and R filters[1]. Will Star 1 have a greater flux through the B filter, or through the R filter? What about Star 2?

(c) Which of the two stars will have a large value of B - R? (B and R are the fluxes through the filters, measured in magnitudes.)

(d) It turns out that Star 1 will be whitish in colour. What would Star 2 look like to your eyes?

Problem 1: Solution

(a) To get the maximum flux from star one in a 1 Å wide wavelength interval, we should use $\lambda_0 \approx 4000$ Å as its flux peaks here.

(b) The B filter is centred around 4450 Å, while the R filter is centred around 6580 Å, both with a width of around 1000 Å. Therefore, Star 1 would have a greater flux in the B than in the R filter, while Star 2 would have a greater flux in the R than in the B filter.

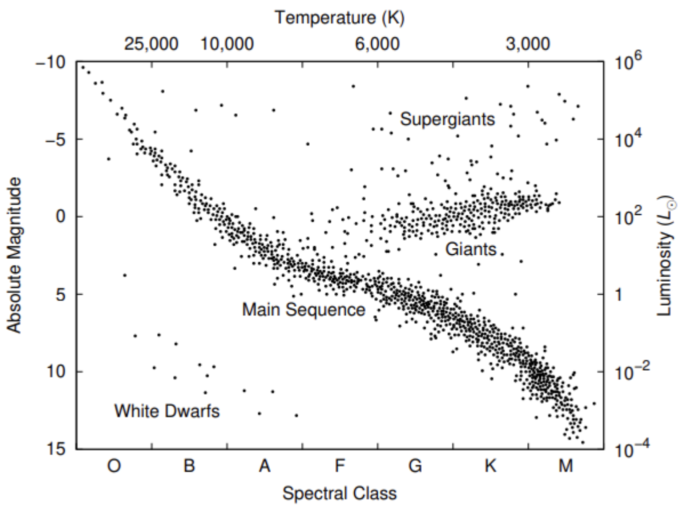
[1]If this has not yet been discussed in the lectures then see, for example, Standard Photometric Systems by Michael S. Bessell, Annual Review of Astronomy and Astrophysics 43, 293 (2005), **accessible here**

(c) The $B - R$ value of Star 1 should be much larger, while the $B - R$ value of Star 2 is slightly negative.

(d) If Star 1 looks whitish in colour, Star 2 should look either yellow, orange or red.

Problem 2: *HRD exploration*

Consider the following Hertzsprung-Russell-Diagram:



(a) You observe a star (Star A) that is bluish in colour. You measure its spectrum, and find that the spectrum F_λ reaches a peak at an ultraviolet wavelength of 1500 Å. What is the temperature and spectral type of this star?

(b) You observe a star (Star B) that is reddish in colour. You measure its spectrum, and find that the spectrum F_λ reaches a peak at a red/infrared wavelength of 7800 Å. What is the temperature and spectral type of this star?

(c) Star A and Star B have the same observed brightness. You measure the parallax of Star A to be $0.0008''$, and the parallax of Star B to be $0.080''$. What is the ratio of luminosities of the two stars?

(d) If you determine that Star A is a main sequence star, what kind of star (white dwarf, main sequence, giant, or supergiant) is Star B? Explain.

Problem 2: Solution

(a) We can use Wien's displacement law to find the peak temperatures of the stars, finding

$$T = 2.897\,\frac{\mu\mathrm{m\ K}}{\lambda} \implies \underline{\underline{T_A = 19\,000\ \mathrm{K}}}. \tag{5.2.1}$$

Therefore, star A should be a B type star.

(b) From Wien's displacement law, we find

$$\underline{\underline{T_B = 3\,700\ \mathrm{K}}}, \tag{5.2.2}$$

so this should be a K star.

(c) Since $m_A = m_B$, we can use that

$$m_A - m_B = 0 = -2.5 \log\left(\frac{F_A}{F_B}\right) \implies F_A = F_B. \tag{5.2.3}$$

Since the luminosity of the stars is proportional via $F = \frac{L}{4\pi d^2}$ where d is the distance, we find that

$$\frac{L_A}{L_B} = \frac{d_A^2}{d_B^2} \approx \underline{\underline{10\,000}}, \tag{5.2.4}$$

where we used the the relation for $d_{[\mathrm{pc}]} = \frac{1\ \mathrm{au}}{p_{['']}}$.

(d) If star A is a main sequence star, its luminosity would be $L_A \approx 10^3 L_\odot \implies L_B \approx 10^{-1} L_\odot$, so star B could also be a main sequence star.

Problem 3: *Absorption by the interstellar medium*

(a) Show that the apparent magnitude m grows linearly with distance r in an absorbing medium if the intensity of the radiation declines exponentially. What is the value for the constant C in the equation below if the intensity of the radiation declines 10 % on a distance of 1 pc?

$$m - m_0 = Cr. \tag{5.3.5}$$

(b) The apparent V-band magnitude of a star is $V = 14.8$, the colour is $B - V = 1.4$ and the absolute magnitude is $M_V = 1.0$. Towards the star the extinction at visual wavelengths is $a_V = 1$ mag/kpc. Calculate the intrinsic (extinction-corrected) colour $(B - V)_0 = (B - V) - E_{B-V}$ of the star. To solve the problem one needs to iterate (you can write a short program for this) or use a numerical equation solver (e. g. Mathematica).

Problem 3: Solution

(a) The intensity after the extinction is $I_1 = 0.9 I_0$ after a decline on a distance of 1 pc, and we have

$$m_1 - m_0 = -2.5 \log\left(\frac{I_0}{I_1}\right). \tag{5.3.6}$$

We can use that the intensity in the absorbing medium depends on r as $I_1 = I_0 e^{-\tau r}$, we have

$$m_1 - m_0 = -2.5 \log\left(e^{-\tau r}\right) = \frac{\tau r}{\ln(10)}. \tag{5.3.7}$$

Putting in the numbers, we find $0.9 I_0 = I_0 e^{-\tau 1\ \mathrm{pc}} \iff \tau = 0.1054 \frac{1}{\mathrm{pc}}$ and therefore

$$C = \frac{\tau}{\ln(10)} = \frac{0.15}{\mathrm{pc}}. \tag{5.3.8}$$

(b) Using $m_V = 14.8$, a colour of $m_B - m_V = 1.4$ and $M_V = 1.0$, and an extinction of $a_V = 1$ mag/kpc, we use the relation

$$M = m_0 + 5 - \log r \tag{5.3.9}$$

to obtain the absolute magnitude, and

$$m_V = m_{V_0} + A_V = m_{V_0} + a_V r \tag{5.3.10}$$

to get the initial extinction corrected magnitude m_{V_0}. Putting these equation together, we have

$$M_V = m_V - a_V r + 5 - \log r. \tag{5.3.11}$$

This equation can iteratively be solved for r, yielding $r \approx 2.14$ kpc and therefore $A_V = a_V r = 2.14$.
Remembering from the lecture that the colour excess E_{B-V} and A_V are related via

$$R = \frac{A_V}{E_{B-V}} = \frac{A_V}{A_B - A_V} \approx 3.1, \tag{5.3.12}$$

we find $A_B - A_V \approx 0.7$.
Therefore, we have

$$(B - V)_0 = m_B - m_V - E_{B-V} = 1.4 - 0.7 = \underline{\underline{0.7}}. \tag{5.3.13}$$

Problem 4: *Timescales*

(a) Hydrogen fusion has an efficiency of about 0.7 % for converting mass into energy. Assume that the Sun will use 10 % of its hydrogen for fusion (and that it is mostly hydrogen). Given the Sun's luminosity, how long will it shine? How many kg of mass is the Sun losing each second by converting mass into energy?

(b) We know that the Sun is not "on fire" because chemical reactions are not nearly efficient enough to keep the Sun shining at its current luminosity for anything like the amount of time we know it's been around. The efficiency of chemical burning of hydrogen in the reaction: $2H_2 + O_2 \rightarrow 2H_2O$ is about 2×10^{-10}: that is the fraction of its mass that gets converted into energy. Assuming that the Sun were made up of oxygen and hydrogen in just the right proportions, and that it was able to burn all of its mass, how long (for how many years) would it be able to shine at its current luminosity using chemical reactions?

(c) Another possible energy source is the energy released from gravitational contraction. When you drop something from a height, energy is released; you may use that energy to make a sound, break something, etc. Suppose that you consider all the mass of the Sun to have been *dropped* from a great distance on to the Sun. The total energy released is approximately GM^2/R, where M is the mass of the Sun and R is the radius of the Sun. If this were where the Sun got its energy, for how long would it have been able to shine at its current luminosity?

Problem 4: Solution

(a) If $f_m = 0.1$ of the mass of the sun is converted to energy with the efficiency $\eta = 0.007$, we the total energy output will be

$$E_{\text{fus}} = \eta f_m M_\odot c^2 \approx 1.25 \times 10^{44}\ \text{J}. \tag{5.4.14}$$

We can estimate the lifetime by using the solar luminosity ($L_\odot = 3.83 \times 10^{26}$ W) and just divide as follows:

$$t_{\text{fus}} = \frac{E_{\text{fus}}}{L_\odot} \approx 10^{10}\ \text{yr}. \tag{5.4.15}$$

We can estimate the mass loss per second using the relation

$$E_{\text{fus}} = mc^2\eta \implies \dot{m} = \frac{\dot{E}}{\eta c^2} = \frac{L_\odot}{\eta c^2} = \underline{\underline{6 \times 10^{11}\ \frac{\text{kg}}{\text{s}}}}. \tag{5.4.16}$$

According to **WolframAlpha**, this is about the amount of methane generated by humans each year every second!

(b) If we *burned* the entire Sun using all hydrogen and oxygen available, and assuming perfect mass ratios, and using the efficiency $\eta_{\text{burn}} = 2 \times 10^{-10}$, we have

$$E_{\text{burn}} = \eta_{\text{burn}} M_\odot c^2 = 3.6 \times 10^{37}\ \text{J} \implies t_{\text{burn}} = \frac{E_{\text{burn}}}{L_\odot} \approx \underline{\underline{3000\ \text{yr}}}. \tag{5.4.17}$$

This is way too short (and also, we had very unrealistic assumptions).

(c) Here, we use the gravitational energy:

$$E_{\text{grav}} = \frac{GM_\odot^2}{R_\odot} = 4 \times 10^{41}\ \text{J} \implies t_{\text{grav}} = \frac{E_{\text{grav}}}{L_\odot} = \underline{\underline{32\ \text{Myr}}}. \tag{5.4.18}$$

This would also be too short to e. g. explain the lifetime of the Earth found via carbon dating.

Problem 5: *The Eddington limit*

(a) By using the appropriate equations of stellar structure and the (isotropic) radiation pressure $P_{\text{rad}} = aT^4/3$ with $a = 4\sigma B/c$, calculate the differential $\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}P_{\text{tot}}}$ as a function of local luminosity $L(r)$ and mass $M(r)$. (Assume radiative energy transport, as (mostly) true for the outer regions of massive stars.) From this result, prove the inequality

$$L < \frac{4\pi cGM}{\kappa} = \frac{4\pi cGM}{\alpha/\rho}, \tag{5.5.19}$$

with total luminosity L, total mass M, the speed of light c, the gravitational constant G and the mean (Rosseland) mass absorption coefficient κ (i. e. opacity per unit mass).

(b) The right-hand side of the above equation is called the Eddington luminosity, and is the maximum luminosity that allows a stable stellar configuration (equality is achieved for

$P_{\rm tot} = P_{\rm rad}$).
Derive this luminosity also from the alternative approach that a star cannot be stable if the radiative acceleration is everywhere larger (with respect to absolute value) than the gravitational one. Assume that $\kappa = const$ and remember that

$$g_{\rm rad} = \frac{\kappa}{c} F \tag{5.5.20}$$

where F is the frequency-integrated radiative flux.

(c) Using the above inequality and the approximate mass-luminosity relation $L/L_\odot \approx (M/M_\odot)^3$, calculate an upper stellar mass limit, with $\kappa = 0.34$ cm^2 g^{-1}, which is the electron scattering opacity in stars of solar composition (with 10 % helium by number) and thus a lower limit for the total opacity.

Problem 5: Solution

(a) The pressure differential can be derived by using the fact that $P_{\rm tot} = P_{\rm gas} + P_{\rm rad}$, and also, due to hydrostatic equilibrium,

$$\frac{\mathrm{d}P_{\rm tot}}{\mathrm{d}r} = -\frac{GM(r)}{r^2}\rho(r). \tag{5.5.21}$$

On the other hand, we know that $P_{\rm rad} = aT^4/3$, so the product rule yields

$$\frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}r} = \frac{4}{3}aT^3\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{5.5.22}$$

where we had already derived

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{3\alpha(r)F(r)}{4acT^3} = \frac{3\kappa\rho(r)L/(4\pi r^2)}{4acT^3} \tag{5.5.23}$$

in the lecture. Overall we then have

$$\frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}P_{\rm tot}} = \frac{4/3aT^3r^2}{GM(r)\rho(r)}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{5.5.24}$$

For the criterion, we use that the radiation pressure should not exceed the total pressure, so **TO DO Describe this a little more extensively (!)**

$$\frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}P_{\rm tot}} = \frac{\mathrm{d}P_{\rm rad}}{\mathrm{d}P_{\rm rad} + \mathrm{d}P_{\rm tot}} < 1 \implies L < \frac{4\pi GM}{\kappa}. \tag{5.5.25}$$

(b) We have

$$g_{\rm rad} = \frac{\kappa}{c}F = \frac{\kappa}{c}\frac{L}{4\pi r^2} < \frac{GM}{r^2} = g_{\rm grav}, \tag{5.5.26}$$

from which we immediately obtain the equation found in (a).

(c) Using the relation (which doesn't hold for very massive stars) we were given ($L \propto M^3$), and using the criterion derived in (a) and (b), we have

$$\frac{M}{M_\odot} < \sqrt{\frac{4\pi GM_\odot}{\kappa L_\odot}} \approx \underline{\underline{196 M_\odot}}. \tag{5.5.27}$$

(here, it's not the square root as the M cancels out once.)

Problem 6: *Binary system*

Visible-light spectroscopic observations of a nearby star similar to the Sun show that it must be part of a binary. A careful study of the spectrum of the star shows that its Doppler shift varies with a period of 10^6 seconds (about a month) and a velocity amplitude of $90\,\mathrm{m/s}$. The Doppler shift of the star varies sinusoidally, suggesting that the star is in a circular orbit.

(a) Estimate the mass of the invisible companion to the star whose spectrum is seen. Speculate on the nature of this companion (i. e. what kind of object is it?)

(b) Further observations show that the secondary eclipses the primary. What is the distance of the star from the secondary? It takes approximately 16 minutes for the brightness of the star to drop at the beginning of each eclipse and to rise at the end of each eclipse. What is the radius of the secondary?

(c) Assuming that the secondary is held up by gas pressure, estimate its central temperature. Is this above or below the hydrogen burning limit?

Problem 6: Solution

Thanks and credits to Iliya Tikhonenko!

(a) The velocity amplitude is just the maximum value of the projected orbital velocity. As the star is on circular orbit, we can estimate its velocity via the following relation (centre of mass + Kepler III for a circular orbit):

$$m_2^3 = \frac{v^3}{\omega \sin^3 i} \frac{(m_1 - m_2)^2}{G} \tag{5.6.28}$$

Here, we can take into account that $\sin i = 1$ and $\omega = \frac{2\pi}{T}$, $m_1 + m_2 \approx M_\odot$ (as $m_1 \approx M_\odot$ and $m_2 \approx m_{\text{Jupiter}}$, as we could derive in hindsight).

(b) We can calculate the semimajor axis using Kepler III, $a = r_1 + r_2$, and $\frac{a^3}{T^2} = \frac{GM_\odot}{4\pi^2} \implies a \approx 0.1$ au.

Also, via trigonometry, the angle that the planet subtends in the moment when it leaves (or comes in front of) the star during the eclipse is

$$\delta_{\text{leave}} = \frac{R_{s1} + R_{s2}}{a} - \frac{R_{s1} - R_{s2}}{a} = \frac{2R_{s2}}{a}. \tag{5.6.29}$$

On the other hand, because we know the time it takes for the leave $t_{\text{leave}} = 16$ min, we can relate this with the period, so we have

$$\frac{\delta_{\text{leave}}}{2\pi} = \frac{t_{\text{leave}}}{T} \iff R_{s2} = \frac{t_{\text{leave}}}{T}\frac{2\pi}{2} a \approx 0.6 R_{\text{Jup}}. \tag{5.6.30}$$

Therefore, we found out that the companion 'star' is actually smaller than Jupiter and comparable in mass.

(c) If the second component is stable due to gas pressure than the hydrostatic equilibrium equation should hold

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{r^2}. \tag{5.6.31}$$

Ideally, we should assume some equation of state and some boundary conditions (e.g. $P = 0$ at the surface) and solve this equation directly. However, this derivation would be a bit tedious, so instead we can use the following very rough estimate of the central pressure:

$$\frac{P_c}{R} = \frac{GM\bar{\rho}}{R^2}. \tag{5.6.32}$$

Assuming the ideal gas law

$$P_c = \frac{\bar{\rho}}{\mu m_{\mathrm{H}}} k_{\mathrm{B}} T, \tag{5.6.33}$$

we arrive at

$$\frac{\bar{\rho} k_{\mathrm{B}} T}{\mu m_{\mathrm{H}} R} = \frac{GM\bar{\rho}}{R^2} \iff T = \frac{GM}{k_{\mathrm{B}} R} \mu m_{\mathrm{H}} \approx 3 \times 10^5 \text{ k} \ll 10^7 \text{ K} = T_{\text{H-fusion}}, \tag{5.6.34}$$

where we used $\mu \approx 1$.