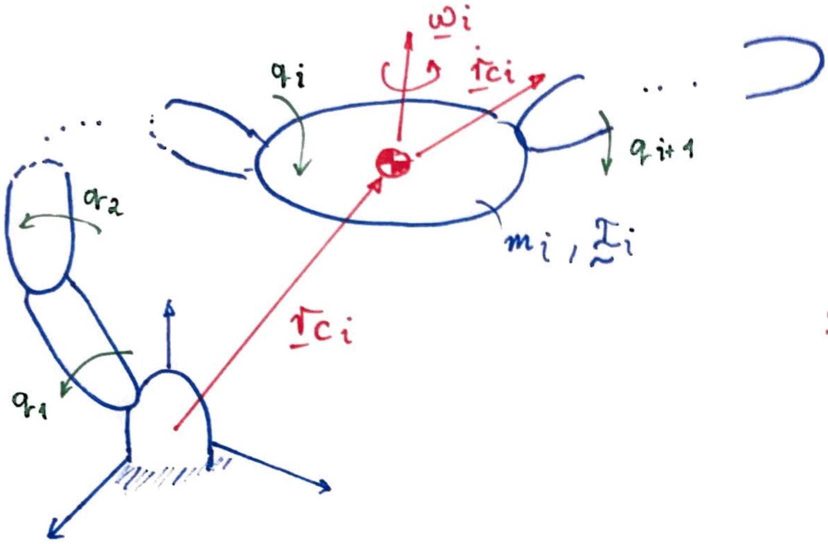
1. Dynamics of a fixed-base manipulator



$$\underline{\vartheta}^{h}_{IC_i} = \begin{bmatrix} \dot{\underline{r}}_{C_i} \\ \underline{\omega}_i \end{bmatrix} := \begin{bmatrix} \dot{\underline{r}}_{C_i} \\ {}_{S}\underline{\omega}_{IC_i} \end{bmatrix}$$

Kinetic energy: $K = \frac{1}{2} \sum_{i=1}^{n} \left(m_i \dot{\underline{r}}_{C_i}^T \dot{\underline{r}}_{C_i} + \underline{\omega}_i^T \underset{\sim}{I}_i \underline{\omega}_i \right)$

Instantaneous rate of change of the generalized coordinates is mapped to the hybrid velocity of each link using Jacobians

$$\dot{\underline{r}}_{C_i} = \frac{\partial \underline{r}_{C_i}}{\partial \underline{q}} \dot{\underline{q}} = \underset{\sim}{J}_{v_i} \dot{\underline{q}}$$

$$\underline{\omega}_i = \underset{\sim}{J}_{\omega_i} \dot{\underline{q}}$$

$$\Rightarrow \quad K = \frac{1}{2} \dot{\underline{q}}^T \underset{\sim}{D}(\underline{q}) \dot{\underline{q}}$$

$$\underset{\sim}{D}(\underline{q}) = \sum_{i=1}^{n} \left(m_i \underset{\sim}{J}_{v_i}^T \underset{\sim}{J}_{v_i} + \underset{\sim}{J}_{\omega_i}^T \underset{\sim}{I}_i \underset{\sim}{J}_{\omega_i} \right)$$

Eq. of motion

$$\underset{\sim}{D}(\underline{q}) \ddot{\underline{q}} + \underset{\sim}{C}(\underline{q}) \dot{\underline{q}} + \underline{g}(\underline{q}) = \underline{\tau}$$

2. Dynamic model in spatial coordinates

- Interaction with the environment via end-links (e.g. end-effector, hands, feet)
- Tasks (desired motion / force) specified in spatial co-ordinates

→ How to express the eq. of motion wrt the end-link coordinates?

assuming $\underset{\sim}{J}$ is invertible, then

$$\dot{\underline{q}} = \underset{\sim}{J}^{-1}\underline{\vartheta}$$

$$\dot{\underline{q}}^T = \underline{\vartheta}^T(\underset{\sim}{J}^{-1})^T = \underline{\vartheta}^T\underset{\sim}{J}^{-T}$$

$$\underset{\sim}{D}\ddot{\underline{q}} + \underset{\sim}{C}\dot{\underline{q}} + \underline{g} = \underline{\tau} \quad \Big| \cdot \underset{\sim}{J}^{-T}$$

$$\underset{\sim}{J}^{-T}\underset{\sim}{D}\ddot{\underline{q}} + \underset{\sim}{J}^{-T}\underset{\sim}{C}\dot{\underline{q}} + \underset{\sim}{J}^{-T}\underline{g} = \underset{\sim}{J}^{-T}\underline{\tau}$$

$$\underline{\vartheta} = \underset{\sim}{J}(\underline{q})\dot{\underline{q}}$$

$$\dot{\underline{\vartheta}} = \underset{\sim}{J}(\underline{q})\ddot{\underline{q}} + \dot{\underset{\sim}{J}}(\underline{q})\dot{\underline{q}}$$

$$\ddot{\underline{q}} = \underset{\sim}{J}^{-1}(\underline{q})\dot{\underline{\vartheta}} - \underset{\sim}{J}^{-1}\dot{\underset{\sim}{J}}\dot{\underline{q}}$$

$$\underset{\sim}{J}^{-T}\underset{\sim}{D}\underset{\sim}{J}^{-1}\dot{\underline{\vartheta}} - \underset{\sim}{J}^{-T}\underset{\sim}{D}\underset{\sim}{J}^{-1}\dot{\underset{\sim}{J}}\dot{\underline{q}} + \ldots$$

$$\ldots \underset{\sim}{J}^{-T}\underset{\sim}{C}\dot{\underline{q}} + \underset{\sim}{J}^{-T}\underline{g} = \underset{\sim}{J}^{-T}\underline{\tau}$$

$\underset{\sim}{J}^{-T}\underset{\sim}{D}\underset{\sim}{J}^{-1} = \underset{\sim}{\Lambda}$ Operational space inertia matrix

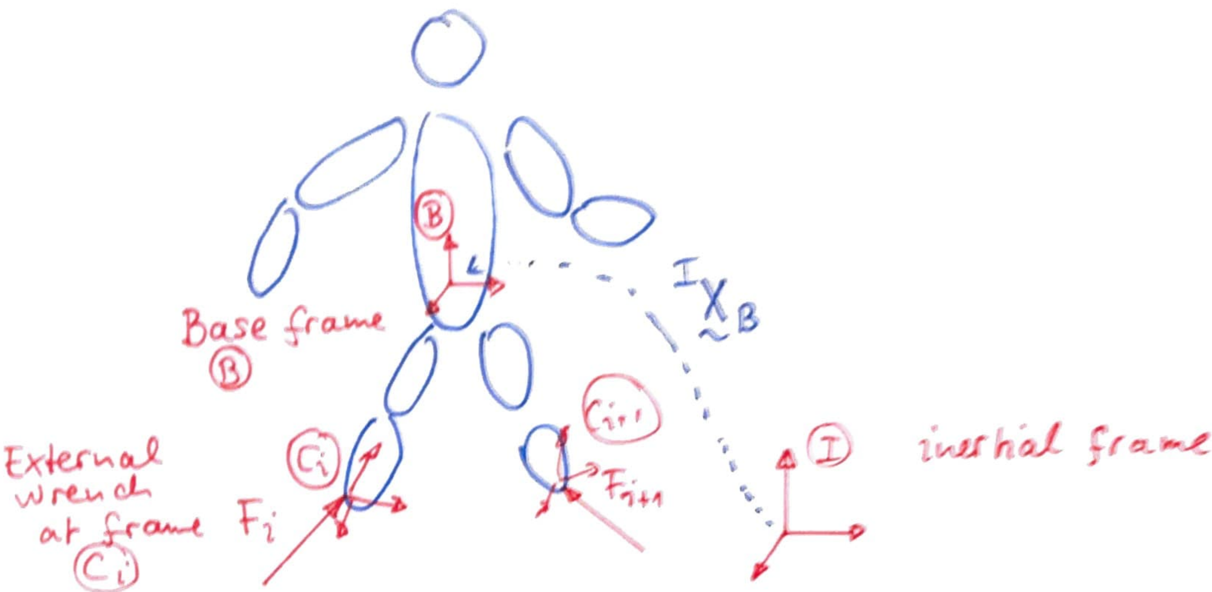
$\underset{\sim}{J}^{-T}(\underset{\sim}{C}\dot{\underline{q}} - \underset{\sim}{D}\underset{\sim}{J}^{-1}\dot{\underset{\sim}{J}}\dot{\underline{q}}) = \underline{\mu}$ Coriolis terms in operational space

$\underset{\sim}{J}^{-T}\underline{g} = \underline{\rho}$ Gravitational terms in op. sp.

$\underline{F} = \underset{\sim}{J}^{-T}\underline{\tau}$ End-effector wrench

$$\Rightarrow \quad \boxed{\underset{\sim}{\Lambda}\dot{\underline{\vartheta}} + \underline{\mu} + \underline{\rho} = \underline{F}}$$

3. Floating-base dynamics



- Humanoid / legged robot : Base not mechanically connected to the inertial frame
- Base link has 6DoF which can be represented by a fictitious 6DoF (virtual) joint
- The floating-base DoFs are not actuated, thus legged robots are part of a special class of systems, namely underactuated systems (the number of control inputs is less than the number of DoFs)

The equations of motion can be obtained similar to the fixed-base case:

$$\begin{bmatrix} D_b & D_{bj} \\ D_{bj}^T & D_j \end{bmatrix} \begin{bmatrix} \dot{\nu}_b \\ \ddot{q}_j \end{bmatrix} + \begin{bmatrix} c_b \\ c_j \end{bmatrix} + \begin{bmatrix} g_b \\ g_j \end{bmatrix} = \begin{bmatrix} 0 \\ \tau \end{bmatrix} + \tau_{ext}$$

$$\tau_{ext} = \sum_{i=1}^{n_C} \begin{bmatrix} Ad_{^B X_{C_i}} \\ J_i^T \end{bmatrix} F_i$$

$Ad_{^B X_{C_i}}$: Adjoint transformation relating an external wrench at C_i to the wrench expressed at the base frame B