Problem set #3 Radiation, dust and IMFs (here)

24.11.2022

Problem 1: *Planck function*

The Planck function, defined here in terms of a specific intensity with respect to frequency (i.e., with units erg cm^{-2} s^{-1} Hz^{-1} sr^{-1}), is given by

$$B_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right) - 1} \tag{3.1.1}$$

with Planck's constant h, frequency ν, speed of light c, Boltzmann's constant k_B, and temperature T.

(a) Derive the expression for the corresponding energy density, u_ν.

(b) What is the corresponding function B_λ, i. e. with units erg cm^{-2} s^{-1} cm^{-1} sr^{-1}?

(c) Show that Planck curves for different T do not cross each other.

(d) Prove the Stefan-Boltzmann law and *calculate* the Stefan-Boltzmann constant σ_B.

(e) Prove Wien's displacement law for both B_ν and B_λ and calculate the corresponding constants.
Hint: Use the fact that

$$3e^{x_1} - 3 - x_1 e^{x_1} = 0 \implies x_1 \approx 2.82144 \tag{3.1.2}$$
$$5e^{x_2} - 5 - x_2 e^{x_2} = 0 \implies x_2 \approx 4.96511. \tag{3.1.3}$$

(f) Plot the Planck functions B_ν and B_λ for $T = 5500$ K – both as functions of wavelength, using the same x-axis – into one figure (normalised such that both curves have the same peak value) and compare with the results from exercise (e).
What do you conclude?

Problem 1: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) The energy density u_ν is given by an integration over the solid angle Ω, so we have

$$u_\nu = \frac{1}{c}\oint B_\nu \mathrm{d}\Omega = \underline{\underline{\frac{4\pi}{c}B_\nu}}. \tag{3.1.4}$$

(b) We know that

$$B_\lambda \mathrm{d}\lambda = -B_\nu \mathrm{d}\nu, \tag{3.1.5}$$

where the minus sign corresponds to the fact that the wavelength decreases with increasing frequency, so with $\nu = \frac{c}{\lambda}$ we have

$$B_\lambda = -\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}B_\nu\left(\nu = \frac{c}{\lambda}\right) = \underline{\underline{\frac{c}{\lambda^2}\frac{2hc}{\lambda^3}\frac{1}{\exp\left(\frac{hc}{\lambda k_\mathrm{B}T}\right) - 1}}}. \tag{3.1.6}$$

(c) Let's assume that two Planck curves at different temperatures $T_1 \neq T_2$ would cross each other.
In this case, there'd be at least one point where $B_\nu(T_1) = B_\nu(T_2)$.
Putting this into the equation, we have

$$\begin{aligned} \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{k_\mathrm{B}T_1}\right)-1} &= \frac{2h\nu^3}{c^2}\frac{1}{\exp\left(\frac{h\nu}{k_\mathrm{B}T_2}\right)-1} \\ \iff \quad \exp\left(\frac{h\nu}{k_\mathrm{B}T_1}\right) &= \exp\left(\frac{h\nu}{k_\mathrm{B}T_2}\right) \\ \iff \quad T_1 &= T_2, \end{aligned}$$

which clashes with our assumption that $T_1 \neq T_2$.
The fact that Planck curves do not cross each other implies that the intensity (and therefore also the flux) is higher at any given frequency for an object hotter than another (as long as T.E. is assumed).
Note:
We could have also derived this by showing that $\dfrac{\partial B_\nu}{\partial T} > 0$ for all T.

(d) We can derive the Stefan-Boltzmann-law by integrating the forward flux term $F_\nu^+ = \pi B_\nu$ over all frequencies:

$$\begin{aligned} F(T) &= \pi \int_{\nu=0}^{\infty} B_\nu(T)\mathrm{d}\nu \\ &= \frac{2h\pi}{c^2}\int_0^\infty \frac{\nu^3}{e^{\frac{h\nu}{k_\mathrm{B}T}}-1}\mathrm{d}\nu \\ &\overset{x:=\frac{h\nu}{k_\mathrm{B}T}}{=} \frac{2k_\mathrm{B}^4T^4\pi}{c^2h^3}\underbrace{\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x}_{=\frac{\pi^4}{15}} \\ &\overset{(3.1.13)}{=} \underbrace{\frac{2k_\mathrm{B}^4\pi^5}{15c^2h^3}}_{:=\sigma_\mathrm{B}}T^4, \end{aligned} \tag{3.1.7}$$

where we used the substitution $x := \frac{h\nu}{k_\mathrm{B}T}$ with $\mathrm{d}\nu = \frac{k_\mathrm{B}T}{h}$, used the integral for $x^3(e^x-1)^{-1}$ (see below), and defined $\sigma_\mathrm{B} \approx 5.67 \times 10^{-5}$ ergs cm^{-2} s^{-1} K^{-4}.

(e) Wien's displacement law is a formula for the peak in the Planck spectrum and can therefore be obtained via

$$\frac{\mathrm{d}B_\nu(T)}{\mathrm{d}\nu} = 0, \quad \frac{\mathrm{d}B_\lambda(T)}{\mathrm{d}\lambda} = 0. \tag{3.1.8}$$

For the frequency domain, we have (using the chain and product rules)

$$0 \overset{!}{=} \frac{\mathrm{d}B_\nu}{\mathrm{d}\nu} = \frac{6h}{c^2}\frac{\nu^2}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)-1} - \frac{2h\nu^3}{c^2}\frac{h}{k_\mathrm{B}T}\frac{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)}{\left(\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)-1\right)^2}$$

$$\iff \quad 3\left(\frac{h\nu}{k_\mathrm{B}T}\right)^2 \frac{k_\mathrm{B}^2T^2}{h^2} = \left(\frac{h\nu}{k_\mathrm{B}T}\right)^3 \frac{k_\mathrm{B}^2T^2}{h^2} \frac{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)}{\exp\left(\frac{h\nu}{k_\mathrm{B}T}\right)-1}$$

$$\overset{x:=\frac{h\nu}{k_\mathrm{B}T}}{\iff} \quad (e^x-1)3x^2 = e^x x^3$$

$$\iff \quad 3e^x - 3 - xe^x = 0 \overset{(3.1.2)}{\implies} x_1 \approx 2.82144$$

$$\iff \quad \nu_\mathrm{max} = 2.82144\frac{k_\mathrm{B}T}{h} = \underline{\underline{58.789\frac{T}{\mathrm{K}}\,\mathrm{GHz}}}. \tag{3.1.9}$$

Doing the same for the wavelength, it gets even uglier for a moment:

$$0 \overset{!}{=} \frac{\mathrm{d}B_\lambda}{\mathrm{d}\lambda} = -\frac{10c^2h}{\lambda^6}\frac{1}{\exp\left(\frac{hc}{\lambda k_\mathrm{B}T}\right)-1} + \frac{2h^2c^3}{\lambda^7}\frac{\exp\left(\frac{hc}{\lambda k_\mathrm{B}T}\right)}{\left(\exp\left(\frac{hc}{\lambda k_\mathrm{B}T}\right)-1\right)^2}$$

$$\overset{x:=\frac{hc}{\lambda k_\mathrm{B}T}}{\iff} \quad (e^x-1)5x^6 = e^x x^7$$

$$\iff \quad 5e^{x_2} - 5 - x_2e^{x_2} = 0 \implies x_2 \approx 4.96511 \tag{3.1.10}$$

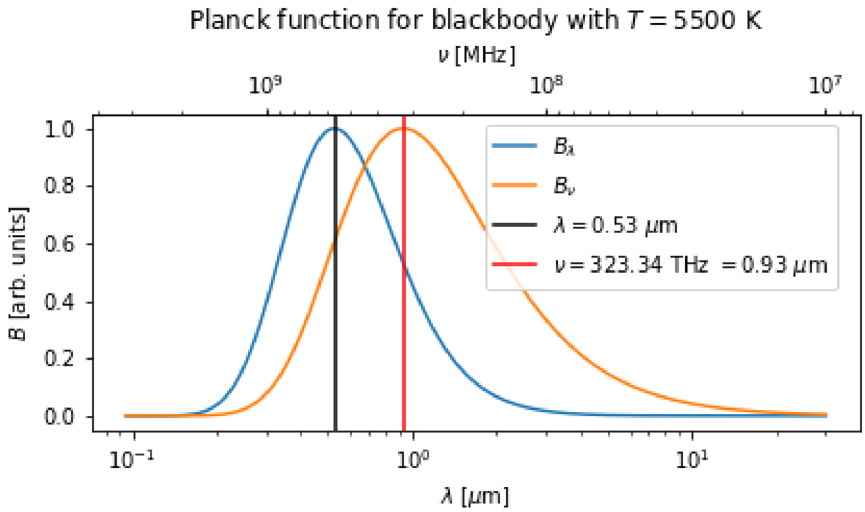
$$\iff \quad \lambda_\mathrm{max}T = \frac{hc}{4.96511k_\mathrm{B}} = \underline{\underline{0.00289777\ \mathrm{m\ K}}}. \tag{3.1.11}$$

We see that they are actually not the same, e. g.

$$\lambda T = \frac{c}{\nu_\mathrm{max}} = 0.0050995\ \mathrm{m\ K}, \tag{3.1.12}$$

which is due to their differential relationship.

(f) As we see, we reproduce the results from exercise (e).



Note: *Derivation of the integral above:*
First we note that we can express the integrand as a geometric series:

$$\frac{x^3}{e^x - 1} = x^3 e^{-x} \frac{1}{1 - e^{-x}} = x^3 e^{-x} \sum_{n=0}^{\infty} \left(e^{-x}\right)^n = x^3 \sum_{n=0}^{\infty} e^{-x(n+1)} = x^3 \sum_{n=1}^{\infty} e^{-nx},$$

where we slurped up the e^{-x} *term and index-shifted For the last equalities.*
Using this trick (and the fact that in this case, we may change the order of summation and integration, let's skip showing that we can...), we have

$$\begin{aligned}
\int_{x=1}^{\infty} \frac{x^3}{e^x - 1}\, dx &= \int_0^{\infty} x^3 \sum_{n=0}^{\infty} e^{-nx}\, dx \\
&= \sum_{n=1}^{\infty} \int_0^{\infty} x^3 e^{-nx}\, dx \\
&\overset{z:=nx}{=} \sum_{n=1}^{\infty} \frac{1}{n^4} \underbrace{\int_0^{\infty} z^3 e^{-z}\, dz}_{=\Gamma(4)=(4-1)!=3!=6} \\
&= 6 \sum_{n=1}^{\infty} \frac{1}{n^4} \\
&= 6 \frac{\pi^4}{90} \\
&= \underline{\underline{\frac{\pi^4}{15}}}. \qquad (3.1.13)
\end{aligned}$$

In this derivation, we used a bunch of other math tricks, including the integral representation of the ***Gamma function*** $\Gamma(n)$*, and the infinite sum* $\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$ *can e. g. be shown using Fourier Series and Parseval's theorem.*

Problem 2: *Accretion disk temperature*

A simple model of an accretion disk approximates the disk as a series of nested rings of radius r with black body spectra of the temperature

$$T(r, M, \dot{M}) = \left(\frac{GM\dot{M}}{8\pi\sigma_{\mathrm{B}} r^3}\right)^{\frac{1}{4}}, \qquad (3.2.14)$$

where M is the mass of the accreting body and $\dot{M}$ is the accretion rate.

- Derive an expression for the total disk luminosity $L_{\mathrm{disk}}(R_{\mathrm{min}}, M, \dot{M})$ from the above formula for $T(r, M, \dot{M})$ using R_{min} as the inner disk radius (near the surface of the accreting body).

Assume accretion disks around a classical T Tauri star, a typical white dwarf, and a neutron star. The values are provided in the table below:

(a) In this model, where does the accretion disk have the highest temperature? Calculate this temperature.
(b) Calculate the luminosity of the disk in ergs/s and in solar luminosities.
(c) Compare this to a black hole accreting $10^{-10}\ M_\odot$/yr and converting 1/16 of this mass into radiation.
(d) Which wavelength range would you use for observations?

Problem 2: Solution

Note: The code used for parts of this exercise is available in **this repository**.

- The luminosity of one ring element is

$$L_{\text{ring}} = 2\sigma_{\text{B}}T^4 = \sigma_{\text{B}}\frac{GM\dot{M}}{8\pi\sigma_{\text{B}}r^3} = \frac{GM\dot{M}}{4\pi r^3}, \tag{3.2.15}$$

where the factor of 2 accounts for the two sides of the disk.
Integrating over all rings, we have to consider that each ring is $2\pi r$ in size, so we have

$$\begin{aligned} L_{\text{disk}} &= 2\int L_{\text{ring}} = \int_{R_{\min}}^{\infty} 2\pi r \frac{GM\dot{M}}{8\pi r^3}\mathrm{d}r \\ &= -\frac{GM\dot{M}}{2}\left(0 - \frac{1}{R_{\min}}\right) \\ &= \underline{\underline{\frac{GM\dot{M}}{2R_{\min}}}}. \end{aligned} \tag{3.2.16}$$

(a) In this model, the since $T \propto r^{-\frac{3}{4}}$, the highest temperature is achieved at $R_{\min}$.
(b) The luminosities can be found in the table below (at the end of this solution).
(c) If the accreting black hole converts $\eta = 1/16$ of the accretion mass into radiation having $\dot{M} = 10^{-10}M_\odot$/yr, its luminosity would be

$$L_{\text{acc}} = \frac{\Delta E}{\Delta t} = \eta\dot{M}c^2 = \underline{\underline{92.46L_\odot}} \tag{3.2.17}$$

The fractional values can be found in the table below.
(d) To find the optimal wavelength range for observations, we can just assume the inner disk's ring as a black body (which is not entirely correct to assume for the entire disk as the temperature drops rapidly, so the maximum of the flux changes) and use Wien's law which we derived in (3.1.11).
Due to the temperature drop, it might be advisable to centre our observations at slightly higher temperatures.
The results are in the table below, but we note that the value for the luminosity for the neutron star disk would be larger by quite a lot.

Employing the values for R_{min}, M and $\dot{M}$ and the equations shown above, we have the following values:

Type	L $[L_\odot]$	L [ergs/s]	L/L_{acc}	λ_{obs} [nm]	Wavelength range
T Tauri	0.07850	3.005×10^{32}	0.000849	1341.33	Far Infrared
White dwarf	0.1334	5.108×10^{32}	0.001443	83.063	Far UV
Neutron star	3058.2	1.171×10^{37}	33.0764	0.255952	X-Rays

Note that the accretion disk for the neutron star would be quite extreme, but maybe our model isn't completely realistic there.

Problem 3: *Dust grain temperature*

(a) What is the dependence of the equilibrium particle temperature on the distance from a star of given luminosity, if the particle absorbs 50 % of the stellar radiation and radiates (mostly infrared) thermal flux with wavelength-independent efficiency $Q_{\text{abs}}^{(\text{IR})} = 1$ (Kirchhoff's law states that absorption and emission efficiencies are equal, hence "abs".). Express your result in the form $T(r) = \text{const}(r/\text{au})^{\text{const}}$ and find T at a distance of $r = 80$ au from a star of luminosity $L = 8L_\odot$.

(b) What is the dependence of temperature of a small dust grain on the distance from the star if the particle absorbs 50 % of the stellar radiation, and radiates the (mostly infrared) ($\lambda > 10$ μm) thermal flux with a wavelength-dependent efficiency given by the formula

$$Q_{\text{abs}}^{(\text{IR})} = \frac{\lambda_0}{\lambda} \tag{3.3.18}$$

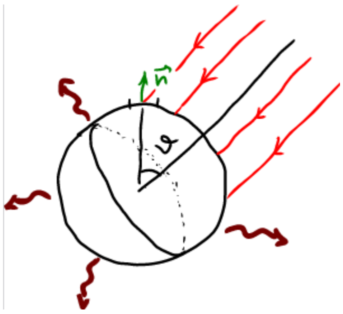
where $\lambda_0 = 10$ μm?
For simplicity, substitute for the emitted λ in this formula an effective wavelength λ_{eff} provided by Wien's law of black body radiation (i.e., the λ at which the Planck curve B_λ with temperature T peaks). Express your result in the form $T(r) = \text{const}(r/\text{au})^{\text{const}}$ and find $T(r = 80\ \text{au})$ for the same star as above.
(This latter problem is motivated by the fact that astronomers are often faced with dust disks in which the size distribution of particles is such that most mass resides in the large particles but most area in the smallest ones, a few micrometers in radius. These small grains have smaller emissivity (absorptivity) at the typical λ_{eff} that follows from Wien's law at the equilibrium temperature. The reason is that they are much smaller than the wavelength of radiation they emit, and in that case the coupling between light and matter is always much weaker.)

Problem 3: Solution

Thanks and credits to Iliya Tikhonenko!

Let a be the distance from the star to the dust grain, r be the size of the grain (to confuse everyone reading this, you know), and R_* be the radius of the star.
From the lectures (and problem set #2) we know that the flux density on a small plate $\mathrm{d}\sigma$ (with its normal pointing towards the source) at the surface of the grain would be

$$F_\lambda = B_\lambda(T_*) \cdot \Omega_*, \quad \text{where } \Omega_* \approx \frac{\pi R_*^2}{a^2} \quad (R \ll d).$$



A lone dust grain basking in the fading light of its host star (and radiating some of it back) somewhere at the end of the Universe.

The total power of absorbed radiation is then (with $\alpha = 0.5$, that's not quite albedo, but still the measure of the surface reflectivity)

$$L_\text{abs} = \alpha \int_0^\lambda \int_\text{hemisph} F_\lambda \cos\vartheta \,\mathrm{d}\sigma\,\mathrm{d}\lambda, \quad \text{where } \mathrm{d}\sigma = r^2 \sin\vartheta\,\mathrm{d}\phi\,\mathrm{d}\vartheta\,.$$

Hence,

$$\begin{aligned}
L_\text{abs} &= \alpha\pi\, r^2 \frac{R_*^2}{a^2} \int_0^\lambda B_\lambda(T_*) \int_0^{\pi/2} \int_0^{2\pi} \cos\vartheta \sin\vartheta \,\mathrm{d}\phi\,\mathrm{d}\vartheta\,\mathrm{d}\lambda \\
&= 2\alpha\pi^2\, r^2 \frac{R_*^2}{a^2} \int_0^\lambda B_\lambda(T_*) \int_0^{\pi/2} \cos\vartheta \sin\vartheta \,\mathrm{d}\vartheta\,\mathrm{d}\lambda \\
&= \alpha\pi^2\, r^2 \frac{R_*^2}{a^2} \int_0^\lambda B_\lambda(T_*) \,\mathrm{d}\lambda \\
&= \alpha\pi\, r^2 \frac{R_*^2}{a^2}\, \sigma T_*^4 \\
&= r^2 \frac{\alpha}{4} \frac{L_*}{a^2}.
\end{aligned}$$

As the grain is in equilibrium with the star radiation, it should emit exactly the same power it receives in all directions (we can now drop the indices of Q for brevity)

$$L_\text{em} = 4\pi r^2\,\pi \int_0^\infty B_\lambda(T_g)\, Q(\lambda)\,\mathrm{d}\lambda.$$

Putting everything together, we get

$$\int_0^\infty B_\lambda(T_g)\, Q(\lambda)\,\mathrm{d}\lambda = \frac{\alpha}{16\pi^2} \frac{L_*}{a^2}.$$

(a) $Q(\lambda) = 1$

$$\int_0^\infty B_\lambda(T_g)\, Q(\lambda)\,\mathrm{d}\lambda = \frac{\sigma T_g^4}{\pi} = \frac{\alpha}{16\pi^2}\frac{L_*}{a^2}$$

$$\implies \quad T_g = \left(\frac{\alpha\, L_*}{16\pi\,\sigma}\right)^{1/4} a^{-1/2} = 234.05\,\mathrm{K} \cdot \left(\frac{L_*}{L_\odot}\right)^{1/4} \left(\frac{a}{\mathrm{AU}}\right)^{-1/2}.$$

So, for the given data we have $T_g \approx 44$ K. Seems reasonable?

(b) $Q(\lambda) = \begin{cases} 0, & \lambda < \lambda_0 \\ \frac{\lambda_0}{\lambda}, & \lambda \geq \lambda_0 \end{cases}$

Let us first roughly estimate λ_{eff} using Wien's law:
$\lambda_{\text{eff}} = 2.9 \times 10^{-3}/44 \approx 66\,\mu\text{m} > \lambda_0 = 10\,\mu\text{m}$. This means that we can not use Rayleigh-Jeans approximation for the Plank curve $\ddot{\frown}$. The sanest (and, probably, expected) way would be to assume that the Plank curve for the grain is zero for all $\lambda < \lambda_0$ and substitute λ in Q with λ_{eff}; In this case

$$\int_0^\infty B_\lambda(T_g)\, Q(\lambda)\, \mathrm{d}\lambda \approx \int_0^\infty B_\lambda(T_g)\, \mathrm{d}\lambda \cdot \frac{\lambda_0}{\lambda_{\text{eff}}} = \frac{\lambda_0\, \sigma T_g^5}{\pi b}.$$

Therefore,

$$T_g = \left(\frac{\alpha\, b\, L_*}{16\pi\, \sigma\, \lambda_0}\right)^{1/5} a^{-2/5} = 244.25\,\text{K} \cdot \left(\frac{L_*}{L_\odot}\right)^{1/5} \left(\frac{a}{\text{AU}}\right)^{-2/5}.$$

So, for the given data we have $T_g \approx 64$ K. Not sure if it is reasonable...
If we try to compute everything exactly, it would be quite scary, but the final result is approximately the same:

$$\int_0^\infty B_\lambda(T_g)\, Q(\lambda)\, \mathrm{d}\lambda = \frac{2\, \lambda_0\, (kT)^5}{h^4\, c^3}\, \zeta(5)\, \Gamma(5)$$

$$\Longrightarrow \quad T_g = \left(\frac{\alpha\, L_*\, h^4\, c^3}{16\pi^2\, \zeta(5)\, \Gamma(5)\, k^5\, 2\lambda_0}\right)^{1/5} a^{-2/5}$$

$$= 257.25\,\text{K} \cdot \left(\frac{L_*}{L_\odot}\right)^{1/5} \left(\frac{a}{\text{AU}}\right)^{-2/5},$$

i. e. for the data in the problem statement it would be ≈ 68 K, which does not differ much from our previous result.

Some things to remember (this was mentioned in the tutorial):
The observers usually use the light reaching the particle, $f = 1 - A$.
If you want to have a flux F at a certain distance d of an object with luminosity L,

$$F = \frac{L}{4\pi d^2}. \tag{3.3.19}$$

Also useful is the relation

$$\mathrm{d}F = \frac{\mathrm{d}E}{\mathrm{d}t\mathrm{d}A}, \tag{3.3.20}$$

and sometimes it's enough to consider the physical size $\mathrm{d}A = \pi s^2$

Problem 4: *Free-fall collapse of a sphere*

The free-fall time is the characteristic time that would take a body to collapse under its own gravitational attraction, if no other forces existed to oppose the collapse.

(a) Consider a homogeneous sphere of radius R_0 and density ρ_0. Set up the equation of motion for a piece of material initially at $r(t=0) = r_0$. Assume (justified a posteriori) that the mass originally contained inside this initial radius remains preserved inside the considered outer edge $r(t)$ of this part of the collapsing sphere,

$$M(r \leq r(t)) = M(r \leq r(t=0)). \tag{3.4.21}$$

(b) Show that the free fall time of a homogeneous sphere with initial density ρ_0 is given by

$$t_{\mathrm{ff}} = \sqrt{\frac{3\pi}{32 G \rho_0}}, \tag{3.4.22}$$

with G the gravitational constant.
Hint: A solution $y = y(t)$ of the differential equation $y'' = f(y)$ can be written as

$$t = \pm \int_{y(0)}^{y(t)} \frac{\mathrm{d}y}{\sqrt{\left(2 \int_{y(0)}^{y(t)} f(y)\mathrm{d}y\right) + (y'(0))^2}} \tag{3.4.23}$$

where the positive sign applies for $y'(0) > 0$ and the negative sign for $y'(0) < 0$.

(c) Discuss under which conditions the above eq. (3.4.22) for the free-fall time can be generalised to *non-homogeneous* spheres (where the density ρ is a function of r), when ρ_0 in eq. (3.4.22) is replaced by the *average* density within r at the initial time, $\langle \rho_0 \rangle$.
Hint: Consider whether shells of material can fall faster or slower than in the case of homogeneous density.

(d) Calculate the free-fall time for a cloud with $\rho_0 = 4 \times 10^{-23}$ g cm^{-3}, corresponding to a slightly enhanced interstellar density.

Problem 4: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) The equation of motion for a test particle at radius r with enclosed mass $M(r)$ is

$$m\ddot{r} = -G\frac{mM(r)}{r^2} \tag{3.4.24}$$

In the freefall-regime, the mass collapses inwards, so the enclosed mass is the same as in the beginning when the sphere had the radius R_0. Thus, we can model the mass as

$$M(r) = M(R_0) = \rho_0 V(R_0) = \rho_0 \frac{4}{3}\pi R_0^3, \tag{3.4.25}$$

which leads us to

$$(3.4.24) \overset{(3.4.25)}{\Longrightarrow} \ddot{r} = -\frac{4}{3} G\pi\bar{\rho}\frac{R^3}{r^2} =: -\frac{\alpha}{r^2}, \tag{3.4.26}$$

where we defined α to make the following derivation less ugly.

(b) To make this equation solvable, we use a trick (which is also hinted at in the exercise, but let's derive it).
According to the product rule, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \frac{\mathrm{d}r}{\mathrm{d}t}\frac{\mathrm{d}^2r}{\mathrm{d}t^2} + \frac{\mathrm{d}^2r}{\mathrm{d}t^2}\frac{\mathrm{d}r}{\mathrm{d}t} \iff \ddot{r} = \frac{\mathrm{d}^2r}{\mathrm{d}t^2} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2. \tag{3.4.27}$$

Using this, we can rewrite our differential equation (3.4.26) and integrate w. r. t. r (using the boundaries $r(t=0) = R_0$ and $r = r(t)$):

$$\frac{\mathrm{d}}{\mathrm{d}r}\dot{r}^2 = -\frac{2\alpha}{r^2} \implies \dot{r}^2 = \frac{2\alpha}{r'}\Big|_{r'=R_0}^{r'=r} = \frac{2\alpha}{r} - \frac{2\alpha}{R_0} \implies \frac{\mathrm{d}r}{\mathrm{d}t} = \pm\sqrt{2\alpha(r^{-1} - R_0^{-1})}. \tag{3.4.28}$$

Here, we also assumed $\dot{r}(t=0) = 0$.
The sign depends on the way we are looking at the problem. If we start at R_0, the collapsing mass (or the particle 'on top of it') moves away from us, so we can choose the negative sign.
Separation of variables then yields

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{2\alpha}\sqrt{\frac{1 - r/R_0}{r}}$$

$$\iff \int_0^{R_0}\sqrt{\frac{r}{1 - r/R_0}}\mathrm{d}r = \int_0^{t_{\mathrm{ff}}}\sqrt{2\alpha}\mathrm{d}t \tag{3.4.29}$$

$$\iff \frac{\pi}{2}R_0\sqrt{R_0} = t_{\mathrm{ff}}\sqrt{2\alpha} \tag{3.4.30}$$

$$\iff t_{\mathrm{ff}} = \frac{\pi}{2}R_0\sqrt{R_0}\sqrt{\frac{3}{8\pi G\bar{\rho}R_0^3}} = \sqrt{\frac{3\pi}{32G\bar{\rho}}}. \tag{3.4.31}$$

In the second step (eq. (3.4.29) to eq. (3.4.30)), we used a very complex integral identity which is shown in the appendix (sec. 3.6).

(c) A typical assumption would be that the inner region of the cloud would have higher densities. Since $t_{\mathrm{ff}} \propto \rho^{-1/2}$, they would have shorter free-fall times. In these cases, it would be alright to just replace ρ_0 with $\langle\rho\rangle$.

(d) The free-fall time of the cloud with $\rho_0 = 4 \times 10^{-23}$ g cm^{-3} is $t_{\mathrm{ff}} = 10.53$ Myr.

Problem 5: *Initial mass function of an open cluster*

When stars form, the distribution of their masses follows an initial mass function (IMF) $\frac{\mathrm{d}N}{\mathrm{d}m}$, which gives the differential number of stars dN in a mass interval $m \ldots m +$ dm) as

$$\mathrm{d}N = \frac{\mathrm{d}N}{\mathrm{d}m}\mathrm{d}m. \tag{3.5.32}$$

Salpeter (1955) determined the initial mass function as a power-law in mass,

$$\frac{\mathrm{d}N}{\mathrm{d}m} = am^{-\alpha}, \tag{3.5.33}$$

where a is an amplitude for normalisation to the total stellar mass inside a volume and the exponent is $-\alpha = -2.35$.
Consider an open cluster, in which a total stellar mass $10^3 M_\odot$ has formed instantaneously following a Salpeter initial mass function in the mass range $0.1M_\odot \ldots 20M_\odot$ (with no stars outside that mass range; why is that a reasonable assumption?).

- Find the normalisation constant a.
- Find the initial total luminosity of the cluster, assuming that all its stars are on the main sequence, and a mass–luminosity relation $L \propto M^4$. What fraction of the initial luminosity is contributed by stars more massive than $5M_\odot$? What fraction of the number of stars is in this high mass end?
- Find the initial mean mass of stars in the cluster.
- Assume that the main-sequence lifetime of a $1M_\odot$ star is 10 Gyr, and main-sequence lifetime scales with mass as M^{-2}. From your observations of the cluster, you know that the brightest main sequence stars that still exist are about 100 times more luminous than the sun. How long ago did the cluster form its stars? What fraction of its initial luminosity does the cluster have today?

Problem 5: Solution

Assuming $M_{\mathrm{min}} = 0.1M_\odot$ and $M_{\mathrm{max}} = 20M_\odot$ is reasonable for an open cluster since stars with lower masses should not be found as open clusters usually disperse after quite a short time and stars with lower mass wouldn't reach the Main Sequence. Higher mass stars could be considered, but then they could maybe contribute too much. This might also be related to the Eddington limit.
Note: If we had set $N_{\mathrm{tot}} = 1$ and used $1 = N_{\mathrm{tot}} = \int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} bm^{-\alpha}\mathrm{d}m$, we would obtain the normalisation constant b for a fractional IMF.

- Since we don't know N_{tot} a priori, we need to employ another way of finding the normalisation constant:
  Because we know the total mass $M_{\mathrm{tot}} = 10^3 M_\odot$ of the cluster, can look at the distribution in terms of mass to determine a:

$$M_{\mathrm{tot}} \stackrel{!}{=} \int_{M_{\mathrm{min}}}^{M_{\mathrm{max}}} amm^{-\alpha}\mathrm{d}m = \frac{a}{2-\alpha}\left(M_{\mathrm{max}}^{2-\alpha} - M_{\mathrm{min}}^{2-\alpha}\right) \tag{3.5.34}$$

$$\iff a = \frac{M_{\mathrm{tot}}}{M_{\mathrm{max}}^{2-\alpha} - M_{\mathrm{min}}^{2-\alpha}}(1-\alpha) = \frac{-0.35M_{\mathrm{tot}}}{20^{-0.35} - 0.1^{-0.35}}M_\odot^{0.35} \tag{3.5.35}$$

$$= 0.18535M_{\mathrm{tot}}M_\odot^{0.35} = 185.35M_\odot^{1.35}. \tag{3.5.36}$$

- Using the scaling relation $L \propto M^4$ that was provided and the fact that the Sun is a Main-Sequence-star as well, we have

$$L_{\text{tot}} = \int_{M_{\min}}^{M_{\max}} am^4 \frac{L_\odot}{M_{\text{odot}}^4} m^{-\alpha} \mathrm{d}m = \frac{a}{5-\alpha}\left(M_{\max}^{5-\alpha} - M_{\min}^{5-\alpha}\right)\frac{L_\odot}{M_\odot} = 1.96 \times 10^5 L_\odot. \tag{3.5.37}$$

To find the luminosity the stars with $M \geq 5M_\odot =: M_0$ contribute, we need to replace the lower boundary of the integral:

$$\frac{L_{\text{massive}}}{L_{\text{tot}}} = \frac{1}{L_{\text{tot}}} \int_{M_0}^{M_{\max}} a \frac{L_\odot}{M_{\text{odot}}^4} m^{4-\alpha} \mathrm{d}m = 0.97. \tag{3.5.38}$$

To find the fraction of stars that make up for this luminosity, we have to employ the relation for N, use the cutoff and also divide by the total number.
Here, we find

$$\frac{N_{M \geq 5M_\odot}}{N_{\text{tot}}} = \frac{\int_{M_0}^{M_{\max}} am^{-\alpha}\mathrm{d}m}{\int_{M_{\min}}^{M_{\max}} am^{-\alpha}\mathrm{d}m} = \frac{13.2}{3071.4} = 0.43\,\%, \tag{3.5.39}$$

so we see that even though these stars contribute a majority of the luminosity, they only make up a very small fraction of the total population.

- The mean mass is simply
$$\bar{m} = \frac{M_{\text{tot}}}{N_{\text{tot}}} = 0.33 M_\odot. \tag{3.5.40}$$
- From the scaling relations for the lifetime τ, the mass M and the luminosity L of main sequence stars, we can derive that
$$L \propto M^4 \iff M \propto L^{\frac{1}{4}}, \quad \tau \propto M^{-2} \implies \tau \propto L^{-\frac{1}{2}}. \tag{3.5.41}$$
Using that a star of solar mass would have one solar luminosity (related to the lifetime $\tau_\odot \approx 10$ Gyr) and that the brightest (and therefore oldest) star of the cluster has $L_{\text{OC,max}} = 100 L_\odot$, for the lifetime of the cluster we have
$$\tau_{\text{OC}} = \frac{\tau_\odot}{\sqrt{L_{\text{OC,max}}/L_\odot}} = \frac{10 \text{ Gyr}}{\sqrt{100}} = 1 \text{ Gyr}. \tag{3.5.42}$$
To find the fraction of its initial luminosity, we can use the relation for τ to first find out the maximum mass of the remaining strs, which is
$$M_{\text{OC,max}} = (\tau_{\text{OC}}/\tau_\odot)^{-1/2} M_\odot = \sqrt{10} M_\odot \approx 3.16. \tag{3.5.43}$$
Now, we simply integrate to this mass to find
$$\frac{L_{\text{now}}}{L_{\text{tot}}} = \frac{1}{L_{\text{tot}}} \int_{M_{\min}}^{M_{\text{OC,max}}} a \frac{L_\odot}{M_{\text{odot}}^4} m^{4-\alpha} \mathrm{d}m = 0.75\,\%, \tag{3.5.44}$$
so the luminosity has (without much surprise) decreased drastically in the last 1 Gyr.

3.6 Appendix

Here, we show that

$$\int_0^{R_0} \sqrt{\frac{r}{1 - r/R_0}} \mathrm{d}r = \frac{\pi}{2} R_0 \sqrt{R_0}. \tag{3.6.1}$$

First, we observe that we can factor out $\sqrt{R_0}$ and substitute $z := \sqrt{r}$ (so $\frac{\mathrm{d}z}{\mathrm{d}r} = \frac{1}{2\sqrt{r}} \iff \mathrm{d}r = 2\sqrt{r}\mathrm{d}z = 2z\mathrm{d}z$):

$$\int_0^{R_0} \sqrt{\frac{r}{1 - r/R_0}} \mathrm{d}r = \sqrt{R_0} \int_0^{R_0} \sqrt{\frac{r}{R_0 - r}} \mathrm{d}r \overset{z=\sqrt{r}}{=} \sqrt{R_0} \int_0^{\sqrt{R_0}} \frac{2z^2 \mathrm{d}z}{\sqrt{R_0 - z^2}}. \tag{3.6.2}$$

The part in the square root looks like a trigonometric substitution could work.
If we set $z = \sqrt{R_0} \sin(u)$[1], the denominator becomes $R_0\sqrt{1 - \sin^2(u)} = r\cos(u)$, while our upper integration border becomes $\arcsin(1) = \frac{\pi}{2}$.
The integral is thus

$$2\sqrt{R_0} \int_0^{\frac{\pi}{2}} \frac{R_0 \sin^2(u) \cos(u) \mathrm{d}u}{\cos(u)} = 2\sqrt{R_0} R_0 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos(2u)) \mathrm{d}u, \tag{3.6.3}$$

where we used the trigonometric identity

$$\sin^2(x) = \frac{1}{(2i)^2}(e^{ix} - e^{-ix})^2 = -\frac{1}{4}(e^{i2x} + e^{-i2x} - 2) = \frac{1}{2}(1 - \cos(2x)). \tag{3.6.4}$$

This last integral is easy:

$$2\sqrt{R_0} R_0 \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - \cos(2u)) \mathrm{d}u = 2\sqrt{R_0} R_0 \frac{1}{2} \left(\frac{\pi}{2} - \frac{\sin(\pi) - \sin(0)}{2} \right) = \frac{\pi}{2} \sqrt{R_0} R_0. \tag{3.6.5}$$

There, we have it! $\square$

[1] so $\mathrm{d}z = \sqrt{R_0} \cos(u) \mathrm{d}u$