Problem set #6 Star slosh (here)

15.12.2022

Problem 1: *Relations between the equations of state and stellar structure*

With respect to stellar structure the equation of state is in its simplest form represented by the ideal gas law

$$P = nkBT \tag{6.1.1}$$

(pressure = particle number density × Boltzmann constant × temperature). Corrections to the ideal gas law are due to electron degeneracy, radiation pressure, and Coulomb interactions.

- Derive a relation $T(\rho)$ for which gas pressure P_{gas} and radiation pressure P_{rad} become equal. For simplicity, assume the gas is pure hydrogen. (Hint: the radiation pressure is $P_{\text{rad}} = 4\sigma_{\text{B}}T^4/(3c)$.)
- The onset of degeneracy can be estimated through the comparison of the Fermi energy to the thermal energy. Derive a relation $T(\rho)$ for which these energies become equal. (Hint: the relation between particle number density and Fermi energy E_f is given by

$$n = \frac{\alpha}{h^3}\frac{2}{\sqrt{\pi}}\left(2\pi m k_{\text{B}}T\right)^{\frac{3}{2}}\frac{2}{3}\left(\frac{E_f}{k_{\text{B}}T}\right)^{\frac{3}{2}} \tag{6.1.2}$$

Why can you assume $n = \rho/m_{\text{p}}$, $\alpha = 2$, $m = m_{\text{e}}$?)
- For higher densities and particularly low temperatures Coulomb interactions of the ions must be considered. The ions tend to form a lattice which minimises their total energy. The onset of this effect can be estimated through a comparison of thermal energy and Coulomb energy. Derive a relation $T(\rho)$ for which these energies become equal. (Again assume the gas is pure hydrogen.)
(Hint: the mean separation r_{sep} of the ions can be approximated and expressed by the ion density n_{ion}, by making use of $V_{\text{ion}} = (4\pi/3)r_{\text{sep}}^3$ and $V_{\text{ion}} = 1/n_{\text{ion}}$.)
- Make a plot of $\log T$ (10^3 to 10^{10} K) on the y-axis versus $\log\rho$ (10^{-7} to 10^7 g/cm^3) on the x-axis. Draw the regions where radiation pressure, degeneracy pressure, and Coulomb interactions become strong perturbations. For comparison, also indicate the central density and temperature for typical stellar objects, e.g., the sun.

Problem 1: Solution

- If we set $P_{\text{rad}} = P_{\text{gas}}$, we obtain

$$n k_{\text{B}} T = \frac{4}{3c}\sigma_{\text{B}}T^4 \tag{6.1.3}$$

$$\iff T = \sqrt[3]{\frac{3}{4}\frac{cnk_{\text{B}}}{\sigma_{\text{B}}}} \tag{6.1.4}$$

$$= \sqrt[3]{\frac{3}{4}\frac{c\rho k_{\text{B}}}{\mu m_{\text{H}}\sigma_{\text{B}}}} \tag{6.1.5}$$

$$\approx 4 \times 10^7 \left(\frac{\rho}{\mathrm{g/cm}^3}\right)^{\frac{1}{3}} \mathrm{~K}, \tag{6.1.6}$$

where we used $n = \frac{\rho}{\mu m_\mathrm{H}}$ for the number density.
For ionised hydrogen, $\mu = \frac{1}{2}$.

- If we set the Fermi energy equal to the thermal energy, $E_f \approx \frac{3}{2} k_\mathrm{B} T$, we have

$$n = \frac{\rho}{\mu m_\mathrm{H}} = \frac{\alpha}{h^3} \frac{2}{\sqrt{\pi}} (2\pi m k_\mathrm{B} T)^{\frac{3}{2}} \frac{2}{3} \cdot \left(\frac{3}{2}\right)^{\frac{3}{2}} \tag{6.1.7}$$

$$\iff T = \frac{1}{3\pi m k_\mathrm{B}} \left(\frac{3}{4} \frac{h^3 \rho \sqrt{\pi}}{\alpha \mu m_\mathrm{H}}\right)^{\frac{2}{3}} \tag{6.1.8}$$

$$= \frac{1}{3\pi m_e k_\mathrm{B}} \left(\frac{3}{8} \frac{h^3 \rho \sqrt{\pi}}{m_\mathrm{H}}\right)^{\frac{2}{3}} \tag{6.1.9}$$

$$\approx 2 \times 10^5 \left(\frac{\rho}{\mathrm{g/cm}^3}\right)^{\frac{2}{3}} \mathrm{~K}, \tag{6.1.10}$$

where we used $\mu = 1$ (since we're looking at the Fermi gas where we only care about the electrons), $\alpha = 2$ (because we're looking at Fermions, where two of them cannot occupy the same energy state), $m = m_e$ because we again only consider electrons.

- The Coulomb energy between two particles with charges q_1 and q_2 at separation r is given by

$$E_\mathrm{Coul} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}, \tag{6.1.11}$$

so if we set $E_\mathrm{Coul} \approx \frac{3}{2} k_\mathrm{B} T$, for two electrons (charge e), we obtain

$$T = \frac{2e^2}{c \cdot 4\pi\epsilon_0 k_\mathrm{B}} \frac{1}{r_\mathrm{sep}} \tag{6.1.12}$$

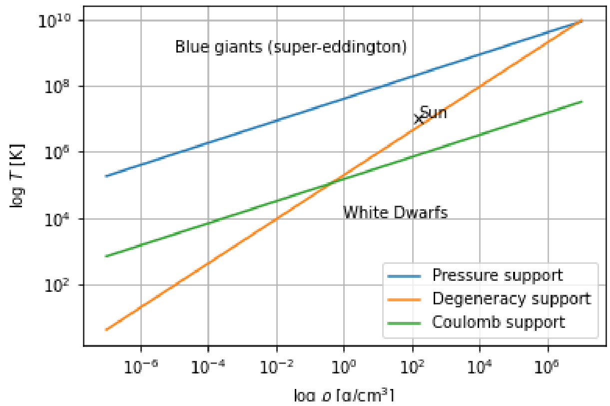
$$= \frac{2e^2}{12\pi\epsilon_0 k_\mathrm{B}} \sqrt[3]{\frac{4\pi n_\mathrm{ion}}{3}} \tag{6.1.13}$$

$$= \frac{2e^2}{12\pi\epsilon_0 k_\mathrm{B}} \sqrt[3]{\frac{4\pi\rho}{3\mu m_\mathrm{H}}} \tag{6.1.14}$$

$$\approx 1.5 \times 10^5 \left(\frac{\rho}{\mathrm{g/cm}^3}\right)^{\frac{1}{3}} \mathrm{~K}, \tag{6.1.15}$$

where we used $V_\mathrm{ion} = \frac{4}{3}\pi r^3 = \frac{1}{n_\mathrm{ion}}$, and can set $\mu = 1$ again.

- The equilibrium lines are plotted below.



Red giants are hot, so radiation pressure dominates. White dwarfs are colder at the same densities, so the electron pressure dominates.

Problem 2: *Mass–Radius relation for White Dwarfs*

Brown dwarfs and low-mass white dwarfs are supported by the degeneracy pressure of nonrelativistic electrons. A fully degenerate (zero temperature, or close enough) electron gas has a pressure which depends only on the electron density,

$$P = \frac{h^2}{20m_\mathrm{e}}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} n_\mathrm{e}^{\frac{5}{3}} \tag{6.2.16}$$

where n_e, the electron density, equals ρ/μ_e if μ_e is the mass per electron. We will assume this to be $\mu_\mathrm{e} = 2m_\mathrm{p}$, which is correct for helium and the lighter heavy elements. Thus the above relation for the pressure is of the form $P = K\rho^{\frac{5}{3}}$, $K = \frac{h^2}{20m_\mathrm{e}}\left(\frac{3}{\pi}\right)^{\frac{2}{3}}(2m_\mathrm{p})^{-\frac{5}{3}}$.
The important point here is that K is a constant [a so-called polytropic constant].

(a) Derive the relation between the mass M of the object and its radius R. (Hints: The central "Fermi pressure" depends on the central density ρ_c, which for each type of polytrope (K constant) is some factor larger than the mean density ρ_m of the star. In our case, $\frac{\rho_c}{\rho_m} = 6$. Moreover, ρ_m is by definition given by the total mass divided by the total volume. As a final consideration, P_c, the pressure at the centre, is obtained from hydrostatic equilibrium $\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}$ assuming an average pressure gradient $\frac{P_0 - P_c}{M - 0}$ where $P_0 = 0$ and by replacing m and r by the mean values $\frac{M}{2}$ and $\frac{R}{2}$.)

(b) For which mass is the radius just about the radius of the Earth?

(c) How does the radius vary with mass? What would you get for a rocky body that can maintain constant density? How can you explain the result obtained?

(d) Bonus question:
How large is the error resulting from our approximation in part (a)? The (numerical) solution of the Lane–Emden equation for this particular polytrope ($\gamma = \frac{5}{3}$, $n = \frac{1}{\gamma-1} = \frac{3}{2}$) gives $R = 3.6537537\alpha$ and $M = 2.71405514\pi\alpha 3\rho_c$, where $\alpha = \sqrt{\frac{K(n+1)}{4\pi G\rho_c^{2-\gamma}}}$.

Problem 2: Solution

(a) We approach this problem via the hints that were provided, starting from the pressure gradient which we approximate as being constant, so

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} \approx \frac{GM/2}{4\pi(R/2)^4} = -\frac{P_c}{M} = \frac{P_0 - P_c}{M - 0} \tag{6.2.17}$$

$$\implies P_c = \underbrace{\frac{2G}{\pi}}_{:=\beta} \frac{M^2}{R^4}, \tag{6.2.18}$$

giving us a nice relation for the central pressure. We have defined the constant β for convenience.
On the other hand, since $\frac{\rho_c}{\rho_m} = 6$, and thus $\rho_c = 6\rho_m = 6\frac{M}{4/3\pi R^3}$, we alternatively find the central pressure to be

$$P_c = K\rho_c^{\frac{5}{3}} = \underbrace{K\left(\frac{9}{2\pi}\right)^{\frac{5}{3}}}_{:=\alpha} M^{\frac{5}{3}} R^{-5}, \tag{6.2.19}$$

where we defined the constant α for convenience.
Requiring these two differently obtained pressures to be equal, we have

$$P_c = P_c \tag{6.2.20}$$

$$\alpha\frac{M^{\frac{5}{3}}}{R^5} = \beta\frac{M^2}{R^4} \tag{6.2.21}$$

$$R = \frac{\alpha}{\beta}M^{-\frac{1}{3}} = K\left(\frac{9}{2\pi}\right)^{\frac{5}{3}}\frac{\pi}{2G}M^{-\frac{1}{3}}, \text{ or} \tag{6.2.22}$$

$$M = \frac{\beta}{\alpha}R^{-3}. \tag{6.2.23}$$

(b) **TO DO Calculate the values (sorrey) (!)**

Problem 3: *Saha equation and pressure ionisation*

The Saha equation describes the fractions of an element in the different ionisation states in thermal equilibrium,

$$\frac{n_{\mathrm{II}} n_e}{n_{\mathrm{I}}} = 2\frac{g_{\mathrm{II}}}{g_{\mathrm{I}}}\left(\frac{2\pi m_e k_{\mathrm{B}} T}{h^2}\right)^{\frac{3}{2}} e^{-E/(k_{\mathrm{B}}T)}, \tag{6.3.24}$$

where n_{I} and n_{II} are the number densities of neutral and ionised atoms, n_e is the number density of electrons, m_e the electron mass, and E the ionisation potential of this element.
In the case of hydrogen, the statistical weights of the ionised and neutral atoms are $g_{\mathrm{II}} = 1$ and $g_{\mathrm{I}} = 2$, respectively.

(a) Assume you have a pure hydrogen medium, where all electrons come from the ionisation of neutral hydrogen, not from metals that have lower ionisation potentials. Use the ionisation fraction

$$x = \frac{n_{\rm II}}{n_{\rm I} + n_{\rm II}} \tag{6.3.25}$$

and the total gas density $n_{\rm g} = n_{\rm I} + n_{\rm II} + n_e$ to rewrite the Saha equation without n_e. Note that

$$n_{\rm g} = n(1 + x) \tag{6.3.26}$$

where n is the density of hydrogen nuclei (neutral hydrogen plus protons).

(b) The gas pressure (from neutral hydrogen, protons, and electrons together) and the temperature at the centre of the Sun are $P = 2.32 \times 10^{17}$ dyn/cm^2 and $T = 1.57 \times 10^7$ K.
Calculate the gas number density $n_{\rm g}$ assuming the medium is an ideal gas.

(c) Calculate the ionisation fraction x using the equation derived at (a). Do you think the result is plausible? Why or why not?

(d) In reality, the centre of the Sun is equivalent to being fully ionised, but in the sense that the electrons in metals are unbound. This effect is called pressure ionisation. Calculate the distance between hydrogen nuclei at the centre of the Sun assuming the above pressure and temperature, but an ionisation fraction of $x = 0$. Compare it with the Bohr radius $r_{\rm B} = 5.29 \times 10^{-9}$ cm.

Problem 3: Solution

TO DO Put in solution I've written down on iPad (!)

Problem 4: *Helium Flash*

When the Sun leaves the main sequence, it will go into its "red giant" phase, with a degenerate inert helium core surrounded by a shell of hydrogen which is fusing to helium. Helium "ash" builds up in the core, contracting it, until it is dense and hot enough to ignite helium fusion. At this point, the helium core has a radius of roughly 7000 km, and a mass of about $1/3$ the mass of the Sun. The excellent thermal conduction of the degenerate helium core allows much of the helium to fuse all at once, resulting in the helium flash. This fuses (say) a tenth of the helium up to carbon, and expands the core out to a radius of about $70\,000$ km, over the course of minutes or hours.

(a) Even though this helium flash happens very quickly, and is a tremendous thermonuclear explosion, no direct effect is seen right away in the luminosity of the star. Where is all the energy from that fusion going?

(b) Do an order-of-magnitude energy calculation to show that your answer to (a) is plausible. Helium is fused to carbon via the "triple-alpha" process. The mass of one helium-4 atom is $4.002\,603\,2u$ and the mass of one carbon-12 atom is $12.0000000u$.

Problem 4: Solution

(a) As soon as the degeneracy of the gas in the core is lifted, the hydrogen shell around the core expands rapidly and decreases its temperature. This is where the energy of the flash goes into.

(b) The energy released in the fusion process can be approximated if we just account for the energy released by three He atoms being fused into one C atom.
If $M_{\text{He, core}} = \frac{1}{3} M_\odot$ and a $\eta = 0.1$ fusion efficiency, we have

$$E_{\text{single}} = (3m_{\text{He}} - m_{\text{C}})c^2 = 1.17 \times 10^{-12} \text{ J}, \tag{6.4.27}$$

$$M_{\text{fused}} = M_{\text{He, core}}\eta \tag{6.4.28}$$

$$\implies E_{\text{tot}} = N_{\text{reaction}} E_{\text{single}} = \frac{M_{\text{fused}}}{3m_{\text{He}}} E_{\text{single}} = 3.87 \times 10^{42} \text{ J}. \tag{6.4.29}$$

Problem 5: *Magnitudes, reddening, and distances*

(a) Express the absolute visual magnitude M_V as a function of the observed visual magnitude V, the distance d (in pc) and the visual extinction, A_V. Explain the relationship between these quantities in physical terms.

(b) Calculate the stellar radius (in solar units) for a star with given absolute visual magnitude M_V, effective temperature T_{eff}, and bolometric correction B.C. The bolometric magnitude of the Sun is $M_{\odot,, \text{bol}} = 4.75$ and its effective temperature is $T_{\odot,\text{eff}} = 5777$ K.

(c) Alternatively, one can calculate the radius from M_V and an adequate theoretical flux distribution $F(\lambda)$. In this case, we need an absolute flux calibration (at least for one star). Remember that the visual magnitude is defined by

$$V = -2.5 \log \left(\frac{R^2}{d^2} \int F(\lambda) S(\lambda) \text{d}\lambda \right) + \text{const} = -2.5 \log \left(\frac{R^2}{d^2} F(\bar{\lambda} \int S(\lambda) \text{d}\lambda \right) + \text{const}. \tag{6.5.30}$$

with photometric response function $S(\lambda)$ and isophotal wavelength $\bar{\lambda} = 5500$ Å for the V-band. Numerous measurements for the standard star Vega have shown that $V = 0$ corresponds to an absolute flux of $(R/d)^2 F(\bar{\lambda}) = 3.69 \times 10 - 9$ erg s^{-1}cm^{-2} Å^{-1} at the isophotal wavelength (and outside the earth's atmosphere), whereas the integrated V-filter response function $RS(\lambda)\text{d}\lambda$ has a value of 2897.5. Derive the "const"!
Show that the stellar radius can be calculated via

$$5 \log \frac{R}{R_\odot} \approx 30.81 + V_{\text{theo}} - M_V, \tag{6.5.31}$$

if $V_{\text{theo}} = -2.5 log(RF_{\text{theo}}(\lambda) S(\lambda) \text{d}\lambda)$, with $F_{\text{theo}}(\lambda)$ the energy flux from a theoretical model.

(d) Cyg OB2 is a heavily reddened OB-association. We will use archival and literature data to derive its distance, from O-stars and appropriate calibrations, via the expression derived in exercise (a).
Look (on the web or in "Simbad/Vizier") for the "Galactic O-star catalog" (by Maíz

Apellániz et al.) and find the O-stars in Cyg OB2. Record the catalog numbers and spectral type and luminosity class information. Then visit "Simbad" and look for the available information on these objects. Use the GOS number from the O-star catalogue as identifier.
Example: for Cyg OB2-7, use "GOS G080.24+00.80 01". Make a table for all stars for which information is available, including spectral type, luminosity class, B- and V-magnitudes.
Look on ADS for the paper by F. Martins et al. (A&A, 2005) which contains the most recent calibration for M_V from O-stars as a function of spectral type and luminosity class. Use this expression to derive the distances of the individual stars. For the extinction, use the relation

$$A_V = RE(B-V), \quad R \approx 3.1 \tag{6.5.32}$$

where $E(B-V)$ is the colour excess. For the intrinsic colours, use the fact that for O-stars these colours are rather constant, with $(B-V)0 \approx -0.28$ for luminosity class I objects and -0.31 else.
Derive the mean and the standard deviation of the individual distances, and compare with the literature value for the distance to Cyg OB2, which is $d \approx 1.7$ kpc.

Problem 5: Solution

(a) The relationship of the magnitudes is

$$m_V = M_V + \underbrace{5\log(d) - 5}_{\text{distance modulus}} + A_V, \tag{6.5.33}$$

as the apparent magnitude increases with extinction, and the distance d is given in terms of parsec.

(b) We know that the luminosity in terms of the effective temperature and radius is given by

$$L_\star = 4\pi R^2 \sigma_\mathrm{B} T_\mathrm{eff}^4. \tag{6.5.34}$$

The bolometric correction is

$$M_\mathrm{bol} - M_{\odot,\,\mathrm{bol}} = -2.5\log\left(\frac{L_\star}{L_odot}\right). \tag{6.5.35}$$

We therefore get

$$M_\mathrm{bol} = M_V + B.C. = -2.5\log\left(\frac{L_\star}{L_\odot}\right) - M_{\odot,\,\mathrm{bol}} = -2.5\log\left(4\pi R^2\sigma_\mathrm{B}T_\mathrm{eff}^4/L_\odot\right) - M_{\odot,\,\mathrm{bol}}. \tag{6.5.36}$$

Solving this for R (and also expressing $L_\odot$ in terms of $R_\odot$ and $T_\odot$), we find

$$R = \left(\frac{T_\odot}{T_\mathrm{eff}}\right)^2 10^{-0.2(M_V+B.C-M_{\odot,\,\mathrm{bol}}} R_\odot. \tag{6.5.37}$$

(c) If we just use the equations given, setting $V = 0$ for the zero-point, we have

$$0 = -2.5 \log \left(3.69 \times 10^{-9} 2897.5\right) + K \implies K = -12.427. \tag{6.5.38}$$

Using the relation from (a) (eq. (6.5.33)), we find

$$m_V + 5 - M_V = 5 \log d \implies \tag{6.5.39}$$