Problem set #4 Exoplanet detection (here)

01.12.2022

Problem 1: *Radial velocity variations caused by planets*

(a) Derive the amplitude of the radial velocity variation of the host star of a transiting extra-solar planet ($i = 90°$) as a function of period and planetary and stellar mass. Use Kepler's third law in the following approximations:

$$a_{\text{star}} \ll a_{\text{planet}} \tag{4.1.1}$$

$$M_{\text{planet}} \ll M_{\text{star}} \tag{4.1.2}$$

(Hint: use the centre of mass theorem and the above approximations to derive a_{planet} as a function of M_{star} and P.)

(b) Calculate the amplitude of the Sun's radial velocity variation caused by Earth, Jupiter, and Saturn.
Use the following values:

$$\begin{aligned}
G &= 6.67 \times 10^{11}\text{m}^3\text{kg}^{-1}\text{s}^{-2} \\
M_{\text{Earth}} &= 5.97 \times 10^{24}\,\text{kg} \\
M_{\text{Saturn}} &= 5.68 \times 10^{26}\,\text{kg} \\
M_{\text{Jupiter}} &= 1.90 \times 10^{27}\,\text{kg} \\
M_{\text{Sun}} &= 1.99 \times 10^{30}\,\text{kg} \\
p_{\text{Jupiter}} &= 11.86\,\text{yr} \\
p_{\text{Saturn}} &= 29.46\,\text{yr}
\end{aligned}$$

Problem 1: Solution

Note: The code used for parts of this exercise is available in **this repository**.

(a) The idea to derive the velocity is to first find the orbital period time P and use this as the orbital period of the star around the common centre of mass.
The orbital period time is given by Kepler's third law:

$$P^2 \approx \frac{4\pi^2 a_{\text{p}}^3}{G(M_{\text{star}} + M_{\text{planet}})} \iff a_{\text{p}} = \sqrt[3]{\frac{G(M_{\text{s}} + M_{\text{p}})P^2}{4\pi^2}}. \tag{4.1.3}$$

The centre of mass of the planet and star is along the line connecting them with distances a_1 and a_2, denoting the distances of the CMS to star and planet, respectively.
Therefore, we have

$$a_1 + a_2 = a_{\text{p}}, \quad M_{\text{s}} a_1 \overset{!}{=} M_{\text{p}} a_2 = (a_{\text{p}} - a_1) M_{\text{p}} \iff a_1 = \frac{M_{\text{p}}}{M_{\text{p}} + M_{\text{s}}} a_{\text{p}}. \tag{4.1.4}$$

Using these two equations and the fact that the change in radial velocity we observe is twice the maximum circular velocity of the star around its centre of mass achieved (since it's both radially receding and approaching[a]), we have

$$\begin{aligned}\Delta v_\mathrm{r} &= 2v_\mathrm{circ} = 2\frac{2\pi a_1}{P} \\ &= \frac{4\pi}{P}\frac{M_\mathrm{p}}{M_\mathrm{p}+M_\mathrm{s}}a_\mathrm{p} \\ &= M_\mathrm{p}\sqrt[3]{\frac{G4^3\pi^3}{4\pi^2(M_\mathrm{s}+M_\mathrm{p})^2P}} \\ &\approx \underline{\underline{M_\mathrm{p}\sqrt[3]{\frac{G16\pi}{M_\mathrm{s}^2P}}}},\end{aligned} \tag{4.1.5}$$

where we used $M_\mathrm{p} \ll M_\mathrm{s}$ for the last approximation.

(b) Putting in the numbers, we find

Name	Mass [kg]	Period [yr]	Δv [m/s]	$\Delta\lambda_{\mathrm{H}\alpha}$ [nm]
Earth	5.97×10^{24}	1	0.1651	3.61×10^{-7}
Jupiter	1.89×10^{27}	11.86	22.99	5.03×10^{-5}
Saturn	5.68×10^{26}	29.46	5.082	1.11×10^{-5}

The last column converts the difference in radial velocity to the difference in observed Hα wavelength (with $\lambda_{\mathrm{H}\alpha} = 656.281$ nm) - it's insanely minute!

[a] in the tutorials, we also discussed this and argued that this factor of 2 might not be justified, but it doesn't change the final result that much.

Problem 2: *Transit detection*

What fraction of sunlight is blocked when Earth passes in front of the Sun and how large is the decrease in brightness expressed in magnitudes? Here it shall be assumed that the observer resides far outside the solar system and that the Sun and the planets appear as uniform disks. How large is the effect for Mercury, Jupiter, and Neptune?
(Magnitudes are a logarithmic measure of radiative flux customarily used in observational astronomy. The magnitude difference between two fluxes F_1 and F_2 is defined as

$$\Delta m_{12} = -2.5\log\left(\frac{F_1}{F_2}\right). \tag{4.2.6}$$

Note that, being logarithmic, a *difference* in magnitude corresponds to a ratio in flux.)

Problem 2: Solution

Note: The code used for parts of this exercise is available in **this repository**.
Assuming we are in a wavelength band the planet is completely opaque to, ignoring edge effects, and assuming that we are sufficiently far away for plane parallel approximation, we can just use the radii of the star and the planet to calculate the circular areas the

discs cover, leading to a fraction of

$$f = \frac{\pi r_\mathrm{p}^2}{\pi r_\mathrm{s}^2}. \tag{4.2.7}$$

Since the flux is proportional to the area, the relative change in flux would just be

$$\frac{F_\text{with planet}}{F_\text{without planet}} = 1 - \frac{r_\mathrm{p}^2}{r_\mathrm{s}^2} \implies \Delta m_\text{planet} = \underline{\underline{-2.5\log\left(1 - \frac{r_\mathrm{p}^2}{r_\mathrm{s}^2}\right)}}. \tag{4.2.8}$$

For Mercury, Earth, Jupiter, and Neptune, we therefore have

Name	Radius [km]	Δm_planet [mag]
Mercury	2.43×10^3	1.33×10^{-5}
Earth	6.37×10^6	9.12×10^{-5}
Jupiter	7.14×10^7	1.15×10^{-2}
Neptune	2.46×10^4	1.36×10^{-3}

The changes in magnitude are quite small!

Problem 3: *Transit probability*

Show that the probability p for a suitable orientation of a planet's orbital plane to allow observing a transit is given by the simple formula

$$p = \frac{R_\star}{a}, \tag{4.3.9}$$

where $R_\star$ is the radius of the star and a is the planet's orbital radius. How large are the probabilities for an alien observer to be in a position allowing the observation of transits of Mercury, Earth, Jupiter, Neptune?

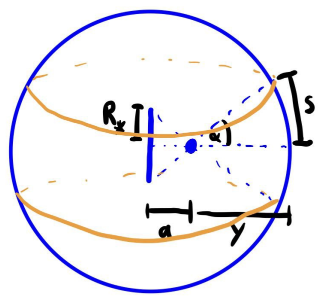
Problem 3: Solution

Note: The code used for parts of this exercise is available in **this repository**.

The transit probability has excellently been derived in **Borucki et al. 1984**, which will be reproduced here:

As the basis for this derivation, we note that the probability of a transit being observed corresponds to the fraction the 'shadow' of the planet covers on the celestial sphere.

Therefore, we can use basic geometry to see that, with the distance between star and planet a, the distance to an observer y and the projected height of the shadow s, the area of the 'ring' the shadow covers on the celestial sphere is $A_\text{Ring} = 2 \cdot 2\pi(a+y)s$, while the celestial sphere itself has an area of $4\pi(a+y)^2$.



Thus, we first have

$$p = \frac{2 \cdot 2\pi(a+y)s}{4\pi(a+y)^2}, \tag{4.3.10}$$

and we can identify $R_\star = a \tan\alpha \approx a\alpha$ and thus $s = y\tan\alpha \approx y\alpha = \frac{yR_\star}{a}$, so we find

$$p = \frac{yR_\star}{(a+y)a} \overset{y\to\infty}{\to} \underline{\underline{\frac{R_\star}{a}}}. \tag{4.3.11}$$

Name	a [m]	probability
Mercury	5.79×10^{10}	1.20×10^{-2}
Earth	1.49×10^{11}	4.65×10^{-3}
Jupiter	7.78×10^{11}	8.93×10^{-4}
Neptune	4.49×10^{12}	1.54×10^{-4}

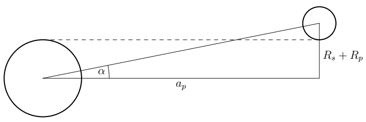
Problem 4: *Transit duration*

Derive a general expression for the duration of a transit across the centre of a star by using Kepler's third law. Assume that the observer is at a very large distance and that the orbits are circular. The formula should give the transit duration t in hours, when the mass M_s of the star is given in $M_\odot$, the radius R_s of the star in $R_\odot$, and the planet's orbital radius a_p in AU. How long does a central transit of Mercury, Earth, Jupiter, and Neptune last? How long does the transit last relative to the total orbital period of the planet?

Problem 4: Solution

Note: The code used for parts of this exercise is available in **this repository**.

Since we assume circular orbits and a central transit, the distance the planet travels in front of the host star is $d = 2R_\mathrm{s} + 2R_\mathrm{p} \approx 2R_\mathrm{s}$ where we can neglect $R_\mathrm{p} \ll R_\mathrm{s}$ (otherwise, we could argue that the transit starts and ends once half of the planet is in front of the star).



Using Kepler's third law, neglecting the planets mass, and assuming a circular orbit, we find the velocity of the planet to be

$$P^2 = \frac{4\pi a_\mathrm{p}^3}{(M_\mathrm{s}+M_\mathrm{p})G} \implies v = \frac{2\pi a_\mathrm{p}}{P} \approx 2\pi a_\mathrm{p}\sqrt{\frac{GM_\mathrm{s}}{4\pi a_\mathrm{p}^3}}. \tag{4.4.12}$$

Therefore, the time of the transit is approximately

$$t = \frac{d}{v} \approx 2R_\mathrm{s}\sqrt{\frac{a_\mathrm{p}}{GM\pi}} \approx 12.976\,(R_s/R_\odot)\sqrt{\frac{a_\mathrm{p}/\mathrm{au}}{M/M_\odot}}\ \mathrm{h}. \tag{4.4.13}$$

Applying this for the planets, we have

Name	a [m]	t [h]	P [h]	t/P
Mercury	5.79×10^{10}	8.074	2.11×10^{3}	3.82×10^{-3}
Earth	1.49×10^{11}	12.98	8.76×10^{3}	1.48×10^{-3}
Jupiter	7.78×10^{11}	29.6	1.04×10^{5}	2.84×10^{-4}
Neptune	4.49×10^{12}	71.16	1.44×10^{6}	4.92×10^{-5}

Problem 5: *Astrometric precision*

How large is the astrometric wobble (in arcseconds) of a Sun-like star at a distance of 10 pc that harbours an Earth-like planet? How large is a structure on the Moon that has the same angular extent when seen from earth? Why can only interferometers making use of *differential techniques* achieve the required precision to detect such planets?
(What is the resolution of a conventional optical interferometer with a baseline of 1 km?)

Problem 5: Solution

Thanks and credits to Iliya Tikhonenko!
If we assume that there are no other planets in the system, we can derive the radius of Sun's "orbit" using the center of mass formula for just two bodies (placing the origin of the coordinate frame at the barycenter):

$$\boldsymbol{r}_c = \frac{\boldsymbol{r}_\odot M_\odot + \boldsymbol{r}_e M_e}{M_\odot + M_e} = 0 \;\Rightarrow\; \Delta = 2a_\odot = 2\, a_e \frac{M_e}{M_\odot} \approx 6 \times 10^{-6}\,\mathrm{AU} \tag{4.5.14}$$

Remembering the definition of 1pc, we can trivially obtain the wobble in arcseconds:

$$\delta = \frac{\Delta}{r} = \frac{6 \times 10^{-6}}{10} = 6 \times 10^{-7} = 0.6\,\mu as, \tag{4.5.15}$$

which is very, *very* small. The structure with this angular size observed on the Moon would be only $6 \times 10^{-7} \cdot 206265^{-1} \cdot 384\,400\,\mathrm{km} \approx 1\,\mathrm{mm}$ long. For example, that one 1 cent coin (16.25 mm in diameter) that you lost somewhere 4 years ago would be about 9 mas from the Moon, i.e. more than 15 times larger.
For a conventional optical interferometer, the angular resolution can be estimated by the following relation:

$$\delta^{(\text{in radians})} = \frac{\lambda}{2b}, \tag{4.5.16}$$

where b stands for baseline; essentially it means that we should have a phase difference equal to π for the longest possible value of the baseline. For an interferometer with 1 km baseline observing at e.g. $\lambda_0 = 600\,\mathrm{nm}$, the resolution would be $3 \times 10^{-10}\,\mathrm{rad} \approx 62\,\mu\mathrm{as}$ which is obviously still not enough for our case. In the differential interferometry the same star is used as a reference for itself at multiple wavelengths, which somehow improves the resolution, but I'm not sure I understand how it works, so maybe we should discuss it further on the tutorial. **Monnier (2003)** gives a nice overview of the subject, but the paper is relatively old.

Problem 6: *The solar nebula*

Note: The code used for parts of this exercise is available in **this repository**. Most of the material within the original solar nebula from which the solar system formed has been lost. Here the original mass of this nebula shall be derived by assuming that the elements are in the cosmic abundance ratios (see table).

	H	He	O	Mg	Si	Fe
Rel. number of atoms	1	0.08	0.0007	4e-05	4e-05	3e-05
Atomic weight	1	4	16	24.3	28.1	55.8

With the further assumption that the heavy elements are completely bound in rocks (represented by $\mathrm{Mg_2SiO_4}$, $\mathrm{Fe_2SiO_4}$, and $\mathrm{SiO_2}$) and water ice ($\mathrm{H_2O}$), determine the ratio between the total masses of rocks, water ice, and gases (H, He).
If the terrestrial planets together contain $2M_\oplus$ of rocks, Jupiter and Saturn both have rocky cores of about $10M_\oplus$ and the rocky masses of Uranus and Neptune are $3M_\oplus$ each, what was the minimum solar nebula mass required?
Compare this with the actual mass of the planets and derive a planet formation efficiency for the solar system.

Problem 6: Solution

To start this analysis, we need to figure out the relative fractions of the rocky elements. This can be done by realising that the number densities of some of the elements are completely locked by one of the atomic species they consist of (e. g. magnesium for $\mathrm{Mg_2SiO_4}$) as these do not appear in any other elements.
Therefore, we can find the relative number densities of the molecules in an iterative way:

$$n_{\mathrm{Mg_2SiO_4}} \overset{!}{=} \frac{1}{2} n_{\mathrm{Mg}} \tag{4.6.17}$$

$$n_{\mathrm{Fe_2SiO_4}} \overset{!}{=} \frac{1}{2} n_{\mathrm{Fe}} \tag{4.6.18}$$

$$\implies n_{\mathrm{SiO_2}} \overset{!}{=} n_{\mathrm{Si}} - n_{\mathrm{Mg_2SiO_4}} - n_{\mathrm{Fe_2SiO_4}} \tag{4.6.19}$$

$$\implies n_{\mathrm{H_2O}} \overset{!}{=} n_{\mathrm{O}} - 2n_{\mathrm{SiO_2}} - 4n_{\mathrm{Mg_2SiO_4}} - 4n_{\mathrm{Fe_2SiO_4}} \tag{4.6.20}$$

$$\implies n_{\mathrm{H}} = n_{\mathrm{H,\ tot}} - 2n_{\mathrm{H_2O}} \tag{4.6.21}$$

TO DO Actually finish this exercise lol (!)