

# Finite temperature and $\delta$ -regime in the Schwinger model

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# Schwinger model

- introduced by [Schwinger, 1961]: *two-dimensional quantum electrodynamics* — fermions coupled to Abelian gauge field
- simple example for chiral anomaly, topology, confinement
- often used as a testbed for conceptual and numerical approaches in lattice field theory
- nevertheless, some of the rich physical properties of the model in anisotropic volumes have not yet been tested:

**Finite temperature** : Hosotani solution has not been compared with the lattice simulation results

**$\delta$ -regime** : conjecture for the residual pion mass

# N-flavor Schwinger model

- massless case has analytic solution [Belvedere et al., 1979]
- one massive boson ("eta") —  $g$  is gauge coupling

$$m_{\eta}^2 = N \frac{g^2}{\pi}$$

- $N - 1$  massless bosons ("pions")
- massive case (fermion mass  $m > 0$ ) has no exact solution
- semiclassical prediction at infinite volume

$$m_{\pi} = \left( 4e^{2\gamma} \sqrt{\frac{2}{\pi}} \right) (m^2 g)^{1/3} = 2.1633... (m^2 g)^{1/3}$$

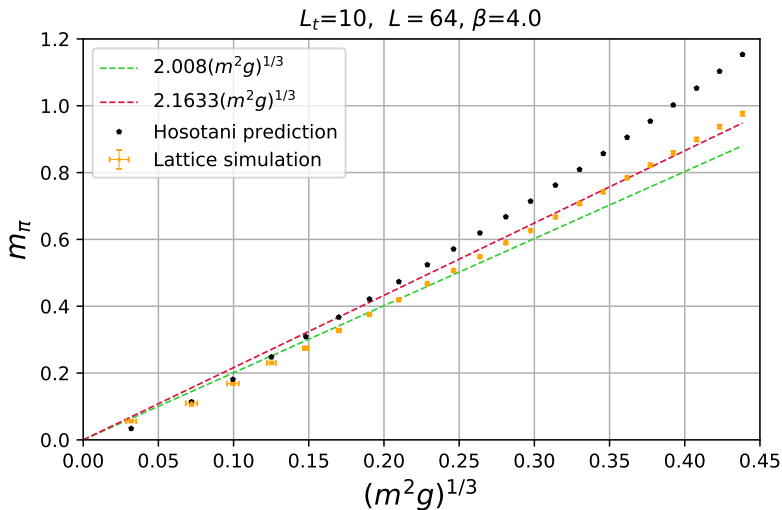
# Finite temperature - Hosotani approximate solution

- approximate solution by Hosotani et al. based on bosonization
- finite temperature massive Schwinger model reduced to quantum mechanical system with  $N - 1$  degrees of freedom
- set of nonlinear equations valid when

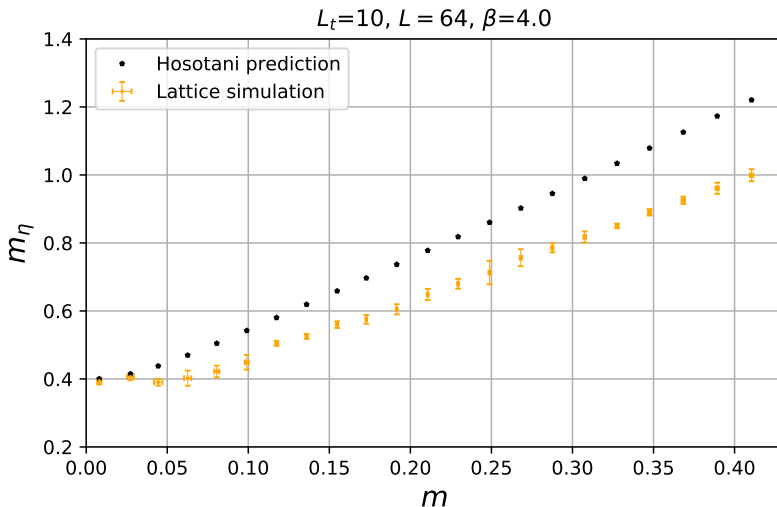
$$m \ll m_\eta$$

- boson masses can be computed by solving this set of equations in a self-consistent way
- we compare the Hosotani predictions for two flavors with the results of the HMC simulation with Wilson fermions

# Pion mass - Hosotani vs. lattice simulation



# Pion mass - Hosotani vs. lattice simulation



# $\delta$ -regime

- spatial volume is small compared to the correlation length

$$\xi = m_{\pi}^{-1}$$

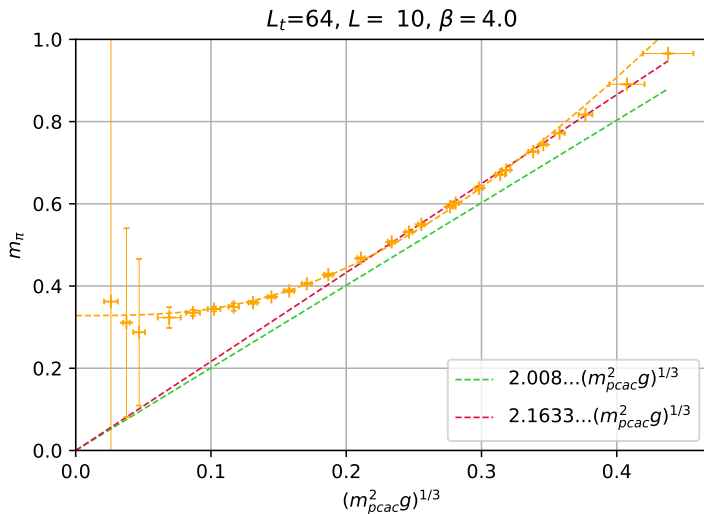
but the Euclidean time extent is large

$$L_t \gg \xi \gtrsim L$$

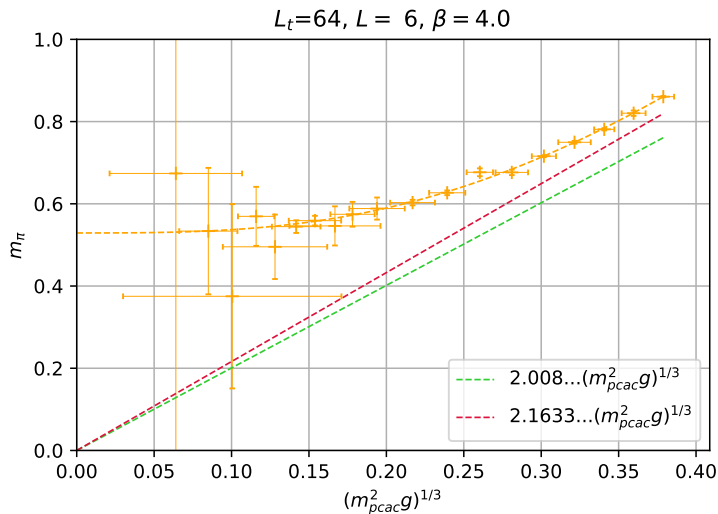
- the system is quasi one dimensional - approximation by quantum rotor [Leutwyler, 1987]
- pion has *residual mass*

$$m_{\pi}^R = \frac{3}{2\Theta}$$

where  $\Theta$  is the moment of inertia

Residual pion mass plateau:  $L_t = 64, L = 10$ 



Residual pion mass plateau:  $L_t = 64, L = 6$ 

# Conjecture

- [Hasenfratz and Niedermayer, 1993] computed  $\Theta$  up to next-to-leading order, for a general dimension  $d > 2$

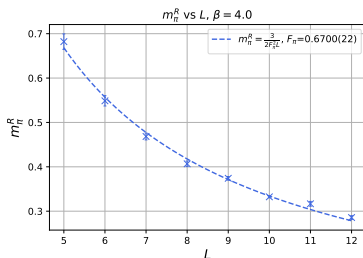
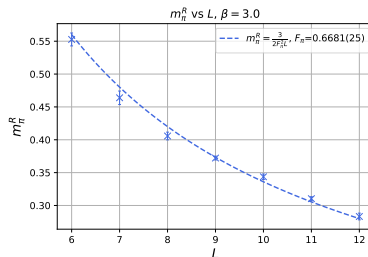
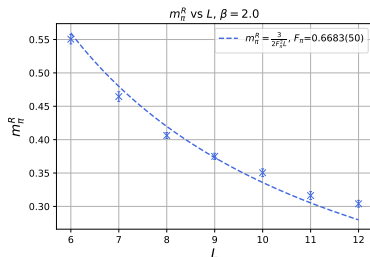
$$\Theta = F_\pi^2 L^{d-1} \left[ 1 + \frac{\mathcal{N} - 2}{4\pi F_\pi^2 L^{d-2}} \left( 2 \frac{d-1}{d-2} + \dots \right) \right]$$

- in two dimensions ( $d = 2$ ) there is a divergence of the next to leading term, so we just consider the leading term

$$m_\pi^R \simeq \frac{3}{2F_\pi^2 L}$$

- we verify the relation  $m_\pi^R \propto 1/L$  with simulation data and extract the value of pion decay constant  $F_\pi$

# 1 / L confirmed by lattice simulation



$\beta$	$F_\pi$
2.0	0.6683(50)
3.0	0.6681(25)
4.0	0.6700(22)

$$F_\pi = 0.6688(5)$$

# Witten-Veneziano formula

- in the chiral  $N$ -flavor Schwinger model the Witten-Veneziano formula is simplified to [Seiler and Stamatescu, 1987]

$$m_\eta^2 = \frac{2N}{F_\eta^2} \chi_T^{que}$$

- mass of the  $\eta$  particle is known analytically [Belvedere et al., 1979]

$$m_\eta^2 = \frac{N}{\pi\beta}$$

- continuum prediction for  $\chi_T^{que}$  [Seiler and Stamatescu, 1987]

$$\beta \chi_T^{que} = \frac{1}{4\pi^2}$$

# Quenched topological susceptibility

- [Bardeen et al., 1998] were able to analytically compute  $\chi_T^{que}$  on the lattice

$$\beta\chi_T^{que} = \frac{I_1(\beta)}{4\pi^2 I_0(\beta)}$$

by using an alternative definition of topological charge

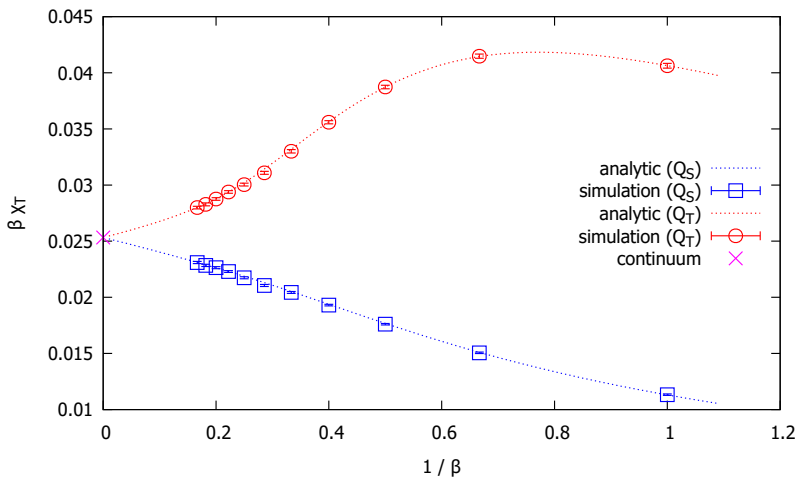
$$Q_S = \frac{1}{2\pi} \sum_P \sin(\theta_P)$$

- for the usual definition of topological charge

$$Q_T = \frac{1}{2\pi} \sum_P \theta_P$$

it is not possible to find analytic expression, but using the same line of reasoning it is possible to numerically compute  $\chi_T^{que}$  to arbitrary precision

# Quenched topological susceptibility



# $F_\eta$ versus $F_\pi$

- in large  $N_c$  QCD, to the order  $1/N_c$

$$F_{\eta'} = F_\pi$$

- in the Schwinger model nothing assures that this relation holds
- inserting the confirmed values for  $m_\eta^2$  and  $\chi_T^{que}$

$$F_\eta^2 = \frac{2N}{m_\eta^2} \chi_T^{que} = 2N \left( \frac{\pi\beta}{N} \right) \left( \frac{1}{4\pi^2\beta} \right) = \frac{1}{2\pi}$$

- our results suggest that in the Schwinger model the two decay constants differ significantly

$$F_\eta = 0.3989 \quad F_\pi = 0.6688(5)$$