Chapter 5

The Witten-Veneziano formula

The Witten-Veneziano formula [1, 2] relates the masses of the η, η' and π mesons with the *topological susceptibility* χ_T and the *pion decay constant* F_π. The latter is dimensionless in two dimensions and it will be introduced in the next chapter when we discuss the chiral Lagrangian. In this chapter we show a method to obtain its value. Meanwhile, χ_T is defined for the Euclidean Schwinger model in the continuum as

$$\chi_T = \int d^2x \, \langle q(x)q(0) \rangle, \tag{5.1}$$

where

$$q(x) = -\frac{g}{2\pi}\epsilon_{\mu\nu}F^{\mu\nu} = \frac{g}{2\pi}F_{12}(x) \tag{5.2}$$

is the *topological charge density*. With $q(x)$ one can introduce the *topological charge* as well

$$Q(x) = \int d^2x \, q(x). \tag{5.3}$$

An important property of the topological charge is that it is an integer number (see refs. [3, 4]). The general version of the Witten-Veneziano formula in QCD reads

$$m_{\eta'}^2 - \frac{1}{2}m_\eta^2 - \frac{1}{2}m_\pi^2 = \frac{2N}{F_\pi^2}\chi_T, \tag{5.4}$$

where N is the number of flavors. In the limit of massless fermions, eq. (5.4) is simplified for the Schwinger model [5]

$$m_{\eta'}^2 = \frac{2N}{F_\pi^2}\chi_T^{\text{que}}, \tag{5.5}$$

where "que" stands for quenched, *i.e.* its value when the fermion mass $m \to \infty$. On the other hand, as we mentioned in chapter 3, one can relate $m_{\eta'}$ with the coupling constant as follows

$$m_{\eta'}^2 = N\frac{g^2}{\pi}. \tag{5.6}$$

Thus, by determining χ_T one can obtain a value for F_π. To measure the topological susceptibility using lattice simulations, we have to discretize the topological charge density. This can be done through the plaquettes defined in chapter 2. From eq. (2.96), we know that for a small lattice spacing a, the plaquette has the following expression

$$U_{\mu\nu}(\vec{n}) = e^{-iga^2F_{\mu\nu}(\vec{n})}. \tag{5.7}$$

Then

$$F_{\mu\nu}(\vec{n}) = -\frac{i}{ga^2}\ln U_{\mu\nu}(\vec{n}). \tag{5.8}$$

That way, we have

$$q(\vec{n}) = -\frac{i}{2\pi a^2} \ln U_{12}(\vec{n}). \tag{5.9}$$

The lattice version of χ_T is

$$\chi_T = \frac{\langle Q^2 \rangle - \langle Q \rangle}{V}, \tag{5.10}$$

where V is the lattice volume and

$$Q = \sum_{\vec{n} \in L} a^2 q(\vec{n}), \tag{5.11}$$

where $L = \{\vec{n} = (n_1, n_2) |\, n_\mu = 0, 1, \ldots, N_\mu - 1; \mu = 1, 2\}$ is the set of points on the lattice.

Turns out that the lattice configurations generated through Monte Carlo algorithms are sorted in different sectors, where each one is characterized by a different topological charge. Furthermore, there is evidence (see e.g. refs. [6, 7]) that the distribution of these configurations corresponds to a Gaussian function, so one would expect that

$$\langle Q \rangle \approx 0. \tag{5.12}$$

Then, one can calculate χ_T using the following equation

$$\chi_T = \frac{\sum_i Q_i \,\#N_i}{V \sum_i \#N_i}, \tag{5.13}$$

where i denotes a sector with topological charge Q_i and $\#N_i$ configurations.

Q and χ_T were measured for several lattice sizes (see chapter 6) using the HMC algorithm. However, here we only show results obtained for high statistics (10,000 measurements), a spatial volume $L = 10$ and a time extension $L_t = 64$, since the other data for χ_T was too "noisy". In figure 5.1 the topological susceptibility is shown as a function of the degenerate fermion mass, while in figure 5.2 the distribution of the configurations is shown.

Now, one can substitute equation (5.6) in (5.5) and solve for F_π

$$F_\pi^2 = \chi_T^{\text{que}} \frac{2\pi}{g^2}. \tag{5.14}$$

From figure 5.1, we have $\chi_T^{\text{que}} = 0.0274(2)g^2$, thus

$$F_\pi = 0.4149(15). \tag{5.15}$$

In the next chapter we use a different method to measure F_π that does not rely on the topological charge and the Witten-Veneziano formula.

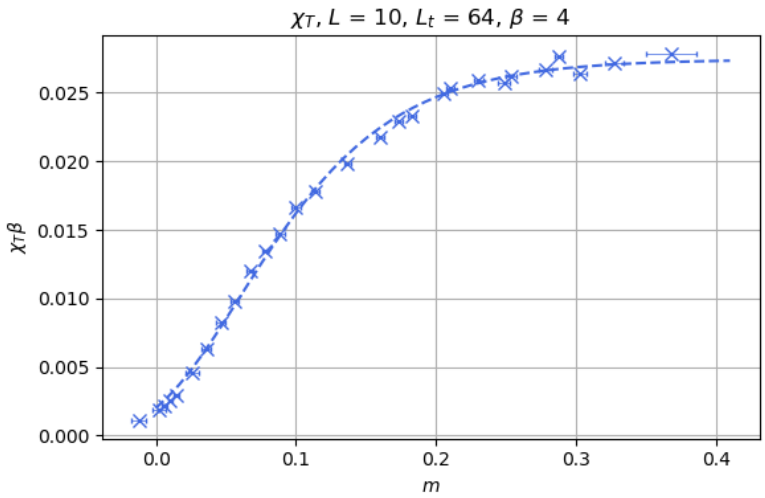
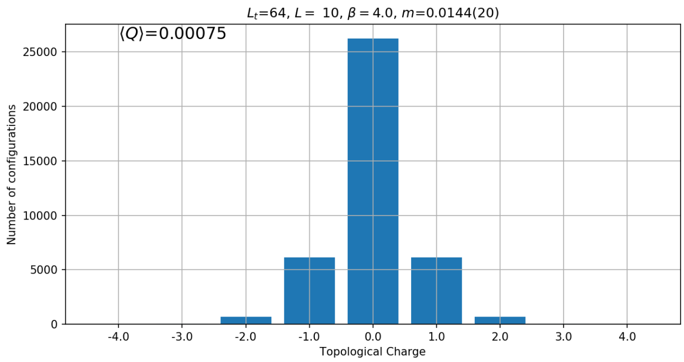
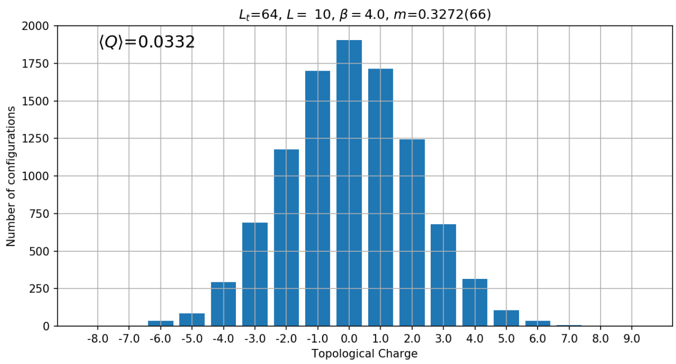


Figure 5.1: Topological susceptibility as a function of the degenerate fermion mass, the latter was measured as well. A function of the form $y = ae^{-be^{-cx}}$ was fitted to the data in order to extract the value of χ_T when $m \to \infty$. The only relevant parameter of the fit is $a = 0.0274(2)$, since it is equal to χ_T^{que}/g^2. The results are in lattice units. Each point was obtained through 10,000 measurements.



(a) Configurations sorted by their topological charge for $m = 0.0144(20)$



(b) Configurations sorted by their topological charge for $m = 0.3272(66)$

Figure 5.2: Distribution of the Monte Carlo configurations in different topological sectors for $\beta = 1/g^2 = 4$. We can see a Gaussian distribution. m denotes the degenerate fermion mass. When the mass is smaller, we have less topological sectors

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