Problem set #8 Galaxy dynamics (here)

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Problem 1: *Inclination angles of spiral galaxies*

Spiral galaxies can be observed under different inclination angles i, where $i = 0$ describes a "face-on" perspective and $i = \pi/2$ describes an "edge-on" perspective. How are the observed inclination angles i of spiral galaxies distributed if the galaxies are oriented randomly in space?

(a) First assume a disk which can be tilted with respect to an axis ("tilt axis" in the figure) perpendicular to the line of sight. (This axis is assumed to be fixed in space.) How are the inclination angles i distributed in this case, if i can take values from 0 to $\pi/2$? Derive the normalised distribution function $p(i)$ so that $\int_0^{\pi/2} p(i)\mathrm{d}i = 1$.

(b) Allowing different tilt axes (still perpendicular to the line of sight) gives an additional degree of freedom. To calculate the true distribution of the inclination angles it is convenient to use the normal vector $\boldsymbol{n}$ of the disk surface and to distribute the end of the vectors uniformly over the surface of a sphere. Convince yourself that the probability $p(i)$ of an inclination angle i must be proportional to the circumference of the circle on the sphere touched by the normal vector $\boldsymbol{n}$. Estimate for which i the probability must be highest and lowest.

(c) Find the functional dependence of the radius $r_{\mathrm{proj}}(i)$ of this circle by plotting r_{proj} against i. Can this be described by a mathematical function? From this derive the true normalised distribution function $p(i)$ of the inclination angle i.

Problem 1: Solution

TO DO Put in once discussed (!)

$$p(i)\mathrm{d}i = \frac{2\pi \sin i \cdot 2}{4\pi}\mathrm{d}i \implies p(i) = \sin i, \tag{8.1.1}$$

Where

Problem 2: *Rotation Curve*

How does the rotational velocity of the Milky Way's disk vary with radius (distance from the galactic centre)? What velocity profile would we expect if the total mass were concentrated in the bulge? What is measured instead, and how is it measured? Derive an equation that allows determining the density distribution $\rho(r)$ from the observed velocity profile $v(r)$, and describe under what assumptions it is valid. What is the resulting "observed" density profile?

Problem 2: Solution

The rotation curve of the Milky Way's disk looks as follows: **TO DO Put in a qualitative plot (!)** To derive the expected velocity profile if all mass were concentrated, we start by using the fact that the centripetal force on a particle should be given by the gravitational force, so we have

$$F_{\text{grav}} = F_{\text{cp}} \tag{8.2.2}$$

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \implies M(r) = \frac{rv^2}{G}. \tag{8.2.3}$$

To get the density profile, we can look at the mass enclosed in a spherical shell of thickness dr, we have **TO DO This looks fishy, but I'm too tired to correct it right now, so I'll need to have another look at this sometime) (!)**

$$\mathrm{d}M = \rho(r)4\pi r^2 \mathrm{d}r \overset{!}{=} \frac{v_{\text{rot}}^2}{G}\mathrm{d}r. \tag{8.2.4}$$

This yields

$$\rho(r) = \frac{v_{\text{rot}}}{4\pi G}\frac{1}{r^2} \propto \frac{1}{r^2} \tag{8.2.5}$$

for a density profile with constant rotation velocity.

Problem 3: *Gravitational potential*

(a) Consider an N-body system of stars with coordinates $\boldsymbol{x}_i$ and masses m_i, $i = 1, \ldots, N$. Show that the negative of the gradient of the quantity

$$\Phi(\boldsymbol{x}) = -\sum_{i=1}^{N} \frac{Gm_i}{|\boldsymbol{x} - \boldsymbol{x}_i|} \tag{8.3.6}$$

is equal to the gravitational force of the system. (In other words, Φ is the gravitational potential of the system.)

(b) Newton proved two results that enable us to calculate the gravitational potential of any spherically symmetric distribution of matter easily:
Newton's first theorem: A body that is inside a spherical shell of matter experiences no net gravitational force from that shell.
Newton's second theorem: The gravitational force on a body that lies outside a spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its centre.

Consider a spherical distribution of mass with density $\rho(r) \propto r^{-n}$, with $-n = \dfrac{\mathrm{d}\ln\rho}{\mathrm{d}\ln r}$.

- Use Newton's theorems to calculate the gravitational force F on a particle of mass m_1 at a position r_1 for $n = 1, 2, 3, 4$.
- How does the total mass of the system behave when going to large radii ($r_1 \to \infty$)?
- At what value for n does this behaviour change?

Problem 3: Solution

(a) We are supposed to show that the gravitational acceleration

$$\boldsymbol{F}_{\text{grav}}/m = -\nabla\Phi(\boldsymbol{x}). \tag{8.3.7}$$

We have

$$-\nabla\Phi(\boldsymbol{x}) = -\nabla\left(-\sum_{i=1}^{N}\frac{Gm_i}{|\boldsymbol{x}-\boldsymbol{x}_i|}\right) \tag{8.3.8}$$

$$= \sum Gm_i\nabla\left(\boldsymbol{x}-\boldsymbol{x}_i\right)^{-\frac{1}{2}} \tag{8.3.9}$$

$$= \sum Gm_i\left(-\frac{1}{2}\left(\boldsymbol{x}-\boldsymbol{x}_i\right)^{-\frac{3}{2}}2\left(\boldsymbol{x}-\boldsymbol{x}_i\right)\right) \tag{8.3.10}$$

$$= -\sum\frac{Gm_i}{|\boldsymbol{x}-\boldsymbol{x}_i|}\left(\boldsymbol{x}-\boldsymbol{x}_i\right), \tag{8.3.11}$$

which is indeed the gravitational acceleration.

(b) Generally, the mass enclosed in inside a sphere with radius r_1 is given by

$$M(<r_1) = \int_{r_0}^{r_1} 4\pi r^2\rho(r)\mathrm{d}r + M_0, \tag{8.3.12}$$

where M_0 is the mass enclosed in the inner part of the sphere with radius r_0 (which we need to consider to not get divergence problems).
The force acting on a particle of mass m_1 is given by

$$F = -\frac{GM(<r_1)m_1}{r_1^2} \tag{8.3.13}$$

We are supposed to look at different values of n. Thus (with $\rho(r) = cr^{-n}$)

- $n = 1$

$$M(<r_1) = c\int_{r_0}^{r_1} 4\pi r\mathrm{d}r + M_0 = 2c\pi(r_1^2 - r_0^2) + M_0 \overset{\infty}{\to} \infty. \tag{8.3.14}$$

The force on the particle therefore is

$$F = -Gcm_1\frac{2\pi(r_1^2 - r_0^2) + M_0}{r_1^2}, \tag{8.3.15}$$

so even at infinite range, the particle would still experience some force.

- $n = 2$

$$M(<r_1) = c\int_{r_0}^{r_1} 4\pi\mathrm{d}r + M_0 = 4c\pi(r_1 - r_0) + M_0 \overset{\infty}{\to} \infty. \tag{8.3.16}$$

The force on the particle therefore is

$$F = -Gcm_1\frac{4\pi(r_1 - r_0) + M_0}{r_1^2}. \tag{8.3.17}$$

- $n = 3$

$$M(< r_1) = c \int_{r_0}^{r_1} \frac{4\pi}{r} \mathrm{d}r + M_0 = 4c\pi \ln\left(\frac{r_1}{r_0}\right) + M_0 \overset{\infty}{\rightarrow} \infty. \tag{8.3.18}$$

The force on the particle therefore is

$$F = -Gcm_1 \frac{4\pi \ln\left(\frac{r_1}{r_0}\right) + M_0}{r_1^2}. \tag{8.3.19}$$

- $n = 4$

$$M(< r_1) = c \int_{r_0}^{r_1} 4\pi \frac{1}{r^2} \mathrm{d}r + M_0 = -4c\pi \left(\frac{1}{r_1} - \frac{1}{r_0}\right) + M_0 \overset{\infty}{\rightarrow} \frac{4c\pi}{r_0} + M_0. \tag{8.3.20}$$

The force on the particle therefore is

$$F = -Gcm_1 \frac{4\pi(1/r_0 - 1/r_1) + M_0}{r_1^2}. \tag{8.3.21}$$

So the behaviour of $M \to \infty$ for $r_1 \to \infty$ only changes going from $n = 3$ to $n = 4$.
TO DO Maybe plots of the distributions would be nice. (!)

Problem 4: *Plummer profile*

In many spherical systems the density is roughly constant near the centre, and falls to zero at large radii. A simple potential with these properties is the Plummer model

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}, \tag{8.4.22}$$

where b is called the Plummer scale length.

(a) Using Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$, calculate $\rho(r)$ for the Plummer potential. (Note: In spherical coordinates, under the assumption of spherical symmetry, we can write: $\nabla^2\Phi(r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\Phi(r)}{\mathrm{d}r}\right)$.)

(b) Show that indeed $\rho(r \to 0) = const$.

(c) What is the total mass of the system (for $r \to \infty$)?

(d) Calculate the escape velocity (as function of r).

(e) Determine the rotation curve $v_{\mathrm{circ}}(r)$ for the Plummer potential.
Where is its maximum $r(v_{\mathrm{circ}}^{\mathrm{max}})$?

(f) A typical globular cluster has a mass of $M = 105 M_\odot$ and a scale length of $b = 10$ pc. How large are the escape velocity and the maximum circular velocity for such a system?

Problem 4: Solution

(a) We do as instructed by using the Poisson equation and assuming spherical symmetry, so we have

$$\nabla^2\Phi(r) = \nabla^2\Phi(r) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}}{\mathrm{d}r}\left(-\frac{GM}{\sqrt{r^2+b^2}}\right)\right) \tag{8.4.23}$$

$$= -GM\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\left(-\frac{1}{2}\right)2r\left(r^2+b^2\right)^{-\frac{2}{3}}\right) \tag{8.4.24}$$

$$= \frac{GM}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^3\left(r^2+b^2\right)^{-\frac{3}{2}}\right) \tag{8.4.25}$$

$$= \frac{GM}{r^2}\left(3r^2\left(r^2+b^2\right)^{-\frac{3}{2}} - \frac{3}{2}2r^4\left(r^2+b^2\right)^{-\frac{5}{2}}\right) \tag{8.4.26}$$

$$= \frac{3GM}{(r^2+b^2)^{\frac{3}{2}}}\left(1-\frac{r^2}{r^2+b^2}\right) \tag{8.4.27}$$

$$= \frac{3GMb^2}{(r^2+b^2)^{\frac{5}{2}}} \stackrel{!}{=} 4\pi G\rho \tag{8.4.28}$$

$$\iff \rho(r) = \frac{3M}{4\pi}\frac{b^2}{(r^2+b^2)^{\frac{5}{2}}}. \tag{8.4.29}$$

(b) Indeed, we find that

$$\lim_{r\to 0}\rho(r) = \frac{M}{4/3\pi b^3}, \tag{8.4.30}$$

which corresponds to the density of a uniform sphere with mass M and radius b.

(c) We can calculate the enclosed mass via integration (and we rename M from the previous parts to M_0):

$$M(R) = \int_0^R 4\pi\rho(r)r^2\mathrm{d}r \tag{8.4.31}$$

$$= M_0\int_0^R \frac{3b^2r^2}{(r^2+b^2)^{\frac{5}{2}}}\mathrm{d}r \tag{8.4.32}$$

$$= M_0\frac{R^3}{(R^2+b^2)^{\frac{3}{2}}} \tag{8.4.33}$$

$$= M_0\left(\frac{1}{1+b^2/R^2}\right)^{\frac{3}{2}} \tag{8.4.34}$$

$$\implies \lim_{R\to\infty} M(R) = M_0. \tag{8.4.35}$$

(d) To derive the escape velocity, we can use the fact that the energy of an escaping test particle of mass m would be 0, so

$$0 \stackrel{!}{=} \frac{1}{2}mv^2 + \Phi(r) = \frac{mv^2}{2} - \frac{Gm}{\sqrt{r^2+b^2}} \tag{8.4.36}$$

$$\iff v_{\rm esc} = \sqrt{\frac{2G}{\sqrt{r^2+b^2}}}. \tag{8.4.37}$$

(e) For the rotation curve, we start with

$$v_{\rm circ}^2 = \frac{GM(r)}{r} = \frac{GMr^2}{(r^2+b^2)^{\frac{3}{2}}}. \tag{8.4.38}$$

We arrive at $r_{\max} = b\sqrt{2}$.
Therefore, we find that

$$v_{\rm circ}^{\max}(r_{\max}) = \sqrt{\frac{2GM}{b3^{\frac{3}{2}}}} = 0.62\frac{GM}{b}. \tag{8.4.39}$$

(f) Plugging in the values, we find

$$v_{\rm esc} \approx 9\frac{\rm km}{\rm s} \tag{8.4.40}$$

$$v_{\rm circ}^{\max} \approx 4\frac{\rm km}{\rm s}. \tag{8.4.41}$$

Problem 5: *Effective potential*

(a) The effective potential is defined as $\Phi_{\rm eff}(r) = \Phi(r) + \frac{l^2}{2r^2}$.
- What motion is described by the effective potential?
- Calculate the effective potential for the Plummer profile.
- Plot this effective potential as a function of radius. Experiment with different values for the parameters. Does the behaviour change qualitatively for particular parameter combinations? If yes, for which ones? If no, why not?
- How does $\Phi_{\rm eff}$ behave for $r \to 0$ and $r \to \infty$?

(b) There are only two potentials for which all bound orbits are closed:
the Kepler potential and the harmonic potential.
- Show that the harmonic potential indeed has closed orbits.
- What density distribution leads to a harmonic potential?

Problem 5: Solution

Thanks and credits to Iliya Tikhonenko!

(a) The energy in generally is

$$\frac{E}{m} = \frac{\dot{\boldsymbol{r}}^2}{2} + \Phi(\boldsymbol{r}). \tag{8.5.42}$$

Due to angular momentum conservation, we can just consider a planar motion and transform into cylindrical coordinates:

Applying the transformations of $R = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$, z (cylindrical coordinates with $x = R\cos\phi$, $y = R\sin\phi$ and therefore

$$\dot{x} = \dot{R}\cos\phi - R\sin\phi\dot{\phi}, \quad \dot{y} = \dot{R}\sin\phi + R\cos\phi\dot{\phi}, \quad \dot{z} = \dot{z}, \tag{8.5.43}$$

so $\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2$, where we can identify $l = R^2\dot{\phi}$, so the transformed energy (using $\dot{z} = 0$ in the plane we are considering) is

$$\frac{E}{m} = \frac{\dot{R}^2}{2} + \frac{l^2}{2R^2} + \Phi, \tag{8.5.44}$$

so we have derived the effective potential.
The combined effective Plummer potential is

$$\Phi_{\mathrm{eff}}(r) = -\frac{GM}{\sqrt{r^2 + b^2}} + \frac{l^2}{2r^2}. \tag{8.5.45}$$

The motion described in the effective potential would be elliptical.
At very large radii ($r \to \infty$), the potential approaches 0.
It diverges at $r = 0$.

(b) For a Kepler potential, $\Phi \propto \frac{1}{r}$, while for a harmonic potential, $\Phi \propto r^2$.**TO DO Do this a little more thorough or find good references (should be undergrad textbook stuff) (!)**
The density distribution for a harmonic potential can once again be derived by the Poisson equation (and we once again assume spherical symmetry):

$$\nabla^2\left(\frac{C}{2}r^2\right) = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 Cr\right) \tag{8.5.46}$$

$$= \frac{1}{r^2}3cr^2 \tag{8.5.47}$$

$$= 3C, \tag{8.5.48}$$

where C is the strength of the harmonic potential.
Therefore, the density distribution required for a harmonic potential would be a homogeneous sphere.

Problem 6: *Andromeda galaxy nucleus*

The absolute magnitude of the nucleus in the Andromeda galaxy (M31) is $M_{\mathrm{bol}}^{\mathrm{nuc}} = -11.7$ and its stellar mass is $M_{\mathrm{nuc}} = 13 \times 10^6 M_\odot$.

(a) With which luminosity does the nucleus radiate?

(b) Calculate the number of stars in the nucleus, if we assume that all stars in the nucleus are identical and that the luminosity of each star can be obtained from the mass-luminosity relation in the form of $\log(L/L_\odot) = 3.5\log(M/M_\odot)$.

(c) Let us assume that there is a super-massive black hole at the centre of the nucleus. How much material would need to be accreted onto the black hole every second if the total luminosity was only due to matter accretion? Assume that 15 % of the rest-mass energy is emitted as radiation when matter is accreted onto the black hole.

Problem 6: Solution

(a) To calculate the luminosity, we take the usual formula (which is also very nicely described **here**, I sometimes use that to remember it better) relating magnitudes and luminosities since we know that $M_{\mathrm{bol}} \approx 4.75$, so we have

$$M_{\mathrm{bol}} - M_{\mathrm{bol},\odot} = -2.5 \log\left(\frac{L}{L_\odot}\right) \tag{8.6.49}$$

$$\iff \quad L_{\mathrm{nuc}} = 10^{0.4\left(M_{\mathrm{bol},\odot} - M_{\mathrm{bol}}\right)} L_\odot \tag{8.6.50}$$

$$= 3.8 \times 10^6 L_\odot. \tag{8.6.51}$$

(b) We approach this problem by writing down what we know:
Since all stars are assumed to be identical with individual masses of $M_\star$ and luminosities of $L_\star$, and since mass and luminosities are additive quantities, we have

$$M_{\mathrm{nuc}} = N M_\star, \quad L_{\mathrm{nuc}} = N L_\star. \tag{8.6.52}$$

Using the mass-luminosity relation that was provided, we find

$$\log\left(\frac{L_\star}{L_\odot}\right) = 3.5 \log\left(\frac{M_\star}{M_\odot}\right) \tag{8.6.53}$$

$$\iff \quad \frac{L_{\mathrm{nuc}}}{N L_\odot} = \left(\frac{M_{\mathrm{nuc}}}{N M_\odot}\right)^{3.5} \tag{8.6.54}$$

$$\iff \quad N^{2.5} = \left(\frac{M_{\mathrm{nuc}}}{M_\odot}\right)^{3.5} \frac{L_\odot}{L_{\mathrm{nuc}}} \tag{8.6.55}$$

$$\iff \quad N = \left(\frac{M_{\mathrm{nuc}}}{M_\odot}\right)^{1.4} \left(\frac{L_{\mathrm{nuc}}}{L_\odot}\right)^{-0.6} \tag{8.6.56}$$

$$= 2.13 \times 10^7. \tag{8.6.57}$$

(c) For this part of the problem, we start with the definition of the luminosity and use $\eta = 0.15$ as the fraction of mass converted into energy when accreted onto the black hole.
Thus, we have[a]

$$L = \frac{\Delta E}{\Delta t} = \eta \dot{M} c^2 \tag{8.6.58}$$

$$\iff \quad \dot{M} = \frac{L_{\mathrm{nuc}}}{\eta c^2} \tag{8.6.59}$$

$$= 1.08 \times 10^{17} \frac{\mathrm{kg}}{\mathrm{s}} = 1.71 \times 10^{-6} \frac{M_\odot}{\mathrm{yr}} \tag{8.6.60}$$

$$= 1.8 \times 10^{13} \, \frac{\text{elephants}}{\text{s}} \tag{8.6.61}$$

$$= 0.278 \, \frac{\text{total mass of Earth's oceans}}{\text{h}} \tag{8.6.62}$$

For these last units, we assumed elephants to weigh about $m_{\text{ele}} = 6000$ kg, and used $m_{\text{ocean}} = 1.4 \times 10^{21}$ kg as the approximate mass of all water in the oceans. That event might be a little steamy...

[a] See **here** for the main WolframAlpha calculation