Problem sets Astrophysics Essentials

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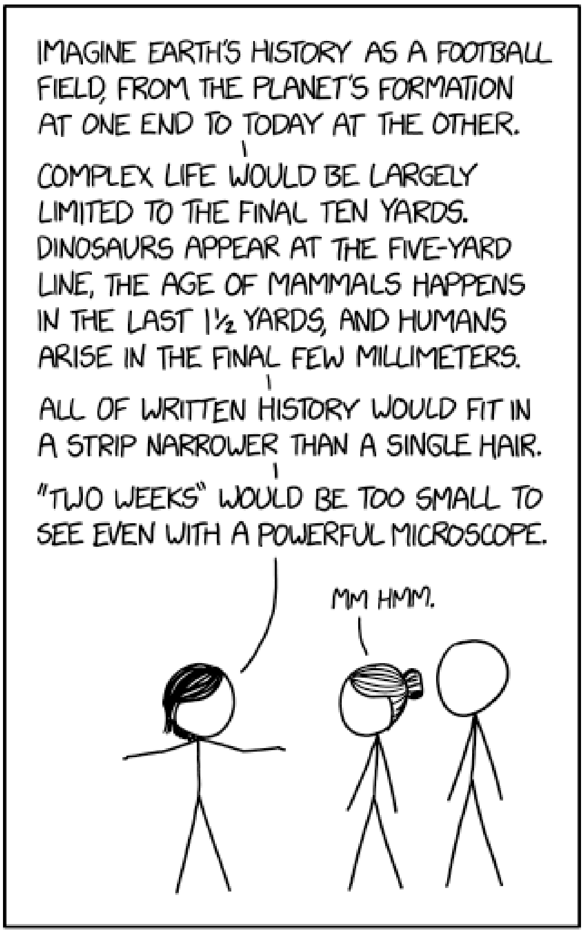
(Version of February 17, 2023)



GEOLOGISTS ALWAYS TRY THIS WHEN THEY'RE LATE TURNING SOMETHING IN.

Source: `https://xkcd.com/2187/`

Moin!
These are solutions of the problem sets for the essentials lecture that I have come up with and refined during the tutorials with Manuel Behrendt.
If you have any remarks, feel free to drop me an email at fbalzer@mpe.mpg.de.
Good luck in the exam and happy studies!
:)

Contents

Problem set #1 Kepler and sky positions (here)

10.11.2022

Problem 1.1: *Stranded*

You wake up one night, and find that you are on a desert island with no idea how you got here. After briefly vowing never to repeat the previous evening's activities, you decide to try and figure out where you are. With nothing but sea around you and the star-filled sky above, you resort to your basic knowledge of astronomy.
In one direction, you see that as the stars rise, they appear to be moving, from your perspective, up and to the left. In which direction are you looking, and what can you conclude about your location on Earth from this observation? Explain.

Problem 1.1: Solution

Because the Earth rotates eastward, the stars moving upwards hints that we are looking in the eastern direction and we are not too close to one of the poles.
Since they seem to be moving left in addition to that, this hints that we are in the southern hemisphere. By measuring the angle the stars appear to move on, we could try to estimate our latitude.
Looking upwards on the north pole, we would see the stars just moving counterclockwise, while on the south pole they appear to move clockwise.

Problem 1.2: *Sky view*

Assume that stars are distributed evenly across the celestial sphere (this assumption isn't quite right, but assume it to be for this problem).

- *At the North Pole.*
  What fraction of the stars would an observer at the North Pole be able to observe over the course of the year?
- *All stars.*
  Where on Earth would you need to be in order to be able to see every star over the course of a year without changing your location? Explain.
- *Wendelstein vs. Paranal.*
  Wendelstein observatory is at a latitude of $+47.5\,\mathrm{deg}$.
  What fraction of the stars would an observer at Wendelstein be able to observe over the course of the year?
  What about an observer at Paranal Observatory in Chile, latitude $-24.5\,\mathrm{deg}$?
  Would they be able to observe all these stars on the same night? Why or why not?

Problem 1.2: Solution

- *At the North Pole.*
  Here, looking up, we would not see any stars below the horizon. Over the course of a night, we would only be able to see half of the whole sky[a].
- *All stars.*

Only at the equator you would be able to see all stars over the course of the year (we would need the whole year for it to be night time for each star).

- *Wendelstein vs. Paranal.*
  We start using the solid angle of the whole sky $\Omega = 4\pi$ as a normalisation constant. The integral for the fraction of stars we can see is

$$f = \frac{1}{4\pi} \int_0^{1-\Theta} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \sin\theta = \frac{\cos\Theta + 1}{2}, \tag{1.2.1}$$

  where we integrated θ from 0 to $1 - \Theta$.
  This function also makes sense as we have $f(\Theta = 0°) = 1$, and $f(\Theta = 90°) = \frac{1}{2}$ (it matches the cases discussed above).
  We can put in the numbers for Wendelstein and Paranal:

$$f_{\text{Wendelstein}} = 0.84, \quad f_{\text{Paranal}} = 0.95. \tag{1.2.2}$$

  Due to the day-night cycle, we could not observe all of them in the same night.

[a] and maybe a little more due to diffraction

Problem 1.3: *Parallax project*

Suppose you are designing a project for observational astronomy. You want to measure the distance to a near-Earth asteroid using parallax. You have two small telescopes you can use, one at Calar Alto Observatory (in Spain) and one at Wendelstein Observatory. The two telescopes are separated by 1662 km.
You take a simultaneous image of the asteroid from each telescope, and compare the position of the asteroid to background stars. This gives you the angular offset between the asteroid as seen from the two locations to a precision of $2''$.
Draw a picture that shows what you've done above. Indicate on this picture what measurement is 1662 km. Label with d the distance from the Earth to the asteroid.

(a) What is the largest distance an asteroid can have without being too far for your measurement given the precision of $2''$? (Hint: if you come up with an answer that is measured in parsecs or tenths of parsecs, you've done something very wrong. [Why?])

(b) How does your answer in (a) compare to the distance from the Earth to the Moon, and to 1 AU (the distance from the Earth to the Sun)?

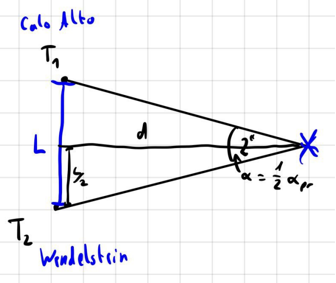
(c) Where do you want the asteroid to be in the sky as seen from each observatory so as to make the best possible measurement? (E.g., near rising, near setting, directly overhead, a little east of overhead, or a little west of overhead?)

Problem 1.3: Solution

We employ the distance between the two telescopes $L = 1662$ km, and the precision α_{pr}, and use d as the distance between Earth and the asteroid.

(a) The furthest distance is directly given by the precision (and related to the optimal case of the asteroid being right in between the two telescopes).

For this case, we can assume a right triangle with $\alpha := \alpha_{\mathrm{pr}}/2$ as the angle, and d and $L/2$ as the legs (see the picture to the right), so we have



$$\tan\alpha = \frac{L}{2d} \iff d_{\max} = \frac{L}{2\tan\alpha} \approx \frac{L}{1''} \approx 1.6{\times}10^{8}\ \mathrm{km}\ \approx 1 \tag{1.3.3}$$

where we used the small-angle approximation for $\tan\alpha \approx \alpha$, and converted the $1''$ to radians via $1\,\mathrm{deg} = \frac{\pi}{180}$, so $1'' = \frac{\pi}{180\cdot 3600}$.

(b) The distance is similar to the distance from the Earth to the Sun, and therefore much further than the distance from the Earth to the Moon.

(c) An observation right overhead should yield the best results, other observations would need different geometrical considerations.

Problem 1.4: *Halley's comet*

The orbits of comets in the Solar system often have very large eccentricities, close to 1. For example, Halley's comet has an orbital period of 76 years and an eccentricity of $e = 0.967$.

(a) Determine the semimajor axis a of Comet Halley's orbit *without using the Sun's mass*. Compare this value with those of the outer planets, in particular Pluto.

(b) Use the above orbital parameters of Halley's comet to determine the mass of the Sun, and discuss your result.

(c) How large is Comet Halley's distance to the sun at its perihelion and at its aphelion? What is its velocity at these points? What is its velocity where it crosses the minor axis of its orbit?

(d) What is the ratio of the comet's kinetic energies at perihelion and aphelion?

Problem 1.4: Solution

We employ $P_{\mathrm{H}} = 76$ yr, $e = 0.967$.

(a) Using Kepler's 3rd law and the parameters of the Earth, we find

$$\frac{a_\oplus^3}{a_{\mathrm{H}}^3} = \frac{P_\oplus^2}{P_{\mathrm{H}}^2} \iff a_{\mathrm{H}} = a_\oplus \sqrt[3]{\frac{P_{\mathrm{H}}^2}{P_\oplus^2}} = \sqrt[3]{76^2}\ \mathrm{au} = 17.94\ \mathrm{au}. \tag{1.4.4}$$

This result compares as follows to $a_{\mathrm{Saturn}} \approx 9.54$ au, $a_{\mathrm{Uranus}} \approx 19.2$ au, and $a_{\mathrm{Pluto}} \approx 39.48$ au:

$$\frac{a_{\mathrm{H}}}{a_{\mathrm{Saturn}}} \approx 1.88, \quad \frac{a_{\mathrm{H}}}{a_{\mathrm{Uranus}}} \approx 0.93, \quad \frac{a_{\mathrm{H}}}{a_{\mathrm{Pluto}}} \approx 0.45. \tag{1.4.5}$$

(b) Using the period-mass-relationship we derived in the lecture and neglecting Halley's mass, we have

$$P^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)} \sim \frac{4\pi^2 a_{\mathrm{H}}^3}{GM_\odot} \implies M_\odot = \frac{4\pi^2 a_{\mathrm{H}}^3}{GP_{\mathrm{H}}^2} = 1.99\times 10^{30} kg, \tag{1.4.6}$$

which is indeed the literature mass of the sun.

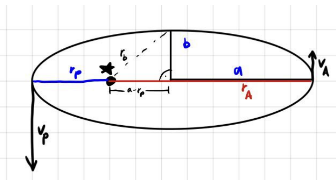
(c) We calculate the elliptical parameters using the semimajor axis a_{H} obtained in (a) and the given ellipticity e:

$$r_{\mathrm{P}} := r_{\mathrm{Perihelion}} = a(1-e) \approx 0.592 \text{ au}, \tag{1.4.7}$$

$$r_{\mathrm{A}} := r_{\mathrm{Aphelion}} = a(1+e) \approx 35.29 \text{ au}, \tag{1.4.8}$$

$$b = a\sqrt{1-e^2} \approx 4.57 \text{ au}, \tag{1.4.9}$$

so the comet comes closer to the Sun than the Earth in perihelion, and stays out almost as far as Pluto in aphelion.



Interestingly, in reality, there are times when Pluto is actually closer than Halley's comet because Pluto is also on an elliptical orbit.

The speed of an object with semi-major axis a at the distance r from the sun can be calculated via the **Vis-viva equation**[a], which can derived from energy conservation (sum of kinetic ($\frac{1}{2}mv_i^2$) and gravitational energies ($-\frac{GMm}{r_i}$) at perihelion and aphelion is conserved), angular momentum conservation ($r_{\mathrm{A}}v_{\mathrm{A}} = r_{\mathrm{P}}v_{\mathrm{P}}$) and the fact that $r_{\mathrm{P}} + r_{\mathrm{A}} = 2a$:

$$v^2 = GM_\odot\left(\frac{2}{r} - \frac{1}{a}\right). \tag{1.4.10}$$

Using this, we have

$$v_{\mathrm{P}} = \sqrt{GM_\odot\left(\frac{2}{r_{\mathrm{P}}} - \frac{1}{a}\right)} \approx 52.29 \,\frac{\mathrm{km}}{\mathrm{s}} \tag{1.4.11}$$

$$v_{\mathrm{A}} \approx 0.91 \,\frac{\mathrm{km}}{\mathrm{s}}. \tag{1.4.12}$$

For the speed at the semi-minor axis, we first need to use trigonometry to determine the distance of the comet to the Sun, we have

$$r_b = \sqrt{(a-r_p)^2 + b^2} \approx 17.94 \text{ au } \sim a, \tag{1.4.13}$$

so we arrive at

$$v_b \approx \sqrt{\frac{GM_\odot}{a}} \approx 7032 \,\frac{\mathrm{m}}{\mathrm{s}}. \tag{1.4.14}$$

(d) The ratio of the comet's kinetic energies is given by

$$\frac{E_{\mathrm{kin,P}}}{E_{\mathrm{kin,A}}} = \frac{v_{\mathrm{P}}^2}{v_{\mathrm{A}}^2} \approx 3302. \tag{1.4.15}$$

Note: Due to angular momentum conservation ($r_{\mathrm{A}}v_{\mathrm{A}} = r_{\mathrm{P}}v_{\mathrm{P}}$), we could have also used

$$\frac{E_{\mathrm{kin,P}}}{E_{\mathrm{kin,A}}} = \frac{r_{\mathrm{A}}^2}{r_{\mathrm{P}}^2}. \tag{1.4.16}$$

[a] see **wikipedia**, also for an in-depth derivation

Problem 1.5: *Neutron Star*

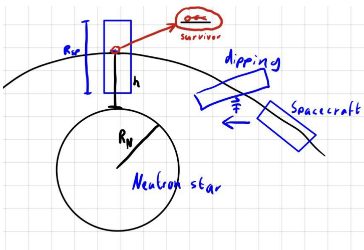
In Larry Niven's short story *Neutron Star*, a research vessel with a hull of indestructible alien material makes a flyby of a neutron star, passing its surface on a hyperbolic orbit in free fall at a minimum distance of one mile. When the ship is later retrieved, the hull is indeed intact, but the crew compartment is wrecked, and the crew is dead. To determine what had happened, the story's protagonist is blackmailed by the ship's manufacturer into repeating the trip with an identical ship (the manufacturer feared bad publicity should it become known that the hull might not have been impregnable after all and that something may have gotten into the ship), and survives.

- Draw a sketch of the situation. Make some plausible assumptions regarding the physical dimensions (explain what your assumptions are) and use what you have already learned in the lecture to *calculate* what happened to the original crew. How could the protagonist have managed to survive in the story? Is this possible?

Problem 1.5: Solution

For the calculations, we assume $R_{\mathrm{N}} \approx 10.4$ km (as the numbers are nicer that way and a radius of ~ 10 km is typical for neutron stars). The minimum distance of the flyby was $h = 1$ mi ≈ 1.6 km, so the distance to the core of the neutron star is

$$r = h + R \approx 12 \text{ km}. \tag{1.5.17}$$



We also apply a mass of $M_{\mathrm{N}} \approx M_{\odot}$.
We can calculate the tidal acceleration at this distance for an object stretched by $R_{\mathrm{Ship}} = 50$ km:

$$\Delta a_{\mathrm{Ship}} = \frac{2GM}{r^3} R_{\mathrm{Ship}} \approx 10^{10}\, \frac{\mathrm{m}}{\mathrm{s}^2} = 10^9 g, \tag{1.5.18}$$

where $g \approx 10$ m/s^2 is the gravitational acceleration on Earth.
For a person standing up relative to the neutron star (so $R_{\mathrm{Stand}} \sim 2$ m, the tidal forces acting on them are

$$\Delta a_{\mathrm{Stand}} \approx 1.5 \times 10^8\, \frac{\mathrm{m}}{\mathrm{s}^2} \implies F_{\mathrm{Stand}} = a_{\mathrm{Stand}} m_{\mathrm{pers}} \approx 10^{10} \text{ N}, \tag{1.5.19}$$

which would wreck any human being apart (we assumed a very lanky ($m_{\mathrm{pers}} \approx 66$ kg) and tall human here).
The protagonist might survive if they try to lie down. Assuming a vertical distance of $R_{\mathrm{Lie}} \sim 10$ cm, the tidal acceleration and force would be

$$\Delta a_{\mathrm{Lie}} \approx 7.7 \times 10^6\, \frac{\mathrm{m}}{\mathrm{s}^2} \implies F_{\mathrm{Lie}} \approx 5 \times 10^8 \text{ N}, \tag{1.5.20}$$

which is around 0.1 times the force applied on the hoover dam.
Assuming an area of ~ 1 m^2 of the poor protagonist's body, the pressure would be... just insane and still probably unbearable.
I honestly am not sure whether this would be survivable, but I'd not sign up for trying

it out.
Since we just saw that the only possibility of standing a chance to survive the spaghettification would be to lie down (and even there the pressure might rip your body apart), it is important to note that one would need to factor in the dip of the (I don't know how) indestructible ship, as the tidal force difference would also bend its nose toward the neutron star - if not accounted for, you could accidentally be pushed to a stand-up-position. RIP.

Problem set #2 Radiation and the ISM (here) 17.11.2022

Problem 2.1: *Radiation through an isothermal layer*

Electromagnetic radiation can be described by the Planck law and its relatives (the Wien displacement law and the Stefan–Boltzmann law). These equations hold strictly in TE ("thermodynamic equilibrium") and reasonably well in most stellar photospheres.
The Planck function specifies the radiation intensity emitted by a gas or a body in TE (a black body, cf. Figure 1).

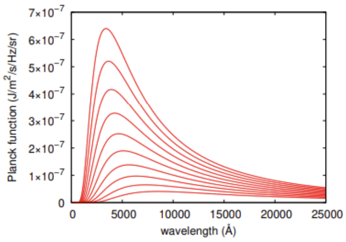


Figure 1: Planck function B_ν for temperatures from 6000 K to 15000 K.

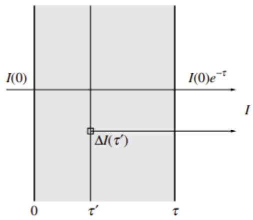


Figure 2: Emergent radiation from a layer consists of transmitted radiation and radiation produced in the layer itself.

Another quantity next to the radiation produced by a gas of temperature T is the amount of absorption. Take the situation sketched in Figure 2.
A beam of radiation with intensity $I(0)$ passes through a layer in which it is attenuated. The weakened intensity I that emerges on the right is given by

$$I = I(0)e^{-\tau} \tag{2.1.1}$$

in which the decay parameter τ specifies the attenuation by absorption in the layer. It is a dimensionless measure of the opaqueness of the layer that is usually called the "optical thickness" because it measures how thick the layer is, not in cm but in terms of its effect on the passing radiation.
Nothing comes through if $\tau \gg 1$ and (almost) everything comes through if $\tau \ll 1$. The next step in computing the total emergent radiation is to add the radiation that originates within the layer itself. The amount ΔI of radiation that is generated locally at position τ' within the layer is equal to $\underline{\hspace{2em}}_1$ (assume the layer has a fixed temperature $T(\tau')$).
This radiation is subsequently attenuated by the remainder of the layer to the right, so that its contribution to the emergent beam is given by $\underline{\hspace{2em}}_2$. The total emergent intensity (containing all contributions) is therefore $\underline{\hspace{2em}}_3$, which for an isothermal layer (one in which T and thus also $B(T)$ is independent of τ') simplifies to $\underline{\hspace{2em}}_4$.

(a) Derive the 4 equations which are marked by "$\underline{\hspace{2em}}$".
(b) Make plots of the emergent intensity I for given values B and $I(0)$ against the total layer thickness τ. Use $B = 2, I(0) = 0, 1, 2, 3, 4$ and $\tau = 0...10$.
(c) How does I depend on τ for $\tau \ll 1$ when $I(0) = 0$?
And when $I(0) > B$? Such a layer is called *optically thin*. Why?
(d) A layer is called *optically thick* when it has $\tau \gg 1$. Why?
The emergent intensity becomes independent of τ for large τ. Explain why this is so in physical terms.

Problem 2.1: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) Let's consider the "$\_\_\_\_$" one by one:

- $\_\_\_\__1$: The amount of radiation generated in one layer is temperature-dependent and given by the Stefan-Boltzmann law:

$$\Delta I = B(\tau')\mathrm{d}\tau'. \tag{2.1.2}$$

- $\_\_\_\__2$: Therefore, the amount of radiation from this layer reaching the observer (and therefore emerging through a layer of thickness τ') after attenuation is given by

$$I'(\tau') = \Delta I e^{-\tau'} = B(\tau')e^{-\tau'}\mathrm{d}\tau'. \tag{2.1.3}$$

- $\_\_\_\__3$: To calculate the total emergent intensity I_{tot}, we need to consider all intensity that has been picked up, which we obtain by integration:

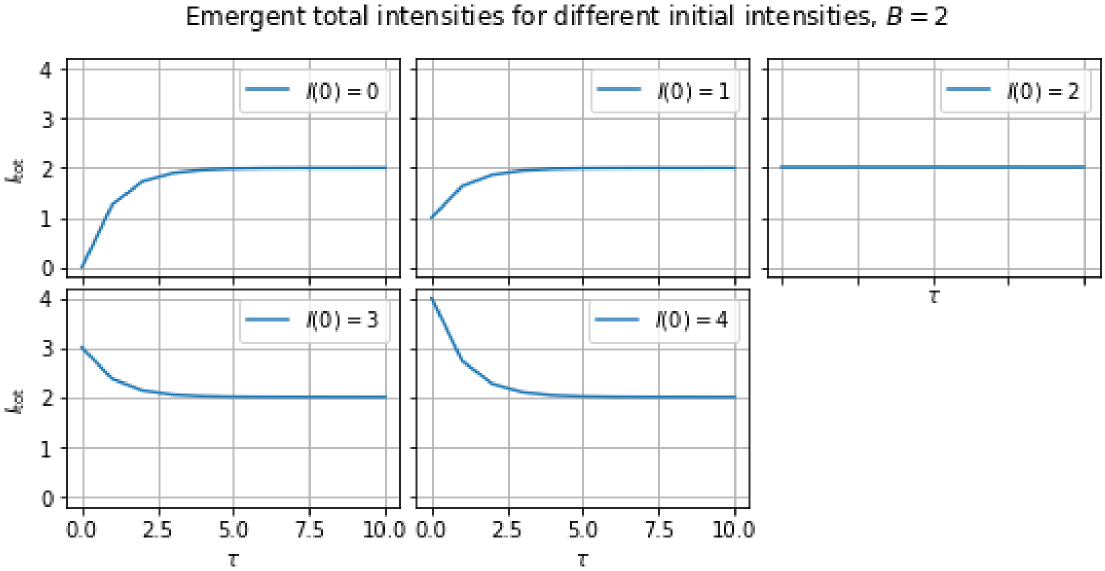
$$I_{\text{tot}} = I(0)e^{-\tau} + \int_0^\tau B(\tau')e^{-(\tau-\tau')}\mathrm{d}\tau'. \tag{2.1.4}$$

  Here, we considered the fact that $\tau = 0$ is defined on the left side.

- $\_\_\_\__4$: For an isothermal layer, $B(\tau') = B$, so the integral can be calculated:

$$I_{\text{tot}} = I(0)e^{-\tau} + \left[-Be^{-\tau'}\right]_0^\tau = \underline{\underline{I(0)e^{-\tau} + (1-e^{-\tau})B}}. \tag{2.1.5}$$

(b) As instructed, we plot the τ-dependence for different values of $I(0)$, keeping $B = 2$ fixed.



As we'll explore in the next two parts of this exercise, we see that for large values of τ, the intensity always approaches B.

(c) For $\tau \ll 1$, we have

$$I_{\text{tot}} = (I(0) - B)e^{-\tau} + B \overset{I(0)=0}{=} (1-e^{-\tau})B \overset{e^x \approx 1+x}{\approx} \tau B. \tag{2.1.6}$$

For $I(0) > B$, we have

$$I_{\mathrm{tot}} \approx \tau B + I(0)(1+\tau) = I(0) + \tau(B - I(0)). \tag{2.1.7}$$

For $\tau \gg 1$, we have $e^{-\tau} \approx 0$, so we arrive at

$$I_{\mathrm{tot}} \approx B. \tag{2.1.8}$$

This makes sense, as the initial intensity is completely attenuated, and we can basically only see the outer layer of the isothermal sheet.

Problem 2.2: *Flux from the sun in a clever way*

As observations show, on Oct. 1 the sun subtends an angular diameter of 32 arcmin.

(a) Calculate the solid angle $\Omega_\odot$ subtended by the sun, in steradians.

(b) Show that the flux (in W m-2s^{-1}) or its cgs equivalent) of solar radiation on Earth is $F = I(T_\odot)\Omega_\odot$ with $T_\odot = 5777$ K, and calculate this value numerically. *Note that to calculate the flux from a blackbody of known temperature (or other source of known specific intensity), you do not need to know the distance or luminosity, but only the temperature and angle subtended! Both of these are direct observables, unlike distance and luminosity.*

(c) Show that the answer to the previous part is the same as you would get by the more obvious but unnecessarily complicated method $F = \frac{L_\odot}{4\pi a^2}$, with $a = 1$ au and $L_\odot = 4\pi R_\odot^2 \sigma T_\odot^4$.

Problem 2.2: Solution

(a) If we consider a circular area on a sphere, its size (given an angle θ) can be integrated as follows:

$$S = \int_0^\theta 2\pi R^2 \sin\theta' \mathrm{d}\theta' = 2\pi R^2(1-\cos\theta). \tag{2.2.9}$$

The solid angle is then

$$\Omega = \frac{S}{R^2}(1-\cos\theta). \tag{2.2.10}$$

Applying $\theta = 16$ arcmin (half of the diameter) as the aperture angle, we have

$$\Omega_\odot = 2\pi\left(1-\cos(\theta)\right)\ \mathrm{sr} = 6.8\times10^{-5}\ \mathrm{sr}. \tag{2.2.11}$$

Approximating small angles θ, we have $\cos\theta \approx 1 - \theta^2/2$ (taylor), so the equation simplified to one of a circle:

$$\Omega_\odot \approx 2\pi(1-(1-\theta^2/2) = \pi\theta^2 = \pi\left(\frac{32/2}{60}\right)^2\ \mathrm{deg}^2 = \pi\left(\frac{4\pi}{15\cdot180}\right)^2\ \mathrm{sr} = \underline{\underline{6.8\times10^{-5}\ \mathrm{sr}}}, \tag{2.2.12}$$

where we again used $\theta = 16$ arcmin as the angular radius and converted it to radians applying $360\ \mathrm{deg} = 2\pi$.

(b) In the lecture, we saw that the forward-flux term is $F^+ = \pi I$, and $\Omega = \frac{A}{d^2} = \frac{\pi R_\odot^2}{d^2}$, so we can derive the flux as follows:

$$F_\oplus = \frac{L_\odot}{4\pi d^2} = \frac{F_\odot^+ A}{4\pi d^2} = \frac{F_\odot^+ 4\pi R_\odot}{4\pi d^2} = \frac{\pi I R_\odot}{d^2} = I(T_\odot)\Omega_\odot. \tag{2.2.13}$$

Applying the Stefan-Boltzmann-Law for the flux density ($I = \frac{\sigma_\mathrm{B} T^4}{\pi}$), we have

$$F = \frac{\sigma_\mathrm{B} T_\odot^4}{\pi}\Omega_\odot = \underline{\underline{1360\ \mathrm{Wm}^{-2}}}. \tag{2.2.14}$$

(c) We find that

$$F = \frac{L_\odot}{4\pi a^2} = \frac{4\pi R_\odot \sigma_\mathrm{B} T_\odot^4}{4\pi a^2} = \underline{\underline{\sigma_\mathrm{B} T_\odot^4 \left(\frac{R_\odot}{a}\right)^2}}. \tag{2.2.15}$$

Problem 2.3: *Mean molecular weight*

The equation of state often depends on the particle density n (e .g., for the ideal gas the pressure is $P = nk_\mathrm{B}T$). One can relate the particle density to the mass density ρ and the mass m_H of the hydrogen atom by introducing the mean molecular weight μ,

$$\mu = \frac{\rho}{n}\frac{1}{m_\mathrm{H}}, \tag{2.3.16}$$

i.e., the average mass of a particle (ρ/n) in units of the hydrogen atom mass, so that

$$n = \frac{\rho}{\mu m_\mathrm{H}}. \tag{2.3.17}$$

- What is the mean molecular weight of a pure hydrogen plasma? Assume that the gas is fully ionised.
- What is the mean molecular weight of a pure helium plasma? Again assume full ionisation.
- What is the analogous expression for heavier elements? Give a simplified approximation. (Hint: how many electrons do you get from carbon? from nitrogen? from oxygen? given full ionisation.)
- Now combine the above and derive an expression for n for a plasma consisting of hydrogen, helium, and heavier elements in mass fractions X, Y , and Z (i.e., in one gram of material, X grams are in the form of (ionised) hydrogen atoms, Y grams are in the form of (ionised) helium atoms, and Z grams are in the form of (ionised) heavier atoms). Express Z in terms of X and Y and derive an equation for μ that depends only on X and Y.

Problem 2.3: Solution

- We can summarise the results (assuming the easiest isotopes):

Atom	Nucleus (corresponding to mass)	# Electrons	# free particles	μ
H	1 p	1	2	$\frac{1}{2}$
He	2 p + 2 n	2	3	$\frac{4}{3}$
C	6 p + 6 n	6	7	$\frac{12}{7}$
N	7 p + 7 n	7	8	$\frac{14}{8}$
O	8 p + 8 n	8	9	$\frac{16}{9}$

  This table includes $\mu_{\mathrm{H}} = \frac{1}{2}$ and $\mu_{\mathrm{He}} = \frac{4}{3}$.
  Generalised, for fully ionised plasma, we have

$$\mu = \frac{A}{Z+1}, \tag{2.3.18}$$

  where A is the atomic mass number and Z is the atomic number, denoting the amount of electrons.
- We see that for heavier elements, μ approaches 2, although we note that the heavier the element, the less likely it is that it is fully ionised.
- To obtain the combined number density n for a plasma of H, He, and heavier elements with mass fractions X, Y, and Z, we first note that the sum of the mass fractions has to be 1:

$$X + Y + Z = 1. \tag{2.3.19}$$

  We can then sum up the individual number densities of the different species:

$$\begin{aligned}
n = \sum_i n_i &\approx 2\frac{\rho}{m_{\mathrm{H}}}X + \frac{3}{4}\frac{\rho}{m_{\mathrm{H}}}Y + \frac{1}{\mu_Z}\frac{\rho}{m_{\mathrm{H}}}Z && (2.3.20)\\
&\approx \left(2X + \frac{3}{4}Y + \frac{1}{2}\right)\frac{\rho}{m_{\mathrm{H}}} && (2.3.21)\\
&= \frac{8X + 3Y + 2Z}{4}\frac{\rho}{m_{\mathrm{H}}} && (2.3.22)\\
&\overset{(2.3.19)}{=} \underline{\underline{\underbrace{\frac{6X + Y + 2}{4}}_{=1/\mu_{\mathrm{tot}}}\frac{\rho}{m_{\mathrm{H}}}}}. && (2.3.23)
\end{aligned}$$

  Here, we have found $\mu_{\mathrm{tot}} = \frac{4}{6X+Y+2}$ as the effective μ of the combined plasma, while assuming that $\mu \approx 2$ for the heavy elements.

Problem 2.4: *Pressures and energies in the ISM*

The interstellar medium consists of five main phases. Recall the typical densities and temperatures of cold molecular gas, cold and warm neutral (atomic) gas, and warm and hot ionised gas. Where do we find the different phases in the Milky Way and other galaxies?
Calculate the gas pressure and the mean energy per particle for each of the five phases.

(Hint: Assume ideal gas in equilibrium at the corresponding temperature and consider the number of degrees of freedom available.)

Problem 2.4: Solution

Note: The code used for parts of this exercise is available in **this repository**.
The different phases can be found in almost all regions of the galaxy, although they are more prevalent near the spiral arms.
The gas pressure P and the mean energy per particle $\langle E \rangle$ (assuming an ideal gas in equilibrium) can be calculated as follows:

$$P = nk_\mathrm{B}T, \quad \langle E \rangle = \frac{f}{2}k_\mathrm{B}T, \tag{2.4.24}$$

where f denotes the degrees of freedom.
All of the monoatomic species only have the three standard degrees of freedom.
For the molecular gas, in principle there would be two additional rotational and two vibrational degrees of freedom possible, but they only become available for temperatures $T \gtrsim 100$ K (rotational) and $T \gtrsim 1000$ K (vibrational), so we have $f = 3$ for all phases in the interstellar medium.

Phase	**Where to find them**	n [cm^{-3}]	T [K]	P [K k_B^{-1}cm^{-3}]	$\langle E \rangle$ [eV]
Molecular	Star formation sites	300.000	10	3000	0.0013
Cold atomic	Sheets and filaments	50.000	80	4000	0.0103
Warm atomic	Photodissociation regions	0.500	8000	4000	1.0341
Warm ionised	Near O and B stars	0.300	8000	2400	1.0341
Hot ionised	Near SF regions and SNe	0.003	500000	1500	64.6300

Problem 2.5: *Sound speeds in the ISM*

What are the typical sound speeds in the five phases of the ISM? Be aware that the average particle mass as well as the adiabatic index γ may differ from one phase to the other. For simplicity you can assume that the ISM consists only of hydrogen.
(Hint: What is the connection between the equation of state and the propagation of small disturbances?)

Problem 2.5: Solution

Note: The code used for parts of this exercise is available in **this repository**.
The sound speed can be obtained via

$$c_\mathrm{s} = \sqrt{\gamma \frac{P}{\rho}}, \quad \text{where } \gamma = 1 + \frac{2}{f}, \tag{2.5.25}$$

so $\gamma = 5/3$ for the monoatomic and the cold (< 100 K) molecular gas.

Therefore, we have (applying the equation in exs. 3)

$$c_\mathrm{s} = \sqrt{\frac{\gamma k_\mathrm{B} T}{\mu m_\mathrm{H}}}. \tag{2.5.26}$$

Note that $\mu = 1$ for neutral and $\mu = \frac{1}{2}$ for ionised hydrogen.

Phase	T [K]	μ	c_s [km/s]
Molecular	10	1.0	0.3709
Cold atomic	80	1.0	1.0491
Warm atomic	8000	1.0	10.4909
Warm ionised	8000	0.5	14.8364
Hot ionised	500000	0.5	117.2917

Problem 2.6: *Virial theorem and Jeans mass*

The Jeans criterion for the collapse of gas clouds is an important concept for interpreting the behaviour of interstellar matter. The corresponding Jeans mass can (among other derivations) be obtained by an application of the virial theorem.

(a) Show that for a self-gravitating spherically-symmetric ideal-gas cloud in (hydrostatic) pressure equilibrium,

$$E_\mathrm{kin} = -\frac{1}{2} E_\mathrm{pot} \tag{2.6.27}$$

(Hint: all the descriptive words above are relevant.

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{GM(r)\rho(r)}{r^2} \tag{2.6.28}$$

Don't compute any numbers here, just show that the corresponding integrals are equal.)

(b) For a cloud to be able to contract, what can we then say about the energies involved? Use simple approximations to evaluate the integrals appearing in (a) above. What minimum mass must a cloud of given mean density and mean temperature therefore at least have in order to contract?
Compare your result to the one from the lectures and discuss possible differences.

Problem 2.6: Solution

(a) We start with the expression given in eq. (2.6.28), which we can assume if we consider hydrostatic equilibrium, in which the pressure gradient should depend on the gravity pulling everything inwards. $M(r)$ is the enclosed mass inside of radius r.

The potential energy can then be obtained as

$$\begin{aligned}
E_{\text{pot}} &= \int_0^M -\frac{Gm(r)}{r}\mathrm{d}m \\
&\overset{\mathrm{d}m=\rho(r)4\pi r^2\mathrm{d}r}{=} \int_0^R -\frac{GM(r)\rho(r)}{r^2}4\pi r^3\mathrm{d}r \\
&= 4\pi \int_0^R \frac{\mathrm{d}p}{\mathrm{d}r} r^3 \mathrm{d}r \\
&\overset{\text{int. by parts}}{=} 4\pi \underbrace{r^3 P(r)\big|_0^R}_{=0,\text{ see below}} - 4\pi \int_0^R P(r) 3r^2 \mathrm{d}r \\
&= -3\int_0^R 4\pi r^2 P(r)\frac{V}{V}\mathrm{d}r \\
&= -3V\langle P\rangle, \qquad (2.6.29)
\end{aligned}$$

where $\langle P\rangle$ is the volume-averaged pressure.
The term $r^3P(r)$ vanishes at $r = 0$ because the pressure has a finite value, and vanishes at $r = R$ because we can assume $P(R) = 0$.
Applying the ideal gas law, for the kinetic energy we find

$$E_{\text{kin}} = \frac{3}{2}Nk_{\text{B}}\langle T\rangle = \frac{3}{2}\langle P\rangle V. \qquad (2.6.30)$$

$$\implies E_{\text{pot}} = -2E_{\text{kin}}. \qquad (2.6.31)$$

(b) The assumption is that the potential energy is approximately $E_{\text{grav}} \sim \frac{GM^2}{R}$, so the criterion for instability becomes

$$\frac{GM^2}{R} > \frac{3}{2}Nk_{\text{B}}T = \frac{3}{2}\frac{M}{\bar{m}}k_{\text{B}}T \qquad (2.6.32)$$

$$\iff M > \frac{3Rk_{\text{B}}T}{2G\bar{m}}. \qquad (2.6.33)$$

This deviates from the known Jeans criterion since the potential depends on the density profile that we apply.

Note: The code used for parts of this exercise is available in **this repository**.

Problem set #3 Radiation, dust and IMFs (here)

24.11.2022

Problem 3.1: *Planck function*

The Planck function, defined here in terms of a specific intensity with respect to frequency (i.e., with units erg cm^{-2} s^{-1} Hz^{-1} sr^{-1}), is given by

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)-1} \tag{3.1.1}$$

with Planck's constant h, frequency ν, speed of light c, Boltzmann's constant k_B, and temperature T.

(a) Derive the expression for the corresponding energy density, u_ν.

(b) What is the corresponding function B_λ, i. e. with units erg cm^{-2} s^{-1} cm^{-1} sr^{-1}?

(c) Show that Planck curves for different T do not cross each other.

(d) Prove the Stefan-Boltzmann law and *calculate* the Stefan-Boltzmann constant σ_B.

(e) Prove Wien's displacement law for both B_ν and B_λ and calculate the corresponding constants.
Hint: Use the fact that

$$3e^{x_1} - 3 - x_1 e^{x_1} = 0 \implies x_1 \approx 2.82144 \tag{3.1.2}$$

$$5e^{x_2} - 5 - x_2 e^{x_2} = 0 \implies x_2 \approx 4.96511. \tag{3.1.3}$$

(f) Plot the Planck functions B_ν and B_λ for $T = 5500$ K – both as functions of wavelength, using the same x-axis – into one figure (normalised such that both curves have the same peak value) and compare with the results from exercise (e).
What do you conclude?

Problem 3.1: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) The energy density u_ν is given by an integration over the solid angle Ω, so we have

$$u_\nu = \frac{1}{c}\oint B_\nu \mathrm{d}\Omega = \underline{\underline{\frac{4\pi}{c}B_\nu}}. \tag{3.1.4}$$

(b) We know that

$$B_\lambda \mathrm{d}\lambda = -B_\nu \mathrm{d}\nu, \tag{3.1.5}$$

where the minus sign corresponds to the fact that the wavelength decreases with increasing frequency, so with $\nu = \frac{c}{\lambda}$ we have

$$B_\lambda = -\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}B_\nu\left(\nu = \frac{c}{\lambda}\right) = \underline{\underline{\frac{c}{\lambda^2}\frac{2hc}{\lambda^3}\frac{1}{\exp\left(\frac{hc}{\lambda k_\mathrm{B}T}\right)-1}}}. \tag{3.1.6}$$

(c) Let's assume that two Planck curves at different temperatures $T_1 \neq T_2$ would cross each other.
In this case, there'd be at least one point where $B_\nu(T_1) = B_\nu(T_2)$.
Putting this into the equation, we have

$$\begin{aligned} \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{k_\mathrm{B}T_1}\right)-1} &= \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{k_\mathrm{B}T_2}\right)-1} \\ \iff \quad \exp\left(\frac{h\nu}{k_\mathrm{B}T_1}\right) &= \exp\left(\frac{h\nu}{k_\mathrm{B}T_2}\right) \\ \iff \quad T_1 &= T_2, \end{aligned}$$

which clashes with our assumption that $T_1 \neq T_2$.
The fact that Planck curves do not cross each other implies that the intensity (and therefore also the flux) is higher at any given frequency for an object hotter than another (as long as T.E. is assumed).
Note:
We could have also derived this by showing that $\dfrac{\partial B_\nu}{\partial T} > 0$ for all T.

(d) We can derive the Stefan-Boltzmann-law by integrating the forward flux term $F_\nu^+ = \pi B_\nu$ over all frequencies:

$$\begin{aligned} F(T) &= \pi \int_{\nu=0}^{\infty} B_\nu(T)\mathrm{d}\nu \\ &= \frac{2h\pi}{c^2}\int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{k_\mathrm{B}T}}-1}\mathrm{d}\nu \\ &\overset{x:=\frac{h\nu}{k_\mathrm{B}T}}{=} \frac{2k_\mathrm{B}^4T^4\pi}{c^2h^3}\underbrace{\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x}_{=\frac{\pi^4}{15}} \\ &\overset{(3.1.13)}{=} \underbrace{\frac{2k_\mathrm{B}^4\pi^5}{15c^2h^3}}_{:=\sigma_\mathrm{B}}T^4, \end{aligned} \tag{3.1.7}$$

where we used the substitution $x := \frac{h\nu}{k_\mathrm{B}T}$ with $\mathrm{d}\nu = \frac{k_\mathrm{B}T}{h}$, used the integral for $x^3(e^x-1)^{-1}$ (see below), and defined $\sigma_\mathrm{B} \approx 5.67 \times 10^{-5}$ ergs cm^{-2} s^{-1} K^{-4}.

(e) Wien's displacement law is a formula for the peak in the Planck spectrum and can therefore be obtained via

$$\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}\nu} = 0, \quad \frac{\mathrm{d}B_\lambda(T)}{\mathrm{d}\lambda} = 0. \tag{3.1.8}$$

For the frequency domain, we have (using the chain and product rules)

$$0 \overset{!}{=} \frac{\mathrm{d}B_\nu}{\mathrm{d}\nu} = \frac{6h}{c^2}\frac{\nu^2}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)-1} - \frac{2h\nu^3}{c^2}\frac{h}{k_\mathrm{B}T}\frac{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)}{\left(\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)-1\right)^2}$$

$$\iff \quad 3\left(\frac{h\nu}{k_\mathrm{B}T}\right)^2 \frac{k_\mathrm{B}^2T^2}{h^2} = \left(\frac{h\nu}{k_\mathrm{B}T}\right)^3 \frac{k_\mathrm{B}^2T^2}{h^2} \frac{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)-1}$$

$$\overset{x:=\frac{h\nu}{k_\mathrm{B}T}}{\iff} \quad (e^x-1)3x^2 = e^x x^3$$

$$\iff \quad 3e^x - 3 - xe^x = 0 \overset{(3.1.2)}{\implies} x_1 \approx 2.82144$$

$$\iff \quad \nu_\mathrm{max} = 2.82144\frac{k_\mathrm{B}T}{h} = \underline{\underline{58.789\frac{T}{\mathrm{K}}\,\mathrm{GHz}}}. \tag{3.1.9}$$

Doing the same for the wavelength, it gets even uglier for a moment:

$$0 \overset{!}{=} \frac{\mathrm{d}B_\lambda}{\mathrm{d}\lambda} = -\frac{10c^2h}{\lambda^6}\frac{1}{\exp\left(\frac{hc}{\lambda k_\mathrm{B}T}\right)-1} + \frac{2h^2c^3}{\lambda^7}\frac{\exp\left(\frac{hc}{\lambda k_\mathrm{B}T}\right)}{\left(\exp\left(\frac{hc}{\lambda k_\mathrm{B}T}\right)-1\right)^2}$$

$$\overset{x:=\frac{hc}{\lambda k_\mathrm{B}T}}{\iff} \quad (e^x-1)5x^6 = e^x x^7$$

$$\iff \quad 5e^{x_2} - 5 - x_2e^{x_2} = 0 \implies x_2 \approx 4.96511 \tag{3.1.10}$$

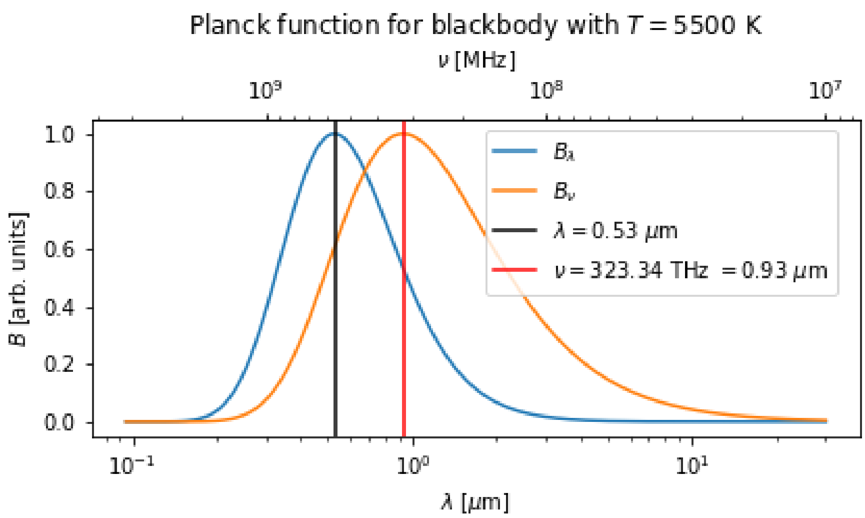
$$\iff \quad \lambda_\mathrm{max}T = \frac{hc}{4.96511k_\mathrm{B}} = \underline{\underline{0.00289777\ \mathrm{m\ K}}}. \tag{3.1.11}$$

We see that they are actually not the same, e. g.

$$\lambda T = \frac{c}{\nu_\mathrm{max}} = 0.0050995\ \mathrm{m\ K}, \tag{3.1.12}$$

which is due to their differential relationship.

(f) As we see, we reproduce the results from exercise (e).



Note: *Derivation of the integral above:*
First we note that we can express the integrand as a geometric series:

$$\frac{x^3}{e^x-1} = x^3 e^{-x}\frac{1}{1-e^{-x}} = x^3 e^{-x}\sum_{n=0}^{\infty}\left(e^{-x}\right)^n = x^3\sum_{n=0}^{\infty} e^{-x(n+1)} = x^3\sum_{n=1}^{\infty} e^{-nx},$$

where we slurped up the e^{-x} *term and index-shifted For the last equalities.*
Using this trick (and the fact that in this case, we may change the order of summation and integration, let's skip showing that we can...), we have

$$\begin{aligned}\int_{x=1}^{\infty}\frac{x^3}{e^x-1}\,dx &= \int_0^{\infty} x^3\sum_{n=0}^{\infty} e^{-nx}\,dx\\ &= \sum_{n=1}^{\infty}\int_0^{\infty} x^3 e^{-nx}\,dx\\ &\overset{z:=nx}{=} \sum_{n=1}^{\infty}\frac{1}{n^4}\underbrace{\int_0^{\infty} z^3 e^{-z}\,dz}_{=\Gamma(4)=(4-1)!=3!=6}\\ &= 6\sum_{n=1}^{\infty}\frac{1}{n^4}\\ &= 6\frac{\pi^4}{90}\\ &= \underline{\underline{\frac{\pi^4}{15}}}.\end{aligned}\tag{3.1.13}$$

In this derivation, we used a bunch of other math tricks, including the integral representation of the ***Gamma function*** $\Gamma(n)$*, and the infinite sum* $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$ *can e. g. be shown using Fourier Series and Parseval's theorem.*

Problem 3.2: *Accretion disk temperature*

A simple model of an accretion disk approximates the disk as a series of nested rings of radius r with black body spectra of the temperature

$$T(r, M, \dot{M}) = \left(\frac{GM\dot{M}}{8\pi\sigma_{\rm B} r^3}\right)^{\frac{1}{4}}, \tag{3.2.14}$$

where M is the mass of the accreting body and $\dot{M}$ is the accretion rate.

- Derive an expression for the total disk luminosity $L_{\rm disk}(R_{\rm min}, M, \dot{M})$ from the above formula for $T(r, M, \dot{M})$ using $R_{\rm min}$ as the inner disk radius (near the surface of the accreting body).

Assume accretion disks around a classical T Tauri star, a typical white dwarf, and a neutron star. The values are provided in the table below:

(a) In this model, where does the accretion disk have the highest temperature? Calculate this temperature.
(b) Calculate the luminosity of the disk in ergs/s and in solar luminosities.
(c) Compare this to a black hole accreting $10^{-10}\ M_\odot$/yr and converting 1/16 of this mass into radiation.
(d) Which wavelength range would you use for observations?

Problem 3.2: Solution

Note: The code used for parts of this exercise is available in **this repository**.

- The luminosity of one ring element is

$$L_{\mathrm{ring}} = 2\sigma_{\mathrm{B}} T^4 = \sigma_{\mathrm{B}} \frac{GM\dot{M}}{8\pi\sigma_{\mathrm{B}} r^3} = \frac{GM\dot{M}}{4\pi r^3}, \tag{3.2.15}$$

where the factor of 2 accounts for the two sides of the disk.
Integrating over all rings, we have to consider that each ring is $2\pi r$ in size, so we have

$$\begin{aligned} L_{\mathrm{disk}} = 2\int L_{\mathrm{ring}} &= \int_{R_{\mathrm{min}}}^{\infty} 2\pi r \frac{GM\dot{M}}{8\pi r^3} \mathrm{d}r \\ &= -\frac{GM\dot{M}}{2}\left(0 - \frac{1}{R_{\mathrm{min}}}\right) \\ &= \underline{\underline{\frac{GM\dot{M}}{2R_{\mathrm{min}}}}}. \end{aligned} \tag{3.2.16}$$

(a) In this model, the since $T \propto r^{-\frac{3}{4}}$, the highest temperature is achieved at R_{min}.
(b) The luminosities can be found in the table below (at the end of this solution).
(c) If the accreting black hole converts $\eta = 1/16$ of the accretion mass into radiation having $\dot{M} = 10^{-10} M_\odot/\mathrm{yr}$, its luminosity would be

$$L_{\mathrm{acc}} = \frac{\Delta E}{\Delta t} = \eta \dot{M} c^2 = \underline{\underline{92.46 L_\odot}} \tag{3.2.17}$$

The fractional values can be found in the table below.
(d) To find the optimal wavelength range for observations, we can just assume the inner disk's ring as a black body (which is not entirely correct to assume for the entire disk as the temperature drops rapidly, so the maximum of the flux changes) and use Wien's law which we derived in (3.1.11).
Due to the temperature drop, it might be advisable to centre our observations at slightly higher temperatures.
The results are in the table below, but we note that the value for the luminosity for the neutron star disk would be larger by quite a lot.

Employing the values for R_{min}, M and $\dot{M}$ and the equations shown above, we have the following values:

Type	L $[L_\odot]$	L [ergs/s]	L/L_{acc}	λ_{obs} [nm]	Wavelength range
T Tauri	0.07850	3.005×10^{32}	0.000849	1341.33	Far Infrared
White dwarf	0.1334	5.108×10^{32}	0.001443	83.063	Far UV
Neutron star	3058.2	1.171×10^{37}	33.0764	0.255952	X-Rays

Note that the accretion disk for the neutron star would be quite extreme, but maybe our model isn't completely realistic there.

Problem 3.3: *Dust grain temperature*

(a) What is the dependence of the equilibrium particle temperature on the distance from a star of given luminosity, if the particle absorbs 50 % of the stellar radiation and radiates (mostly infrared) thermal flux with wavelength-independent efficiency $Q_{\text{abs}}^{(\text{IR})} = 1$ (Kirchhoff's law states that absorption and emission efficiencies are equal, hence "abs".). Express your result in the form $T(r) = \text{const}(r/\text{au})^{\text{const}}$ and find T at a distance of $r = 80$ au from a star of luminosity $L = 8L_\odot$.

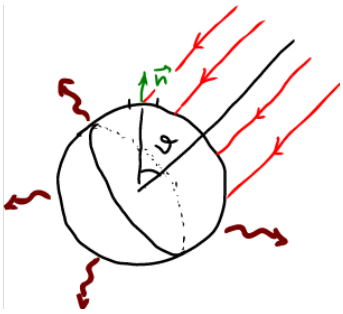
(b) What is the dependence of temperature of a small dust grain on the distance from the star if the particle absorbs 50 % of the stellar radiation, and radiates the (mostly infrared) ($\lambda > 10\ \mu$m) thermal flux with a wavelength-dependent efficiency given by the formula

$$Q_{\text{abs}}^{(\text{IR})} = \frac{\lambda_0}{\lambda} \tag{3.3.18}$$

where $\lambda_0 = 10\ \mu$m?
For simplicity, substitute for the emitted λ in this formula an effective wavelength λ_{eff} provided by Wien's law of black body radiation (i.e., the λ at which the Planck curve B_λ with temperature T peaks). Express your result in the form $T(r) = \text{const}(r/\text{au})^{\text{const}}$ and find $T(r = 80\ \text{au})$ for the same star as above.
(This latter problem is motivated by the fact that astronomers are often faced with dust disks in which the size distribution of particles is such that most mass resides in the large particles but most area in the smallest ones, a few micrometers in radius. These small grains have smaller emissivity (absorptivity) at the typical λ_{eff} that follows from Wien's law at the equilibrium temperature. The reason is that they are much smaller than the wavelength of radiation they emit, and in that case the coupling between light and matter is always much weaker.)

Problem 3.3: Solution

Thanks and credits to Iliya Tikhonenko!



A lone dust grain basking in the fading light of its host star (and radiating some of it back) somewhere at the end of the Universe.

Let a be the distance from the star to the dust grain, r be the size of the grain (to confuse everyone reading this, you know), and R_* be the radius of the star.
From the lectures (and problem set #2) we know that the flux density on a small plate $d\sigma$ (with its normal pointing towards the source) at the surface of the grain would be

$$F_\lambda = B_\lambda(T_*) \cdot \Omega_*, \quad \text{where } \Omega_* \approx \frac{\pi R_*^2}{a^2} \quad (R \ll d).$$

The total power of absorbed radiation is then (with $\alpha = 0.5$, that's not quite albedo, but still the measure of the surface reflectivity)

$$L_{\text{abs}} = \alpha \int_0^\lambda \int_{\text{hemisph}} F_\lambda \cos\vartheta \, d\sigma \, d\lambda, \quad \text{where } d\sigma = r^2 \sin\vartheta \, d\phi \, d\vartheta \, .$$

Hence,

$$\begin{aligned} L_{\text{abs}} &= \alpha\pi \, r^2 \frac{R_*^2}{a^2} \int_0^\lambda B_\lambda(T_*) \int_0^{\pi/2} \int_0^{2\pi} \cos\vartheta \sin\vartheta \, d\phi \, d\vartheta \, d\lambda \\ &= 2\alpha\pi^2 \, r^2 \frac{R_*^2}{a^2} \int_0^\lambda B_\lambda(T_*) \int_0^{\pi/2} \cos\vartheta \sin\vartheta \, d\vartheta \, d\lambda \\ &= \alpha\pi^2 \, r^2 \frac{R_*^2}{a^2} \int_0^\lambda B_\lambda(T_*) \, d\lambda \\ &= \alpha\pi \, r^2 \frac{R_*^2}{a^2} \, \sigma T_*^4 \\ &= r^2 \, \frac{\alpha}{4} \, \frac{L_*}{a^2}. \end{aligned}$$

As the grain is in equilibrium with the star radiation, it should emit exactly the same power it receives in all directions (we can now drop the indices of Q for brevity)

$$L_{\text{em}} = 4\pi r^2 \, \pi \int_0^\infty B_\lambda(T_g) \, Q(\lambda) \, d\lambda.$$

Putting everything together, we get

$$\int_0^\infty B_\lambda(T_g) \, Q(\lambda) \, d\lambda = \frac{\alpha}{16\pi^2} \frac{L_*}{a^2}.$$

(a) $Q(\lambda) = 1$

$$\begin{aligned} \int_0^\infty B_\lambda(T_g) \, Q(\lambda) \, d\lambda &= \frac{\sigma \, T_g^4}{\pi} = \frac{\alpha}{16\pi^2} \frac{L_*}{a^2} \\ \implies \quad T_g &= \left(\frac{\alpha \, L_*}{16\pi \, \sigma} \right)^{1/4} a^{-1/2} = 234.05 \, \text{K} \cdot \left(\frac{L_*}{L_\odot} \right)^{1/4} \left(\frac{a}{\text{AU}} \right)^{-1/2}. \end{aligned}$$

So, for the given data we have $T_g \approx 44$ K. Seems reasonable?

(b) $Q(\lambda) = \begin{cases} 0, & \lambda < \lambda_0 \\ \frac{\lambda_0}{\lambda}, & \lambda \geq \lambda_0 \end{cases}$

Let us first roughly estimate λ_{eff} using Wien's law:
$\lambda_{\text{eff}} = 2.9 \times 10^{-3}/44 \approx 66\,\mu\text{m} > \lambda_0 = 10\,\mu\text{m}$. This means that we can not use Rayleigh-Jeans approximation for the Plank curve $\ddot{\frown}$. The sanest (and, probably, expected) way would be to assume that the Plank curve for the grain is zero for all $\lambda < \lambda_0$ and substitute λ in Q with λ_{eff}; In this case

$$\int_0^\infty B_\lambda(T_g)\,Q(\lambda)\,\mathrm{d}\lambda \approx \int_0^\infty B_\lambda(T_g)\,\mathrm{d}\lambda \cdot \frac{\lambda_0}{\lambda_{\text{eff}}} = \frac{\lambda_0\,\sigma T_g^5}{\pi b}.$$

Therefore,

$$T_g = \left(\frac{\alpha\, b\, L_*}{16\pi\,\sigma\,\lambda_0}\right)^{1/5} a^{-2/5} = 244.25\,\text{K} \cdot \left(\frac{L_*}{L_\odot}\right)^{1/5} \left(\frac{a}{\text{AU}}\right)^{-2/5}.$$

So, for the given data we have $T_g \approx 64$ K. Not sure if it is reasonable...
If we try to compute everything exactly, it would be quite scary, but the final result is approximately the same:

$$\int_0^\infty B_\lambda(T_g)\,Q(\lambda)\,\mathrm{d}\lambda = \frac{2\,\lambda_0\,(kT)^5}{h^4\,c^3}\,\zeta(5)\,\Gamma(5)$$

$$\implies T_g = \left(\frac{\alpha\,L_*\,h^4\,c^3}{16\pi^2\,\zeta(5)\,\Gamma(5)\,k^5\,2\lambda_0}\right)^{1/5} a^{-2/5}$$

$$= 257.25\,\text{K} \cdot \left(\frac{L_*}{L_\odot}\right)^{1/5} \left(\frac{a}{\text{AU}}\right)^{-2/5},$$

i. e. for the data in the problem statement it would be ≈ 68 K, which does not differ much from our previous result.

Some things to remember (this was mentioned in the tutorial):
The observers usually use the light reaching the particle, $f = 1 - A$.
If you want to have a flux F at a certain distance d of an object with luminosity L,

$$F = \frac{L}{4\pi d^2}. \tag{3.3.19}$$

Also useful is the relation

$$\mathrm{d}F = \frac{\mathrm{d}E}{\mathrm{d}t\mathrm{d}A}, \tag{3.3.20}$$

and sometimes it's enough to consider the physical size $\mathrm{d}A = \pi s^2$

Problem 3.4: *Free-fall collapse of a sphere*

The free-fall time is the characteristic time that would take a body to collapse under its own gravitational attraction, if no other forces existed to oppose the collapse.

(a) Consider a homogeneous sphere of radius R_0 and density ρ_0. Set up the equation of motion for a piece of material initially at $r(t=0)=r_0$. Assume (justified a posteriori) that the mass originally contained inside this initial radius remains preserved inside the considered outer edge $r(t)$ of this part of the collapsing sphere,

$$M(r \leq r(t)) = M(r \leq r(t=0)). \tag{3.4.21}$$

(b) Show that the free fall time of a homogeneous sphere with initial density ρ_0 is given by

$$t_{\text{ff}} = \sqrt{\frac{3\pi}{32G\rho_0}}, \tag{3.4.22}$$

with G the gravitational constant.
Hint: A solution $y=y(t)$ of the differential equation $y''=f(y)$ can be written as

$$t = \pm \int_{y(0)}^{y(t)} \frac{\mathrm{d}y}{\sqrt{\left(2\int_{y(0)}^{y(t)} f(y)\mathrm{d}y\right) + (y'(0))^2}} \tag{3.4.23}$$

where the positive sign applies for $y'(0)>0$ and the negative sign for $y'(0)<0$.

(c) Discuss under which conditions the above eq. (3.4.22) for the free-fall time can be generalised to *non-homogeneous* spheres (where the density ρ is a function of r), when ρ_0 in eq. (3.4.22) is replaced by the *average* density within r at the initial time, $\langle\rho_0\rangle$.
Hint: Consider whether shells of material can fall faster or slower than in the case of homogeneous density.

(d) Calculate the free-fall time for a cloud with $\rho_0 = 4\times10^{-23}$ g cm^{-3}, corresponding to a slightly enhanced interstellar density.

Problem 3.4: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) The equation of motion for a test particle at radius r with enclosed mass $M(r)$ is

$$m\ddot{r} = -G\frac{mM(r)}{r^2} \tag{3.4.24}$$

In the freefall-regime, the mass collapses inwards, so the enclosed mass is the same as in the beginning when the sphere had the radius R_0. Thus, we can model the mass as

$$M(r) = M(R_0) = \rho_0 V(R_0) = \rho_0 \frac{4}{3}\pi R_0^3, \tag{3.4.25}$$

which leads us to

$$(3.4.24) \overset{(3.4.25)}{\Longrightarrow} \ddot{r} = -\frac{4}{3}G\pi\bar{\rho}\frac{R^3}{r^2} =: -\frac{\alpha}{r^2}, \tag{3.4.26}$$

where we defined α to make the following derivation less ugly.

(b) To make this equation solvable, we use a trick (which is also hinted at in the exercise, but let's derive it).
According to the product rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}^2r}{\mathrm{d}t^2} + \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\frac{\mathrm{d}r}{\mathrm{d}t} \iff \ddot{r} = \frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2. \tag{3.4.27}$$

Using this, we can rewrite our differential equation (3.4.26) and integrate w. r. t. r (using the boundaries $r(t=0) = R_0$ and $r = r(t)$):

$$\frac{\mathrm{d}}{\mathrm{d}r}\dot{r}^2 = -\frac{2\alpha}{r^2} \implies \dot{r}^2 = \frac{2\alpha}{r'}\Big|_{r'=R_0}^{r'=r} = \frac{2\alpha}{r} - \frac{2\alpha}{R_0} \implies \frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{2\alpha(r^{-1} - R_0^{-1})}. \tag{3.4.28}$$

Here, we also assumed $\dot{r}(t=0) = 0$.
The sign depends on the way we are looking at the problem. If we start at R_0, the collapsing mass (or the particle 'on top of it') moves away from us, so we can choose the negative sign.
Separation of variables then yields

$$\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}t} &= \sqrt{2\alpha}\sqrt{\frac{1 - r/R_0}{r}} \\
\iff \quad \int_0^{R_0}\sqrt{\frac{r}{1 - r/R_0}}\mathrm{d}r &= \int_0^{t_{\mathrm{ff}}}\sqrt{2\alpha}\mathrm{d}t && (3.4.29) \\
\iff \quad \frac{\pi}{2}R_0\sqrt{R_0} &= t_{\mathrm{ff}}\sqrt{2\alpha} && (3.4.30) \\
\iff \quad t_{\mathrm{ff}} &= \frac{\pi}{2}R_0\sqrt{R_0}\sqrt{\frac{3}{8\pi G\bar{\rho}R_0^3}} = \sqrt{\frac{3\pi}{32G\bar{\rho}}}. && (3.4.31)
\end{aligned}$$

In the second step (eq. (3.4.29) to eq. (3.4.30)), we used a very complex integral identity which is shown in the appendix (sec. 3.6).

(c) A typical assumption would be that the inner region of the cloud would have higher densities. Since $t_{\mathrm{ff}} \propto \rho^{-1/2}$, they would have shorter free-fall times. In these cases, it would be alright to just replace ρ_0 with $\langle\rho\rangle$.

(d) The free-fall time of the cloud with $\rho_0 = 4 \times 10^{-23}$ g cm^{-3} is $t_{\mathrm{ff}} = 10.53$ Myr.

Problem 3.5: *Initial mass function of an open cluster*

When stars form, the distribution of their masses follows an initial mass function (IMF) $\frac{\mathrm{d}N}{\mathrm{d}m}$, which gives the differential number of stars dN in a mass interval $m \ldots m + \mathrm{d}m$) as

$$\mathrm{d}N = \frac{\mathrm{d}N}{\mathrm{d}m}\mathrm{d}m. \tag{3.5.32}$$

Salpeter (1955) determined the initial mass function as a power-law in mass,

$$\frac{\mathrm{d}N}{\mathrm{d}m} = am^{-\alpha}, \tag{3.5.33}$$

where a is an amplitude for normalisation to the total stellar mass inside a volume and the exponent is $-\alpha = -2.35$.
Consider an open cluster, in which a total stellar mass $10^3 M_\odot$ has formed instantaneously following a Salpeter initial mass function in the mass range $0.1M_\odot \dots 20M_\odot$ (with no stars outside that mass range; why is that a reasonable assumption?).

- Find the normalisation constant a.
- Find the initial total luminosity of the cluster, assuming that all its stars are on the main sequence, and a mass–luminosity relation $L \propto M^4$. What fraction of the initial luminosity is contributed by stars more massive than $5M_\odot$? What fraction of the number of stars is in this high mass end?
- Find the initial mean mass of stars in the cluster.
- Assume that the main-sequence lifetime of a $1M_\odot$ star is 10 Gyr, and main-sequence lifetime scales with mass as M^{-2}. From your observations of the cluster, you know that the brightest main sequence stars that still exist are about 100 times more luminous than the sun. How long ago did the cluster form its stars? What fraction of its initial luminosity does the cluster have today?

Problem 3.5: Solution

Assuming $M_{\mathrm{min}} = 0.1M_\odot$ and $M_{\mathrm{max}} = 20M_\odot$ is reasonable for an open cluster since stars with lower masses should not be found as open clusters usually disperse after quite a short time and stars with lower mass wouldn't reach the Main Sequence. Higher mass stars could be considered, but then they could maybe contribute too much. This might also be related to the Eddington limit.
Note: If we had set $N_{\mathrm{tot}} = 1$ and used $1 = N_{\mathrm{tot}} = \int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} bm^{-\alpha}\mathrm{d}m$, we would obtain the normalisation constant b for a fractional IMF.

- Since we don't know N_{tot} a priori, we need to employ another way of finding the normalisation constant:
  Because we know the total mass $M_{\mathrm{tot}} = 10^3 M_\odot$ of the cluster, can look at the distribution in terms of mass to determine a:

$$M_{\mathrm{tot}} \overset{!}{=} \int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} amm^{-\alpha}\mathrm{d}m = \frac{a}{2-\alpha}\left(M_{\mathrm{max}}^{2-\alpha} - M_{\mathrm{min}}^{2-\alpha}\right) \tag{3.5.34}$$

$$\iff a = \frac{M_{\mathrm{tot}}}{M_{\mathrm{max}}^{2-\alpha} - M_{\mathrm{min}}^{2-\alpha}}(1-\alpha) = \frac{-0.35M_{\mathrm{tot}}}{20^{-0.35} - 0.1^{-0.35}} M_\odot^{0.35} \tag{3.5.35}$$

$$= 0.18535 M_{\mathrm{tot}} M_\odot^{0.35} = 185.35 M_\odot^{1.35}. \tag{3.5.36}$$

- Using the scaling relation $L \propto M^4$ that was provided and the fact that the Sun is a Main-Sequence-star as well, we have

$$L_{\text{tot}} = \int_{M_{\min}}^{M_{\max}} am^4 \frac{L_\odot}{M_{\text{odot}}^4} m^{-\alpha} \mathrm{d}m = \frac{a}{5-\alpha} \left(M_{\max}^{5-\alpha} - M_{\min}^{5-\alpha} \right) \frac{L_\odot}{M_\odot} = 1.96 \times 10^5 L_\odot. \tag{3.5.37}$$

To find the luminosity the stars with $M \geq 5M_\odot =: M_0$ contribute, we need to replace the lower boundary of the integral:

$$\frac{L_{\text{massive}}}{L_{\text{tot}}} = \frac{1}{L_{\text{tot}}} \int_{M_0}^{M_{\max}} a \frac{L_\odot}{M_{\text{odot}}^4} m^{4-\alpha} \mathrm{d}m = 0.97. \tag{3.5.38}$$

To find the fraction of stars that make up for this luminosity, we have to employ the relation for N, use the cutoff and also divide by the total number.
Here, we find

$$\frac{N_{M \geq 5M_\odot}}{N_{\text{tot}}} = \frac{\int_{M_0}^{M_{\max}} am^{-\alpha} \mathrm{d}m}{\int_{M_{\min}}^{M_{\max}} am^{-\alpha} \mathrm{d}m} = \frac{13.2}{3071.4} = 0.43\,\%, \tag{3.5.39}$$

so we see that even though these stars contribute a majority of the luminosity, they only make up a very small fraction of the total population.

- The mean mass is simply

$$\bar{m} = \frac{M_{\text{tot}}}{N_{\text{tot}}} = 0.33 M_\odot. \tag{3.5.40}$$

- From the scaling relations for the lifetime τ, the mass M and the luminosity L of main sequence stars, we can derive that

$$L \propto M^4 \iff M \propto L^{\frac{1}{4}}, \quad \tau \propto M^{-2} \implies \tau \propto L^{-\frac{1}{2}}. \tag{3.5.41}$$

Using that a star of solar mass would have one solar luminosity (related to the lifetime $\tau_\odot \approx 10$ Gyr) and that the brightest (and therefore oldest) star of the cluster has $L_{\text{OC,max}} = 100 L_\odot$, for the lifetime of the cluster we have

$$\tau_{\text{OC}} = \frac{\tau_\odot}{\sqrt{L_{\text{OC,max}}/L_\odot}} = \frac{10 \text{ Gyr}}{\sqrt{100}} = 1 \text{ Gyr}. \tag{3.5.42}$$

To find the fraction of its initial luminosity, we can use the relation for τ to first find out the maximum mass of the remaining strs, which is

$$M_{\text{OC,max}} = (\tau_{\text{OC}}/\tau_\odot)^{-1/2} M_\odot = \sqrt{10} M_\odot \approx 3.16. \tag{3.5.43}$$

Now, we simply integrate to this mass to find

$$\frac{L_{\text{now}}}{L_{\text{tot}}} = \frac{1}{L_{\text{tot}}} \int_{M_{\min}}^{M_{\text{OC,max}}} a \frac{L_\odot}{M_{\text{odot}}^4} m^{4-\alpha} \mathrm{d}m = 0.75\,\%, \tag{3.5.44}$$

so the luminosity has (without much surprise) decreased drastically in the last 1 Gyr.

3.6 Appendix

Here, we show that

$$\int_0^{R_0} \sqrt{\frac{r}{1 - r/R_0}} \mathrm{d}r = \frac{\pi}{2} R_0 \sqrt{R_0}. \tag{3.6.1}$$

First, we observe that we can factor out $\sqrt{R_0}$ and substitute $z := \sqrt{r}$ (so $\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{1}{2\sqrt{r}} \iff \mathrm{d}r = 2\sqrt{r}\mathrm{d}z = 2z\mathrm{d}z$):

$$\int_0^{R_0} \sqrt{\frac{r}{1 - r/R_0}} \mathrm{d}r = \sqrt{R_0} \int_0^{R_0} \sqrt{\frac{r}{R_0 - r}} \mathrm{d}r \overset{z=\sqrt{r}}{=} \sqrt{R_0} \int_0^{\sqrt{R_0}} \frac{2z^2 \mathrm{d}z}{\sqrt{R_0 - z^2}}. \tag{3.6.2}$$

The part in the square root looks like a trigonometric substitution could work.
If we set $z = \sqrt{R_0}\sin(u)$[1], the denominator becomes $R_0\sqrt{1 - \sin^2(u)} = r\cos(u)$, while our upper integration border becomes $\arcsin(1) = \frac{\pi}{2}$.
The integral is thus

$$2\sqrt{R_0} \int_0^{\frac{\pi}{2}} \frac{R_0 \sin^2(u)\cos(u)\mathrm{d}u}{\cos(u)} = 2\sqrt{R_0}R_0 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos(2u))\mathrm{d}u, \tag{3.6.3}$$

where we used the trigonometric identity

$$\sin^2(x) = \frac{1}{(2i)^2}(e^{ix} - e^{-ix})^2 = -\frac{1}{4}(e^{i2x} + e^{-i2x} - 2) = \frac{1}{2}(1 - \cos(2x)). \tag{3.6.4}$$

This last integral is easy:

$$2\sqrt{R_0}R_0 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos(2u))\mathrm{d}u = 2\sqrt{R_0}R_0 \frac{1}{2}\left(\frac{\pi}{2} - \frac{\sin(\pi) - \sin(0)}{2}\right) = \frac{\pi}{2}\sqrt{R_0}R_0. \tag{3.6.5}$$

There, we have it! □

[1] so $\mathrm{d}z = \sqrt{R_0}\cos(u)\mathrm{d}u$

Problem set #4 Exoplanet detection (here)

01.12.2022

Problem 4.1: *Radial velocity variations caused by planets*

(a) Derive the amplitude of the radial velocity variation of the host star of a transiting extra-solar planet ($i = 90°$) as a function of period and planetary and stellar mass. Use Kepler's third law in the following approximations:

$$a_{\text{star}} \ll a_{\text{planet}} \tag{4.1.1}$$

$$M_{\text{planet}} \ll M_{\text{star}} \tag{4.1.2}$$

(Hint: use the centre of mass theorem and the above approximations to derive a_{planet} as a function of M_{star} and P.)

(b) Calculate the amplitude of the Sun's radial velocity variation caused by Earth, Jupiter, and Saturn.
Use the following values:

$$\begin{aligned} G &= 6.67 \times 10^{11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2} \\ M_{\text{Earth}} &= 5.97 \times 10^{24}\,\text{kg} \\ M_{\text{Saturn}} &= 5.68 \times 10^{26}\,\text{kg} \\ M_{\text{Jupiter}} &= 1.90 \times 10^{27}\,\text{kg} \\ M_{\text{Sun}} &= 1.99 \times 10^{30}\,\text{kg} \\ p_{\text{Jupiter}} &= 11.86\,\text{yr} \\ p_{\text{Saturn}} &= 29.46\,\text{yr} \end{aligned}$$

Problem 4.1: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) The idea to derive the velocity is to first find the orbital period time P and use this as the orbital period of the star around the common centre of mass.
The orbital period time is given by Kepler's third law:

$$P^2 \approx \frac{4\pi^2 a_{\text{p}}^3}{G(M_{\text{star}} + M_{\text{planet}})} \iff a_{\text{p}} = \sqrt[3]{\frac{G(M_{\text{s}} + M_{\text{p}})P^2}{4\pi^2}}. \tag{4.1.3}$$

The centre of mass of the planet and star is along the line connecting them with distances a_1 and a_2, denoting the distances of the CMS to star and planet, respectively.
Therefore, we have

$$a_1 + a_2 = a_{\text{p}}, \quad M_{\text{s}} a_1 \stackrel{!}{=} M_{\text{p}} a_2 = (a_{\text{p}} - a_1) M_{\text{p}} \iff a_1 = \frac{M_{\text{p}}}{M_{\text{p}} + M_{\text{s}}} a_{\text{p}}. \tag{4.1.4}$$

Using these two equations and the fact that the change in radial velocity we observe is twice the maximum circular velocity of the star around its centre of mass achieved (since it's both radially receding and approaching[a]), we have

$$\begin{aligned}
\Delta v_\mathrm{r} &= 2v_\mathrm{circ} = 2\frac{2\pi a_1}{P} \\
&= \frac{4\pi}{P}\frac{M_\mathrm{p}}{M_\mathrm{p}+M_\mathrm{s}}a_\mathrm{p} \\
&= M_\mathrm{p}\sqrt[3]{\frac{G4^3\pi^3}{4\pi^2(M_\mathrm{s}+M_\mathrm{p})^2P}} \\
&\approx \underline{\underline{M_\mathrm{p}\sqrt[3]{\frac{G16\pi}{M_\mathrm{s}^2P}}}},
\end{aligned} \tag{4.1.5}$$

where we used $M_\mathrm{p} \ll M_\mathrm{s}$ for the last approximation.

(b) Putting in the numbers, we find

Name	Mass [kg]	Period [yr]	Δv [m/s]	$\Delta\lambda_{\mathrm{H}\alpha}$ [nm]
Earth	5.97×10^{24}	1	0.1651	3.61×10^{-7}
Jupiter	1.89×10^{27}	11.86	22.99	5.03×10^{-5}
Saturn	5.68×10^{26}	29.46	5.082	1.11×10^{-5}

The last column converts the difference in radial velocity to the difference in observed Hα wavelength (with $\lambda_{\mathrm{H}\alpha} = 656.281$ nm) - it's insanely minute!

[a] in the tutorials, we also discussed this and argued that this factor of 2 might not be justified, but it doesn't change the final result that much.

Problem 4.2: *Transit detection*

What fraction of sunlight is blocked when Earth passes in front of the Sun and how large is the decrease in brightness expressed in magnitudes? Here it shall be assumed that the observer resides far outside the solar system and that the Sun and the planets appear as uniform disks. How large is the effect for Mercury, Jupiter, and Neptune?
(Magnitudes are a logarithmic measure of radiative flux customarily used in observational astronomy. The magnitude difference between two fluxes F_1 and F_2 is defined as

$$\Delta m_{12} = -2.5\log\left(\frac{F_1}{F_2}\right). \tag{4.2.6}$$

Note that, being logarithmic, a *difference* in magnitude corresponds to a ratio in flux.)

Problem 4.2: Solution

Note: The code used for parts of this exercise is available in **this repository**.
Assuming we are in a wavelength band the planet is completely opaque to, ignoring edge effects, and assuming that we are sufficiently far away for plane parallel approximation, we can just use the radii of the star and the planet to calculate the circular areas the

discs cover, leading to a fraction of

$$f = \frac{\pi r_\mathrm{p}^2}{\pi r_\mathrm{s}^2}. \tag{4.2.7}$$

Since the flux is proportional to the area, the relative change in flux would just be

$$\frac{F_\text{with planet}}{F_\text{without planet}} = 1 - \frac{r_\mathrm{p}^2}{r_\mathrm{s}^2} \implies \Delta m_\text{planet} = \underline{\underline{-2.5 \log\left(1 - \frac{r_\mathrm{p}^2}{r_\mathrm{s}^2}\right)}}. \tag{4.2.8}$$

For Mercury, Earth, Jupiter, and Neptune, we therefore have

Name	Radius [km]	Δm_planet [mag]
Mercury	2.43×10^3	1.33×10^{-5}
Earth	6.37×10^6	9.12×10^{-5}
Jupiter	7.14×10^7	1.15×10^{-2}
Neptune	2.46×10^4	1.36×10^{-3}

The changes in magnitude are quite small!

Problem 4.3: *Transit probability*

Show that the probability p for a suitable orientation of a planet's orbital plane to allow observing a transit is given by the simple formula

$$p = \frac{R_\star}{a}, \tag{4.3.9}$$

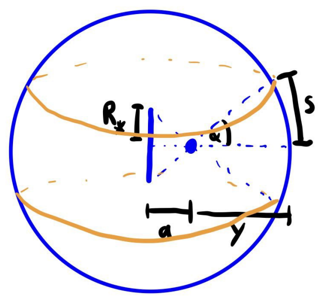
where $R_\star$ is the radius of the star and a is the planet's orbital radius. How large are the probabilities for an alien observer to be in a position allowing the observation of transits of Mercury, Earth, Jupiter, Neptune?

Problem 4.3: Solution

Note: The code used for parts of this exercise is available in **this repository**.

The transit probability has excellently been derived in **Borucki et al. 1984**, which will be reproduced here:

As the basis for this derivation, we note that the probability of a transit being observed corresponds to the fraction the 'shadow' of the planet covers on the celestial sphere.



Therefore, we can use basic geometry to see that, with the distance between star and planet a, the distance to an observer y and the projected height of the shadow s, the area of the 'ring' the shadow covers on the celestial sphere is $A_\text{Ring} = 2 \cdot 2\pi(a+y)s$, while the celestial sphere itself has an area of $4\pi(a+y)^2$.

Thus, we first have

$$p = \frac{2 \cdot 2\pi(a+y)s}{4\pi(a+y)^2}, \tag{4.3.10}$$

and we can identify $R_\star = a\tan\alpha \approx a\alpha$ and thus $s = y\tan\alpha \approx y\alpha = \frac{yR_\star}{a}$, so we find

$$p = \frac{yR_\star}{(a+y)a} \overset{y\to\infty}{\to} \underline{\underline{\frac{R_\star}{a}}}. \tag{4.3.11}$$

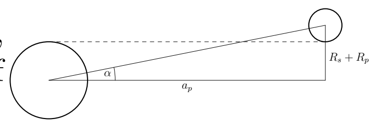
Name	a [m]	probability
Mercury	5.79×10^{10}	1.20×10^{-2}
Earth	1.49×10^{11}	4.65×10^{-3}
Jupiter	7.78×10^{11}	8.93×10^{-4}
Neptune	4.49×10^{12}	1.54×10^{-4}

Problem 4.4: *Transit duration*

Derive a general expression for the duration of a transit across the centre of a star by using Kepler's third law. Assume that the observer is at a very large distance and that the orbits are circular. The formula should give the transit duration t in hours, when the mass M_s of the star is given in $M_\odot$, the radius R_s of the star in $R_\odot$, and the planet's orbital radius a_p in AU. How long does a central transit of Mercury, Earth, Jupiter, and Neptune last? How long does the transit last relative to the total orbital period of the planet?

Problem 4.4: Solution

Note: The code used for parts of this exercise is available in **this repository**.
Since we assume circular orbits and a central transit, the distance the planet travels in front of the host star is $d = 2R_\mathrm{s} + 2R_\mathrm{p} \approx 2R_\mathrm{s}$ where we can neglect $R_\mathrm{p} \ll R_\mathrm{s}$ (otherwise, we could argue that the transit starts and ends once half of the planet is in front of the star).



Using Kepler's third law, neglecting the planets mass, and assuming a circular orbit, we find the velocity of the planet to be

$$P^2 = \frac{4\pi a_\mathrm{p}^3}{(M_\mathrm{s}+M_\mathrm{p})G} \implies v = \frac{2\pi a_\mathrm{p}}{P} \approx 2\pi a_\mathrm{p}\sqrt{\frac{GM_\mathrm{s}}{4\pi a_\mathrm{p}^3}}. \tag{4.4.12}$$

Therefore, the time of the transit is approximately

$$t = \frac{d}{v} \approx 2R_\mathrm{s}\sqrt{\frac{a_\mathrm{p}}{GM\pi}} \approx 12.976\,(R_s/R_\odot)\sqrt{\frac{a_\mathrm{p}/\mathrm{au}}{M/M_\odot}}\ \mathrm{h}. \tag{4.4.13}$$

Applying this for the planets, we have

Name	a [m]	t [h]	P [h]	t/P
Mercury	5.79×10^{10}	8.074	2.11×10^{3}	3.82×10^{-3}
Earth	1.49×10^{11}	12.98	8.76×10^{3}	1.48×10^{-3}
Jupiter	7.78×10^{11}	29.6	1.04×10^{5}	2.84×10^{-4}
Neptune	4.49×10^{12}	71.16	1.44×10^{6}	4.92×10^{-5}

Problem 4.5: *Astrometric precision*

How large is the astrometric wobble (in arcseconds) of a Sun-like star at a distance of 10 pc that harbours an Earth-like planet? How large is a structure on the Moon that has the same angular extent when seen from earth? Why can only interferometers making use of *differential techniques* achieve the required precision to detect such planets?
(What is the resolution of a conventional optical interferometer with a baseline of 1 km?)

Problem 4.5: Solution

Thanks and credits to Iliya Tikhonenko!
If we assume that there are no other planets in the system, we can derive the radius of Sun's "orbit" using the center of mass formula for just two bodies (placing the origin of the coordinate frame at the barycenter):

$$\boldsymbol{r}_c = \frac{\boldsymbol{r}_\odot M_\odot + \boldsymbol{r}_e M_e}{M_\odot + M_e} = 0 \;\Rightarrow\; \Delta = 2a_\odot = 2\,a_e\,\frac{M_e}{M_\odot} \approx 6 \times 10^{-6}\,\mathrm{AU} \tag{4.5.14}$$

Remembering the definition of 1pc, we can trivially obtain the wobble in arcseconds:

$$\delta = \frac{\Delta}{r} = \frac{6 \times 10^{-6}}{10} = 6 \times 10^{-7} = 0.6\,\mu as, \tag{4.5.15}$$

which is very, *very* small. The structure with this angular size observed on the Moon would be only $6 \times 10^{-7} \cdot 206265^{-1} \cdot 384\,400\,\mathrm{km} \approx 1\,\mathrm{mm}$ long. For example, that one 1 cent coin (16.25 mm in diameter) that you lost somewhere 4 years ago would be about 9 mas from the Moon, i.e. more than 15 times larger.
For a conventional optical interferometer, the angular resolution can be estimated by the following relation:

$$\delta^{(\text{in radians})} = \frac{\lambda}{2b}, \tag{4.5.16}$$

where b stands for baseline; essentially it means that we should have a phase difference equal to π for the longest possible value of the baseline. For an interferometer with 1 km baseline observing at e.g. $\lambda_0 = 600\,\mathrm{nm}$, the resolution would be $3 \times 10^{-10}\,\mathrm{rad} \approx 62\,\mu\mathrm{as}$ which is obviously still not enough for our case. In the differential interferometry the same star is used as a reference for itself at multiple wavelengths, which somehow improves the resolution, but I'm not sure I understand how it works, so maybe we should discuss it further on the tutorial. **Monnier (2003)** gives a nice overview of the subject, but the paper is relatively old.

Problem 4.6: *The solar nebula*

Note: The code used for parts of this exercise is available in **this repository**. Most of the material within the original solar nebula from which the solar system formed has been lost. Here the original mass of this nebula shall be derived by assuming that the elements are in the cosmic abundance ratios (see table).

	H	He	O	Mg	Si	Fe
Rel. number of atoms	1	0.08	0.0007	4e-05	4e-05	3e-05
Atomic weight	1	4	16	24.3	28.1	55.8

With the further assumption that the heavy elements are completely bound in rocks (represented by $\mathrm{Mg_2SiO_4}$, $\mathrm{Fe_2SiO_4}$, and $\mathrm{SiO_2}$) and water ice ($\mathrm{H_2O}$), determine the ratio between the total masses of rocks, water ice, and gases (H, He).
If the terrestrial planets together contain $2M_\oplus$ of rocks, Jupiter and Saturn both have rocky cores of about $10M_\oplus$ and the rocky masses of Uranus and Neptune are $3M_\oplus$ each, what was the minimum solar nebula mass required?
Compare this with the actual mass of the planets and derive a planet formation efficiency for the solar system.

Problem 4.6: Solution

To start this analysis, we need to figure out the relative fractions of the rocky elements. This can be done by realising that the number densities of some of the elements are completely locked by one of the atomic species they consist of (e. g. magnesium for $\mathrm{Mg_2SiO_4}$) as these do not appear in any other elements.
Therefore, we can find the relative number densities of the molecules in an iterative way:

$$n_{\mathrm{Mg_2SiO_4}} \overset{!}{=} \frac{1}{2} n_{\mathrm{Mg}} \tag{4.6.17}$$

$$n_{\mathrm{Fe_2SiO_4}} \overset{!}{=} \frac{1}{2} n_{\mathrm{Fe}} \tag{4.6.18}$$

$$\implies n_{\mathrm{SiO_2}} \overset{!}{=} n_{\mathrm{Si}} - n_{\mathrm{Mg_2SiO_4}} - n_{\mathrm{Fe_2SiO_4}} \tag{4.6.19}$$

$$\implies n_{\mathrm{H_2O}} \overset{!}{=} n_{\mathrm{O}} - 2n_{\mathrm{SiO_2}} - 4n_{\mathrm{Mg_2SiO_4}} - 4n_{\mathrm{Fe_2SiO_4}} \tag{4.6.20}$$

$$\implies n_{\mathrm{H}} = n_{\mathrm{H,\ tot}} - 2n_{\mathrm{H_2O}} \tag{4.6.21}$$

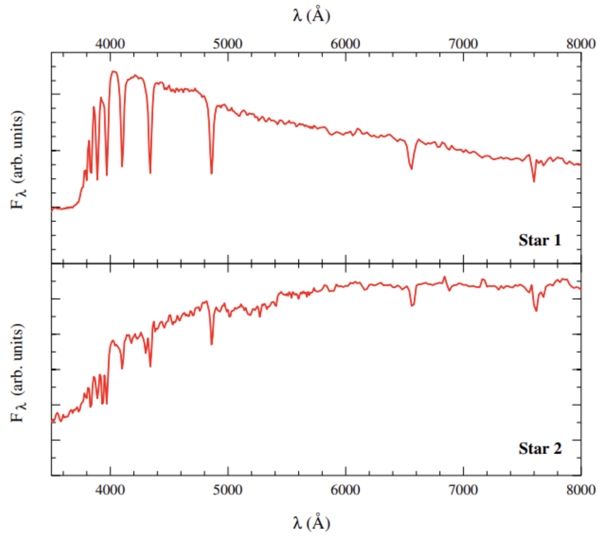
TO DO Actually finish this exercise lol (!)

Problem set #5 I am a star! ⋆ (here)

08.12.2022

Problem 5.1: *Flux and colour with filters*

Remember that $F_\lambda(\lambda)$ is defined such that $F_\lambda(\lambda)\Delta\lambda$ is equal to the flux (rate that energy is collected by a 1 m^2 telescope) arriving at wavelengths between λ and $\lambda + \Delta\lambda$. Consider the two stars whose spectra are plotted below.



(a) Suppose you can make a filter that is exactly 1 Å wide, that is, it will only transmit colours between λ_0 and $\lambda_0 + 1$ Å. If you were designing this filter to transmit the maximum amount of energy possible from Star 1, at what wavelength would you choose to make the λ_0 of this filter?

(b) Consider the B and R filters[1]. Will Star 1 have a greater flux through the B filter, or through the R filter? What about Star 2?

(c) Which of the two stars will have a large value of B - R? (B and R are the fluxes through the filters, measured in magnitudes.)

(d) It turns out that Star 1 will be whitish in colour. What would Star 2 look like to your eyes?

Problem 5.1: Solution

(a) To get the maximum flux from star one in a 1 Å wide wavelength interval, we should use $\lambda_0 \approx 4000$ Å as its flux peaks here.

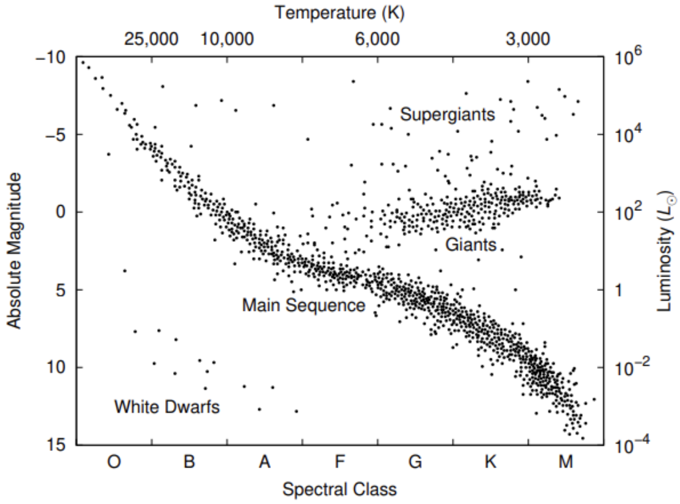
(b) The B filter is centred around 4450 Å, while the R filter is centred around 6580 Å, both with a width of around 1000 Å. Therefore, Star 1 would have a greater flux in the B than in the R filter, while Star 2 would have a greater flux in the R than in the B filter.

[1] If this has not yet been discussed in the lectures then see, for example, Standard Photometric Systems by Michael S. Bessell, Annual Review of Astronomy and Astrophysics 43, 293 (2005), **accessible here**

(c) The $B - R$ value of Star 1 should be much larger, while the $B - R$ value of Star 2 is slightly negative.
(d) If Star 1 looks whitish in colour, Star 2 should look either yellow, orange or red.

Problem 5.2: *HRD exploration*

Consider the following Hertzsprung-Russell-Diagram:



(a) You observe a star (Star A) that is bluish in colour. You measure its spectrum, and find that the spectrum F_λ reaches a peak at an ultraviolet wavelength of 1500 Å. What is the temperature and spectral type of this star?
(b) You observe a star (Star B) that is reddish in colour. You measure its spectrum, and find that the spectrum F_λ reaches a peak at a red/infrared wavelength of 7800 Å. What is the temperature and spectral type of this star?
(c) Star A and Star B have the same observed brightness. You measure the parallax of Star A to be $0.0008''$, and the parallax of Star B to be $0.080''$. What is the ratio of luminosities of the two stars?
(d) If you determine that Star A is a main sequence star, what kind of star (white dwarf, main sequence, giant, or supergiant) is Star B? Explain.

Problem 5.2: Solution

(a) We can use Wien's displacement law to find the peak temperatures of the stars, finding

$$T = 2.897\,\frac{\mu\text{m K}}{\lambda} \implies \underline{\underline{T_A = 19\,000\ \text{K}}}. \tag{5.2.1}$$

Therefore, star A should be a B type star.
(b) From Wien's displacement law, we find

$$\underline{\underline{T_B = 3\,700\ \text{K}}}, \tag{5.2.2}$$

so this should be a K star.

(c) Since $m_A = m_B$, we can use that

$$m_A - m_B = 0 = -2.5 \log\left(\frac{F_A}{F_B}\right) \implies F_A = F_B. \tag{5.2.3}$$

Since the luminosity of the stars is proportional via $F = \frac{L}{4\pi d^2}$ where d is the distance, we find that

$$\frac{L_A}{L_B} = \frac{d_A^2}{d_B^2} \approx \underline{\underline{10\,000}}, \tag{5.2.4}$$

where we used the the relation for $d_{[\mathrm{pc}]} = \frac{1\,\mathrm{au}}{p_{['']}}$.

(d) If star A is a main sequence star, its luminosity would be $L_A \approx 10^3 L_\odot \implies L_B \approx 10^{-1} L_\odot$, so star B could also be a main sequence star.

Problem 5.3: *Absorption by the interstellar medium*

(a) Show that the apparent magnitude m grows linearly with distance r in an absorbing medium if the intensity of the radiation declines exponentially. What is the value for the constant C in the equation below if the intensity of the radiation declines 10 % on a distance of 1 pc?

$$m - m_0 = Cr. \tag{5.3.5}$$

(b) The apparent V-band magnitude of a star is $V = 14.8$, the colour is $B - V = 1.4$ and the absolute magnitude is $M_V = 1.0$. Towards the star the extinction at visual wavelengths is $a_V = 1$ mag/kpc. Calculate the intrinsic (extinction-corrected) colour $(B - V)_0 = (B - V) - E_{B-V}$ of the star. To solve the problem one needs to iterate (you can write a short program for this) or use a numerical equation solver (e. g. Mathematica).

Problem 5.3: Solution

(a) The intensity after the extinction is $I_1 = 0.9 I_0$ after a decline on a distance of 1 pc, and we have

$$m_1 - m_0 = -2.5 \log\left(\frac{I_0}{I_1}\right). \tag{5.3.6}$$

We can use that the intensity in the absorbing medium depends on r as $I_1 = I_0 e^{-\tau r}$, we have

$$m_1 - m_0 = -2.5 \log\left(e^{-\tau r}\right) = \frac{\tau r}{\ln(10)}. \tag{5.3.7}$$

Putting in the numbers, we find $0.9 I_0 = I_0 e^{-\tau 1\,\mathrm{pc}} \iff \tau = 0.1054 \frac{1}{\mathrm{pc}}$ and therefore

$$C = \frac{\tau}{\ln(10)} = \frac{0.15}{\mathrm{pc}}. \tag{5.3.8}$$

(b) Using $m_V = 14.8$, a colour of $m_B - m_V = 1.4$ and $M_V = 1.0$, and an extinction of $a_V = 1$ mag/kpc, we use the relation

$$M = m_0 + 5 - \log r \tag{5.3.9}$$

to obtain the absolute magnitude, and

$$m_V = m_{V_0} + A_V = m_{V_0} + a_V r \tag{5.3.10}$$

to get the initial extinction corrected magnitude m_{V_0}. Putting these equation together, we have

$$M_V = m_V - a_V r + 5 - \log r. \tag{5.3.11}$$

This equation can iteratively be solved for r, yielding $r \approx 2.14$ kpc and therefore $A_V = a_V r = 2.14$.
Remembering from the lecture that the colour excess E_{B-V} and A_V are related via

$$R = \frac{A_V}{E_{B-V}} = \frac{A_V}{A_B - A_V} \approx 3.1, \tag{5.3.12}$$

we find $A_B - A_V \approx 0.7$.
Therefore, we have

$$(B - V)_0 = m_B - m_V - E_{B-V} = 1.4 - 0.7 = \underline{\underline{0.7}}. \tag{5.3.13}$$

Problem 5.4: *Timescales*

(a) Hydrogen fusion has an efficiency of about 0.7 % for converting mass into energy. Assume that the Sun will use 10 % of its hydrogen for fusion (and that it is mostly hydrogen). Given the Sun's luminosity, how long will it shine? How many kg of mass is the Sun losing each second by converting mass into energy?

(b) We know that the Sun is not "on fire" because chemical reactions are not nearly efficient enough to keep the Sun shining at its current luminosity for anything like the amount of time we know it's been around. The efficiency of chemical burning of hydrogen in the reaction: $2H_2 + O_2 \rightarrow 2H_2O$ is about 2×10^{-10}: that is the fraction of its mass that gets converted into energy. Assuming that the Sun were made up of oxygen and hydrogen in just the right proportions, and that it was able to burn all of its mass, how long (for how many years) would it be able to shine at its current luminosity using chemical reactions?

(c) Another possible energy source is the energy released from gravitational contraction. When you drop something from a height, energy is released; you may use that energy to make a sound, break something, etc. Suppose that you consider all the mass of the Sun to have been *dropped* from a great distance on to the Sun. The total energy released is approximately GM^2/R, where M is the mass of the Sun and R is the radius of the Sun. If this were where the Sun got its energy, for how long would it have been able to shine at its current luminosity?

Problem 5.4: Solution

(a) If $f_m = 0.1$ of the mass of the sun is converted to energy with the efficiency $\eta = 0.007$, we the total energy output will be

$$E_{\text{fus}} = \eta f_m M_\odot c^2 \approx 1.25 \times 10^{44}\ \text{J}. \tag{5.4.14}$$

We can estimate the lifetime by using the solar luminosity ($L_\odot = 3.83 \times 10^{26}$ W) and just divide as follows:

$$t_{\text{fus}} = \frac{E_{\text{fus}}}{L_\odot} \approx 10^{10}\ \text{yr}. \tag{5.4.15}$$

We can estimate the mass loss per second using the relation

$$E_{\text{fus}} = mc^2\eta \implies \dot{m} = \frac{\dot{E}}{\eta c^2} = \frac{L_\odot}{\eta c^2} = \underline{\underline{6 \times 10^{11}\ \frac{\text{kg}}{\text{s}}}}. \tag{5.4.16}$$

According to **WolframAlpha**, this is about the amount of methane generated by humans each year every second!

(b) If we *burned* the entire Sun using all hydrogen and oxygen available, and assuming perfect mass ratios, and using the efficiency $\eta_{\text{burn}} = 2 \times 10^{-10}$, we have

$$E_{\text{burn}} = \eta_{\text{burn}} M_\odot c^2 = 3.6 \times 10^{37}\ \text{J} \implies t_{\text{burn}} = \frac{E_{\text{burn}}}{L_\odot} \approx \underline{\underline{3000\ \text{yr}}}. \tag{5.4.17}$$

This is way too short (and also, we had very unrealistic assumptions).

(c) Here, we use the gravitational energy:

$$E_{\text{grav}} = \frac{GM_\odot^2}{R_\odot} = 4 \times 10^{41}\ \text{J} \implies t_{\text{grav}} = \frac{E_{\text{grav}}}{L_\odot} = \underline{\underline{32\ \text{Myr}}}. \tag{5.4.18}$$

This would also be too short to e. g. explain the lifetime of the Earth found via carbon dating.

Problem 5.5: *The Eddington limit*

(a) By using the appropriate equations of stellar structure and the (isotropic) radiation pressure $P_{\text{rad}} = aT^4/3$ with $a = 4\sigma B/c$, calculate the differential $\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}P_{\text{tot}}}$ as a function of local luminosity $L(r)$ and mass $M(r)$. (Assume radiative energy transport, as (mostly) true for the outer regions of massive stars.) From this result, prove the inequality

$$L < \frac{4\pi c G M}{\kappa} = \frac{4\pi c G M}{\alpha/\rho}, \tag{5.5.19}$$

with total luminosity L, total mass M, the speed of light c, the gravitational constant G and the mean (Rosseland) mass absorption coefficient κ (i. e. opacity per unit mass).

(b) The right-hand side of the above equation is called the Eddington luminosity, and is the maximum luminosity that allows a stable stellar configuration (equality is achieved for

$P_{\text{tot}} = P_{\text{rad}}$).

Derive this luminosity also from the alternative approach that a star cannot be stable if the radiative acceleration is everywhere larger (with respect to absolute value) than the gravitational one. Assume that $\kappa = const$ and remember that

$$g_{\text{rad}} = \frac{\kappa}{c} F \tag{5.5.20}$$

where F is the frequency-integrated radiative flux.

(c) Using the above inequality and the approximate mass-luminosity relation $L/L_\odot \approx (M/M_\odot)^3$, calculate an upper stellar mass limit, with $\kappa = 0.34$ cm^2 g^{-1}, which is the electron scattering opacity in stars of solar composition (with 10 % helium by number) and thus a lower limit for the total opacity.

Problem 5.5: Solution

(a) The pressure differential can be derived by using the fact that $P_{\text{tot}} = P_{\text{gas}} + P_{\text{rad}}$, and also, due to hydrostatic equilibrium,

$$\frac{\mathrm{d}P_{\text{tot}}}{\mathrm{d}r} = -\frac{GM(r)}{r^2}\rho(r). \tag{5.5.21}$$

On the other hand, we know that $P_{\text{rad}} = aT^4/3$, so the product rule yields

$$\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}r} = \frac{4}{3}aT^3\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{5.5.22}$$

where we had already derived

$$\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{3\alpha(r)F(r)}{4acT^3} = \frac{3\kappa\rho(r)L/(4\pi r^2)}{4acT^3} \tag{5.5.23}$$

in the lecture. Overall we then have

$$\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}P_{\text{tot}}} = \frac{4/3aT^3r^2}{GM(r)\rho(r)}\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{5.5.24}$$

For the criterion, we use that the radiation pressure should not exceed the total pressure, so **TO DO Describe this a little more extensively (!)**

$$\frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}P_{\text{tot}}} = \frac{\mathrm{d}P_{\text{rad}}}{\mathrm{d}P_{\text{rad}} + \mathrm{d}P_{\text{tot}}} < 1 \implies L < \frac{4\pi GM}{\kappa}. \tag{5.5.25}$$

(b) We have

$$g_{\text{rad}} = \frac{\kappa}{c}F = \frac{\kappa}{c}\frac{L}{4\pi r^2} < \frac{GM}{r^2} = g_{\text{grav}}, \tag{5.5.26}$$

from which we immediately obtain the equation found in (a).

(c) Using the relation (which doesn't hold for very massive stars) we were given ($L \propto M^3$), and using the criterion derived in (a) and (b), we have

$$\frac{M}{M_\odot} < \sqrt{\frac{4\pi GM_\odot}{\kappa L_\odot}} \approx \underline{\underline{196M_\odot}}. \tag{5.5.27}$$

(here, it's not the square root as the M cancels out once.)

Problem 5.6: *Binary system*

Visible-light spectroscopic observations of a nearby star similar to the Sun show that it must be part of a binary. A careful study of the spectrum of the star shows that its Doppler shift varies with a period of 10^6 seconds (about a month) and a velocity amplitude of $90\,\mathrm{m/s}$. The Doppler shift of the star varies sinusoidally, suggesting that the star is in a circular orbit.

(a) Estimate the mass of the invisible companion to the star whose spectrum is seen. Speculate on the nature of this companion (i. e. what kind of object is it?)

(b) Further observations show that the secondary eclipses the primary. What is the distance of the star from the secondary? It takes approximately 16 minutes for the brightness of the star to drop at the beginning of each eclipse and to rise at the end of each eclipse. What is the radius of the secondary?

(c) Assuming that the secondary is held up by gas pressure, estimate its central temperature. Is this above or below the hydrogen burning limit?

Problem 5.6: Solution

Thanks and credits to Iliya Tikhonenko!

(a) The velocity amplitude is just the maximum value of the projected orbital velocity. As the star is on circular orbit, we can estimate its velocity via the following relation (centre of mass + Kepler III for a circular orbit):

$$m_2^3 = \frac{v^3}{\omega \sin^3 i} \frac{(m_1 - m_2)^2}{G} \tag{5.6.28}$$

Here, we can take into account that $\sin i = 1$ and $\omega = \frac{2\pi}{T}$, $m_1 + m_2 \approx M_\odot$ (as $m_1 \approx M_\odot$ and $m_2 \approx m_{\mathrm{Jupiter}}$, as we could derive in hindsight).

(b) We can calculate the semimajor axis using Kepler III, $a = r_1 + r_2$, and $\frac{a^3}{T^2} = \frac{GM_\odot}{4\pi^2} \implies a \approx 0.1$ au.
Also, via trigonometry, the angle that the planet subtends in the moment when it leaves (or comes in front of) the star during the eclipse is

$$\delta_{\mathrm{leave}} = \frac{R_{\mathrm{s1}} + R_{\mathrm{s2}}}{a} - \frac{R_{\mathrm{s1}} - R_{\mathrm{s2}}}{a} = \frac{2R_{\mathrm{s2}}}{a}. \tag{5.6.29}$$

On the other hand, because we know the time it takes for the leave $t_{\mathrm{leave}} = 16$ min, we can relate this with the period, so we have

$$\frac{\delta_{\mathrm{leave}}}{2\pi} = \frac{t_{\mathrm{leave}}}{T} \iff R_{\mathrm{s2}} = \frac{t_{\mathrm{leave}}}{T}\frac{2\pi}{2} a \approx 0.6 R_{\mathrm{Jup}}. \tag{5.6.30}$$

Therefore, we found out that the companion 'star' is actually smaller than Jupiter and comparable in mass.

(c) If the second component is stable due to gas pressure than the hydrostatic equilibrium equation should hold

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM\rho}{r^2}. \tag{5.6.31}$$

Ideally, we should assume some equation of state and some boundary conditions (e.g. $P = 0$ at the surface) and solve this equation directly. However, this derivation would be a bit tedious, so instead we can use the following very rough estimate of the central pressure:

$$\frac{P_c}{R} = \frac{GM\bar{\rho}}{R^2}. \tag{5.6.32}$$

Assuming the ideal gas law

$$P_c = \frac{\bar{\rho}}{\mu m_{\mathrm{H}}} k_{\mathrm{B}} T, \tag{5.6.33}$$

we arrive at

$$\frac{\bar{\rho} k_{\mathrm{B}} T}{\mu m_{\mathrm{H}} R} = \frac{GM\bar{\rho}}{R^2} \iff T = \frac{GM}{k_{\mathrm{B}} R} \mu m_{\mathrm{H}} \approx 3 \times 10^5 \text{ k} \ll 10^7 \text{ K} = T_{\text{H-fusion}}, \tag{5.6.34}$$

where we used $\mu \approx 1$.

Problem set #6 Star slosh (here)

15.12.2022

Problem 6.1: *Relations between the equations of state and stellar structure*

With respect to stellar structure the equation of state is in its simplest form represented by the ideal gas law

$$P = nkBT \tag{6.1.1}$$

(pressure = particle number density × Boltzmann constant × temperature). Corrections to the ideal gas law are due to electron degeneracy, radiation pressure, and Coulomb interactions.

- Derive a relation $T(\rho)$ for which gas pressure P_{gas} and radiation pressure P_{rad} become equal. For simplicity, assume the gas is pure hydrogen. (Hint: the radiation pressure is $P_{\mathrm{rad}} = 4\sigma_{\mathrm{B}}T^4/(3c)$.)
- The onset of degeneracy can be estimated through the comparison of the Fermi energy to the thermal energy. Derive a relation $T(\rho)$ for which these energies become equal. (Hint: the relation between particle number density and Fermi energy E_f is given by

$$n = \frac{\alpha}{h^3}\frac{2}{\sqrt{\pi}}\left(2\pi m k_{\mathrm{B}}T\right)^{\frac{3}{2}}\frac{2}{3}\left(\frac{E_f}{k_{\mathrm{B}}T}\right)^{\frac{3}{2}} \tag{6.1.2}$$

  Why can you assume $n = \rho/m_{\mathrm{p}}$, $\alpha = 2$, $m = m_{\mathrm{e}}$?)
- For higher densities and particularly low temperatures Coulomb interactions of the ions must be considered. The ions tend to form a lattice which minimises their total energy. The onset of this effect can be estimated through a comparison of thermal energy and Coulomb energy. Derive a relation $T(\rho)$ for which these energies become equal. (Again assume the gas is pure hydrogen.)
  (Hint: the mean separation r_{sep} of the ions can be approximated and expressed by the ion density n_{ion}, by making use of $V_{\mathrm{ion}} = (4\pi/3)r_{\mathrm{sep}}^3$ and $V_{\mathrm{ion}} = 1/n_{\mathrm{ion}}$.)
- Make a plot of $\log T$ (10^3 to 10^{10} K) on the y-axis versus $\log\rho$ (10^{-7} to 10^7 g/cm^3) on the x-axis. Draw the regions where radiation pressure, degeneracy pressure, and Coulomb interactions become strong perturbations. For comparison, also indicate the central density and temperature for typical stellar objects, e.g., the sun.

Problem 6.1: Solution

- If we set $P_{\mathrm{rad}} = P_{\mathrm{gas}}$, we obtain

$$n k_{\mathrm{B}} T = \frac{4}{3c}\sigma_{\mathrm{B}} T^4 \tag{6.1.3}$$

$$\Longleftrightarrow \quad T = \sqrt[3]{\frac{3}{4}\frac{c n k_{\mathrm{B}}}{\sigma_{\mathrm{B}}}} \tag{6.1.4}$$

$$= \sqrt[3]{\frac{3}{4}\frac{c\rho k_{\mathrm{B}}}{\mu m_{\mathrm{H}}\sigma_{\mathrm{B}}}} \tag{6.1.5}$$

$$\approx 4 \times 10^7 \left(\frac{\rho}{\mathrm{g/cm^3}}\right)^{\frac{1}{3}} \text{ K}, \tag{6.1.6}$$

where we used $n = \frac{\rho}{\mu m_\mathrm{H}}$ for the number density.
For ionised hydrogen, $\mu = \frac{1}{2}$.

- If we set the Fermi energy equal to the thermal energy, $E_f \approx \frac{3}{2} k_\mathrm{B} T$, we have

$$n = \frac{\rho}{\mu m_\mathrm{H}} = \frac{\alpha}{h^3} \frac{2}{\sqrt{\pi}} (2\pi m k_\mathrm{B} T)^{\frac{3}{2}} \frac{2}{3} \cdot \left(\frac{3}{2}\right)^{\frac{3}{2}} \tag{6.1.7}$$

$$\iff T = \frac{1}{3\pi m k_\mathrm{B}} \left(\frac{3}{4} \frac{h^3 \rho \sqrt{\pi}}{\alpha \mu m_\mathrm{H}}\right)^{\frac{2}{3}} \tag{6.1.8}$$

$$= \frac{1}{3\pi m_e k_\mathrm{B}} \left(\frac{3}{8} \frac{h^3 \rho \sqrt{\pi}}{m_\mathrm{H}}\right)^{\frac{2}{3}} \tag{6.1.9}$$

$$\approx 2 \times 10^5 \left(\frac{\rho}{\mathrm{g/cm^3}}\right)^{\frac{2}{3}} \text{ K}, \tag{6.1.10}$$

where we used $\mu = 1$ (since we're looking at the Fermi gas where we only care about the electrons), $\alpha = 2$ (because we're looking at Fermions, where two of them cannot occupy the same energy state), $m = m_e$ because we again only consider electrons.

- The Coulomb energy between two particles with charges q_1 and q_2 at separation r is given by

$$E_\mathrm{Coul} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r}, \tag{6.1.11}$$

so if we set $E_\mathrm{Coul} \approx \frac{3}{2} k_\mathrm{B} T$, for two electrons (charge e), we obtain

$$T = \frac{2e^2}{c \cdot 4\pi\epsilon_0 k_\mathrm{B}} \frac{1}{r_\mathrm{sep}} \tag{6.1.12}$$

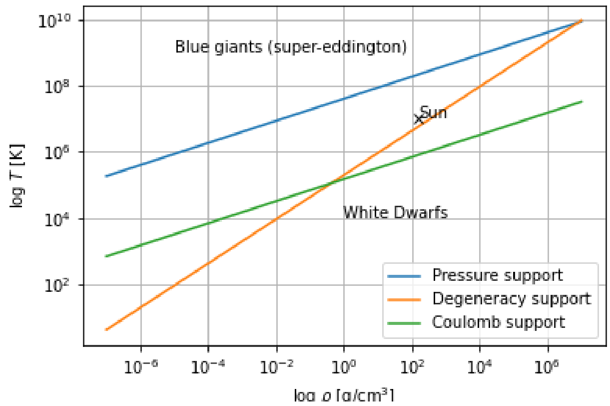
$$= \frac{2e^2}{12\pi\epsilon_0 k_\mathrm{B}} \sqrt[3]{\frac{4\pi n_\mathrm{ion}}{3}} \tag{6.1.13}$$

$$= \frac{2e^2}{12\pi\epsilon_0 k_\mathrm{B}} \sqrt[3]{\frac{4\pi\rho}{3\mu m_\mathrm{H}}} \tag{6.1.14}$$

$$\approx 1.5 \times 10^5 \left(\frac{\rho}{\mathrm{g/cm^3}}\right)^{\frac{1}{3}} \text{ K}, \tag{6.1.15}$$

where we used $V_\mathrm{ion} = \frac{4}{3}\pi r^3 = \frac{1}{n_\mathrm{ion}}$, and can set $\mu = 1$ again.

- The equilibrium lines are plotted below.



Red giants are hot, so radiation pressure dominates. White dwarfs are colder at the same densities, so the electron pressure dominates.

Problem 6.2: *Mass–Radius relation for White Dwarfs*

Brown dwarfs and low-mass white dwarfs are supported by the degeneracy pressure of nonrelativistic electrons. A fully degenerate (zero temperature, or close enough) electron gas has a pressure which depends only on the electron density,

$$P = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{\frac{2}{3}} n_e^{\frac{5}{3}} \tag{6.2.16}$$

where n_e, the electron density, equals ρ/μ_e if μ_e is the mass per electron. We will assume this to be $\mu_e = 2m_p$, which is correct for helium and the lighter heavy elements. Thus the above relation for the pressure is of the form $P = K\rho^{\frac{5}{3}}$, $K = \frac{h^2}{20m_e}\left(\frac{3}{\pi}\right)^{\frac{2}{3}}(2m_p)^{-\frac{5}{3}}$.
The important point here is that K is a constant [a so-called polytropic constant].

(a) Derive the relation between the mass M of the object and its radius R. (Hints: The central "Fermi pressure" depends on the central density ρ_c, which for each type of polytrope (K constant) is some factor larger than the mean density ρ_m of the star. In our case, $\frac{\rho_c}{\rho_m} = 6$. Moreover, ρ_m is by definition given by the total mass divided by the total volume. As a final consideration, P_c, the pressure at the centre, is obtained from hydrostatic equilibrium $\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4}$ assuming an average pressure gradient $\frac{P_0 - P_c}{M-0}$ where $P_0 = 0$ and by replacing m and r by the mean values $\frac{M}{2}$ and $\frac{R}{2}$.)

(b) For which mass is the radius just about the radius of the Earth?

(c) How does the radius vary with mass? What would you get for a rocky body that can maintain constant density? How can you explain the result obtained?

(d) Bonus question:
How large is the error resulting from our approximation in part (a)? The (numerical) solution of the Lane–Emden equation for this particular polytrope ($\gamma = \frac{5}{3}$, $n = \frac{1}{\gamma-1} = \frac{3}{2}$) gives $R = 3.6537537\alpha$ and $M = 2.71405514\pi\alpha3\rho_c$, where $\alpha = \sqrt{\frac{K(n+1)}{4\pi G\rho_c^{2-\gamma}}}$.

Problem 6.2: Solution

(a) We approach this problem via the hints that were provided, starting from the pressure gradient which we approximate as being constant, so

$$\frac{\mathrm{d}P}{\mathrm{d}m} = -\frac{Gm}{4\pi r^4} \approx \frac{GM/2}{4\pi(R/2)^4} = -\frac{P_c}{M} = \frac{P_0 - P_c}{M - 0} \tag{6.2.17}$$

$$\implies P_c = \underbrace{\frac{2G}{\pi}}_{:=\beta} \frac{M^2}{R^4}, \tag{6.2.18}$$

giving us a nice relation for the central pressure. We have defined the constant β for convenience.
On the other hand, since $\frac{\rho_c}{\rho_m} = 6$, and thus $\rho_c = 6\rho_m = 6\frac{M}{4/3\pi R^3}$, we alternatively find the central pressure to be

$$P_c = K\rho_c^{\frac{5}{3}} = \underbrace{K\left(\frac{9}{2\pi}\right)^{\frac{5}{3}}}_{:=\alpha} M^{\frac{5}{3}} R^{-5}, \tag{6.2.19}$$

where we defined the constant α for convenience.
Requiring these two differently obtained pressures to be equal, we have

$$P_c = P_c \tag{6.2.20}$$

$$\alpha \frac{M^{\frac{5}{3}}}{R^5} = \beta \frac{M^2}{R^4} \tag{6.2.21}$$

$$R = \frac{\alpha}{\beta} M^{-\frac{1}{3}} = K\left(\frac{9}{2\pi}\right)^{\frac{5}{3}} \frac{\pi}{2G} M^{-\frac{1}{3}}, \text{ or} \tag{6.2.22}$$

$$M = \frac{\beta}{\alpha} R^{-3}. \tag{6.2.23}$$

(b) **TO DO Calculate the values (sorrey) (!)**

Problem 6.3: *Saha equation and pressure ionisation*

The Saha equation describes the fractions of an element in the different ionisation states in thermal equilibrium,

$$\frac{n_{\mathrm{II}} n_e}{n_{\mathrm{I}}} = 2\frac{g_{\mathrm{II}}}{g_{\mathrm{I}}} \left(\frac{2\pi m_e k_{\mathrm{B}} T}{h^2}\right)^{\frac{3}{2}} e^{-E/(k_{\mathrm{B}}T)}, \tag{6.3.24}$$

where n_{I} and n_{II} are the number densities of neutral and ionised atoms, n_e is the number density of electrons, m_e the electron mass, and E the ionisation potential of this element.
In the case of hydrogen, the statistical weights of the ionised and neutral atoms are $g_{\mathrm{II}} = 1$ and $g_{\mathrm{I}} = 2$, respectively.

(a) Assume you have a pure hydrogen medium, where all electrons come from the ionisation of neutral hydrogen, not from metals that have lower ionisation potentials. Use the ionisation fraction

$$x = \frac{n_{\mathrm{II}}}{n_{\mathrm{I}} + n_{\mathrm{II}}} \tag{6.3.25}$$

and the total gas density $n_{\mathrm{g}} = n_{\mathrm{I}} + n_{\mathrm{II}} + n_e$ to rewrite the Saha equation without n_e. Note that

$$n_{\mathrm{g}} = n(1 + x) \tag{6.3.26}$$

where n is the density of hydrogen nuclei (neutral hydrogen plus protons).

(b) The gas pressure (from neutral hydrogen, protons, and electrons together) and the temperature at the centre of the Sun are $P = 2.32 \times 10^{17}$ dyn/cm^2 and $T = 1.57 \times 10^7$ K.
Calculate the gas number density n_{g} assuming the medium is an ideal gas.

(c) Calculate the ionisation fraction x using the equation derived at (a). Do you think the result is plausible? Why or why not?

(d) In reality, the centre of the Sun is equivalent to being fully ionised, but in the sense that the electrons in metals are unbound. This effect is called pressure ionisation. Calculate the distance between hydrogen nuclei at the centre of the Sun assuming the above pressure and temperature, but an ionisation fraction of $x = 0$. Compare it with the Bohr radius $r_{\mathrm{B}} = 5.29 \times 10^{-9}$ cm.

Problem 6.3: Solution

TO DO Put in solution I've written down on iPad (!)

Problem 6.4: *Helium Flash*

When the Sun leaves the main sequence, it will go into its "red giant" phase, with a degenerate inert helium core surrounded by a shell of hydrogen which is fusing to helium. Helium "ash" builds up in the core, contracting it, until it is dense and hot enough to ignite helium fusion. At this point, the helium core has a radius of roughly 7000 km, and a mass of about $1/3$ the mass of the Sun. The excellent thermal conduction of the degenerate helium core allows much of the helium to fuse all at once, resulting in the helium flash. This fuses (say) a tenth of the helium up to carbon, and expands the core out to a radius of about $70\,000$ km, over the course of minutes or hours.

(a) Even though this helium flash happens very quickly, and is a tremendous thermonuclear explosion, no direct effect is seen right away in the luminosity of the star. Where is all the energy from that fusion going?

(b) Do an order-of-magnitude energy calculation to show that your answer to (a) is plausible. Helium is fused to carbon via the "triple-alpha" process. The mass of one helium-4 atom is $4.002\,603\,2u$ and the mass of one carbon-12 atom is $12.0000000u$.

Problem 6.4: Solution

(a) As soon as the degeneracy of the gas in the core is lifted, the hydrogen shell around the core expands rapidly and decreases its temperature. This is where the energy of the flash goes into.

(b) The energy released in the fusion process can be approximated if we just account for the energy released by three He atoms being fused into one C atom.
If $M_{\text{He, core}} = \frac{1}{3}M_\odot$ and a $\eta = 0.1$ fusion efficiency, we have

$$E_{\text{single}} = (3m_{\text{He}} - m_{\text{C}})c^2 = 1.17 \times 10^{-12}\ \text{J}, \tag{6.4.27}$$

$$M_{\text{fused}} = M_{\text{He, core}}\eta \tag{6.4.28}$$

$$\Longrightarrow\ E_{\text{tot}} = N_{\text{reaction}}E_{\text{single}} = \frac{M_{\text{fused}}}{3m_{\text{He}}}E_{\text{single}} = 3.87 \times 10^{42}\ \text{J}. \tag{6.4.29}$$

Problem 6.5: *Magnitudes, reddening, and distances*

(a) Express the absolute visual magnitude M_V as a function of the observed visual magnitude V, the distance d (in pc) and the visual extinction, A_V. Explain the relationship between these quantities in physical terms.

(b) Calculate the stellar radius (in solar units) for a star with given absolute visual magnitude M_V, effective temperature T_{eff}, and bolometric correction B.C. The bolometric magnitude of the Sun is $M_{\odot,,\ \text{bol}} = 4.75$ and its effective temperature is $T_{\odot,\text{eff}} = 5777$ K.

(c) Alternatively, one can calculate the radius from M_V and an adequate theoretical flux distribution $F(\lambda)$. In this case, we need an absolute flux calibration (at least for one star). Remember that the visual magnitude is defined by

$$V = -2.5\log\left(\frac{R^2}{d^2}\int F(\lambda)S(\lambda)\mathrm{d}\lambda\right) + \text{const} = -2.5\log\left(\frac{R^2}{d^2}F(\bar{\lambda}\int S(\lambda)\mathrm{d}\lambda\right) + \text{const}. \tag{6.5.30}$$

with photometric response function $S(\lambda)$ and isophotal wavelength $\bar{\lambda} = 5500$ Å for the V-band. Numerous measurements for the standard star Vega have shown that $V = 0$ corresponds to an absolute flux of $(R/d)^2F(\bar{\lambda}) = 3.69 \times 10 - 9$ erg s^{-1}cm^{-2} Å^{-1} at the isophotal wavelength (and outside the earth's atmosphere), whereas the integrated V-filter response function $RS(\lambda)\mathrm{d}\lambda$ has a value of 2897.5. Derive the "const"!
Show that the stellar radius can be calculated via

$$5\log\frac{R}{R_\odot} \approx 30.81 + V_{\text{theo}} - M_V, \tag{6.5.31}$$

if $V_{\text{theo}} = -2.5log(RF_{\text{theo}}(\lambda)S(\lambda)\mathrm{d}\lambda)$, with $F_{\text{theo}}(\lambda)$ the energy flux from a theoretical model.

(d) Cyg OB2 is a heavily reddened OB-association. We will use archival and literature data to derive its distance, from O-stars and appropriate calibrations, via the expression derived in exercise (a).
Look (on the web or in "Simbad/Vizier") for the "Galactic O-star catalog" (by Maíz

Apellániz et al.) and find the O-stars in Cyg OB2. Record the catalog numbers and spectral type and luminosity class information. Then visit "Simbad" and look for the available information on these objects. Use the GOS number from the O-star catalogue as identifier.
Example: for Cyg OB2-7, use "GOS G080.24+00.80 01". Make a table for all stars for which information is available, including spectral type, luminosity class, B- and V-magnitudes.
Look on ADS for the paper by F. Martins et al. (A&A, 2005) which contains the most recent calibration for M_V from O-stars as a function of spectral type and luminosity class. Use this expression to derive the distances of the individual stars. For the extinction, use the relation

$$A_V = RE(B-V), \quad R \approx 3.1 \tag{6.5.32}$$

where $E(B-V)$ is the colour excess. For the intrinsic colours, use the fact that for O-stars these colours are rather constant, with $(B-V)0 \approx -0.28$ for luminosity class I objects and -0.31 else.
Derive the mean and the standard deviation of the individual distances, and compare with the literature value for the distance to Cyg OB2, which is $d \approx 1.7$ kpc.

Problem 6.5: Solution

(a) The relationship of the magnitudes is

$$m_V = M_V + \underbrace{5\log(d) - 5}_{\text{distance modulus}} + A_V, \tag{6.5.33}$$

as the apparent magnitude increases with extinction, and the distance d is given in terms of parsec.

(b) We know that the luminosity in terms of the effective temperature and radius is given by

$$L_\star = 4\pi R^2 \sigma_B T_{\text{eff}}^4. \tag{6.5.34}$$

The bolometric correction is

$$M_{\text{bol}} - M_{\odot,\,\text{bol}} = -2.5\log\left(\frac{L_\star}{L_odot}\right). \tag{6.5.35}$$

We therefore get

$$M_{\text{bol}} = M_V + B.C. = -2.5\log\left(\frac{L_\star}{L_\odot}\right) - M_{\odot,\,\text{bol}} = -2.5\log\left(4\pi R^2\sigma_B T_{\text{eff}}^4/L_\odot\right) - M_{\odot,\,\text{bol}}. \tag{6.5.36}$$

Solving this for R (and also expressing $L_\odot$ in terms of $R_\odot$ and $T_\odot$), we find

$$R = \left(\frac{T_\odot}{T_{\text{eff}}}\right)^2 10^{-0.2(M_V + B.C - M_{\odot,\,\text{bol}}} R_\odot. \tag{6.5.37}$$

(c) If we just use the equations given, setting $V = 0$ for the zero-point, we have

$$0 = -2.5 \log \left(3.69 \times 10^{-9} 2897.5\right) + K \implies K = -12.427. \tag{6.5.38}$$

Using the relation from (a) (eq. (6.5.33)), we find

$$m_V + 5 - M_V = 5 \log d \implies \tag{6.5.39}$$

Problem set #7 Special Holiday Edition (here)

12.01.2023

Because who needs holidays anyways...

Problem 7.1: *Critical density for forbidden lines*

The spectra of gaseous nebulae show prominent emission from forbidden transitions, whereas these forbidden lines are absent from the spectra of stars.

- What is the critical density (order of magnitude) above which these forbidden lines will not appear?

Make a rough approximation regarding the velocity of the electrons and the interaction cross-section between electrons and ions. The Einstein "A" coefficient of the forbidden O[III] transition at $4959, 5007$ Å is $A = 2.6 \times 10^{-2}$/s. Assume the nebular kinetic temperature to be $T = 8000$ K.

Problem 7.1: Solution

We assume the Maxwell-Boltzmann distribution, so the average velocity of the electrons is

$$\langle v \rangle = \sqrt{\frac{2k_\mathrm{B}T}{m_e}} = 2.4 \times 10^4 \text{ m/s} \tag{7.1.1}$$

We use a geometric approach to estimate the cross-section of the ions and electrons, where we can use only the radius of the ion and neglect the extent of the electron, so we have

$$\sigma = \pi d^2 \approx \pi a_0^2 \tag{7.1.2}$$

where d denotes the distance between the 'centres' of electron and ion, and a_0 is the Bohr radius which we can assume as the typical radius of the ions.
For the gas to be able to emit the 'forbidden emission', the radiation rate $R_\mathrm{rad} = A$ (which in this case corresponds to the spontaneous emission Einstein coefficient A) needs to be higher than the average collision rate R_coll, which we can estimate to be

$$R_\mathrm{coll} = \sigma \langle v \rangle n_\mathrm{e}. \tag{7.1.3}$$

Applying the criterion $R_\mathrm{coll} < R_\mathrm{rad}$ and solving for the number density, we find

$$n_\mathrm{e,\ crit} < \frac{A}{\pi a_0^2}\sqrt{\frac{m_e}{2k_\mathrm{B}T}} \approx \underline{\underline{10^6}}\frac{1}{\mathrm{cm}^3} \tag{7.1.4}$$

to still be able to see the emission lines.

Problem 7.2: *H II regions around hot stars*

Hot stars emit enough UV photons to ionise the surrounding interstellar medium, forming a sharply-bounded "Strömgren sphere", inside of which the material is practically fully ionised, and outside of which it is essentially neutral.

- How large is this H[II] region for a typical 40000 K main-sequence star, if the interstellar medium is pure hydrogen with a number density of $n_{\mathrm{H}} = 102/\mathrm{cm}^3$? What is its mass? (What is the star's mass?)

Approximate the star by a blackbody with a radius of $10R_{\odot}$, and assume that the ionised region has a temperature of $T = 8000$ K. At this temperature, the recombination rate coefficient for hydrogen is about $\alpha_{\mathrm{B}} = 3.5 \times 10^{-13}$ cm^3/s.

(Hint: In equilibrium, the number of ionising photons emitted by the star per second must equal the number of recombinations per second in the H[II] region. [Why?] The number of Photons in the range $[\nu, \nu + \mathrm{d}\nu]$ emitted per second per unit area by a blackbody is $\frac{\pi B_\nu}{h\nu}$. What is the ionisation energy of hydrogen? Is the Wien approximation applicable?)

Problem 7.2: Solution

If the gas is ionised inside of the region and not ionised outside of it, there has to be photoionisation equilibrium in the sphere for it to be able to be ionised.

Therefore, $P_\gamma \overset{!}{=} R_{\mathrm{rec}}$, the photon production rate from the star has to equal the recombination rate.

Since $P_\gamma = \int_{\nu_0}^{\infty} \frac{\pi B_\nu}{h\nu} \mathrm{d}\nu$ (where ν_0 corresponds to $E = 13.6$ eV as the frequency needed at minimum to ionise the photons, so we have

$$P_\gamma = \int_{\nu_0}^{\infty} \frac{2\pi\nu^2}{c^2} \exp\left(\frac{-h\nu}{k_{\mathrm{B}}T}\right) \mathrm{d}\nu = 1.2 \times 10^{49} \frac{1}{\mathrm{s}}, \tag{7.2.5}$$

while the recombination rate gives us an angle at the stromgren radius:

$$R_{\mathrm{rec}} = n^2 \alpha_{\mathrm{B}} \frac{4\pi}{3} R_{\mathrm{S}}^3 \iff R_{\mathrm{S}} = \sqrt[3]{\frac{3P_\gamma}{n^2 \alpha_{\mathrm{B}} 4\pi}} = 3\,\mathrm{pc}. \tag{7.2.6}$$

Via the density of the HII region we were given, we can determine its mass to be

$$M_{\mathrm{S}} = \frac{\bar{\rho}}{\frac{4}{3}\pi R_{\mathrm{S}}^3} = \frac{n_{\mathrm{H}} m_{\mathrm{H}}}{\frac{4}{3}\pi R_{\mathrm{S}}^3} \approx 290 M_{\odot}. \tag{7.2.7}$$

Using $T_{\mathrm{eff}} = 40000$ K **TO DO Once we talk about the solution (we never discussed in the end, sorry!) (!)**

Problem 7.3: *Type Ia supernova brightness*

Type Ia supernovae (SN Ia) are stellar explosions with no signs of hydrogen and helium in their spectra, but showing lines of intermediate mass elements such as Si, S, Ca, and Mg near the maximum of their light curves, and many Fe lines at later times. In contrast to massive stars which are the progenitors of type II supernovae (SN II), SN Ia progenitors are thought to be white dwarfs (WDs) expected to consist mainly of carbon and oxygen. In the canonical model the white dwarf accretes mass from a companion star. When the white dwarf approaches the Chandrasekhar mass, thermonuclear burning ignites in its core, and the white dwarf is disrupted by an explosion, leaving no compact remnant.

The luminosity of the ejecta is powered by the radioactive decay of ^{56}Ni. This is observationally supported by the exponential luminosity decline, suggesting that unstable ^{56}Ni decays into ^{56}Co and then into ^{56}Fe. The initial decay of ^{56}Ni into ^{56}Co emits gamma-rays with an average energy of 1.71 MeV and a half-life of 6.1 days.
Supernova 1991T in NGC 4527 showed an absolute brightness of -19.55 mag in the visual band. The total mass of ^{56}Ni produced by thermonuclear burning is estimated to be $1.1M_\odot$.

- How many ^{56}Ni atoms can decay?
- What percentage of ^{56}Ni has decayed after 20 days? What is the average luminosity of the supernova during these 20 days, in solar luminosities? (Assume that all produced energy is converted to light and radiated away.)
- The absolute magnitude of the sun is $M_{V,\odot} = 4.83$ mag. Calculate the brightness of SN 1991T based on the decay of ^{56}Ni above, and compare your result to the observed value. Discuss possible reasons for discrepancies.
- How does the luminosity of SN 1991T compare with the luminosity of the Milky Way?

Problem 7.3: Solution

- How many ^{56}Ni atoms can decay:
  If we assume the thermonuclear ^{56}Ni mass to be $M_0 = 1.1M_\odot$, and a mass[a] of $m_{\text{Ni-56}} = 55.942128u$, we have[b]

$$N_{\text{Ni-56}} = \frac{M_0}{m_{\text{Ni-56}}} = 2.35 \times 10^{55} \tag{7.3.8}$$

  ^{56}Ni atoms that could possibly decay.
- What percentage of ^{56}Ni has decayed after 20 days?
  Using the formula

$$N(t) = N_0 0.5^{\frac{t}{t_{\text{half}}}}, \tag{7.3.9}$$

  we find (with $t_{\text{half}} = 6.1$ days) that

$$\frac{N(20\text{ d})}{N_0} = 0.5^{\frac{20}{6.1}} = 0.103043, \tag{7.3.10}$$

  so $\eta_{20} = 89.69\,\%$ of the initial ^{56}Ni atoms have decayed after 20 days.
  What is the average luminosity of the supernova during these 20 days, in solar luminosities?
  Assume that all produced energy is converted to light and radiated away, we can just calculate[c] the average luminosity as follows:

$$\langle L\rangle_{20} = \frac{\Delta E}{\Delta t} = \frac{\Delta N_{\text{Ni-56}} E_{\text{dec}}}{20\text{ d}} = \frac{\eta_{20} N_{\text{Ni-56}} E_{\text{dec}}}{20\text{ d}} = 8.685 \times 10^9 L_\odot, \tag{7.3.11}$$

  where we used $E_{\text{dec}} = 1.71$ MeV as the energy per decay, and assumed that the fraction η_{20} of ^{56}Ni that we derived above decayed.
- The absolute magnitude of the sun is $M_{V,\odot} = 4.83$ mag. Calculate the brightness of SN 1991T based on the decay of ^{56}Ni above, and compare your result to the

observed value:
To convert the given luminosity to an absolute magnitude, we can compare it to the sun's absolute magnitude, so we have[d]

$$M_{\text{abs}} = M_{\text{abs},\odot} - 2.5 \log\left(\frac{\langle L\rangle_{20}}{L_\odot}\right) = 4.83 - \frac{2.5\ln(8.685)}{\ln(10)} - 2.5\cdot 9 = -20.017 \quad (7.3.12)$$

This is brighter than the absolute brightness $M_{V,\,\text{abs}} = -19.55$ mag that was reported above for the visual band.
The reason for this discrepancy is primarily that we weren't careful about the band that $\langle L\rangle_{20}$ was emitted in - gamma-rays are usually not covered in the V band, therefore we'd have to estimate the amount of flux only in that band (or, more precisely, the $V - \gamma_{\text{ray}}$ colour).

- How does the luminosity of SN 1991T compare with the luminosity of the Milky Way?
  The Milky Way luminosity is[e] around $M_V = -20.74$, so the supernova is almost as bright as the whole Milky Way over the course of a few days. Wow.

[a] according to **this website**
[b] see **here** for my WolframAlpha calculation
[c] see **here** for my WolframAlpha calculation
[d] see **here** for my WolframAlpha calculation
[e] according to **this** paper by Licquia et al, table 3

Problem 7.4: *How old is Ann?*

Mary is 24 years old. She is twice as old as Ann was when Mary was as old as Ann is now. How old is Ann?

Problem 7.4: Solution

Let's rewrite this in terms of equations.
Let $A_{\text{M}} = 24$ be Mary's age right now, and A_{A} be Ann's age now.
We know that at some time in the past Δt, Mary was as old as Ann is now, so

$$A_{\text{M}} - \Delta t = A_{\text{A}}. \quad (7.4.13)$$

Also, we know that at that point of time in the past, Ann *was* half as old as Mary is now, so

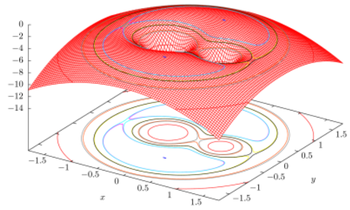
$$A_{\text{M}} = 2(A_{\text{A}} - \Delta t). \quad (7.4.14)$$

Solving these equations yields

$$\begin{aligned} A_{\text{A}} + \Delta t &= 24, & (7.4.15)\\ A_{\text{A}} + \Delta t &= 2(A_{\text{A}} - \Delta t) \iff 3\Delta t = A_{\text{A}} & (7.4.16)\\ \iff \quad 4\Delta t &= 24 \implies \Delta t = 6 & (7.4.17)\\ \implies \quad A_{\text{A}} &= 3\Delta t = 18. & (7.4.18) \end{aligned}$$

Therefore, Ann is now 18 years and was 12 (half the age of Mary now) when Mary was 18 and therefore her current age. Checks out, but can cause some headache.

Problem 7.5: *Roche potential*



Use a computer program (for example, gnuplot) to draw a surface plot of the Roche potential, as in the figure on the right. Experiment with the parameters. For the final plot use masses of $M_1 = 2$ and $M_2 = 1$ and a distance of $d = 1$ between them. Use a gravitational constant of $G = 1$.

- Why are we allowed to rescale the units this way? How many independent dimensional physical parameters does the problem have? Why does the shape of the potential depend only on the mass ratio, not the absolute masses?
- Numerically calculate the potential at the Lagrange points and draw contour lines at (or a bit above and below) these values.
- Why are the Lagrange points L4 and L5 usually considered as stable in the literature, even though they correspond to local maxima of the potential?
  Extra credit: Calculate why this is so.
  (Hint: the potential actually depends on four coordinates. Linearise the equation of motion around the Lagrange points and compute the eigenvalues.)

Problem 7.5: Solution

Thanks and credits to Iliya Tikhonenko!
The usual way to start solving an average classical mechanics problem is to write down the Lagrangian. Let's assume that $\Phi(x)$ is the gravitational potential in the rest frame and then derive equations of motions in the rotating coordinate frame.

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}\left|\dot{\boldsymbol{r}} + \boldsymbol{\Omega}\times\boldsymbol{r}\right|^2 - \Phi(\boldsymbol{r}) \\ &= \tfrac{1}{2}\left|\dot{\boldsymbol{r}}\right|^2 + \dot{\boldsymbol{r}}\cdot(\boldsymbol{\Omega}\times\boldsymbol{r}) - \underbrace{\left(\Phi(\boldsymbol{r}) - \tfrac{1}{2}\left|\boldsymbol{\Omega}\times\boldsymbol{r}\right|^2\right)}_{\Phi_{\text{eff}}}\end{aligned} \tag{7.5.19}$$

Thus, Euler-Lagrange equations for the system would look like:

$$\frac{\partial\mathcal{L}}{\partial\boldsymbol{r}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\boldsymbol{r}}} = 0 \quad\Longrightarrow\quad \ddot{\boldsymbol{r}} = -\nabla\Phi_{\text{eff}} - 2\boldsymbol{\Omega}\times\dot{\boldsymbol{r}}. \tag{7.5.20}$$

If $\boldsymbol{\Omega} = \boldsymbol{k}\Omega$ the vector product from the effective potential definition can be simplified to $\Omega^2 R^2$, where $R = \sqrt{x^2 + y^2}$. When all motion is constrained in xy plane, r would coincide with R. Thus, one can write the effective potential for a test mass in the gravitational field of two bodies as

$$\Phi_{\text{eff}} = \Phi(r) - \tfrac{1}{2}\Omega^2 r^2 = -\sum_{i=1}^{2}\frac{GM_i}{r_i} - \tfrac{1}{2}\Omega^2 r^2 \tag{7.5.21}$$

Using 3rd Kepler's law we can get rid of Ω:

$$\frac{G(M_1+M_2)}{4\pi^2}=\frac{a^3}{T^2}\implies\Omega^2=\frac{G(M_1+M_2)}{a^3},\tag{7.5.22}$$

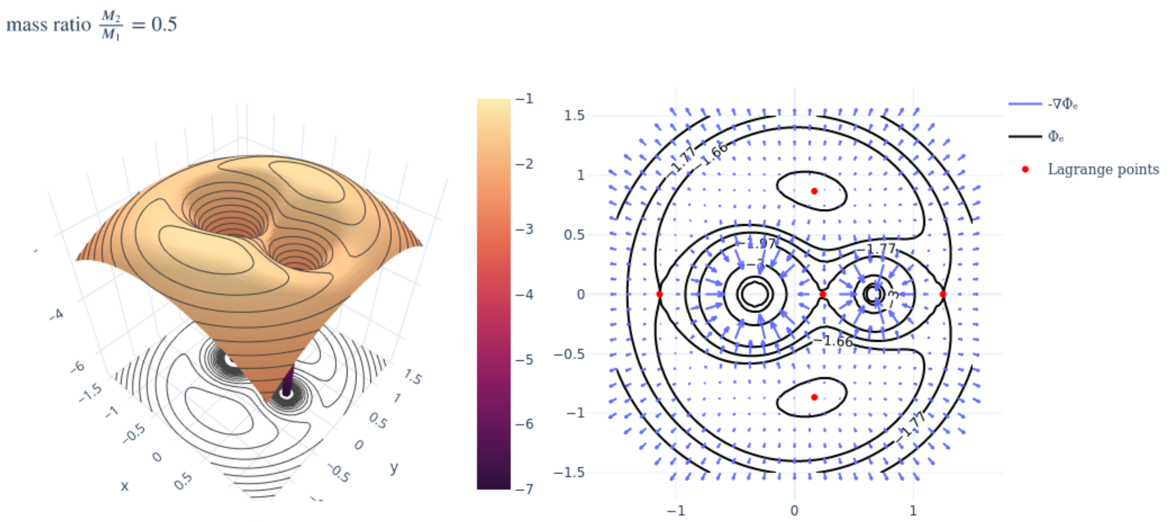
so, plugging this expression inside eq. (7.5.21) we get

$$\Phi_{\text{eff}}=-GM_1\left(\frac{1}{r_1}+\frac{\mu}{r_2}+\frac{1}{2}\frac{(1+\mu)r^2}{a^3}\right),\tag{7.5.23}$$

where a is the distance between two bodies with masses M_1 and M_2, r_1 and r_2 are the distance from the test particle to body #1 and #2 respectively, and $\mu=\frac{M_2}{M_1}$.
Assuming that both bodies lie on x axis at $x_1<0<x_2$, we get

$$r_1=\sqrt{(x-x_1)^2+y^2},\quad x_1=-\frac{a\mu}{1+\mu}\tag{7.5.24}$$

$$r_2=\sqrt{(x-x_2)^2+y^2},\quad x_2=\frac{a}{1+\mu}.\tag{7.5.25}$$



We can also factor a out (effectively expressing all lengths in units of a), so

$$\Phi_{\text{eff}}=-\frac{GM_1}{a}\left(\frac{1}{r_1/a}+\frac{\mu}{r_2/a}+\frac{(1+\mu)\,(r/a)^2}{2}\right),\tag{7.5.26}$$

- In this form, it is immediately visible that we are allowed to rescale the units freely: the actual shape of the potential depends only on μ. You can see it in action in the **interactive version** of the plot above.
- Lagrange points are solutions of $-\nabla\Phi_{\text{eff}}=0$, which in our case means that

$$\frac{\boldsymbol{r}_1}{r_1^3}+\frac{\mu\boldsymbol{r}_2}{r_2^3}+\frac{(1+\mu)\boldsymbol{r}}{a^3}=0.\tag{7.5.27}$$

This equation has 5 roots in the complex plane, but we can almost immediately find two of them(L_4 and L_5) by assuming $r_1=a=r_2$:

$$\frac{\boldsymbol{r}_1}{a^3}+\frac{\mu\boldsymbol{r}_2}{a^3}+\frac{(1+\mu)\boldsymbol{r}}{a^3}=0,\tag{7.5.28}$$

which holds because in our coordinate frame the center of mass of the system coincides with 0:

$$\boldsymbol{r} = \frac{\boldsymbol{r}_1 + \mu \boldsymbol{r}_1}{1+\mu} = \frac{M_1 \boldsymbol{r}_1 + M_2 \boldsymbol{r}_2}{M_1 + M_2}. \tag{7.5.29}$$

The other 3 solutions ($L1$, L_2, and L_3) lie on x and cannot be expressed in an analytical form in the general case (see the right side of the plot above).

Assuming $G = 1$, $a = 1$, $M_1 + M_2 = 1$, and $\mu = \frac{1}{2}$, we can compute the values of the effective potential at Lagrange points:

point	x	y	Φ_{eff}
L_1	0.237	0	−1.973
L_2	1.249	0	−1.774
L_3	−1.136	0	−1.661
L_4	1/6	$\sqrt{3}/2$	−1.389
L_5	1/6	$-\sqrt{3}/2$	−1.389

- Considering the motion around L4 and L5, we need to take into account the second term in eq. (7.5.20): the Coriolis force. If a test particle tries to move away from one of the stable points, the Coriolis force will act in the direction orthogonal to the particle velocity, effectively diverging it from the initial trajectory.

  A more rigorous proof can be done in the following way.

  First we linearise Φ_{eff} near (say) L_4. All first order terms would be zero, due to L_4 being an equilibrium point, thus

$$\Phi_{\text{eff}}(x,y) = \Phi_0 + \underbrace{\frac{\partial^2 \Phi}{\partial x^2}}_{\Phi_{xx}} (\underbrace{x - x_{\text{L4}}}_{\xi})^2 + \underbrace{\frac{\partial^2 \Phi}{\partial y^2}}_{\Phi_{yy}} (\underbrace{y - y_{\text{L4}}}_{\eta})^2, \tag{7.5.30}$$

  so we have

$$\nabla \Phi_{\text{eff}} = 0 + 2\Phi_{xx}\xi\, \boldsymbol{i} + 2\Phi_{yy}\eta\, \boldsymbol{j}. \tag{7.5.31}$$

  On the other hand, we have

$$\Omega \times \dot{\boldsymbol{r}} = \det \begin{pmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & 0 & \Omega \\ \dot{x} & \dot{y} & \dot{z} \end{pmatrix} = \boldsymbol{i}(-\Omega \dot{y}) + \boldsymbol{j}(\Omega \dot{x}) = \boldsymbol{i}(-\Omega \dot{\eta}) + \boldsymbol{j}(\Omega \dot{\xi}), \tag{7.5.32}$$

  so the EOM simplify to

$$\ddot{\xi} = -2\left(\Phi_{xx}\xi - \Omega\dot{\eta}\right) \tag{7.5.33}$$

$$\ddot{\eta} = -2\left(\Phi_{yy}\eta - \Omega\dot{\xi}\right), \tag{7.5.34}$$

  which are solved with the ansatz $\xi = Ae^{\lambda t}$, $\eta = Be^{\lambda t}$.

  TO DO expand.. :((!) We arrive at an equation for λ^2, requiring it to be negative, and have the following boundary conditions:

$$\Phi_{xx}\Phi_{yy} > 0 \tag{7.5.35}$$

$$\Phi_{xx} + \Phi_{yy} + 4\Omega^2 > 0 \tag{7.5.36}$$

$$\Phi_{xx} + \Phi_{yy} + 4\Omega^2 > 4\Phi_{xx}\Phi_{yy}. \tag{7.5.37}$$

The derivation can also be found in *Galactic dynamics*, page 182.

Problem 7.6: *Pulsar periods*

(a) Some of the fastest pulsars have pulse frequencies of around $f = 700$ Hz. Assuming that the pulses are correlated with a "hot spot" on a rigidly spinning sphere, what is the maximum size of that sphere? If the equator spins at $v = 0.2c$, what mass must the sphere at least have so it does not break up? (Make the crude assumption that the sphere isn't deformed by centrifugal forces before it breaks up.) Derive an expression for the density of the sphere. What parameters does it depend on? Compare your calculated density with that of nuclear material. (A uranium nucleus has a radius of about 10 fm and consists of 238 nucleons.)

(b) A certain pulsar has a frequency of 6.6 Hz. Assuming a homogeneous rotating sphere of mass $M = 1.4M_\odot$ and radius $R = 10$ km, what is the rotational kinetic energy of the sphere? The pulsar has a period rate of change of $\dot{P} = 1.54 \times 10^{-12}$ s/s. Compute the energy loss per unit time and compare it to the solar luminosity. What is the current best explanation for this spindown? Discuss the mechanism.

Problem 7.6: Solution

(a) We can just assume for a point at the surface of the sphere having to be slower than the speed of light, so the maximum size of such a sphere is

$$\omega R < c \iff R < \frac{c}{2\pi f} = 68 \text{ km}. \tag{7.6.38}$$

If the equator spins at $0.2c = \omega R$, we find $R = \frac{0.2c}{2\pi f} = 14$ km.
The mass then can be derived from an equilibrium of centrifugal force and gravity (assuming classical mechanics (which arguably might not be appropriate in this case)):

$$\omega^2 R \leq G\frac{M}{R^2} \tag{7.6.39}$$

which we can rewrite to

$$R \leq \left(\frac{G}{(2\pi f)^2}M\right)^{\frac{1}{3}} \lesssim 20 \text{ km}\left(\frac{M}{M_\odot}\right)^{\frac{1}{3}}, \tag{7.6.40}$$

so we find that

$$\frac{M}{M_\odot} \geq \left(\frac{14 \text{ km}}{20 \text{ km}}\right)^3 = 0.4. \tag{7.6.41}$$

Alternatively, we find $\frac{M}{R^3} \geq \frac{\omega^2}{G}$ Therefore, the mean density required is

$$\bar{\rho} = \frac{M}{V} \geq \frac{3\omega^2}{4\pi G} \approx 10^{14}\frac{\text{g}}{\text{cm}^3}. \tag{7.6.42}$$

Comparing this to the approximate density of nuclear material, we find

$$\rho_{\text{nuc}} = \frac{238 m_{\text{p}}}{r_{\text{nuc}}^3 4\pi/3} \approx 10^{14} \frac{\text{g}}{\text{cm}^3}, \tag{7.6.43}$$

so we see that the required density of the neutron star is of the same order of magnitude (we used $r_{\text{nuc}} = 10^{-12}$ m).

(b) For this pulsar, we have $f = 6.6$ Hz, $M = 1.4 M_\odot$, $R = 10$ km and $\dot{P} = 1.54 \times 10^{12}$ s/s.
To find the energy loss per unit time, we have to take a look at the rotational energy possessed by the neutron star, which is

$$E_{\text{rot}} = \frac{1}{2}\omega = \frac{1}{5} M R^2 \left(\frac{2\pi}{P}\right)^2 = 9.6 \times 10^{40} \text{ J}, \tag{7.6.44}$$

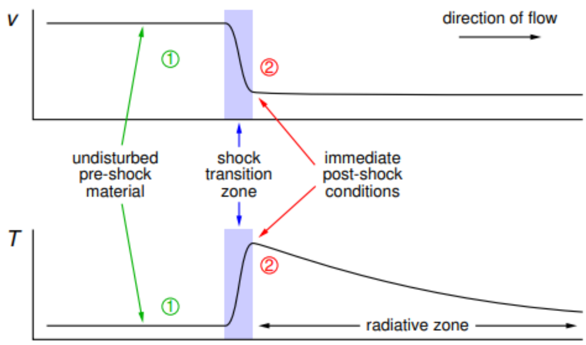
where we assumed the neutron star to be as sphere, having a moment of inertia of $I = \frac{2}{5} M R^2$.
Thus,

$$\dot{E_{\text{rot}}} = \frac{(2\pi)^2}{5} M R^2 \left(-\frac{2}{P^3}\dot{P}\right) = E_{\text{rot}} \left(-2\frac{\dot{P}}{P}\right) = 2 \times 10^{30} \text{ W} \approx 10^4 L_\odot. \tag{7.6.45}$$

Due to the strong magnetic field of the pulsar, currents inside it are induced, which leads to energy loss (magnetorotational breaking), and some energy is lost due to particles being accelerated by the magnetic fields.

Problem 7.7: *Simple one-dimensional shock model*

Shock waves play an important role in astrophysics. They occur on all scales in the ISM, for example in supernova blast waves, cloud-cloud collisions, H[II] regions, interstellar bubbles (very fast winds), accretion and outflow phenomena (jets), spiral shocks in the galactic disk, galaxy mergers, and even at the edges of galaxy clusters.
From a kinetic viewpoint, shocks convert much of the ordered motion of the pre-shock gas into random, thermal motion. If the shock speed is sufficiently high, the hot post-shock gas radiates and heats more distant gas both upstream and downstream from the front itself.



The conservation of mass flow, momentum flow, and energy flow through the shock zone are described by

$$\rho_1 v_1 = \rho_2 v_2 \tag{7.7.46}$$
$$P_1 + \rho_1 v_1^2 = P_2 = \rho_2 v_2^2 \tag{7.7.47}$$
$$v_1\left(\frac{\rho_1 v_1^2}{2} + \rho_1 U_1 + P_1\right) = v_2\left(\frac{\rho_2 v_2^2}{2} + \rho_2 U_2 + P_2\right), \tag{7.7.48}$$

where ρ, v, and P are the mass density, speed, and pressure, and the subscripts 1 and 2 refer to the pre- and post-shock gas, respectively.
The internal energy per unit mass is

$$U = \frac{P}{(\gamma - 1)\rho} \tag{7.7.49}$$

where γ is the adiabatic index of the gas.
The preceding set of equations can be rewritten as the *Rankine-Hugoniot jump conditions*:

$$\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \frac{(\gamma+1)M_1^2}{2 + (\gamma-1)M_1^2} \tag{7.7.50}$$
$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 - \gamma + 1}{\gamma + 1} \tag{7.7.51}$$
$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{v_2}{v_1} \tag{7.7.52}$$

with $M_1 = \frac{v_1}{c_1}$ the *Mach number* and $c_1 = \sqrt{\frac{\gamma P_1}{\rho_1}}$ the sound speed in the pre-shock region. (You can derive these equations yourself. Only basic algebra is involved, but finding the correct order of operations is not entirely trivial.)

(a) Why are there only *compression shocks* [in which fast, thin gas runs into the shock and slower, denser gas leaves it] and no *expansion shocks* [in which slow, dense gas enters the shock and fast, thin gas runs out of it], subject to those same conservation laws?

(b) For fast shocks ($M_1 \gg 1$), how much denser is the post-shock gas than the pre-shock material? Assume a non-relativistic, monatomic gas ($\gamma = \frac{5}{3}$).

(c) For a shock with $v_1 = 1000$ km/s (a typical value for the speed of the blast wave around a supernova remnant) and an ambient ISM temperature of 1000 K, what is the Mach number? What is the temperature of the shocked material? (You can simplify the Rankine–Hugoniot conditions when $M_1 \gg 1$.) What is the Mach number in the post-shock gas?

Problem 7.7: Solution

TO DO Once we get the solution (we never discussed it in the end... sorrey) (!)

Problem set #8 Galaxy dynamics (here)

19.01.2023

Problem 8.1: *Inclination angles of spiral galaxies*

Spiral galaxies can be observed under different inclination angles i, where $i = 0$ describes a "face-on" perspective and $i = \pi/2$ describes an "edge-on" perspective. How are the observed inclination angles i of spiral galaxies distributed if the galaxies are oriented randomly in space?

(a) First assume a disk which can be tilted with respect to an axis ("tilt axis" in the figure) perpendicular to the line of sight. (This axis is assumed to be fixed in space.) How are the inclination angles i distributed in this case, if i can take values from 0 to $\pi/2$? Derive the normalised distribution function $p(i)$ so that $\int_0^{\pi/2} p(i)\mathrm{d}i = 1$.

(b) Allowing different tilt axes (still perpendicular to the line of sight) gives an additional degree of freedom. To calculate the true distribution of the inclination angles it is convenient to use the normal vector $\boldsymbol{n}$ of the disk surface and to distribute the end of the vectors uniformly over the surface of a sphere. Convince yourself that the probability $p(i)$ of an inclination angle i must be proportional to the circumference of the circle on the sphere touched by the normal vector $\boldsymbol{n}$. Estimate for which i the probability must be highest and lowest.

(c) Find the functional dependence of the radius $r_{\mathrm{proj}}(i)$ of this circle by plotting r_{proj} against i. Can this be described by a mathematical function? From this derive the true normalised distribution function $p(i)$ of the inclination angle i.

Problem 8.1: Solution

TO DO Put in once discussed (!)

$$p(i)\mathrm{d}i = \frac{2\pi \sin i \cdot 2}{4\pi}\mathrm{d}i \implies p(i) = \sin i, \tag{8.1.1}$$

Where

Problem 8.2: *Rotation Curve*

How does the rotational velocity of the Milky Way's disk vary with radius (distance from the galactic centre)? What velocity profile would we expect if the total mass were concentrated in the bulge? What is measured instead, and how is it measured? Derive an equation that allows determining the density distribution $\rho(r)$ from the observed velocity profile $v(r)$, and describe under what assumptions it is valid. What is the resulting "observed" density profile?

Problem 8.2: Solution

The rotation curve of the Milky Way's disk looks as follows: **TO DO Put in a qualitative plot (!)** To derive the expected velocity profile if all mass were concentrated, we start by using the fact that the centripetal force on a particle should be given by the gravitational force, so we have

$$F_{\text{grav}} = F_{\text{cp}} \tag{8.2.2}$$
$$\frac{GMm}{r^2} = \frac{mv^2}{r} \implies M(r) = \frac{rv^2}{G}. \tag{8.2.3}$$

To get the density profile, we can look at the mass enclosed in a spherical shell of thickness $\mathrm{d}r$, we have **TO DO This looks fishy, but I'm too tired to correct it right now, so I'll need to have another look at this sometime) (!)**

$$\mathrm{d}M = \rho(r) 4\pi r^2 \mathrm{d}r \stackrel{!}{=} \frac{v_{\text{rot}}^2}{G} \mathrm{d}r. \tag{8.2.4}$$

This yields

$$\rho(r) = \frac{v_{\text{rot}}}{4\pi G} \frac{1}{r^2} \propto \frac{1}{r^2} \tag{8.2.5}$$

for a density profile with constant rotation velocity.

Problem 8.3: *Gravitational potential*

(a) Consider an N-body system of stars with coordinates $\boldsymbol{x}_i$ and masses m_i, $i = 1, \ldots, N$. Show that the negative of the gradient of the quantity

$$\Phi(\boldsymbol{x}) = -\sum_{i=1}^{N} \frac{Gm_i}{|\boldsymbol{x} - \boldsymbol{x}_i|} \tag{8.3.6}$$

is equal to the gravitational force of the system. (In other words, Φ is the gravitational potential of the system.)

(b) Newton proved two results that enable us to calculate the gravitational potential of any spherically symmetric distribution of matter easily:
Newton's first theorem: A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.
Newton's second theorem: The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its centre.

Consider a spherical distribution of mass with density $\rho(r) \propto r^{-n}$, with $-n = \dfrac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r}$.

- Use Newton's theorems to calculate the gravitational force F on a particle of mass m_1 at a position r_1 for $n = 1, 2, 3, 4$.
- How does the total mass of the system behave when going to large radii ($r_1 \to \infty$)?
- At what value for n does this behaviour change?

Problem 8.3: Solution

(a) We are supposed to show that the gravitational acceleration

$$\boldsymbol{F}_{\text{grav}}/m = -\nabla\Phi(\boldsymbol{x}). \tag{8.3.7}$$

We have

$$-\nabla\Phi(\boldsymbol{x}) = -\nabla\left(-\sum_{i=1}^{N}\frac{Gm_i}{|\boldsymbol{x}-\boldsymbol{x}_i|}\right) \tag{8.3.8}$$

$$= \sum Gm_i\nabla\left(\boldsymbol{x}-\boldsymbol{x}_i\right)^{-\frac{1}{2}} \tag{8.3.9}$$

$$= \sum Gm_i\left(-\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{x}_i\right)^{-\frac{3}{2}}2\left(\boldsymbol{x}-\boldsymbol{x}_i\right)\right) \tag{8.3.10}$$

$$= -\sum\frac{Gm_i}{|\boldsymbol{x}-\boldsymbol{x}_i|}\left(\boldsymbol{x}-\boldsymbol{x}_i\right), \tag{8.3.11}$$

which is indeed the gravitational acceleration.

(b) Generally, the mass enclosed in inside a sphere with radius r_1 is given by

$$M(<r_1) = \int_{r_0}^{r_1} 4\pi r^2\rho(r)\mathrm{d}r + M_0, \tag{8.3.12}$$

where M_0 is the mass enclosed in the inner part of the sphere with radius r_0 (which we need to consider to not get divergence problems).
The force acting on a particle of mass m_1 is given by

$$F = -\frac{GM(<r_1)m_1}{r_1^2} \tag{8.3.13}$$

We are supposed to look at different values of n. Thus (with $\rho(r) = cr^{-n}$)

- $n = 1$

$$M(<r_1) = c\int_{r_0}^{r_1} 4\pi r\mathrm{d}r + M_0 = 2c\pi(r_1^2 - r_0^2) + M_0 \overset{\infty}{\to} \infty. \tag{8.3.14}$$

The force on the particle therefore is

$$F = -Gcm_1\frac{2\pi(r_1^2 - r_0^2) + M_0}{r_1^2}, \tag{8.3.15}$$

so even at infinite range, the particle would still experience some force.

- $n = 2$

$$M(<r_1) = c\int_{r_0}^{r_1} 4\pi\mathrm{d}r + M_0 = 4c\pi(r_1 - r_0) + M_0 \overset{\infty}{\to} \infty. \tag{8.3.16}$$

The force on the particle therefore is

$$F = -Gcm_1\frac{4\pi(r_1 - r_0) + M_0}{r_1^2}. \tag{8.3.17}$$

- $n = 3$

$$M(< r_1) = c \int_{r_0}^{r_1} \frac{4\pi}{r} \mathrm{d}r + M_0 = 4c\pi \ln\left(\frac{r_1}{r_0}\right) + M_0 \overset{\infty}{\to} \infty. \tag{8.3.18}$$

The force on the particle therefore is

$$F = -Gcm_1 \frac{4\pi \ln\left(\frac{r_1}{r_0}\right) + M_0}{r_1^2}. \tag{8.3.19}$$

- $n = 4$

$$M(< r_1) = c \int_{r_0}^{r_1} 4\pi \frac{1}{r^2} \mathrm{d}r + M_0 = -4c\pi \left(\frac{1}{r_1} - \frac{1}{r_0}\right) + M_0 \overset{\infty}{\to} \frac{4c\pi}{r_0} + M_0. \tag{8.3.20}$$

The force on the particle therefore is

$$F = -Gcm_1 \frac{4\pi(1/r_0 - 1/r_1) + M_0}{r_1^2}. \tag{8.3.21}$$

So the behaviour of $M \to \infty$ for $r_1 \to \infty$ only changes going from $n = 3$ to $n = 4$. **TO DO Maybe plots of the distributions would be nice. (!)**

Problem 8.4: *Plummer profile*

In many spherical systems the density is roughly constant near the centre, and falls to zero at large radii. A simple potential with these properties is the Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}, \tag{8.4.22}$$

where b is called the Plummer scale length.

(a) Using Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$, calculate $\rho(r)$ for the Plummer potential. (Note: In spherical coordinates, under the assumption of spherical symmetry, we can write: $\nabla^2\Phi(r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi(r)}{\mathrm{d}r}\right)$.)

(b) Show that indeed $\rho(r \to 0) = const.$

(c) What is the total mass of the system (for $r \to \infty$)?

(d) Calculate the escape velocity (as function of r).

(e) Determine the rotation curve $v_{\mathrm{circ}}(r)$ for the Plummer potential. Where is its maximum $r(v_{\mathrm{circ}}^{\max})$?

(f) A typical globular cluster has a mass of $M = 105 M_\odot$ and a scale length of $b = 10$ pc. How large are the escape velocity and the maximum circular velocity for such a system?

Problem 8.4: Solution

(a) We do as instructed by using the Poisson equation and assuming spherical symmetry, so we have

$$\nabla^2\Phi(r) = \nabla^2\Phi(r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{GM}{\sqrt{r^2+b^2}}\right)\right) \tag{8.4.23}$$

$$= -GM\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\left(-\frac{1}{2}\right)2r\left(r^2+b^2\right)^{-\frac{2}{3}}\right) \tag{8.4.24}$$

$$= \frac{GM}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3\left(r^2+b^2\right)^{-\frac{3}{2}}\right) \tag{8.4.25}$$

$$= \frac{GM}{r^2}\left(3r^2\left(r^2+b^2\right)^{-\frac{3}{2}} - \frac{3}{2}2r^4\left(r^2+b^2\right)^{-\frac{5}{2}}\right) \tag{8.4.26}$$

$$= \frac{3GM}{\left(r^2+b^2\right)^{\frac{3}{2}}}\left(1-\frac{r^2}{r^2+b^2}\right) \tag{8.4.27}$$

$$= \frac{3GMb^2}{\left(r^2+b^2\right)^{\frac{5}{2}}} \stackrel{!}{=} 4\pi G\rho \tag{8.4.28}$$

$$\iff \rho(r) = \frac{3M}{4\pi}\frac{b^2}{\left(r^2+b^2\right)^{\frac{5}{2}}}. \tag{8.4.29}$$

(b) Indeed, we find that

$$\lim_{r\to 0}\rho(r) = \frac{M}{4/3\pi b^3}, \tag{8.4.30}$$

which corresponds to the density of a uniform sphere with mass M and radius b.

(c) We can calculate the enclosed mass via integration (and we rename M from the previous parts to M_0):

$$M(R) = \int_0^R 4\pi\rho(r)r^2\mathrm{d}r \tag{8.4.31}$$

$$= M_0\int_0^R \frac{3b^2r^2}{\left(r^2+b^2\right)^{\frac{5}{2}}}\mathrm{d}r \tag{8.4.32}$$

$$= M_0\frac{R^3}{\left(R^2+b^2\right)^{\frac{3}{2}}} \tag{8.4.33}$$

$$= M_0\left(\frac{1}{1+b^2/R^2}\right)^{\frac{3}{2}} \tag{8.4.34}$$

$$\implies \lim_{R\to\infty} M(R) = M_0. \tag{8.4.35}$$

(d) To derive the escape velocity, we can use the fact that the energy of an escaping test particle of mass m would be 0, so

$$0 \stackrel{!}{=} \frac{1}{2}mv^2 + \Phi(r) = \frac{mv^2}{2} - \frac{Gm}{\sqrt{r^2+b^2}} \tag{8.4.36}$$

$$\iff v_{\text{esc}} = \sqrt{\frac{2G}{\sqrt{r^2+b^2}}}. \tag{8.4.37}$$

(e) For the rotation curve, we start with

$$v_{\text{circ}}^2 = \frac{GM(r)}{r} = \frac{GMr^2}{(r^2+b^2)^{\frac{3}{2}}}. \tag{8.4.38}$$

We arrive at $r_{\text{max}} = b\sqrt{2}$.
Therefore, we find that

$$v_{\text{circ}}^{\text{max}}(r_{\text{max}}) = \sqrt{\frac{2GM}{b3^{\frac{3}{2}}}} = 0.62\frac{GM}{b}. \tag{8.4.39}$$

(f) Plugging in the values, we find

$$v_{\text{esc}} \approx 9\frac{\text{km}}{\text{s}} \tag{8.4.40}$$

$$v_{\text{circ}}^{\text{max}} \approx 4\frac{\text{km}}{\text{s}}. \tag{8.4.41}$$

Problem 8.5: *Effective potential*

(a) The effective potential is defined as $\Phi_{\text{eff}}(r) = \Phi(r) + \frac{l^2}{2r^2}$.
- What motion is described by the effective potential?
- Calculate the effective potential for the Plummer profile.
- Plot this effective potential as a function of radius. Experiment with different values for the parameters. Does the behaviour change qualitatively for particular parameter combinations? If yes, for which ones? If no, why not?
- How does Φ_{eff} behave for $r \to 0$ and $r \to \infty$?

(b) There are only two potentials for which all bound orbits are closed: the Kepler potential and the harmonic potential.
- Show that the harmonic potential indeed has closed orbits.
- What density distribution leads to a harmonic potential?

Problem 8.5: Solution

Thanks and credits to Iliya Tikhonenko!

(a) The energy in generally is

$$\frac{E}{m} = \frac{\dot{\boldsymbol{r}}^2}{2} + \Phi(\boldsymbol{r}). \tag{8.5.42}$$

Due to angular momentum conservation, we can just consider a planar motion and transform into cylindrical coordinates:

Applying the transformations of $R = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$, z (cylindrical coordinates with $x = R\cos\phi$, $y = R\sin\phi$ and therefore

$$\dot{x} = \dot{R}\cos\phi - R\sin\phi\dot{\phi}, \quad \dot{y} = \dot{R}\sin\phi + R\cos\phi\dot{\phi}, \quad \dot{z} = \dot{z}, \tag{8.5.43}$$

so $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2$, where we can identify $l = R^2\dot{\phi}$, so the transformed energy (using $\dot{z} = 0$ in the plane we are considering) is

$$\frac{E}{m} = \frac{\dot{R}^2}{2} + \frac{l^2}{2R^2} + \Phi, \tag{8.5.44}$$

so we have derived the effective potential.
The combined effective Plummer potential is

$$\Phi_{\text{eff}}(r) = -\frac{GM}{\sqrt{r^2 + b^2}} + \frac{l^2}{2r^2}. \tag{8.5.45}$$

The motion described in the effective potential would be elliptical.
At very large radii ($r \to \infty$), the potential approaches 0.
It diverges at $r = 0$.

(b) For a Kepler potential, $\Phi \propto \frac{1}{r}$, while for a harmonic potential, $\Phi \propto r^2$.**TO DO Do this a little more thorough or find good references (should be undergrad textbook stuff) (!)**
The density distribution for a harmonic potential can once again be derived by the Poisson equation (and we once again assume spherical symmetry):

$$\begin{aligned} \nabla^2\left(\frac{C}{2}r^2\right) &= \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 Cr\right) && (8.5.46)\\ &= \frac{1}{r^2}3cr^2 && (8.5.47)\\ &= 3C, && (8.5.48)\end{aligned}$$

where C is the strength of the harmonic potential.
Therefore, the density distribution required for a harmonic potential would be a homogeneous sphere.

Problem 8.6: *Andromeda galaxy nucleus*

The absolute magnitude of the nucleus in the Andromeda galaxy (M31) is $M_{\text{bol}}^{\text{nuc}} = -11.7$ and its stellar mass is $M_{\text{nuc}} = 13 \times 10^6 M_\odot$.

(a) With which luminosity does the nucleus radiate?

(b) Calculate the number of stars in the nucleus, if we assume that all stars in the nucleus are identical and that the luminosity of each star can be obtained from the mass-luminosity relation in the form of $\log(L/L_\odot) = 3.5\log(M/M_\odot)$.

(c) Let us assume that there is a super-massive black hole at the centre of the nucleus. How much material would need to be accreted onto the black hole every second if the total luminosity was only due to matter accretion? Assume that 15 % of the rest-mass energy is emitted as radiation when matter is accreted onto the black hole.

Problem 8.6: Solution

(a) To calculate the luminosity, we take the usual formula (which is also very nicely described **here**, I sometimes use that to remember it better) relating magnitudes and luminosities since we know that $M_{\mathrm{bol}} \approx 4.75$, so we have

$$M_{\mathrm{bol}} - M_{\mathrm{bol},\odot} = -2.5 \log\left(\frac{L}{L_\odot}\right) \tag{8.6.49}$$

$$\iff \quad L_{\mathrm{nuc}} = 10^{0.4\left(M_{\mathrm{bol},\odot} - M_{\mathrm{bol}}\right)} L_\odot \tag{8.6.50}$$

$$= 3.8 \times 10^6 L_\odot. \tag{8.6.51}$$

(b) We approach this problem by writing down what we know:
Since all stars are assumed to be identical with individual masses of $M_\star$ and luminosities of $L_\star$, and since mass and luminosities are additive quantities, we have

$$M_{\mathrm{nuc}} = N M_\star, \quad L_{\mathrm{nuc}} = N L_\star. \tag{8.6.52}$$

Using the mass-luminosity relation that was provided, we find

$$\log\left(\frac{L_\star}{L_\odot}\right) = 3.5 \log\left(\frac{M_\star}{M_\odot}\right) \tag{8.6.53}$$

$$\iff \quad \frac{L_{\mathrm{nuc}}}{N L_\odot} = \left(\frac{M_{\mathrm{nuc}}}{N M_\odot}\right)^{3.5} \tag{8.6.54}$$

$$\iff \quad N^{2.5} = \left(\frac{M_{\mathrm{nuc}}}{M_\odot}\right)^{3.5} \frac{L_\odot}{L_{\mathrm{nuc}}} \tag{8.6.55}$$

$$\iff \quad N = \left(\frac{M_{\mathrm{nuc}}}{M_\odot}\right)^{1.4} \left(\frac{L_{\mathrm{nuc}}}{L_\odot}\right)^{-0.6} \tag{8.6.56}$$

$$= 2.13 \times 10^7. \tag{8.6.57}$$

(c) For this part of the problem, we start with the definition of the luminosity and use $\eta = 0.15$ as the fraction of mass converted into energy when accreted onto the black hole.
Thus, we have[a]

$$L = \frac{\Delta E}{\Delta t} = \eta \dot{M} c^2 \tag{8.6.58}$$

$$\iff \quad \dot{M} = \frac{L_{\mathrm{nuc}}}{\eta c^2} \tag{8.6.59}$$

$$= 1.08 \times 10^{17} \frac{\mathrm{kg}}{\mathrm{s}} = 1.71 \times 10^{-6} \frac{M_\odot}{\mathrm{yr}} \tag{8.6.60}$$

$$= 1.8 \times 10^{13}\, \frac{\text{elephants}}{\text{s}} \tag{8.6.61}$$

$$= 0.278\, \frac{\text{total mass of Earth's oceans}}{\text{h}} \tag{8.6.62}$$

For these last units, we assumed elephants to weigh about $m_{\text{ele}} = 6000$ kg, and used $m_{\text{ocean}} = 1.4 \times 10^{21}$ kg as the approximate mass of all water in the oceans. That event might be a little steamy...

[a]See **here** for the main WolframAlpha calculation

Problem set #9 Orbital mechanics and more (here) 26.01.2023

Problem 9.1: *Jaffe density distribution*

The spherically symmetric density distribution of the singular isothermal sphere,

$$\rho_{\text{sis}}(r) = \frac{\sigma^2}{2\pi G r^2} \tag{9.1.1}$$

is only of limited use for describing astronomical objects such as galaxies or globular because the volume integral (the total mass) diverges. Jaffe (1983) derived a modified density distribution which is more suitable for many analytic studies:

$$\rho_{\text{J}}(r) = \left(\frac{M_{\text{J}}}{4\pi r_{\text{J}}^3}\right)\frac{r_{\text{J}}^4}{r^2(r+r_{\text{J}})^2}. \tag{9.1.2}$$

(a) Prove the divergence of the total mass for the isothermal sphere.
(b) How large is the total mass of the Jaffe density distribution? Which radius contains half of the total mass? Give an expression for the circular velocity as a function of r for this density distribution. Compare this with the circular velocity profile of a point mass.
(c) Calculate the potential that corresponds to the Jaffe density distribution.

Problem 9.1: Solution

(a) Since the density of the isothermal sphere profile goes with $\rho \propto r^{-2}$, the mass contained in it diverges if we integrate over it (due to the $4\pi r^2$ factor in the integral):

$$M_{\text{sis}} = \int_0^\infty 4\pi^2 r^2 \frac{\sigma^2}{2\pi G r^2}\mathrm{d}r = \frac{2\sigma^2}{G} r\Big|_0^\infty \to \infty. \tag{9.1.3}$$

(b) For the Jaffe density distribution, the included mass is

$$\begin{aligned} M_{\text{J}} &= \int_0^\infty \frac{4\pi r^2 M_{\text{J}}}{4\pi r_{\text{J}}^3}\frac{r_{\text{J}}^4}{r^2(r+r_{\text{J}})^2}\mathrm{d}r && (9.1.4)\\ &= M_{\text{J}} r_{\text{J}} \int_0^\infty \frac{1}{(r+r_{\text{J}})^2}\mathrm{d}r && (9.1.5)\\ &= -M_{\text{J}} r_{\text{J}} \frac{1}{r+r_{\text{J}}}\Big|_0^\infty && (9.1.6)\\ &= M_{\text{J}}\frac{r_{\text{J}}}{r_{\text{J}}} = M_{\text{J}}. && (9.1.7) \end{aligned}$$

To find the radius R where the enclosed mass is $M_{\text{J}}/2$, we solve the equation:

$$\begin{aligned} &\int_0^R 4\pi r^2 \rho_{\text{J}}(r) = \frac{M_{\text{J}}}{2} && (9.1.8)\\ \iff\quad & M_{\text{J}} r_{\text{J}}\left(\frac{1}{r_{\text{J}}} - \frac{1}{R+r_{\text{J}}}\right) = \frac{M_{\text{J}}}{2} && (9.1.9) \end{aligned}$$

$$\iff \frac{R}{R+r_\mathrm{J}} = \frac{1}{2} \tag{9.1.10}$$

$$\iff R = r_\mathrm{J}, \tag{9.1.11}$$

so conveniently, the Jaffe density profile already contains the half-mass radius and the total enclosed mass.
To find the circular velocity profile, we use the well-known relation where the centripetal acceleration is given by the gravitational acceleration:

$$\frac{GM(r)}{r^2} = \frac{v_\mathrm{rot}^2}{r} \tag{9.1.12}$$

$$\iff v_\mathrm{rot} = \sqrt{\frac{GM_\mathrm{J}r_\mathrm{J}}{r}\left(1 - \frac{r_\mathrm{J}}{r+r_\mathrm{J}}\right)} = \sqrt{\frac{GM_\mathrm{J}}{r+r_\mathrm{J}}}. \tag{9.1.13}$$

If we set $r_\mathrm{J} = 0$, we reproduce the circular velocity profile for a point mass.
It makes sense that compared to the circular velocity profile of a point mass, the circular velocity in a Jaffe distribution of the same mass is always lower, which makes sense as the mass is more distributed and the enclosed mass is always lower.
For $r \gg r_\mathrm{J}$, the velocity profile approaches the one of a point mass and tends to 0.

(c) Due to spherical symmetry, we know that the following equation holds:

$$\frac{GM(R)}{R^2} = -\frac{\mathrm{d}\Phi}{\mathrm{d}r}\Big|_R \tag{9.1.14}$$

$$\implies \Phi(R) = -\int_R^\infty \frac{GM_\mathrm{J}}{r(r+r_\mathrm{J})}\mathrm{d}r \tag{9.1.15}$$

$$\overset{6}{=} -GM_\mathrm{J}\int_R^\infty \frac{1}{r_\mathrm{J}xr_\mathrm{J}(1+x)}\mathrm{d}xr_\mathrm{J} \tag{9.1.16}$$

$$= -\frac{GM_\mathrm{J}}{r_\mathrm{J}}\int_{R/r_\mathrm{J}}^\infty \left(\frac{1}{x} - \frac{1}{1+x}\right)\mathrm{d}x \tag{9.1.17}$$

$$= -\frac{GM_\mathrm{J}}{R_\mathrm{J}}\ln\left(\frac{x}{1+x}\right)\Big|_{R/r_\mathrm{J}}^\infty \tag{9.1.18}$$

$$= \frac{GM_\mathrm{J}}{r_\mathrm{J}}\ln\left(\frac{R}{R+R_\mathrm{J}}\right). \tag{9.1.19}$$

[6] Substituting $x = \frac{r}{r_\mathrm{J}}$, so $\mathrm{d}x = \frac{\mathrm{d}r}{r_\mathrm{J}}$

Problem 9.2: *Fundamental orbit equation*

The fundamental orbit equation (for the unit mass $m = 1$) is given by

$$\frac{l^2}{r^2}\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right) - \frac{l^2}{r^3} = f(r). \tag{9.2.20}$$

(a) Why is it called and "orbit equation"?

(b) For which force fields $f(r)$ is the circular orbit a solution of the fundamental orbit equation?
(c) Show that Kepler ellipses are a solution of the fundamental orbit equation.
(d) Derive the force law for Kepler ellipses.
(e) Show that the total energy $E = T + \Phi$, with the kinetic energy per unit mass T and the potential Φ, is an integral of the fundamental orbit equation for any force law $f(r)$.

Problem 9.2: Solution

(a) This equation is an orbital equation because it is a time-independent differential equation for r which is only dependent on φ.
This can be shown using Lagrangian mechanics as an ansatz.

(b) For a circular orbit, the solution should be independent of φ ($r(\varphi) = c$, so we require

$$f(r) = -\frac{l^2}{r^3}. \tag{9.2.21}$$

This leads to

$$\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{l^2}{r^3} \implies \varphi = l^2 \int_R^\infty \frac{1}{r^3} = -\frac{l^2}{2R^2}, \tag{9.2.22}$$

so $\varphi_{\mathrm{eff}} = \varphi + \frac{l^2}{2R^2}$.

(c) A Kepler ellipse is given by

$$r = \frac{p}{1 + \epsilon \cos\varphi}, \tag{9.2.23}$$

where p is a constant (some scale factor) and ϵ is the eccentricity.
The derivative is given by

$$\frac{\mathrm{d}r}{\mathrm{d}\varphi} \frac{p\epsilon \sin\varphi}{(1 + \epsilon\cos\varphi)^2}, \tag{9.2.24}$$

so multiplying this with $1/r^2 = \frac{(1+\epsilon\cos\varphi)^2}{p^2}$ and taking the derivative again, we have

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\left(\frac{p\epsilon\sin\varphi}{r^2(1+\epsilon\cos\varphi)^2}\right) = p^3\epsilon\cos\varphi, \tag{9.2.25}$$

so this plugging this into the fundamental orbit equation, we arrive at

$$\frac{l^2(1+\epsilon\cos\varphi)^2}{p^2} p^3\epsilon\cos\varphi - \frac{l^2(1+\epsilon\cos\varphi)^3}{p^3} = f(r) \tag{9.2.26}$$

$$\iff \frac{c}{r^2} = f(r), \tag{9.2.27}$$

which is indeed the proportionality we know for the Kepler potential.

(d) Starting with $f(r) = -\dfrac{\mathrm{d}\varphi}{\mathrm{d}r}$ and the following equation (which comes out of the Lagrangian equations of motion)

$$\ddot{r} - \frac{l^2}{r^3} = f(r), \tag{9.2.28}$$

we can multiply it by $\dot{r}$ and integrate to have the following

$$\frac{\dot{r}^2}{2} + \frac{l^2}{r^2} = \int f(r)\dot{r}\mathrm{d}t \tag{9.2.29}$$

$$= \int -\frac{\mathrm{d}\Phi}{\mathrm{d}r}\dot{r}\mathrm{d}\Phi = -\Phi. \tag{9.2.30}$$

Thus, we have the kinetic energy and potential in polar coordinates:

$$\frac{\dot{r}^2}{2} + \frac{l^2}{r^2} + \Phi = const = \frac{T+\Phi}{m} \tag{9.2.31}$$

Problem 9.3: *Timescales and collisionless systems*

A globular cluster contains about 10^6 stars that move with a typical velocity of 10 km/s. It has a half-light diameter of approximately 10 pc, and thus has a mean density of about $n = 10$ stars/pc^3.

(a) Calculate the timescale τ_c for direct collisions of stars in the globular cluster ($\tau_c = 1/(nv\pi d^2)$) for stars of the diameter of our sun ($R_\odot$) and for red giant stars ($100R_\odot$).

(b) In the centre of a globular cluster, the stellar density can be as high as 500 stars/pc^3. How does this change the collision timescales?

(c) The dynamical timescale of a systems gives the time it takes to go around one orbit in the system. Calculate the dynamical timescale for the cluster.

(d) The relaxation timescale measures the time it takes for one object in a system to be significantly perturbed by the other objects in the system. It is given by

$$T_{\text{relax}} = \frac{v^3}{8\pi G^2 m^2 n \ln\Lambda}, \tag{9.3.32}$$

where $\ln\Lambda$ is called the Coulomb logarithm, and is usually in the range $14...22$. Here we will assume $\ln\Lambda = 15$. Calculate the relaxation time for the globular cluster, and compare it to the dynamical time.

(e) What does this mean for the dynamical state of the cluster?

Problem 9.3: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) Using $d_{\text{sun-like}} = R_\odot$ and $d_{\text{rg-like}} = 100R_\odot$ (and $v = 10$ km/s), $n_{\text{stars}} = 10$ stars/pc^3, we have

$$\tau_{c,\,\text{sun-like}} = 1.53 \times 10^9 \text{ Gyr} \tag{9.3.33}$$

$$\tau_{c,\,\text{sun-like}} = 1.53 \times 10^5 \text{ Gyr} \tag{9.3.34}$$

for the collisional timescales, which is longer than the age of the universe ($T_{\text{universe}} \approx 13$ Gyr), so collisions are near impossible.

(b) Changing $n_{\text{stars}} = 500$ stars/pc^3, we have

$$\tau_{c,\text{ sun-like}} = 3.06 \times 10^7 \text{ Gyr} \tag{9.3.35}$$
$$\tau_{c,\text{ sun-like}} = 3.06 \times 10^3 \text{ Gyr}. \tag{9.3.36}$$

Still longer than the age of the universe.

(c) The dynamical timescale of the cluster can be calculated by using the distance where one typical orbit takes place (e. g. the half-mass radius $r_{1/2} \approx 5$ pc), dividing it by the typical speed:

$$\tau_{\text{dyn}} = \frac{2\pi r_{1/2}}{v} = 3 \times 10^6 \text{ years}. \tag{9.3.37}$$

This is less than the age of the universe (yay!).

(d) Using $v = 10$ km/s as the average velocity again, $m \approx 5M_\odot$ (the average mass of a star in the cluster, which we had derived on an earlier problem sheet), $n = 10$ stars/pc^3, $\ln\Lambda = 15$, we arrive at

$$T_{\text{relax}} = 1.4 \times 10^{10} \text{ yr} \tag{9.3.38}$$

Problem 9.4: *The black-hole-mass-velocity dispersion relation*

There is observational evidence that most elliptical galaxies harbour supermassive black holes at their centres. The observations indicate that the black hole masses correlate with global properties of the galaxies. In the table below you will find measurements of the black hole masses M_{BH} and the velocity dispersions σ of the stellar component for a number of elliptical galaxies. Plot the black hole mass against the velocity dispersion and verify that a correlation exists.

The data can be found in the this repository (note: not by the lecturers, see the data folder) along with some code.

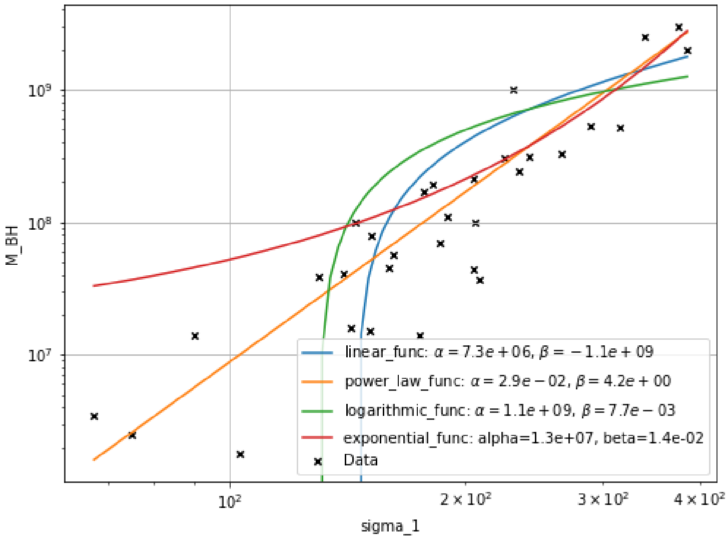
Which of the following functions represents the data best?

- $M_{\text{BH}} = \alpha\sigma + \beta$ (linear)
- $M_{\text{BH}} = \alpha \exp(\beta\sigma)$ (exponential)
- $M_{\text{BH}} = \alpha\sigma^\beta$ (power law)
- $M_{\text{BH}} = \alpha\sigma \log(\beta\sigma)$ (logarithmic)

Determine the parameters α and β of that function.

Problem 9.4: Solution

Note: The code used for parts of this exercise is available in **this repository**.



Both power law fits the data best, which can be seen from the log-log-plot above. Obviously, it has been fitted perfectly.
Here are the fit parameters:

- Power law:

$$M_{\rm BH} = \alpha\sigma^{\beta} : \tag{9.4.39}$$
$$\alpha = 7.33 \times 10^{6} \quad \beta = -1.07 \times 10^{9} \tag{9.4.40}$$

- Exponential:

$$M_{\rm BH} = \alpha \exp(\beta\sigma) : \tag{9.4.41}$$
$$\alpha = 1.3 \times 10^{7} \quad \beta = 0.0139 \tag{9.4.42}$$

- Power law:

$$M_{\rm BH} = \alpha\sigma^{\beta} \tag{9.4.43}$$
$$\alpha = 0.0288 \quad \beta = 4.24 \tag{9.4.44}$$

- Logarithmic:

$$M_{\rm BH} = \alpha\sigma \log(\beta\sigma) \tag{9.4.45}$$
$$\alpha = 1.15 \times 10^{9} \quad \beta = 0.00769 \tag{9.4.46}$$

Problem 9.5: *Major mergers of elliptical galaxies*

Elliptical galaxies can theoretically be described as collisionless stellar systems. Explain what we mean by this. Observations and evolutionary models for elliptical galaxies indicate they can grow by mergers with other small and big ellipticals. In this exercise we will use the virial theorem to estimate how the size, the velocity dispersion, and the mean density of an elliptical galaxy will change if it accretes stars.

- Assuming the systems have isotropic velocity dispersions and no rotation, give expressions for the total kinetic and gravitational energy (and the total energy) of a galaxy as function of Mass M, stellar mean square velocity $\langle v^2 \rangle$, and effective gravitational radius r_{g}.
  (*Hint:* How could an effective radius be suitably defined to describe a galaxy's binding energy?)

Let $\eta = M_{\mathrm{a}}/M_{\mathrm{i}}$ be the ratio of the accreted mass M_{a} to the initial mass M_{i} (so that the final galaxy has a mass $M_{\mathrm{f}} = M_{\mathrm{i}} + M_{\mathrm{a}} = (1+\eta)M_{\mathrm{i}}$), and let $\epsilon = \langle v_{\mathrm{a}}^2 \rangle / \langle v_{\mathrm{i}}^2 \rangle$ be the ratio of the initial average square speeds to the average square speeds of the accreted material.

- What is the energy of the merged galaxy?
- How do the velocities, sizes, and densities change when merging two systems? Compute the final-to-initial ratios of these quantities. Give the general expressions as functions of η and ϵ and numerical values for two identical systems.
- What happens if you have accretion of smaller systems with $\langle v_{\mathrm{a}}^2 \rangle \ll \langle v_{\mathrm{i}}^2 \rangle$? Can the first galaxies have formed through the mergers of single stars?

Problem 9.5: Solution

For this exercise, we only need the virial theorem and the conservation of energy.
Starting with an initial galaxy with parameters $r_{\mathrm{i}}, \langle v_{\mathrm{i}}^2 \rangle, M_{\mathrm{i}}$ and an accreted galaxy with parameters $r_{\mathrm{a}}, \langle v_{\mathrm{a}}^2 \rangle, M_{\mathrm{a}}$, they end up in the final galaxy with subscript f.
We assume that initially, their respective and rotational velocities are 0.
We use the ratios of mass and velocities, again given by

$$\eta = M_{\mathrm{a}}/M_{\mathrm{i}}, \quad \epsilon = \langle v_{\mathrm{a}}^2 \rangle / \langle v_{\mathrm{i}}^2 \rangle, \tag{9.5.47}$$

and also $M_{\mathrm{f}} = (1+\eta)M_{\mathrm{i}}$.
The virial theorem states that

$$\frac{E_{\mathrm{pot}}}{2} = -E_{\mathrm{kin}} \implies -\frac{GM^2}{2r} = -\frac{M\langle v^2 \rangle}{2} \implies \langle v^2 \rangle = \frac{GM}{r}. \tag{9.5.48}$$

Energy conservation also states that the total energy is

$$E_{\mathrm{tot}} = E_{\mathrm{kin}} + E_{\mathrm{pot}} = -\frac{1}{2}M\langle v^2 \rangle = -\frac{1}{2}\frac{GM^2}{r}. \tag{9.5.49}$$

We know that the total energy of the final galaxy has to be the cumulative energy of the initial and the accreted galaxy, so we have

$$E_{\mathrm{tot,\,f}} = E_{\mathrm{tot,\,i}} + E_{\mathrm{tot,\,a}} = -\frac{1}{2}M_{\mathrm{i}}\langle v_{\mathrm{i}}^2 \rangle - \frac{1}{2}M_{\mathrm{a}}\langle v_{\mathrm{a}}^2 \rangle = -\frac{1}{2}M_{\mathrm{i}}\langle v_{\mathrm{i}}^2 \rangle(1+\eta\epsilon). \tag{9.5.50}$$

This has to be equal to the total energy of the final galaxy, so we find

$$-\frac{1}{2}M_\mathrm{i}\langle v_\mathrm{i}^2\rangle(1+\eta\epsilon) \overset{!}{=} -\frac{1}{2}M_\mathrm{f}\langle v_\mathrm{f}^2\rangle \iff \frac{\langle v_\mathrm{f}^2\rangle}{\langle v_\mathrm{i}^2\rangle} = \frac{1+\eta\epsilon}{1+\eta}. \tag{9.5.51}$$

On the other hand, we know that

$$E_\text{tot, f} == -\frac{1}{2}M_\mathrm{i}\langle v_\mathrm{i}^2\rangle(1+\eta\epsilon) = -\frac{1}{2}\frac{GM_\mathrm{f}^2}{r_\mathrm{f}} = -\frac{1}{2}G\frac{M_\mathrm{i}^2(1+\eta)^2}{r_\mathrm{f}}, \tag{9.5.52}$$

so using $\langle v_\mathrm{i}^2\rangle = \frac{GM_\mathrm{i}}{r_\mathrm{i}}$ from the virial theorem, we have

$$\frac{r_\mathrm{f}}{r_\mathrm{i}} = \frac{(1+\eta)^2}{1+\eta\epsilon}. \tag{9.5.53}$$

Assuming spherical distributions, the ratio of densities (using $\rho \propto \frac{M}{r^3}$) is

$$\frac{\rho_\mathrm{f}}{\rho_\mathrm{i}} = \frac{M_\mathrm{f}}{M_\mathrm{i}}\frac{r_\mathrm{i}^3}{r_\mathrm{f}^3} = \frac{(1+\eta)(1+\eta\epsilon)^3}{(1+\eta)^6} = \frac{(1+\eta\epsilon)^3}{(1+\eta)^5}. \tag{9.5.54}$$

We thus have derived three relations for the ratios of velocities, radii and densities.

- If two identical galaxies merge, we have $M_\mathrm{i} = M_\mathrm{a}$, $\langle v_\mathrm{i}^2\rangle = \langle v_\mathrm{a}^2\rangle$, so $\eta = 1$ and $\epsilon = 1$. Therefore, the velocity dispersion of the final galaxy is the same as the one of the initial galaxy since

$$\frac{\langle v_\mathrm{f}^2\rangle}{\langle v\rangle_\mathrm{i}^2} = \frac{1+1}{1+1} = 1. \tag{9.5.55}$$

  For the radius, we find that

$$r_\mathrm{f} = \frac{(1+1)^2}{(1+1)}r_\mathrm{i} = 2r_\mathrm{i}. \tag{9.5.56}$$

  Finally, for the density, we find that

$$\rho_\mathrm{f} = \frac{2^3}{2^5}\rho_\mathrm{i} = \frac{1}{4}\rho_\mathrm{i}, \tag{9.5.57}$$

  so the density is lower by a factor of four.
- If $\langle v_\mathrm{a}^2\rangle \ll \langle v_\mathrm{i}^2\rangle$ with a mass ratio of $\eta = 1$, $\epsilon \approx 0$, leading to

$$\frac{\langle v_\mathrm{f}^2\rangle}{\langle v\rangle_\mathrm{i}^2} = \frac{1}{1+1} = \frac{1}{2} \tag{9.5.58}$$

  for the velocity,

$$r_\mathrm{f} = \frac{(1+1)^2}{(1+0)}r_\mathrm{i} = 4r_\mathrm{i} \tag{9.5.59}$$

  for the radius and

$$\rho_\mathrm{f} = \frac{1^3}{2^5}\rho_\mathrm{i} = \frac{1}{32}\rho_\mathrm{i} \tag{9.5.60}$$

  for the density.

Problem set #10 Galaxy evolution (here)

02.12.2023

Problem 10.1: *Chemical enrichment*

In the first generations of galactic stars the fraction of massive stars ($> 10M_\odot$) which explode as SN II is relatively large. If you assume an average supernova progenitor mass of $25M_\odot$, about $10M_\odot$ per supernova are returned as oxygen. How many one-solar-mass stars can be enriched by one such supernova to a value of [O/H] $= -2$ if you assume that the material is metal-free in the beginning? (The solar oxygen abundance is $\log(\epsilon(\mathrm{O})) = 8.65$.) How many SN II would you need to bring the total baryonic mass of the Milky Way ($\approx 6 \times 10^{10} M_\odot$) to [O/H] $= -2$?
(*Note*:

$$\log(\epsilon(X)) := 12 + \log\left(\frac{N(X)}{N(\mathrm{H})}\right) \tag{10.1.1}$$

$$[X/Y] := \log\left(\frac{N(X)/N(Y)}{(N(X)/N(Y))_\odot}\right), \tag{10.1.2}$$

where the abundance ratios $N(X)/N(Y)$ are number ratios, not mass ratios.

Problem 10.1: Solution

For this exercise, we are provided with a wealth of information, we just need to sort it properly.
First, we are given that $\log(\epsilon(\mathrm{O})) = 8.65$, which leads us to the equation

$$8.65 = 12 + \log\left(\frac{N(\mathrm{O})}{N(\mathrm{H})}\right) \implies \left(\frac{N(\mathrm{O})}{N(\mathrm{H})}\right)_\odot := \eta_{\mathrm{OH},\odot} = 10^{-3.35}. \tag{10.1.3}$$

Also, the requirement is to enrich the stars to $[\mathrm{O/H}] = -2$, leading us to the equation

$$-2 = \log\left(\frac{N(\mathrm{O})}{N(\mathrm{H})}\right) - \log(\eta_{\mathrm{OH},\odot}) = \log(\eta_{\mathrm{OH}}) + 3.35 \implies \eta_{\mathrm{OH}} = 10^{-5.35}. \tag{10.1.4}$$

Also, we can calculate the number of atoms of a species given its total mass via

$$N(X) = \frac{M_X}{m_X}, \tag{10.1.5}$$

where m_X is the specific atomic weight of the species.
Employing this equation for hydrogen ($m_\mathrm{H} = 1u$) and oxygen ($m_\mathrm{O} \approx 16u$), we find that

$$M_\mathrm{H} = N(\mathrm{H})m_\mathrm{H} = \frac{N(O)}{\eta_{\mathrm{OH}}}m_\mathrm{H} = \frac{M_\mathrm{O}}{\eta_{\mathrm{OH}}m_\mathrm{O}}m_\mathrm{H} = 10^{5.35}\frac{1u}{16u}10M_\odot \approx \underline{\underline{1.4 \times 10^5 M_\odot}}, \tag{10.1.6}$$

where we used $M_\mathrm{O} = 10M_\odot$ as the mass returned as oxygen, so 1.4×10^5 solar mass stars could be enriched by such a supernova. Whew.

Now we want to calculate the number of SN II $N_{\rm SNII}$ that are necessary to bring the total baryonic mass to [O/H]=-2.
Luckily, most of the equations above still hold, we just have to adjust the number of oxygen we need, where

$$N(\mathrm{O}) = \frac{N_{\rm SNII} 10 M_\odot}{m_{\rm O}}. \tag{10.1.7}$$

For the Milky Way with $M_{\rm H} \approx 6 \times 10^{10} M_\odot$ (let's neglect all other species), we need

$$N(\mathrm{O}) = \eta_{\rm OH} N(\mathrm{H}) \implies N_{\rm SNII} = \frac{m_{\rm O}}{m_{\rm H}} \eta_{\rm OH} \frac{M_{\rm H}}{10 M_\odot} \approx \underline{\underline{4.3 \times 10^6 \text{ SN II}}}. \tag{10.1.8}$$

Problem 10.2: *Closed box model*

In the closed box model, stars are formed from initially metal-free gas in a confined spatial region. As time goes on, gas is converted into stars which in the next step enrich the ISM homogeneously with heavy elements.
Here we will consider the interstellar gas being consumed and enriched with metals to be a continuous process. Show that for any time t, the relation between the metallicity of the gas and the total gas mass is

$$Z(t) = -Y \ln \left(\frac{M_{\rm g}(t)}{M_{\rm g}(0)} \right), \tag{10.2.9}$$

where Y is the *yield*, i.e., the extra mass of metals produced by a generation of stars and returned to the ISM divided by the mass that remains bound in stars. The mass of the metals in the ISM is $M_{\rm Z}$ and the mass of the interstellar gas is $M_{\rm g}$, with a metallicity defined as $Z := M_{\rm Z}/M_{\rm g}$.
(*Hint*: Use the **instantaneous recycling approximation** where the delay between the formation of a generation of stars and the ejection of metals by SN II can be neglected[a]. If the increase of mass in stars after the massive stars have died is $\mathrm{d}M_{\rm s}$, then the increase of mass in metals in the ISM arising from these stars is

$$\mathrm{d}M_{\rm Z} = Y \mathrm{d}M_{\rm s} - Z \mathrm{d}M_{\rm s} = (Y - Z) \mathrm{d}M_{\rm s}. \tag{10.2.10}$$

Use mass conservation ($\mathrm{d}M_{\rm s} = -\mathrm{d}M_{\rm g}$) and integrate over $\mathrm{d}Z$.

[a] as the SN timescale $\approx 10^7$ yr $\ll t_{\rm Hubble} \approx$ age of the universe

Problem 10.2: Solution

Here again we just need to sort the equations we were given to derive eq. (10.2.9).
Let's first notice that the definition of the metallicity $Z = \frac{M_{\rm Z}}{M_{\rm g}}$ provides us with the following differential representation:

$$\mathrm{d}Z = \frac{1}{M_{\rm g}} \mathrm{d}M_{\rm Z} - \frac{M_{\rm Z}}{M_{\rm g}^2} \mathrm{d}M_{\rm g}. \tag{10.2.11}$$

We can use this as a starting point to insert $M_{\rm Z}$ from eq. (10.2.10), in which we additionally employ mass conservation ($\mathrm{d}M_\mathrm{s} = -\mathrm{d}M_\mathrm{g}$):

$$\begin{aligned}
\mathrm{d}Z &= \frac{1}{M_\mathrm{g}}\left(\mathrm{d}M_\mathrm{Z} - Z\mathrm{d}M_\mathrm{g}\right) \\
&\overset{(10.2.10)}{=} \frac{1}{M_\mathrm{g}}\left((Y-Z)\mathrm{d}M_\mathrm{s} - Z\mathrm{d}M_\mathrm{g}\right) \\
&= \frac{1}{M_\mathrm{g}}\left((Z-Y)\mathrm{d}M_\mathrm{g} - Z\mathrm{d}M_\mathrm{g}\right) \\
&= -Y\frac{\mathrm{d}M_\mathrm{g}}{M_\mathrm{g}}.
\end{aligned}$$

Integrating this differential equation yields

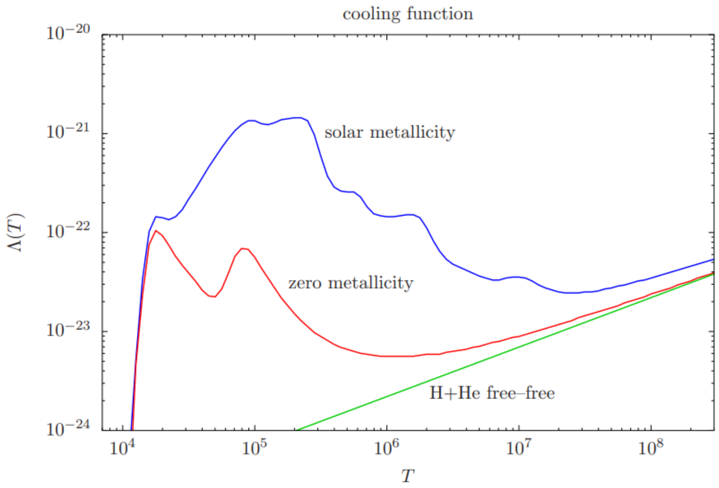
$$Z(t) - Z(0) = -y\left(\ln(M_\mathrm{g}(t)) - \ln(M_\mathrm{g}(0))\right) \tag{10.2.12}$$

$$Z(t) = \underline{\underline{-y\ln\left(\frac{M_\mathrm{g}(t)}{M_\mathrm{g}(0)}\right)}}, \tag{10.2.13}$$

where we used the fact that $Z(0) = 0$ (no chemical enrichment in the beginning).

Problem 10.3: *Radiation from cooling halos*

The cooling rate of hot gas (rate of energy loss per unit volume) can be written as $n^2\Lambda(T)$, where n is the total particle number density (in units of $1/\mathrm{cm}^3$) and $\Lambda(T)$ is the cooling function (the power of 2 on n^2 arises because cooling occurs at a rate that is proportional to the collisions between particles, so long as the gas is optically thin).



For low-metallicity gas hotter than 10^6 K, $\Lambda(T)$ is dominated by free–free cooling and can be approximated by

$$\Lambda(T) \approx 10^{-23}\sqrt{T_7}\ \mathrm{erg\ cm^3/s} \tag{10.3.14}$$

where $T_7 = T/(10^7\ \mathrm{K})$. This behaviour can be applied to the evolution of hot gas in the halo of the Milky Way, where $T_7 = 0.18$ K and n can be approximated by

$$n = 10r_{\text{kpc}}^{-2}\ \text{cm}^{-3}, \tag{10.3.15}$$

with $r_{\text{kpc}} = r/(1\ \text{kpc})$. Outside of a radius of about $r \approx 90$ kpc, radiative losses cause gas to condense and move inward rather than actually becoming colder.
This is referred to as a cooling flow (inside of this radius the temperature of the gas does decrease considerably).

- Estimate the flux of cooling radiation (in ergs per second per cm^2 per steradian) an observer at the centre of the cooling halo would expect to see.
  How does this compare with the observed value of about 2×10^{-8} erg cm^{-2}sr^{-1}s^{-1} from X-ray observations[a]?

(*Hint*: An answer can be obtained from the theory of radiative transfer, in which the sky brightness from an extended optically thin emitting source is $1/(4\pi)$ times the integral of its volume emissivity along the line of sight.)

[a] the observed value is from Cui et al. 1996.

Problem 10.3: Solution

This exercise offers some confusing unit tricks, but is overall on the easier side.
The hint gives us all the ideas we need: For the observer, the only cooling radiation that can be seen emerges from a radius > 90 kpc, and is given by the integral of the cooling rate over the line of sight, normalised by $1/(4\pi)$, which also introduces the unit of sr^{-1}:

$$F = \frac{1}{4\pi}\int_{r_{\text{min}}}^{\infty} \Gamma(r, T)\text{d}r\ \text{sr}^{-1}. \tag{10.3.16}$$

The volume emissivity is in this case given by $\Gamma(r, T) = n^2(r)\Lambda(T)$, where T will now be assumed to be constant as $T = 1.8 \times 10^6$ K in the halo of the Milky Way.
Putting the values and the two relations for Λ and n into this expression, we have

$$\begin{aligned}
F &= \frac{1}{4\pi}\int_{r_{\text{min}}}^{\infty} (10r_{\text{kpc}}^{-2}\text{cm}^{-3})^2\Lambda(t)\text{d}r\ \text{sr}^{-1} && (10.3.17)\\
&= \frac{1}{4\pi}100\cdot 10^{-23}\sqrt{T_7}\frac{\text{erg cm}^3}{\text{s sr cm}^6}\left[-\frac{1}{3r_{\text{kpc}}^3}\text{kpc}\right]_{r_{\text{min}}}^{\infty} && (10.3.18)\\
&= \frac{10^{-21}}{12\pi}\sqrt{0.18}\frac{1}{90^3}\frac{\text{erg kpc}}{\text{s sr cm}^3} && (10.3.19)\\
&= \underline{\underline{4.76 \times 10^{-8}\ \text{erg cm}^{-2}\text{sr}^{-1}\text{s}^{-1}}}. && (10.3.20)
\end{aligned}$$

One dangerous part to note that due to our weird unit handling, we had to introduce the factor of kpc after the integration over r as we basically had to perform a substitution with $r_{\text{kpc}} = r/(1\text{kpc})$.
This is about double the observed value from X-ray observations, which can be explained by the fact that we're considering the integrated flux from all wavelengths here.

Problem 10.4: *Clusters of galaxies in virial equilibrium*

Consider a galaxy cluster in virial equilibrium. Using spectroscopic redshifts, you measure the central velocity dispersion of cluster member galaxies along the line of sight as σ_r.

- Assuming that the cluster is a homogeneous sphere of mass M_{vir} and radius R, derive the relation

$$\sigma_r^2 = \frac{GM_{\text{vir}}}{5R}. \tag{10.4.21}$$

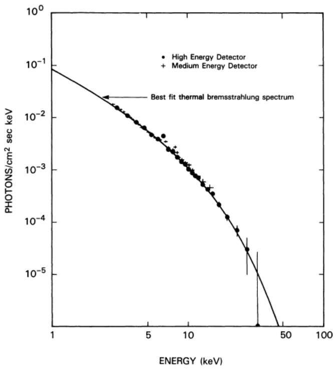
  What further conditions must you assume for this relation to be valid?

- Use the values from the following table to determine the virial masses M_{vir} of the clusters. Compare your result with the stellar mass of both systems, assuming a stellar mass-to-light ratio of 3 in solar units.

Cluster	distance (Mpc)	angular diameter	L_{V} ($L_\odot$)	σ_r (km/s)	L_{X} (erg/s)	T_{X} (K)
Virgo	17	10°	1.2×10^{12}	670	1.5×10^{43}	7.0×10^{7}
Coma	90	4°	5.0×10^{12}	980	5×10^{44}	8.8×10^{7}

- Both clusters have a significant X-ray luminosity.

  The figure shows the measured X-ray spectral energy distribution of the Coma cluster. This is well-fit as thermal bremsstrahlung from hot intra-cluster gas with an emission temperature of $T_{\text{X}} = 8.8 \times 10^7$ K.

  The X-ray luminosity of an isothermal sphere of radius R, temperature T_{X}, and an electron density n_e is

$$L_{\text{X}} = \alpha n_e^2 \sqrt{T_{\text{X}}} R^3, \tag{10.4.22}$$

  where $\alpha = 5.95 \times 10^{-27} \frac{\text{cm}^5\text{g}}{\text{s}^3\text{K}^{1/2}}$.
  Calculate the electron density for both clusters in this approximation. From this, calculate the total mass of X-ray emitting hot gas.
  Compare this mass to the virial and stellar mass. What percentage of the virial mass must be in the form of dark matter?

- Assume that the hot gas is losing energy with constant luminosity L_{X} only by thermal bremsstrahlung. How long does it take until all thermal energy is radiated away? Compare this value to the age of the universe.

Problem 10.4: Solution

Note: The code used for parts of this exercise is available in **this repository**.
Assuming that the cluster is a homogeneous sphere, the total kinetic energy and total potential energy (**TO DO Put in a derivation of spherical kinetic energy (!)**) is

$$E_{\text{kin}} = \frac{M\sigma^2}{2}, \quad E_{\text{pot}} = -\frac{3}{5}\frac{M^2 G}{R}. \tag{10.4.23}$$

Here, σ is the mean velocity of the galaxies, and we can assume $\sigma^2 = 3\sigma_r^2$ for the radial velocity dispersion.

The virial theorem gives us the relation $E_{\text{kin}} = -\frac{1}{2}E_{\text{pot}}$, so we find

$$\frac{3M\sigma_r^2}{2} = \frac{1}{2}\frac{3M^2G}{5R} \iff \underline{\underline{\sigma_r^2 = \frac{GM_{\text{vir}}}{5R}}}. \tag{10.4.24}$$

The values can be found in the table below:

Cluster	Virgo	Coma
Distance [kpc]	17	90
Angle	$10°$	$4°$
L [$L_\odot$]	1.200×10^{12}	5.000×10^{12}
σ_r [km/s]	670	980
L_X [erg/s]	1.5×10^{43}	5.0×10^{44}
T_X [K]	7.00×10^{7}	8.80×10^{7}
R [Mpc]	1.49	3.14
M_{vir}	7.762×10^{14}	3.509×10^{15}
M_{lum}	4.00×10^{12}	1.5×10^{13}
n_e [cm^{-3}]	5.58×10^{-5}	9.91×10^{-5}
M_X [$M_\odot$]	1.902×10^{13}	3.185×10^{14}
$M_{\text{bary}}/M_{\text{vir}}$	0.029	0.095
E_{therm} [erg]	3.277×10^{62}	6.900×10^{63}
t_{brems} [yr]	6.92×10^{11}	4.37×10^{11}
t_{cross} [yr]	2.171×10^{09}	3.136×10^{09}

Assuming constant cooling by bremsstrahlung with constant luminosity of L_X, we find that the values ($\sim 7 \times 10^{11}$ yr) are much more than the age of the universe.

Problem 10.5: *Velocity dispersion and virialisation*

Clusters of galaxies have virialisation timescales that correspond to several times their dynamic free-fall times. Instead of the free-fall time, one can also use the velocity dispersion of member galaxies to draw conclusions about the virialisation timescale. For this purpose, assume that a galaxy is moving with a mean velocity σ through a cluster.

- How long does it take for a galaxy to cross the Virgo or Coma cluster? (Assume cluster sizes and velocity dispersions of approximately $R_{\text{V}} = 1.5$ Mpc and $R_{\text{V}} = 3.1$ Mpc, and $\sigma_{\text{V}} = 670$ km/s and $\sigma_{\text{C}} = 980$ km/s for Virgo and Coma, respectively.) Compare to the age of the universe.
- Superclusters and similar structures have sizes of 100 Mpc and larger. Can you use the assumption of virialisation for determining their masses, if the velocity dispersions are of the same order (~ 1000 km/s) as for galaxy clusters? Why or why not?
- How about giant elliptical galaxies ($R = 10 \ldots 100$ kpc, $\sigma_0 = 250$ km/s)?

Problem 10.5: Solution

Note: The code used for parts of this exercise is available in **this repository**.
The values can be found in the table above.
To calculate the crossing time t_{cross}, we simply divide the distance by the velocity dispersion (which isn't the mean velocity, but a good estimate:

$$t_{\text{cross}} \approx \frac{2R}{\sigma}. \tag{10.5.25}$$

- For the Virgo and the Coma cluster, we get

$$t_{\text{cross, V}} = 4.38 \times 10^9 \text{ yr}, \quad t_{\text{cross, C}} 6.19 \times 10^9 \text{ yr}, \tag{10.5.26}$$

  so maybe our assumption of virialisation is not right since the galaxies cannot have crossed the cluster more than twice during the Hubble time.
- It's even worse for superclusters, where we use $d = 2R = 1000$ MpC

$$t_{\text{cross}} = 9.5 \times 10^{11} \text{ yr}, \tag{10.5.27}$$

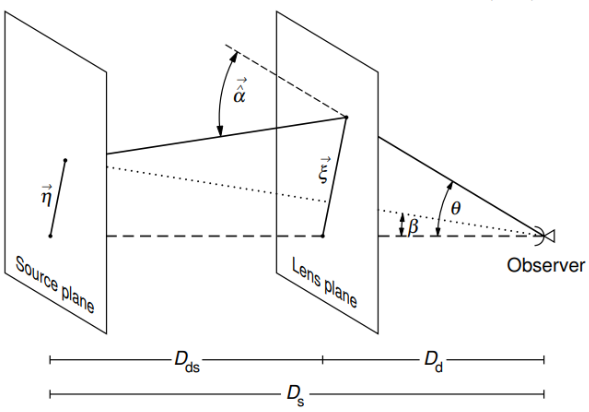
  so the assumption of virialisation should be incorrect.
- For $R = 10$ kpc and $\sigma \approx 250$ km/s, we have

$$t_{\text{cross}} = 3.8 \times 10^7 \text{ yr}, \tag{10.5.28}$$

  and a factor of 10 more for $R = 10$ kpc.
  Depending on when the elliptical galaxies formed, virialisation might be a valid assumption for them.

Problem 10.6: *Gravitational lens equations*

The gravitational lens equation maps observed image positions θ to (true) source positions β.



The lens equation can be written as (note that this is a two-dimensional vector equation)

$$\boldsymbol{\eta} = \frac{D_\mathrm{s}}{D_\mathrm{d}}\boldsymbol{\xi} - D_\mathrm{ds}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \tag{10.6.29}$$

where $\boldsymbol{eta}$ and $\boldsymbol{\xi}$ are vectors in the source and lens/image plane (see figure) and D_s, D_d, and D_ds are the distances between observer and source, observer and lens, and lens and source, respectively. The angle $\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ is the light deflection an observer would measure if they were "sitting in the lens",

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2}\int \Sigma(\boldsymbol{\xi}')\frac{\boldsymbol{\xi}-\boldsymbol{\xi}'}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^2}\mathrm{d}^2\boldsymbol{\xi}', \tag{10.6.30}$$

where $\Sigma(\boldsymbol{\xi})$ is the surface mass density (the mass density of the lens integrated in the direction of the line of sight) as a function of position $\boldsymbol{\xi}$ in the lens plane.

- From this general representation, derive the equation for $\beta(\theta)$ for the special case of a point mass M at $\boldsymbol{\xi} = 0$.
  (*Hint*: Look at the sketch above. Note also that in this case the lens configuration is symmetric with respect to rotation along the line of sight to the lens centre, and the vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ can therefore be represented as scalars measuring the distance from this line of sight.)
- Solve this equation for the Einstein angle $\theta_\mathrm{E} = \theta$ obtained for $\beta = 0$. Give a geometric equation of the Einstein radius R_E.
- Write the lens equation $\beta(\theta)$ in terms of the Einstein angle θ_E.
  When does it have one solution, and when does it have two? Write down the solutions. What does the sign of each of the solutions signify?
  Sketch what the observer sees. For the situation where two solutions exist, calculate the angular separation of the two images.

Problem 10.6: Solution

TO DO Transcribe what I've written down on my iPad (!)

Problem set #11 Cosmology and expanding confusion (here)

10.02.2023

Problem 11.1: *Recombination redshift*

When the universe expanded and cooled, the electrons ("re") combined with baryons and formed neutral atoms.

- Determine at which redshift z this recombination occurred in the early universe. For simplicity, assume the universe was pure hydrogen (no helium or metal nuclei) at that time. The average baryon density today is $n_0 = 2 \times 10{-}7$ cm^{-3}, the temperature of the microwave background is $T_0 = 2.73$ K.

(*Hint*: How do density and radiation temperature vary with redshift? What were the conditions in the universe before recombination?)

Problem 11.1: Solution

For temperature and number density, we note that

$$n = n_0 \left(1 + z\right)^3 , \quad T = T_0 \left(1 + z\right) . \tag{11.1.1}$$

Assuming local thermodynamical equilibrium, the fraction of ionisation x is given by the Saha equation:

$$x = \tag{11.1.2}$$

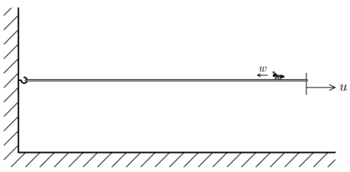
Solving for an ionisation fraction of $x = 0.5$, we find that

$$z_{\rm rec} = 1400 \tag{11.1.3}$$

See also **here**TO DO Actually calculate stuff. Sorry (!)

Problem 11.2: *Ant on a rubber rope*

One end of an initially 1 m long infinitely stretchable rubber rope is anchored to a wall, while the other end is pulled away from the wall with a constant velocity of $u = 1$ cm/s. Starting out at the moving end of the rope is an ant that walks on the rope with a velocity of $w = 1$ mm/s (relative to the piece of rope it is currently on) towards the wall (see sketch).



- Derive an equation describing the motion of the ant as seen from a stationary observer, and compute the position of the ant as a function of time. Will the ant reach the wall? If yes, how long will it take? If no, how must the parameters of the system (length of the rope, velocities of the ant and the pull on the rope) be modified so that it can?

Note that an analytic solution exists for the above case of $u = const$. You can solve the equation numerically to explore different scenarios of $u = u(t)$ varying in time. What cosmological problem is this situation an analogy of?

Problem 11.2: Solution

To determine the time it takes for the ant to reach the end of the expanding rope, we consider the equation for its velocity $\dot{x}$.
Obviously, it has some constant component w in negative x-direction, but how do we factor in the stretch?
Let the total length of the rope be $X(t) = x_0 + ut$, where $x_0 = 1$ m is the initial length. Then, the velocity of the ant has an additional component that is determined by the expansion velocity u and the fraction of the rope the ant is currently located at.
Thus, $\dot{x}$ is given by

$$\dot{x} = -w + u\frac{x}{X} = -w + u\frac{x}{x_0 + ut}. \tag{11.2.4}$$

This is a linear ordinary differential equation (LODE) of the form $\dot{x} = p(t)x + q(t)$, which can be solved as these equations are usually solved:

- **Homogeneous solution**:
  First, we set $q(t) = -w$ to 0 and simply integrate the remaining equation, yielding

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{u}{x_0 + ut}x \tag{11.2.5}$$

$$\iff \int \frac{\mathrm{d}x}{x} = \int \frac{\mathrm{d}t}{t + x_0/u} \tag{11.2.6}$$

$$\iff \ln(x) = \ln(t + x_0/u) + c_0 \quad (c_0 \in \mathbb{R}) \tag{11.2.7}$$

$$\iff x(t) = c_1(t + x_0/u), \quad (c_1 \in \mathbb{R}) \tag{11.2.8}$$

  where c_0 and c_1 are constants.
- **Inhomogeneous solution** (variation of constants):
  For the inhomogeneous solution, we assume the 'constant' c_1 of eq. (11.2.8) to be time-dependent ($c_1(t)$), differentiate the solution, and plug in eq. (11.2.4):

$$\dot{x} = \dot{c}_1(t + x_0/u) + c_1 \tag{11.2.9}$$

$$\overset{(11.2.4)}{\iff} -w + u\overbrace{\frac{x}{x_0 + ut}}^{=\frac{c_1(t+x_0/u)}{t+x_0/u}} = \dot{c}_1(t + x_0/u) + c_1 \tag{11.2.10}$$

$$\overset{(11.2.8)}{\iff} \dot{c}_1 = -\frac{w}{t + x_0/u} + c_1 - c_1 \tag{11.2.11}$$

$$\implies c_1(t) = -w\ln(t + x_0/u) + c_2, \quad (c_2 \in \mathbb{R}), \tag{11.2.12}$$

  where we pick up c_2 as the independent integration constant.
- **Full general solution**:
  The full solution of the LODE is then simply given by

$$x(t) = (t + x_0/u)(c_2 - w\ln(t + x_0/u)). \tag{11.2.13}$$

- **Taking care of initial conditions**:
  In our case, we have the initial condition that $x(t = 0) \overset{!}{=} x_0$, enabling us to determine c_2:

$$x_0 = \frac{x_0}{u}\left(c_2 - w \ln\left(0 + \frac{x_0}{u}\right)\right) \tag{11.2.14}$$

$$\iff c_2 = u + w \ln\left(\frac{x_0}{u}\right). \tag{11.2.15}$$

  Thus, the full solution, considering the initial conditions, is given by

$$x(t) = (ut + x_0)\left(1 - \frac{w}{u}\ln\left(\frac{ut}{x_0} + 1\right)\right). \tag{11.2.16}$$

- Sanity checks:
  This equation surprisingly even recovers the solution we'd expect for $\lim_{u\to 0}$:
  We need to know that we can expand

$$\ln(y+1) = \sum_{k=1} \frac{(-1)^{k+1} y^k}{k} = y - \frac{y^2}{2} + \frac{y^3}{3} + \dots, \tag{11.2.17}$$

  so $\ln(ut/x_0 + 1) \approx ut/x_0$ for small u, so the solution becomes[a]

$$\lim_{u\to 0} x(t) \approx \lim_{u\to 0}(ut + x_0)\left(1 - w\frac{ut}{ux_0}\right) \tag{11.2.18}$$

$$= x_0 - wt. \tag{11.2.19}$$

Now that we have a nice solution for $x(t)$ describing the position of the ant on the rope at all times, we can solve for $x(t) = 0$ (i. e. the time when the ant would reach the start):

- We find

$$0 = \overbrace{(ut + x_0)}^{=0 \quad (i)} \overbrace{\left(1 - \frac{w}{u}\ln\left(\frac{ut}{x_0} + 1\right)\right)}^{=0 \quad (ii)} \tag{11.2.20}$$

$$(i) \qquad t_1 = -\frac{x_0}{u} \quad \text{(not physical as } t > 0\text{)} \tag{11.2.21}$$

$$(ii) \qquad \frac{u}{w} = \ln\left(\frac{ut_2}{x_0} + 1\right) \tag{11.2.22}$$

$$\iff t_2 = \frac{x_0}{u}\left(e^{\frac{u}{w}} - 1\right). \tag{11.2.23}$$

  The surprising result is therefore that no matter how great the difference of u and w, the ant will always reach its goal (although we might have to provide it with some snacks along the way).

- Sanity check:
  For $u \to 0$, we can expand $\exp(u/w) \approx 1 + u/w$, so

$$\lim_{u\to 0} t_2 \approx \frac{x_0}{u}(1 + u/w - 1) = \frac{x_0}{w}, \tag{11.2.24}$$

  so again we recovered the solution we'd expect.

- Numerical values:
  Plugging in the numbers, we obtain $x_0/u = 100$ s, $u/w = 10$, so

$$t_2 = 100\left(e^{10} - 1\right) \text{ s} = 2.2 \times 10^6 \text{ s } = 25.5 \text{ days.} \tag{11.2.25}$$

  By that time, the rope will have grown to a whopping

$$X(t_2) = x_0 + ut_1 = 22 \text{ km...} \tag{11.2.26}$$

  That must be some sturdy material, where can I get it?

Alright, bad remarks aside, let's discuss the last part of the problem:
If $u = u(t)$ isn't constant but rather grows, the ant might not reach its goal at all. This is reminiscent of the accelerated expansion of space, implying that we might never be able to see light sent out by stars too far away due to the constant speed the light can travel.

TO DO Numerical analysis and plotting (!)

[a] I know this isn't mathematically rigorous, but I just wanted to show that this ugly beast can be tamed.

Problem 11.3: *Expansion law for a flat universe*

The parameterised Friedmann equation is

$$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_\mathrm{r}}{a^4} + \frac{\Omega_\mathrm{m}}{a^3} + \frac{1 - \Omega_\mathrm{m} - \Omega_\Lambda - \Omega_\mathrm{r}}{a^2} + \Omega_\Lambda. \tag{11.3.27}$$

(a) Assume a flat universe without radiation. Rewrite the Friedmann equation for this case substituting $y = a^{\frac{3}{2}}$.
(b) Introduce the scaled time variable $x = t/t_\mathrm{acc}$, where t_acc is some characteristic time, and replace the time derivative in the above equation with $\dfrac{\mathrm{d}}{\mathrm{d}x}$.
By appropriately choosing t_acc, write the equation in the form

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = C + y^2, \tag{11.3.28}$$

where C is a constant.
(c) Solve the differential equation for y (*Hint*: $\cosh^2 x - \sinh^2 x = 1$).
Derive the expansion law $a(t)$.
(d) Determine the asymptotic form of $a(t)$ for early and late times[a].
(e) Plot $a(t)$ for different cosmological parameter combinations.

[a] These limits should be consistent with a matter-dominated universe in the first case and a Λ-dominated one in the second.

Problem 11.3: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) For a flat universe without radiation, $\Omega_\mathrm{r} = 0$ and $\Omega_\mathrm{m} + \Omega_\Lambda = 1$.
Let us also already substitute $H = \frac{\dot{a}}{a}$, so the equation simplifies to

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left(\Omega_\mathrm{m} a^{-3} + \Omega_\Lambda\right). \tag{11.3.29}$$

Thus, substituting $y = a^{\frac{3}{2}}$ (or rather, $a = y^{\frac{2}{3}}$) and using the chain rule with $\dot{a} = \frac{2}{3} y^{-\frac{1}{3}} \dot{y}$, we obtain

$$\left(\frac{2}{3} \frac{\dot{y}}{y^{\frac{2}{3}} y^{\frac{1}{3}}}\right)^2 = H_0^2 \left(\Omega_\mathrm{m} y^{-2} + \Omega_\Lambda\right). \tag{11.3.30}$$

Solving for $\dot{y}^2$, we have

$$\dot{y}^2 = \frac{9}{4} H_0^2 \left(\Omega_\mathrm{m} + y^2 \Omega_\Lambda\right). \tag{11.3.31}$$

(b) Let us rewrite the equation once again to see that

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t} \frac{2}{3H_0\sqrt{\Omega_\Lambda}}\right)^2 = \frac{\Omega_\mathrm{m}}{\Omega_\Lambda} + y^2. \tag{11.3.32}$$

Thus, we have found that $t_\mathrm{acc} = \frac{2}{3H_0\sqrt{\Omega_\Lambda}}$, and with $x = t/t_\mathrm{acc} \implies \mathrm{d}x = \mathrm{d}t/t_\mathrm{acc}$, we arrive at

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = y^2 + \frac{\Omega_\mathrm{m}}{\Omega_\Lambda}, \tag{11.3.33}$$

so the constant C in the above solution to derive is $C = \frac{\Omega_\mathrm{m}}{\Omega_\Lambda}$.

(c) To solve this equation, we simply integrate it after separation and substitute with a sinh to use $(*) : 1 + \sinh^2(x) = \cosh^2(x)$ as this alleviates the nasty term in the denominator:

$$\int \frac{\mathrm{d}y}{\sqrt{C + y^2}} = \int \mathrm{d}x \tag{11.3.34}$$

$$\iff \quad \frac{1}{\sqrt{C}} \int \frac{\mathrm{d}y}{\sqrt{1 + (y/\sqrt{C})^2}} = x - x_0 \tag{11.3.35}$$

$$\overset{y=\sqrt{C}\sinh(z)}{\iff} \quad \frac{1}{\sqrt{C}} \int \frac{\mathrm{d}z \sqrt{C}\cosh(z)}{\sqrt{1 + \sinh^2(z)}} = x - x_0 \tag{11.3.36}$$

$$\overset{(*),\, \cosh(z)>0\, \forall z}{\iff} \quad \int \mathrm{d}z = x - x_0 \tag{11.3.37}$$

$$\overset{z=\operatorname{arsinh}(y/\sqrt{C})}{\iff} \quad \operatorname{arsinh}(y/\sqrt{C}) = x - c_1 \quad (c_1 \in \mathbb{R}) \tag{11.3.38}$$

$$\iff \quad y(x) = \sqrt{C}\sinh(x - c_1), \tag{11.3.39}$$

where $c_1 = x_0 - \mathrm{arsinh}(y_0/\sqrt{C})$ is a constant depending on the initial conditions. To derive the expansion law, we have to resubstitute $y = a^{\frac{3}{2}}$ and $x = t/t_{\mathrm{acc}}$, finding

$$a(t) = \left(\frac{\Omega_{\mathrm{m}}}{\Omega_\Lambda}\right)^{\frac{1}{3}} \left(\sinh\left(\frac{t}{t_{\mathrm{acc}}} - c_1\right)\right)^{\frac{2}{3}} . \tag{11.3.40}$$

Kind of ugly, I'd say...

The initial conditions for our problem are $a(t \to 0) = 0$, implying that $c_1 \overset{!}{=} 0$ as $\mathrm{arsinh}(0) = 0$, so the final answer is

$$a(t) = \left(\frac{\Omega_{\mathrm{m}}}{\Omega_\Lambda}\right)^{\frac{1}{3}} \left(\sinh\left(\frac{t}{t_{\mathrm{acc}}}\right)\right)^{\frac{2}{3}} . \tag{11.3.41}$$

(d) To study the behaviour at early and late times, we note that

$$\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right) , \tag{11.3.42}$$

which has the following behaviour:

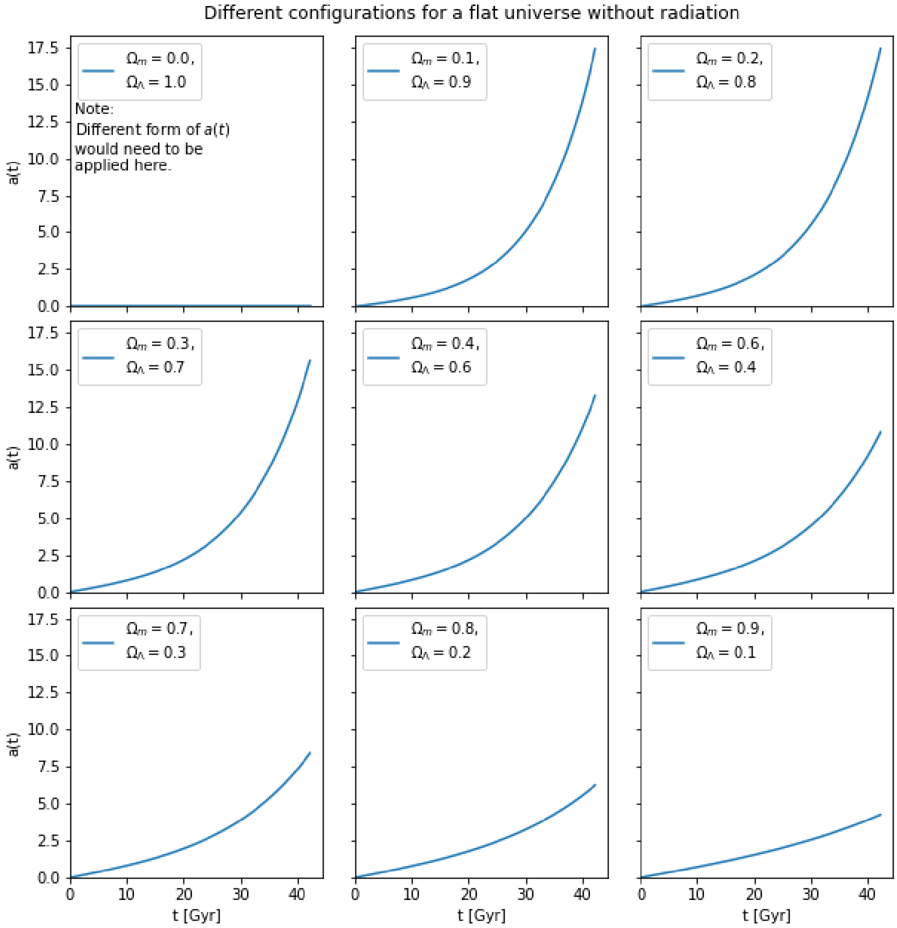
- **Early times**:
  For $x \to 0$, $\sinh(x) \approx \frac{1}{2}(1 + x - (1 - x)) = x$, so

  $$t \to 0 \implies a(t) \approx \left(\frac{\Omega_{\mathrm{m}}}{\Omega_\Lambda t_{\mathrm{acc}}^2}\right)^{\frac{1}{3}} t^{\frac{2}{3}} = \left(\frac{3}{2} H_0 \sqrt{\Omega_{\mathrm{m}}} t\right)^{\frac{2}{3}} , \tag{11.3.43}$$

  which is independent of Ω_Λ (since $t_{\mathrm{acc}}^2 = \frac{2^2}{3^2 H_0^2 \Omega_\Lambda}$, it cancels out).
  This indeed reproduced the solution for the matter-dominated universe.
- **Late times**:
  For $x \to \infty$, the exponential term of the sinh dominates, so $\sinh(x) \approx \frac{1}{2}e^x$, leading to

  $$t \to \infty \implies a(t) \approx \left(\frac{\Omega_{\mathrm{m}}}{8\Omega_\Lambda}\right)^{\frac{1}{3}} \exp\left(\frac{2}{3} t/t_{\mathrm{acc}}\right) = \left(\frac{\Omega_{\mathrm{m}}}{8\Omega_\Lambda}\right)^{\frac{1}{3}} \exp\left(H_0 \sqrt{\Omega_\Lambda} t\right) . \tag{11.3.44}$$

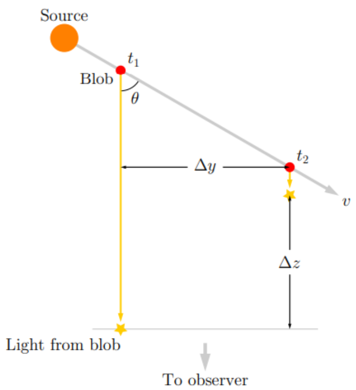
  This is the exponential expansion we'd indeed expect from a Λ-dominated universe (we note that for $\Omega_{\mathrm{m}} = 0$, the differential equation would already look a little different).
- For the timesteps in the plot, we use t in terms of $1/H_0$ from 0 to 3 Hubble times:



We see that for higher values of Ω_Λ, the exponential growth dominates earlier.

Problem 11.4: *Superluminal motion*

Jets from AGN sometimes show superluminal motion that appears to move on the sky faster than the speed of light.



A source ejects a blob in a direction tilted from the observer's line of sight by the angle θ. The blob moves with speed v.

(a) The light emitted by the blob at t_1 precedes the light emitted by the blob at t_2 by Δz. Show that

$$\Delta z = c(t_2 - t_1) - v(t_2 - t_1)\cos\theta, \tag{11.4.45}$$

and therefore

$$\Delta t = (t_2 - t_1) - \frac{v}{c}(t_2 - t_1)\cos\theta \tag{11.4.46}$$

in time.

(b) Show that the apparent speed of the blob that the observer sees is

$$v_y = \frac{\Delta y}{\Delta t} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta}. \tag{11.4.47}$$

(c) If you see superluminal motion at all, you can tell how much the minimum actual speed of the blob must be, regardless of your viewing angle. How much is it?

Problem 11.4: Solution

(a) From trigonometric considerations and by labelling the sides of the triangle as follows: $s := v(t_2 - t_1)$ as the distance between t_1 adn t_2, a as the vertical distance between the blob at t_1 and t_2, and l as the vertical distance between the Blob at t_1 and the observer.
In this case, we simply find that

$$\Delta z = l - a = c(t_2 - t_1) - \cos\theta v(t_2 - t_1). \tag{11.4.48}$$

Using $\Delta t = \frac{\Delta z}{c}$, we find the results given in the assignment as well:

$$\Delta t = (t_2 - t_1) - \frac{v}{c}(t_2 - t_1)\cos\theta. \tag{11.4.49}$$

(b) The apparent speed of the blob at the observer is simply

$$v_y = \frac{\Delta y}{\Delta t} = \frac{\sin\theta v(t_2 - t_1)}{(t_2 - t_1) - \frac{v}{c}(t_2 - t_1)\cos\theta} = \frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta}. \tag{11.4.50}$$

(c) For us to be see any superluminal motion, the observed v_y has to be greater than the speed of light.

$$v_y > c \tag{11.4.51}$$

$$\frac{v\sin\theta}{1 - \frac{v}{c}\cos\theta} > c \tag{11.4.52}$$

$$\frac{v}{c}\sin\theta > (1 - \frac{v}{c}\cos\theta) \tag{11.4.53}$$

$$\frac{v}{c}(\sin\theta + \cos\theta) > 1 \tag{11.4.54}$$

$$v > \frac{c}{\sin\theta + \cos\theta} \tag{11.4.55}$$

$$\implies v_{\min} = \frac{c}{\sqrt{2}}. \tag{11.4.56}$$

Problem 11.5: *Lorentz transformation and beaming*

K and K' are inertial frames; K' moves with a constant velocity v with respect to K along the x axis. The Lorentz transformation from K to K' is given by

$$x' = \gamma(x - vt) \tag{11.5.57}$$
$$y' = y \tag{11.5.58}$$
$$z' = z \tag{11.5.59}$$
$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \tag{11.5.60}$$

where $\gamma := \frac{1}{\sqrt{1-v^2/c^2}}$.

(a) Find the forms for x, y, z, and t as functions of x', y', z', and t'.
(b) Find the differentials $\mathrm{d}x, \mathrm{d}y, \mathrm{d}z$, and $\mathrm{d}t$ as functions of $\mathrm{d}x', \mathrm{d}y', \mathrm{d}z'$, and $\mathrm{d}t'$.
(c) Find the velocities u_x, u_y, and u_z as functions of u'_x, u'_y, and u'_z.
(d) Generalise the velocities above to the velocities parallel and perpendicular to the motion of the frame K',

$$u_\parallel = \frac{u'_\parallel + v}{1 + vu'_\parallel/c^2}, \quad u_\perp = \frac{u'_\perp}{\gamma\left(1 + vu'_\parallel/c^2\right)}. \tag{11.5.61}$$

Define θ by $\tan\theta = \frac{u_\perp}{u_\parallel}$.
Draw a sketch of the geometry, indicating θ.
(e) Find $\tan\theta$ as a function of θ' and $u' := |\boldsymbol{u}'|$ (i. e., $u'_\perp = u'\sin\theta'$ and $u'_\parallel = u'\cos\theta'$).
(f) Find $\tan\theta$ and $\cos\theta$ where $u' = c$.
(g) Rewrite $\tan\theta$ for $\theta' = \frac{\pi}{2}$. Discuss qualitatively what this means.
(h) Find the form of θ when $\gamma \gg 1$.
(i) Explain qualitatively why we often miss counter jets in AGN.

Problem 11.5: Solution

TO DO Do the first parts of this exercise (!)

(a)
(b)
(c)
(d)
(e) Using $\theta' = \frac{u'_\perp}{u'_\parallel}$, we can show that

$$\tan\theta = \frac{u'_\perp}{\gamma(u'_\parallel + v)} = \frac{c\sin\theta'}{\gamma(c\cos\theta' + v}. \tag{11.5.62}$$

(f) For $\theta' = \frac{\pi}{2}$, we have $\tan\theta = \frac{c}{\gamma v}$.
(g) For low γ, it approaches $\frac{c}{v}$, but for high γ (where $v \to c$, $\tan\theta = \frac{\tan(\theta'/2)}{\gamma} \implies \theta \approx \frac{\tan(\theta'/2)}{\gamma}$, which means that the light would be in a very small cone in the direction of motion.

(h) Thus, the higher the velocity of the jets, the smaller the emission cone becomes, reducing the likelihood for us to be able to see it. ,