Problem set #1 Kepler and sky positions (here)

10.11.2022

Problem 1: *Stranded*

You wake up one night, and find that you are on a desert island with no idea how you got here. After briefly vowing never to repeat the previous evening's activities, you decide to try and figure out where you are. With nothing but sea around you and the star-filled sky above, you resort to your basic knowledge of astronomy.
In one direction, you see that as the stars rise, they appear to be moving, from your perspective, up and to the left. In which direction are you looking, and what can you conclude about your location on Earth from this observation? Explain.

Problem 1: Solution

Because the Earth rotates eastward, the stars moving upwards hints that we are looking in the eastern direction and we are not too close to one of the poles.
Since they seem to be moving left in addition to that, this hints that we are in the southern hemisphere. By measuring the angle the stars appear to move on, we could try to estimate our latitude.
Looking upwards on the north pole, we would see the stars just moving counterclockwise, while on the south pole they appear to move clockwise.

Problem 2: *Sky view*

Assume that stars are distributed evenly across the celestial sphere (this assumption isn't quite right, but assume it to be for this problem).

- *At the North Pole.*
  What fraction of the stars would an observer at the North Pole be able to observe over the course of the year?
- *All stars.*
  Where on Earth would you need to be in order to be able to see every star over the course of a year without changing your location? Explain.
- *Wendelstein vs. Paranal.*
  Wendelstein observatory is at a latitude of $+47.5\,\mathrm{deg}$.
  What fraction of the stars would an observer at Wendelstein be able to observe over the course of the year?
  What about an observer at Paranal Observatory in Chile, latitude $-24.5\,\mathrm{deg}$?
  Would they be able to observe all these stars on the same night? Why or why not?

Problem 2: Solution

- *At the North Pole.*
  Here, looking up, we would not see any stars below the horizon. Over the course of a night, we would only be able to see half of the whole sky[a].
- *All stars.*

Only at the equator you would be able to see all stars over the course of the year (we would need the whole year for it to be night time for each star).

- *Wendelstein vs. Paranal.*
  We start using the solid angle of the whole sky $\Omega = 4\pi$ as a normalisation constant. The integral for the fraction of stars we can see is

$$f = \frac{1}{4\pi}\int_0^{1-\Theta} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi \sin\theta = \frac{\cos\Theta + 1}{2}, \tag{1.2.1}$$

where we integrated θ from 0 to $1 - \Theta$.
This function also makes sense as we have $f(\Theta = 0°) = 1$, and $f(\Theta = 90°) = \frac{1}{2}$ (it matches the cases discussed above).
We can put in the numbers for Wendelstein and Paranal:

$$f_{\text{Wendelstein}} = 0.84, \quad f_{\text{Paranal}} = 0.95. \tag{1.2.2}$$

Due to the day-night cycle, we could not observe all of them in the same night.

[a] and maybe a little more due to diffraction

Problem 3: *Parallax project*

Suppose you are designing a project for observational astronomy. You want to measure the distance to a near-Earth asteroid using parallax. You have two small telescopes you can use, one at Calar Alto Observatory (in Spain) and one at Wendelstein Observatory. The two telescopes are separated by 1662 km.
You take a simultaneous image of the asteroid from each telescope, and compare the position of the asteroid to background stars. This gives you the angular offset between the asteroid as seen from the two locations to a precision of $2''$.
Draw a picture that shows what you've done above. Indicate on this picture what measurement is 1662 km. Label with d the distance from the Earth to the asteroid.

(a) What is the largest distance an asteroid can have without being too far for your measurement given the precision of $2''$? (Hint: if you come up with an answer that is measured in parsecs or tenths of parsecs, you've done something very wrong. [Why?])

(b) How does your answer in (a) compare to the distance from the Earth to the Moon, and to 1 AU (the distance from the Earth to the Sun)?

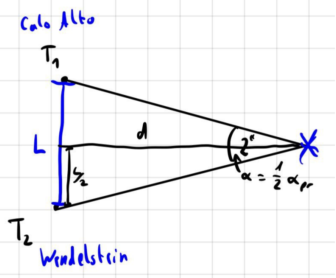
(c) Where do you want the asteroid to be in the sky as seen from each observatory so as to make the best possible measurement? (E.g., near rising, near setting, directly overhead, a little east of overhead, or a little west of overhead?)

Problem 3: Solution

We employ the distance between the two telescopes $L = 1662$ km, and the precision α_{pr}, and use d as the distance between Earth and the asteroid.

(a) The furthest distance is directly given by the precision (and related to the optimal case of the asteroid being right in between the two telescopes).

For this case, we can assume a right triangle with $\alpha := \alpha_{pr}/2$ as the angle, and d and $L/2$ as the legs (see the picture to the right), so we have



$$\tan\alpha = \frac{L}{2d} \iff d_{\max} = \frac{L}{2\tan\alpha} \approx \frac{L}{1''} \approx 1.6{\times}10^{8}\ \text{km} \ \approx 1 \tag{1.3.3}$$

where we used the small-angle approximation for $\tan\alpha \approx \alpha$, and converted the $1''$ to radians via $1\,\text{deg} = \frac{\pi}{180}$, so $1'' = \frac{\pi}{180 \cdot 3600}$.

(b) The distance is similar to the distance from the Earth to the Sun, and therefore much further than the distance from the Earth to the Moon.

(c) An observation right overhead should yield the best results, other observations would need different geometrical considerations.

Problem 4: *Halley's comet*

The orbits of comets in the Solar system often have very large eccentricities, close to 1. For example, Halley's comet has an orbital period of 76 years and an eccentricity of $e = 0.967$.

(a) Determine the semimajor axis a of Comet Halley's orbit *without using the Sun's mass*. Compare this value with those of the outer planets, in particular Pluto.

(b) Use the above orbital parameters of Halley's comet to determine the mass of the Sun, and discuss your result.

(c) How large is Comet Halley's distance to the sun at its perihelion and at its aphelion? What is its velocity at these points? What is its velocity where it crosses the minor axis of its orbit?

(d) What is the ratio of the comet's kinetic energies at perihelion and aphelion?

Problem 4: Solution

We employ $P_\text{H} = 76$ yr, $e = 0.967$.

(a) Using Kepler's 3rd law and the parameters of the Earth, we find

$$\frac{a_\oplus^3}{a_\text{H}^3} = \frac{P_\oplus^2}{P_\text{H}^2} \iff a_\text{H} = a_\oplus \sqrt[3]{\frac{P_\text{H}^2}{P_\oplus^2}} = \sqrt[3]{76^2}\ \text{au} = 17.94\ \text{au}. \tag{1.4.4}$$

This result compares as follows to $a_\text{Saturn} \approx 9.54$ au, $a_\text{Uranus} \approx 19.2$ au, and $a_\text{Pluto} \approx 39.48$ au:

$$\frac{a_\text{H}}{a_\text{Saturn}} \approx 1.88, \qquad \frac{a_\text{H}}{a_\text{Uranus}} \approx 0.93, \qquad \frac{a_\text{H}}{a_\text{Pluto}} \approx 0.45. \tag{1.4.5}$$

(b) Using the period-mass-relationship we derived in the lecture and neglecting Halley's mass, we have

$$P^2 = \frac{4\pi^2 a^3}{G(m_1+m_2)} \sim \frac{4\pi^2 a_\text{H}^3}{GM_\odot} \implies M_\odot = \frac{4\pi^2 a_\text{H}^3}{GP_\text{H}^2} = 1.99 \times 10^{30} kg, \tag{1.4.6}$$

which is indeed the literature mass of the sun.

(c) We calculate the elliptical parameters using the semimajor axis a_H obtained in (a) and the given ellipticity e:

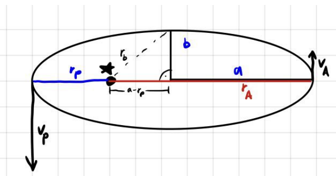
$$r_\mathrm{P} := r_\mathrm{Perihelion} = a(1-e) \approx 0.592 \text{ au}, \tag{1.4.7}$$

$$r_\mathrm{A} := r_\mathrm{Aphelion} = a(1+e) \approx 35.29 \text{ au}, \tag{1.4.8}$$

$$b = a\sqrt{1-e^2} \approx 4.57 \text{ au}, \tag{1.4.9}$$

so the comet comes closer to the Sun than the Earth in perihelion, and stays out almost as far as Pluto in aphelion.

Interestingly, in reality, there are times when Pluto is actually closer than Halley's comet because Pluto is also on an elliptical orbit.



The speed of an object with semi-major axis a at the distance r from the sun can be calculated via the **Vis-viva equation**[a], which can derived from energy conservation (sum of kinetic ($\frac{1}{2}mv_i^2$) and gravitational energies ($-\frac{GMm}{r_i}$) at perihelion and aphelion is conserved), angular momentum conservation ($r_\mathrm{A}v_\mathrm{A} = r_\mathrm{P}v_\mathrm{P}$) and the fact that $r_\mathrm{P} + r_\mathrm{A} = 2a$:

$$v^2 = GM_\odot \left(\frac{2}{r} - \frac{1}{a}\right). \tag{1.4.10}$$

Using this, we have

$$v_\mathrm{P} = \sqrt{GM_\odot \left(\frac{2}{r_\mathrm{P}} - \frac{1}{a}\right)} \approx 52.29 \,\frac{\text{km}}{\text{s}} \tag{1.4.11}$$

$$v_\mathrm{A} \approx 0.91 \,\frac{\text{km}}{\text{s}}. \tag{1.4.12}$$

For the speed at the semi-minor axis, we first need to use trigonometry to determine the distance of the comet to the Sun, we have

$$r_b = \sqrt{(a-r_p)^2 + b^2} \approx 17.94 \text{ au } \sim a, \tag{1.4.13}$$

so we arrive at

$$v_b \approx \sqrt{\frac{GM_\odot}{a}} \approx 7032 \,\frac{\text{m}}{\text{s}}. \tag{1.4.14}$$

(d) The ratio of the comet's kinetic energies is given by

$$\frac{E_\mathrm{kin,P}}{E_\mathrm{kin,A}} = \frac{v_\mathrm{P}^2}{v_\mathrm{A}^2} \approx 3302. \tag{1.4.15}$$

Note: Due to angular momentum conservation ($r_\mathrm{A}v_\mathrm{A} = r_\mathrm{P}v_\mathrm{P}$), we could have also used

$$\frac{E_\mathrm{kin,P}}{E_\mathrm{kin,A}} = \frac{r_\mathrm{A}^2}{r_\mathrm{P}^2}. \tag{1.4.16}$$

[a] see **wikipedia**, also for an in-depth derivation

Problem 5: *Neutron Star*

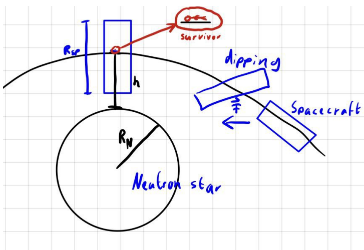
In Larry Niven's short story *Neutron Star*, a research vessel with a hull of indestructible alien material makes a flyby of a neutron star, passing its surface on a hyperbolic orbit in free fall at a minimum distance of one mile. When the ship is later retrieved, the hull is indeed intact, but the crew compartment is wrecked, and the crew is dead. To determine what had happened, the story's protagonist is blackmailed by the ship's manufacturer into repeating the trip with an identical ship (the manufacturer feared bad publicity should it become known that the hull might not have been impregnable after all and that something may have gotten into the ship), and survives.

- Draw a sketch of the situation. Make some plausible assumptions regarding the physical dimensions (explain what your assumptions are) and use what you have already learned in the lecture to *calculate* what happened to the original crew. How could the protagonist have managed to survive in the story? Is this possible?

Problem 5: Solution

For the calculations, we assume $R_\mathrm{N} \approx 10.4$ km (as the numbers are nicer that way and a radius of ~ 10 km is typical for neutron stars). The minimum distance of the flyby was $h = 1$ mi ≈ 1.6 km, so the distance to the core of the neutron star is

$$r = h + R \approx 12 \text{ km}. \tag{1.5.17}$$



We also apply a mass of $M_\mathrm{N} \approx M_\odot$.
We can calculate the tidal acceleration at this distance for an object stretched by $R_\mathrm{Ship} = 50$ km:

$$\Delta a_\mathrm{Ship} = \frac{2GM}{r^3} R_\mathrm{Ship} \approx 10^{10}\,\frac{\mathrm{m}}{\mathrm{s}^2} = 10^9 g, \tag{1.5.18}$$

where $g \approx 10$ m/s^2 is the gravitational acceleration on Earth.
For a person standing up relative to the neutron star (so $R_\mathrm{Stand} \sim 2$ m, the tidal forces acting on them are

$$\Delta a_\mathrm{Stand} \approx 1.5 \times 10^8\,\frac{\mathrm{m}}{\mathrm{s}^2} \implies F_\mathrm{Stand} = a_\mathrm{Stand} m_\mathrm{pers} \approx 10^{10}\text{ N}, \tag{1.5.19}$$

which would wreck any human being apart (we assumed a very lanky ($m_\mathrm{pers} \approx 66$ kg) and tall human here).
The protagonist might survive if they try to lie down. Assuming a vertical distance of $R_\mathrm{Lie} \sim 10$ cm, the tidal acceleration and force would be

$$\Delta a_\mathrm{Lie} \approx 7.7 \times 10^6\,\frac{\mathrm{m}}{\mathrm{s}^2} \implies F_\mathrm{Lie} \approx 5 \times 10^8\text{ N}, \tag{1.5.20}$$

which is around 0.1 times the force applied on the hoover dam.
Assuming an area of ~ 1 m^2 of the poor protagonist's body, the pressure would be... just insane and still probably unbearable.
I honestly am not sure whether this would be survivable, but I'd not sign up for trying

it out.
Since we just saw that the only possibility of standing a chance to survive the spaghettification would be to lie down (and even there the pressure might rip your body apart), it is important to note that one would need to factor in the dip of the (I don't know how) indestructible ship, as the tidal force difference would also bend its nose toward the neutron star - if not accounted for, you could accidentally be pushed to a stand-up-position. RIP.