# Problem set #10 Galaxy evolution (here)

02.12.2023

## Problem 1: *Chemical enrichment*

In the first generations of galactic stars the fraction of massive stars ($> 10M_\odot$) which explode as SN II is relatively large. If you assume an average supernova progenitor mass of $25M_\odot$, about $10M_\odot$ per supernova are returned as oxygen. How many one-solar-mass stars can be enriched by one such supernova to a value of $[\mathrm{O/H}] = -2$ if you assume that the material is metal-free in the beginning? (The solar oxygen abundance is $\log(\epsilon(\mathrm{O})) = 8.65$.) How many SN II would you need to bring the total baryonic mass of the Milky Way ($\approx 6 \times 10^{10} M_\odot$) to $[\mathrm{O/H}] = -2$?
(*Note*:

$$\log(\epsilon(X)) := 12 + \log\left(\frac{N(X)}{N(\mathrm{H})}\right) \tag{10.1.1}$$

$$[X/Y] := \log\left(\frac{N(X)/N(Y)}{(N(X)/N(Y))_\odot}\right), \tag{10.1.2}$$

where the abundance ratios $N(X)/N(Y)$ are number ratios, not mass ratios.

## Problem 1: Solution

For this exercise, we are provided with a wealth of information, we just need to sort it properly.
First, we are given that $\log(\epsilon(\mathrm{O})) = 8.65$, which leads us to the equation

$$8.65 = 12 + \log\left(\frac{N(\mathrm{O})}{N(\mathrm{H})}\right) \implies \left(\frac{N(\mathrm{O})}{N(\mathrm{H})}\right)_\odot := \eta_{\mathrm{OH},\odot} = 10^{-3.35}. \tag{10.1.3}$$

Also, the requirement is to enrich the stars to $[\mathrm{O/H}] = -2$, leading us to the equation

$$-2 = \log\left(\frac{N(\mathrm{O})}{N(\mathrm{H})}\right) - \log(\eta_{\mathrm{OH},\odot}) = \log(\eta_{\mathrm{OH}}) + 3.35 \implies \eta_{\mathrm{OH}} = 10^{-5.35}. \tag{10.1.4}$$

Also, we can calculate the number of atoms of a species given its total mass via

$$N(X) = \frac{M_X}{m_X}, \tag{10.1.5}$$

where $m_X$ is the specific atomic weight of the species.
Employing this equation for hydrogen ($m_\mathrm{H} = 1u$) and oxygen ($m_\mathrm{O} \approx 16u$), we find that

$$M_\mathrm{H} = N(\mathrm{H})m_\mathrm{H} = \frac{N(O)}{\eta_{\mathrm{OH}}} m_\mathrm{H} = \frac{M_\mathrm{O}}{\eta_{\mathrm{OH}} m_\mathrm{O}} m_\mathrm{H} = 10^{5.35} \frac{1u}{16u} 10M_\odot \approx \underline{\underline{1.4 \times 10^5 M_\odot}}, \tag{10.1.6}$$

where we used $M_\mathrm{O} = 10M_\odot$ as the mass returned as oxygen, so $1.4 \times 10^5$ solar mass stars could be enriched by such a supernova. Whew.

Now we want to calculate the number of SN II $N_{\text{SNII}}$ that are necessary to bring the total baryonic mass to [O/H]=-2.
Luckily, most of the equations above still hold, we just have to adjust the number of oxygen we need, where

$$N(\mathrm{O}) = \frac{N_{\text{SNII}} 10 M_\odot}{m_{\mathrm{O}}}. \tag{10.1.7}$$

For the Milky Way with $M_{\mathrm{H}} \approx 6 \times 10^{10} M_\odot$ (let's neglect all other species), we need

$$N(\mathrm{O}) = \eta_{\mathrm{OH}} N(\mathrm{H}) \implies N_{\text{SNII}} = \frac{m_{\mathrm{O}}}{m_{\mathrm{H}}} \eta_{\mathrm{OH}} \frac{M_{\mathrm{H}}}{10 M_\odot} \approx \underline{\underline{4.3 \times 10^6 \text{ SN II}}}. \tag{10.1.8}$$

## Problem 2: *Closed box model*

In the closed box model, stars are formed from initially metal-free gas in a confined spatial region. As time goes on, gas is converted into stars which in the next step enrich the ISM homogeneously with heavy elements.
Here we will consider the interstellar gas being consumed and enriched with metals to be a continuous process. Show that for any time $t$, the relation between the metallicity of the gas and the total gas mass is

$$Z(t) = -Y \ln\left(\frac{M_{\mathrm{g}}(t)}{M_{\mathrm{g}}(0)}\right), \tag{10.2.9}$$

where $Y$ is the *yield*, i.e., the extra mass of metals produced by a generation of stars and returned to the ISM divided by the mass that remains bound in stars. The mass of the metals in the ISM is $M_{\mathrm{Z}}$ and the mass of the interstellar gas is $M_{\mathrm{g}}$, with a metallicity defined as $Z := M_{\mathrm{Z}}/M_{\mathrm{g}}$.
(*Hint*: Use the **instantaneous recycling approximation** where the delay between the formation of a generation of stars and the ejection of metals by SN II can be neglected[a]. If the increase of mass in stars after the massive stars have died is $\mathrm{d}M_{\mathrm{s}}$, then the increase of mass in metals in the ISM arising from these stars is

$$\mathrm{d}M_{\mathrm{Z}} = Y \mathrm{d}M_{\mathrm{s}} - Z \mathrm{d}M_{\mathrm{s}} = (Y - Z)\mathrm{d}M_{\mathrm{s}}. \tag{10.2.10}$$

Use mass conservation ($\mathrm{d}M_{\mathrm{s}} = -\mathrm{d}M_{\mathrm{g}}$) and integrate over $\mathrm{d}Z$.

[a] as the SN timescale $\approx 10^7$ yr $\ll t_{\text{Hubble}} \approx$ age of the universe

## Problem 2: Solution

Here again we just need to sort the equations we were given to derive eq. (10.2.9).
Let's first notice that the definition of the metallicity $Z = \frac{M_{\mathrm{Z}}}{M_{\mathrm{g}}}$ provides us with the following differential representation:

$$\mathrm{d}Z = \frac{1}{M_{\mathrm{g}}} \mathrm{d}M_{\mathrm{Z}} - \frac{M_{\mathrm{Z}}}{M_{\mathrm{g}}^2} \mathrm{d}M_{\mathrm{g}}. \tag{10.2.11}$$

We can use this as a starting point to insert $M_{\mathrm{Z}}$ from eq. (10.2.10), in which we additionally employ mass conservation ($\mathrm{d}M_\mathrm{s} = -\mathrm{d}M_\mathrm{g}$):

$$\begin{aligned}
\mathrm{d}Z &= \frac{1}{M_\mathrm{g}}\left(\mathrm{d}M_\mathrm{Z} - Z\mathrm{d}M_\mathrm{g}\right)\\
&\overset{(10.2.10)}{=} \frac{1}{M_\mathrm{g}}\left((Y-Z)\mathrm{d}M_\mathrm{s} - Z\mathrm{d}M_\mathrm{g}\right)\\
&= \frac{1}{M_\mathrm{g}}\left((Z-Y)\mathrm{d}M_\mathrm{g} - Z\mathrm{d}M_\mathrm{g}\right)\\
&= -Y\frac{\mathrm{d}M_\mathrm{g}}{M_\mathrm{g}}.
\end{aligned}$$

Integrating this differential equation yields

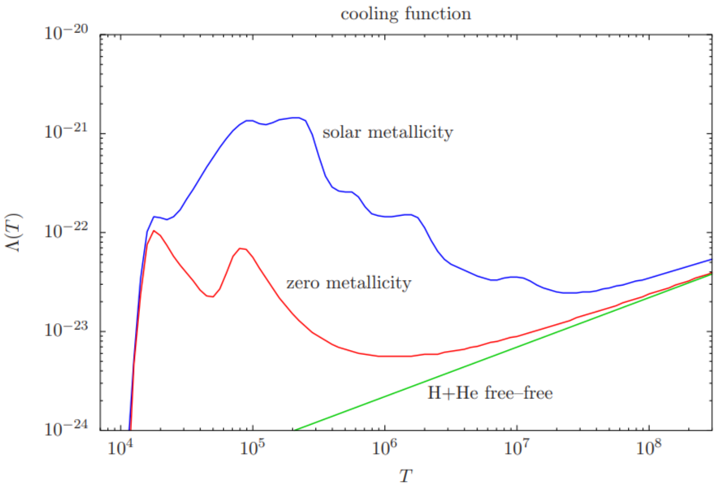
$$Z(t) - Z(0) = -y\left(\ln(M_\mathrm{g}(t)) - \ln(M_\mathrm{g}(0))\right) \tag{10.2.12}$$

$$Z(t) = \underline{\underline{-y\ln\left(\frac{M_\mathrm{g}(t)}{M_\mathrm{g}(0)}\right)}}, \tag{10.2.13}$$

where we used the fact that $Z(0) = 0$ (no chemical enrichment in the beginning).

## Problem 3: *Radiation from cooling halos*

The cooling rate of hot gas (rate of energy loss per unit volume) can be written as $n^2\Lambda(T)$, where $n$ is the total particle number density (in units of $1/\mathrm{cm}^3$) and $\Lambda(T)$ is the cooling function (the power of 2 on $n^2$ arises because cooling occurs at a rate that is proportional to the collisions between particles, so long as the gas is optically thin).




For low-metallicity gas hotter than $10^6$ K, $\Lambda(T)$ is dominated by free–free cooling and can be approximated by

$$\Lambda(T) \approx 10^{-23}\sqrt{T_7}\ \mathrm{erg\ cm^3/s} \tag{10.3.14}$$

where $T_7 = T/(10^7\ \mathrm{K})$. This behaviour can be applied to the evolution of hot gas in the halo of the Milky Way, where $T_7 = 0.18$ K and $n$ can be approximated by

$$n = 10 r_{\text{kpc}}^{-2} \text{ cm}^{-3}, \tag{10.3.15}$$

with $r_{\text{kpc}} = r/(1 \text{ kpc})$. Outside of a radius of about $r \approx 90$ kpc, radiative losses cause gas to condense and move inward rather than actually becoming colder.
This is referred to as a cooling flow (inside of this radius the temperature of the gas does decrease considerably).

- Estimate the flux of cooling radiation (in ergs per second per cm$^2$ per steradian) an observer at the centre of the cooling halo would expect to see.
  How does this compare with the observed value of about $2 \times 10^{-8}$ erg cm$^{-2}$sr$^{-1}$s$^{-1}$ from X-ray observations[a]?

(*Hint*: An answer can be obtained from the theory of radiative transfer, in which the sky brightness from an extended optically thin emitting source is $1/(4\pi)$ times the integral of its volume emissivity along the line of sight.)

[a] the observed value is from Cui et al. 1996.

## Problem 3: Solution

This exercise offers some confusing unit tricks, but is overall on the easier side.
The hint gives us all the ideas we need: For the observer, the only cooling radiation that can be seen emerges from a radius $> 90$ kpc, and is given by the integral of the cooling rate over the line of sight, normalised by $1/(4\pi)$, which also introduces the unit of sr$^{-1}$:

$$F = \frac{1}{4\pi} \int_{r_{\min}}^{\infty} \Gamma(r, T) \mathrm{d}r \text{ sr}^{-1}. \tag{10.3.16}$$

The volume emissivity is in this case given by $\Gamma(r, T) = n^2(r)\Lambda(T)$, where $T$ will now be assumed to be constant as $T = 1.8 \times 10^6$ K in the halo of the Milky Way.
Putting the values and the two relations for $\Lambda$ and $n$ into this expression, we have

$$\begin{aligned}
F &= \frac{1}{4\pi} \int_{r_{\min}}^{\infty} (10 r_{\text{kpc}}^{-2} \text{cm}^{-3})^2 \Lambda(t) \mathrm{d}r \text{ sr}^{-1} && (10.3.17)\\
&= \frac{1}{4\pi} 100 \cdot 10^{-23} \sqrt{T_7} \frac{\text{erg cm}^3}{\text{s sr cm}^6} \left[ -\frac{1}{3 r_{\text{kpc}}^3} \text{kpc} \right]_{r_{\min}}^{\infty} && (10.3.18)\\
&= \frac{10^{-21}}{12\pi} \sqrt{0.18} \frac{1}{90^3} \frac{\text{erg kpc}}{\text{s sr cm}^3} && (10.3.19)\\
&= \underline{\underline{4.76 \times 10^{-8} \text{ erg cm}^{-2} \text{sr}^{-1} \text{s}^{-1}}}. && (10.3.20)
\end{aligned}$$

One dangerous part to note that due to our weird unit handling, we had to introduce the factor of kpc after the integration over $r$ as we basically had to perform a substitution with $r_{\text{kpc}} = r/(1\text{kpc})$.
This is about double the observed value from X-ray observations, which can be explained by the fact that we're considering the integrated flux from all wavelengths here.

## Problem 4: *Clusters of galaxies in virial equilibrium*

Consider a galaxy cluster in virial equilibrium. Using spectroscopic redshifts, you measure the central velocity dispersion of cluster member galaxies along the line of sight as $\sigma_r$.

- Assuming that the cluster is a homogeneous sphere of mass $M_{\mathrm{vir}}$ and radius $R$, derive the relation

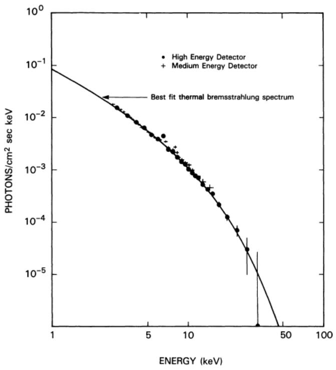
$$\sigma_r^2 = \frac{GM_{\mathrm{vir}}}{5R}. \tag{10.4.21}$$

  What further conditions must you assume for this relation to be valid?

- Use the values from the following table to determine the virial masses $M_{\mathrm{vir}}$ of the clusters. Compare your result with the stellar mass of both systems, assuming a stellar mass-to-light ratio of 3 in solar units.

| Cluster | distance (Mpc) | angular diameter | $L_{\mathrm{V}}$ ($L_\odot$) | $\sigma_r$ (km/s) | $L_{\mathrm{X}}$ (erg/s) | $T_{\mathrm{X}}$ (K) |
|---|---|---|---|---|---|---|
| Virgo | 17 | $10°$ | $1.2 \times 10^{12}$ | 670 | $1.5 \times 10^{43}$ | $7.0 \times 10^7$ |
| Coma | 90 | $4°$ | $5.0 \times 10^{12}$ | 980 | $5 \times 10^{44}$ | $8.8 \times 10^7$ |

- Both clusters have a significant X-ray luminosity.


  The figure shows the measured X-ray spectral energy distribution of the Coma cluster. This is well-fit as thermal bremsstrahlung from hot intra-cluster gas with an emission temperature of $T_{\mathrm{X}} = 8.8 \times 10^7$ K.
  The X-ray luminosity of an isothermal sphere of radius $R$, temperature $T_{\mathrm{X}}$, and an electron density $n_e$ is

$$L_{\mathrm{X}} = \alpha n_e^2 \sqrt{T_{\mathrm{X}}} R^3, \tag{10.4.22}$$

  where $\alpha = 5.95 \times 10^{-27} \frac{\mathrm{cm^5 g}}{\mathrm{s^3 K^{1/2}}}$.
  Calculate the electron density for both clusters in this approximation. From this, calculate the total mass of X-ray emitting hot gas.
  Compare this mass to the virial and stellar mass. What percentage of the virial mass must be in the form of dark matter?

- Assume that the hot gas is losing energy with constant luminosity $L_{\mathrm{X}}$ only by thermal bremsstrahlung. How long does it take until all thermal energy is radiated away? Compare this value to the age of the universe.

## Problem 4: Solution

**Note:** The code used for parts of this exercise is available in **this repository**.
Assuming that the cluster is a homogeneous sphere, the total kinetic energy and total potential energy (**TO DO Put in a derivation of spherical kinetic energy (!)**) is

$$E_{\mathrm{kin}} = \frac{M\sigma^2}{2}, \quad E_{\mathrm{pot}} = -\frac{3}{5}\frac{M^2 G}{R}. \tag{10.4.23}$$

Here, $\sigma$ is the mean velocity of the galaxies, and we can assume $\sigma^2 = 3\sigma_r^2$ for the radial velocity dispersion.

The virial theorem gives us the relation $E_{\text{kin}} = -\frac{1}{2}E_{\text{pot}}$, so we find

$$\frac{3M\sigma_r^2}{2} = \frac{1}{2}\frac{3M^2G}{5R} \iff \underline{\underline{\sigma_r^2 = \frac{GM_{\text{vir}}}{5R}}}. \tag{10.4.24}$$

The values can be found in the table below:

| Cluster | Virgo | Coma |
|---|---|---|
| Distance [kpc] | 17 | 90 |
| Angle | 10° | 4° |
| $L$ [$L_\odot$] | $1.200 \times 10^{12}$ | $5.000 \times 10^{12}$ |
| $\sigma_r$ [km/s] | 670 | 980 |
| $L_X$ [erg/s] | $1.5 \times 10^{43}$ | $5.0 \times 10^{44}$ |
| $T_X$ [K] | $7.00 \times 10^{7}$ | $8.80 \times 10^{7}$ |
| $R$ [Mpc] | 1.49 | 3.14 |
| $M_{\text{vir}}$ | $7.762 \times 10^{14}$ | $3.509 \times 10^{15}$ |
| $M_{\text{lum}}$ | $4.00 \times 10^{12}$ | $1.5 \times 10^{13}$ |
| $n_e$ [cm$^{-3}$] | $5.58 \times 10^{-5}$ | $9.91 \times 10^{-5}$ |
| $M_X$ [$M_\odot$] | $1.902 \times 10^{13}$ | $3.185 \times 10^{14}$ |
| $M_{\text{bary}}/M_{\text{vir}}$ | 0.029 | 0.095 |
| $E_{\text{therm}}$ [erg] | $3.277 \times 10^{62}$ | $6.900 \times 10^{63}$ |
| $t_{\text{brems}}$ [yr] | $6.92 \times 10^{11}$ | $4.37 \times 10^{11}$ |
| $t_{\text{cross}}$ [yr] | $2.171 \times 10^{09}$ | $3.136 \times 10^{09}$ |

Assuming constant cooling by bremsstrahlung with constant luminosity of $L_X$, we find that the values ($\sim 7 \times 10^{11}$ yr) are much more than the age of the universe.

## Problem 5: *Velocity dispersion and virialisation*

Clusters of galaxies have virialisation timescales that correspond to several times their dynamic free-fall times. Instead of the free-fall time, one can also use the velocity dispersion of member galaxies to draw conclusions about the virialisation timescale. For this purpose, assume that a galaxy is moving with a mean velocity $\sigma$ through a cluster.

- How long does it take for a galaxy to cross the Virgo or Coma cluster? (Assume cluster sizes and velocity dispersions of approximately $R_{\text{V}} = 1.5$ Mpc and $R_{\text{V}} = 3.1$ Mpc, and $\sigma_{\text{V}} = 670$ km/s and $\sigma_{\text{C}} = 980$ km/s for Virgo and Coma, respectively.) Compare to the age of the universe.
- Superclusters and similar structures have sizes of 100 Mpc and larger. Can you use the assumption of virialisation for determining their masses, if the velocity dispersions are of the same order ($\sim 1000$ km/s) as for galaxy clusters? Why or why not?
- How about giant elliptical galaxies ($R = 10 \ldots 100$ kpc, $\sigma_0 = 250$ km/s)?

## Problem 5: Solution

**Note:** The code used for parts of this exercise is available in **this repository**.
The values can be found in the table above.
To calculate the crossing time $t_{\text{cross}}$, we simply divide the distance by the velocity dispersion (which isn't the mean velocity, but a good estimate:

$$t_{\text{cross}} \approx \frac{2R}{\sigma}. \tag{10.5.25}$$

- For the Virgo and the Coma cluster, we get

$$t_{\text{cross, V}} = 4.38 \times 10^9 \text{ yr}, \quad t_{\text{cross, C}} 6.19 \times 10^9 \text{ yr}, \tag{10.5.26}$$

  so maybe our assumption of virialisation is not right since the galaxies cannot have crossed the cluster more than twice during the Hubble time.
- It's even worse for superclusters, where we use $d = 2R = 1000$ MpC

$$t_{\text{cross}} = 9.5 \times 10^{11} \text{ yr}, \tag{10.5.27}$$

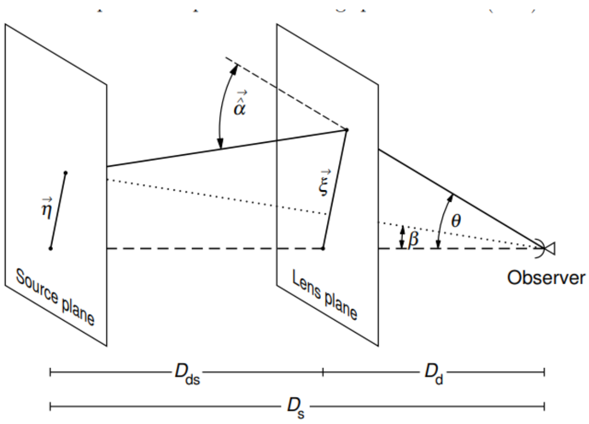
  so the assumption of virialisation should be incorrect.
- For $R = 10$ kpc and $\sigma \approx 250$ km/s, we have

$$t_{\text{cross}} = 3.8 \times 10^7 \text{ yr}, \tag{10.5.28}$$

  and a factor of 10 more for $R = 10$ kpc.
  Depending on when the elliptical galaxies formed, virialisation might be a valid assumption for them.

## Problem 6: *Gravitational lens equations*

The gravitational lens equation maps observed image positions $\theta$ to (true) source positions $\beta$.



The lens equation can be written as (note that this is a two-dimensional vector equation)

$$\boldsymbol{\eta} = \frac{D_\mathrm{s}}{D_\mathrm{d}}\boldsymbol{\xi} - D_\mathrm{ds}\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) \tag{10.6.29}$$

where $\boldsymbol{eta}$ and $\boldsymbol{\xi}$ are vectors in the source and lens/image plane (see figure) and $D_\mathrm{s}$, $D_\mathrm{d}$, and $D_\mathrm{ds}$ are the distances between observer and source, observer and lens, and lens and source, respectively. The angle $\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi})$ is the light deflection an observer would measure if they were "sitting in the lens",

$$\hat{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \frac{4G}{c^2}\int \Sigma(\boldsymbol{\xi}')\frac{\boldsymbol{\xi}-\boldsymbol{\xi}'}{|\boldsymbol{\xi}-\boldsymbol{\xi}'|^2}\mathrm{d}^2\boldsymbol{\xi}', \tag{10.6.30}$$

where $\Sigma(\boldsymbol{\xi})$ is the surface mass density (the mass density of the lens integrated in the direction of the line of sight) as a function of position $\boldsymbol{\xi}$ in the lens plane.

- From this general representation, derive the equation for $\beta(\theta)$ for the special case of a point mass $M$ at $\boldsymbol{\xi} = 0$.
  (*Hint*: Look at the sketch above. Note also that in this case the lens configuration is symmetric with respect to rotation along the line of sight to the lens centre, and the vectors $\boldsymbol{\eta}$ and $\boldsymbol{\xi}$ can therefore be represented as scalars measuring the distance from this line of sight.)
- Solve this equation for the Einstein angle $\theta_\mathrm{E} = \theta$ obtained for $\beta = 0$. Give a geometric equation of the Einstein radius $R_\mathrm{E}$.
- Write the lens equation $\beta(\theta)$ in terms of the Einstein angle $\theta_\mathrm{E}$.
  When does it have one solution, and when does it have two? Write down the solutions. What does the sign of each of the solutions signify?
  Sketch what the observer sees. For the situation where two solutions exist, calculate the angular separation of the two images.

## Problem 6: Solution

**TO DO Transcribe what I've written down on my iPad (!)**