Chapter 1

The Schwinger model

The Schwinger model represents Quantum Electrodynamics in 1+1 dimensions [1]. It is used as a toy model for Quantum Chromodynamics (QCD), because it has similar properties, such as: confinement, chiral symmetry breaking and topology. In contrast to QCD, however, this model does not have a running coupling constant. Its Lagrangian in Minkowski space-time (in natural units) for one flavor is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \gamma^\mu (i\partial_\mu - g A_\mu)\psi - m\overline{\psi}\psi, \tag{1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A_\mu(x)$ is the U(1) gauge field, g is the gauge coupling constant, ψ and $\overline{\psi}$ are independent Grassmann fields in the functional integral formulation (see Chapter 2) and γ^μ are the Dirac matrices. They can be chosen as

$$\gamma^0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{1.2}$$

which satisfy $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with $g_{\mu\nu} = \mathrm{diag}(1, -1)$. We assume $A_\mu(x)$ to be dimensionless and g to have dimension mass. With the γ matrices, we can define one more matrix

$$\gamma_5 \equiv \gamma^0\gamma^1, \text{ which implies } \quad \{\gamma^\mu, \gamma_5\} = 0, \quad \gamma_5^2 = \mathbb{I}, \quad \gamma_5^\dagger = \gamma_5. \tag{1.3}$$

The equations of motion can be obtained through the Euler-Lagrange equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 \quad \Rightarrow \quad \partial_\nu F^{\nu\mu} = g J^\mu, \quad J^\mu \equiv \overline{\psi}\gamma^\mu\psi, \tag{1.4}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad \Rightarrow \quad i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = -g\gamma^\mu A_\mu \overline{\psi}, \tag{1.5}$$

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0 \quad \Rightarrow \quad i\gamma^\mu\partial_\mu\psi - m\psi = g\gamma^\mu A_\mu \psi. \tag{1.6}$$

Since $F^{\mu\nu}$ is antisymmetric, eq. (1.4) implies that J^μ is conserved

$$\partial_\mu J^\mu = 0. \tag{1.7}$$

If one applies a global axial transformation to the fields $\overline{\psi}$ and ψ

$$\psi \to \psi' = e^{i\alpha\gamma_5}\psi, \quad \overline{\psi} \to \overline{\psi}' = \overline{\psi} e^{i\alpha\gamma_5}, \quad \alpha \in \mathbb{R}, \tag{1.8}$$

the Lagrangian in eq. (1.1) transforms to

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}e^{i\alpha\gamma_5}\gamma^\mu(i\partial_\mu - gA_\mu)e^{i\alpha\gamma_5}\psi - m\overline{\psi}e^{2i\alpha\gamma_5}\psi. \tag{1.9}$$

Since $\{\gamma^\mu, \gamma_5\} = 0$, it follows that

$$e^{-i\alpha\gamma_5}\gamma^\mu = (\mathbb{I} - i\alpha\gamma_5 + \cdots)\,\gamma^\mu = \gamma^\mu\,(\mathbb{I} + i\alpha\gamma_5 + \cdots) = \gamma^\mu e^{i\alpha\gamma_5}, \tag{1.10}$$

and therefore

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\gamma^\mu(i\partial_\mu - gA_\mu)\psi - m\overline{\psi}e^{2i\alpha\gamma_5}\psi. \tag{1.11}$$

We see that for $m = 0$, the Lagrangian has a symmetry under the transformation given in eq. (1.8). The Noether current of this symmetry, known as *axial current*, is

$$J_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi. \tag{1.12}$$

Let us compute its divergence by taking into account the mass, using eq. (1.3) and relying on the equations of motion (1.5) and (1.6)

$$\begin{aligned}
\partial_\mu J_5^\mu &= \partial_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + \overline{\psi}\gamma^\mu\gamma_5\partial_\mu\psi \\
&= \partial_\mu\overline{\psi}\gamma^\mu\gamma_5\psi - \overline{\psi}\gamma_5\gamma^\mu\partial_\mu\psi \\
&= i(gA_\mu\overline{\psi}\gamma^\mu + m\overline{\psi})\gamma_5\psi + i\overline{\psi}\gamma_5(gA_\mu\gamma^\mu\psi + m\psi) \\
&= igA_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + im\overline{\psi}\gamma_5\psi - igA_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + im\overline{\psi}\gamma_5\psi \\
&= 2im\overline{\psi}\gamma_5\psi.
\end{aligned} \tag{1.13}$$

Hence, one would expect in the massless model J_5^μ to be conserved. However, it was proved that J_5^μ exhibits an anomaly at the quantum level [2, 3]. When $m = 0$ one actually has

$$\partial_\mu J_5^\mu = -\frac{g}{\pi}\frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu}. \tag{1.14}$$

This equation is known as the *axial anomaly.* In order to show that the theory is sensitive to this expression, we define

$${}^*F \equiv \frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu} = F^{01} = -F_{01} = -E. \tag{1.15}$$

In 1+1 dimensions the Abelian strength field tensor is given by

$$F_{\mu\nu}(x) = \begin{pmatrix} 0 & E(x) \\ -E(x) & 0 \end{pmatrix}, \tag{1.16}$$

which confirms ${}^*F = -E$. Furthermore, $F_{\mu\nu} = \epsilon_{\mu\nu}F_{01} = \epsilon_{\mu\nu}E$, hence

$$F_{\mu\nu} = -\epsilon_{\mu\nu}{}^*F. \tag{1.17}$$

Let us note that

$$\begin{aligned}
\epsilon^{01}\gamma_1 = -\epsilon_{01}\gamma_1 = -\gamma_1 = \gamma^1 = \gamma^0\gamma^0\gamma^1 = \gamma^0\gamma_5, \\
\epsilon^{10}\gamma_0 = -\epsilon_{10}\gamma_0 = \gamma_0 = \gamma^0 = -\gamma^0\gamma^1\gamma^1 = \gamma^1\gamma^0\gamma^1 = \gamma^1\gamma_5,
\end{aligned} \tag{1.18}$$

therefore $\epsilon^{\mu\nu}\gamma_\nu = \gamma^\mu\gamma_5$. With this expression we can rewrite eq. (1.12) as

$$J_5^\mu = \epsilon^{\mu\nu}J_\nu. \tag{1.19}$$

If we multiply by $\epsilon_{\sigma\mu}$ and use the property $\epsilon^{\nu\mu}\epsilon_{\mu\sigma} = \delta^{\nu}_{\sigma}$, eq. (1.19) takes the form

$$J_\sigma = \epsilon_{\sigma\mu} J_5^\mu, \qquad J^\sigma = \epsilon^{\sigma\mu} J_{5\,\mu}. \tag{1.20}$$

Substituting eq. (1.17) in eq. (1.4) leads to

$$-\partial_\mu \epsilon^{\mu\nu}\, {}^*F = gJ^\nu \tag{1.21}$$

and by using eq. (1.20) we have

$$-\partial_\mu \epsilon^{\mu\nu}\, {}^*F = g\epsilon^{\nu\mu} J_{5\,\mu}. \tag{1.22}$$

Multiplying by $\epsilon_{\nu\rho}$ yields

$$\partial_\rho\, {}^*F = gJ_{5\,\rho}. \tag{1.23}$$

We can take the derivative on both sides of the equation and rename the dummy index

$$\partial^\mu \partial_\mu\, {}^*F = g\partial^\mu J_{5\,\mu} = -\frac{g^2}{\pi}\, {}^*F. \tag{1.24}$$

Finally, substituting eq. (1.15) gives

$$\left(\partial^2 + \frac{g^2}{\pi}\right) E = 0, \tag{1.25}$$

which is the Klein-Gordon equation of a scalar field with the mass μ, $\mu^2 = g^2/\pi$. Therefore, in the massless one flavor Schwinger model, a boson of mass μ appears. This result has been generalized to an arbitrary number of N massless flavors [4], where a boson of mass $\mu^2 = Ng^2/\pi$ appears, along with $N-1$ massless bosons. For massive fermions no general solution exists, although there are several approaches. We will review one of those approaches in Chapter 3. Deeper discussions of QED in 1+1 dimensions can be found in refs. [5, 6].

1.1 Confinement

As we mentioned before, the Schwinger model exhibits confinement. We can illustrate this fact by analyzing the classical equations of motion

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{1.26}$$

Let us fix the gauge by setting $A_0 = 0$ and suppose that we place a charge g at the origin,

$$\partial_1 F^{10}(x) = g\delta(x) \quad \Longrightarrow \quad \partial_x E(x) = g\delta(x) \quad \Longrightarrow \quad E(x) = g\theta(x) + E_0, \tag{1.27}$$

where $\theta(x)$ is the Heaviside function and E_0 is a constant electric field. If we calculate the energy of this configuration, we see that it diverges

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\, E^2 \to \infty. \tag{1.28}$$

This means that the finite energy states must be charge neutral. Now, let us consider two charges $\pm g$ at $x = \mp L/2$. The equation of motion reads

$$\partial_x E(x) = g\,\delta\left(x + \frac{L}{2}\right) - g\,\delta\left(x - \frac{L}{2}\right) \quad \Longrightarrow \quad E(x) = g\,\theta\left(x + \frac{L}{2}\right) - g\,\theta\left(x - \frac{L}{2}\right) + E_0. \tag{1.29}$$

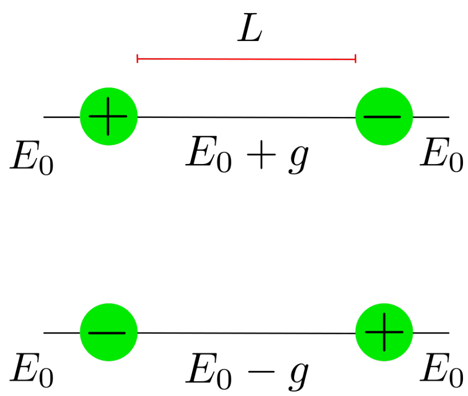

Figure 1.1: Electric field between an electron-positron pair in QED$_2$, considering the background field E_0.

If we set $E_0 = 0$, the electric field is

$$E(x) = \begin{cases} g & |x| < \frac{L}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{1.30}$$

We can calculate the energy of this configuration,

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\, E^2 = \frac{1}{2}\int_{-L/2}^{L/2} dx\, g^2 = \frac{g^2 L}{2}. \tag{1.31}$$

We see that the energy grows linearly with the separation of the charges, illustrating confinement. This property holds at the quantum level as well [7].

1.2 Vacuum angle

If we do not fix the background field E_0 to zero, it is possible to generate electron-positron pairs when the difference of the energy between both particles together and the background field is negative

$$\Delta H = \frac{1}{2}\int_{-L/2}^{L/2} dx\, \left[E(x)^2 - E_0^2\right] < 0. \tag{1.32}$$

The electric field $E(x)$ between the particles is now given by (see figure 1.1)

$$E(x) = E_0 \pm g, \quad -\frac{L}{2} \leq x \leq \frac{L}{2}. \tag{1.33}$$

Pairs can be created when

$$\begin{aligned} \Delta H &= \frac{L}{2}\left(g^2 \pm 2gE_0\right) < 0 \\ &\Leftrightarrow \begin{cases} \frac{g}{2} < E_0 & \text{for } E(x) = E_0 - g \\ E_0 < -\frac{g}{2} & \text{for } E(x) = E_0 + g \end{cases} \\ &\Leftrightarrow \frac{g}{2} < |E_0|. \end{aligned} \tag{1.34}$$

In this context, the *vacuum angle* θ is introduced as

$$\theta = \frac{2\pi E_0}{g}. \tag{1.35}$$

Whenever $|\theta| > \pi$, pair production is favorable. $\theta = 0$ refers to confinement. This parameter was introduced to the Schwinger model by Coleman [8] and it adds the following term to the Lagrangian

$$\mathcal{L}_\theta = \frac{g\theta}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{1.36}$$

We can rewrite $\epsilon^{\mu\nu}F_{\mu\nu}$ as

$$\epsilon^{\mu\nu}F_{\mu\nu} = \partial_\mu(2\epsilon^{\mu\nu}A_\nu), \tag{1.37}$$

which is a divergence. Therefore, $\mathcal{L}_\theta$ does not affect the equations of motion. In QCD a similar parameter appears.

1.3 Chiral symmetry breaking

As we will revise in a more detailed manner in Chapter 4, if one applies the chiral projection operators

$$P_L = \frac{\mathbb{I} - \gamma_5}{2}, \quad P_R = \frac{\mathbb{I} + \gamma_5}{2}, \tag{1.38}$$

to the fields ψ and $\overline{\psi}$, we can write the Lagrangian as

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}_L\gamma^\mu(i\partial_\mu - gA_\mu)\psi_L + \overline{\psi}_R\gamma^\mu(i\partial_\mu - gA_\mu)\psi_R - m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R),$$
$$\psi_R = P_R\psi, \quad \psi_L = P_L\psi, \quad \overline{\psi}_R = \overline{\psi}P_L, \quad \overline{\psi}_L = \overline{\psi}P_R, \tag{1.39}$$

which has a global symmetry under the transformations

$$\psi_L \to \psi'_L = e^{i\varphi_L}\psi_L, \quad \overline{\psi}_L \to \overline{\psi}'_L = \overline{\psi}_L e^{-i\varphi_L}, \quad e^{i\varphi_L} \in \mathrm{U}(1)_L, \tag{1.40}$$
$$\psi_R \to \psi'_R = e^{i\varphi_R}\psi_R, \quad \overline{\psi}_R \to \overline{\psi}'_R = \overline{\psi}_R e^{-i\varphi_R}, \quad e^{i\varphi_R} \in \mathrm{U}(1)_R \tag{1.41}$$

when $m = 0$. However, the *chiral condensate*, *i.e.* the vacuum expectation value $\langle\overline{\psi}\psi\rangle$ transforms as

$$\langle\overline{\psi}'\psi'\rangle = \left\langle \left(\overline{\psi}_R e^{i(\varphi_L-\varphi_R)}\psi_L + \overline{\psi}_L e^{i(\varphi_R-\varphi_L)}\psi_R\right) \right\rangle. \tag{1.42}$$

We see that it is invariant only when $\varphi_L = \varphi_R$, so $\mathrm{U}(1)_L \otimes \mathrm{U}(1)_R$ breaks to $\mathrm{U}(1)_{L=R}$.

In the N-flavor Schwinger model with degenerate fermion mass m, it has been shown [9] that the chiral condensate has the following dependence on m and θ when $m/\mu \ll 1$

$$\langle\overline{\psi}\psi\rangle = -\frac{\mu}{4\pi}\left(2e^\gamma\cos\frac{\theta}{2}\right)^{\frac{2N}{N+1}}\left(\frac{m}{\mu}\right)^{\frac{N-1}{N+1}}, \quad \mu = \frac{Ng^2}{\pi} \tag{1.43}$$

where γ is the Euler-Mascheroni constant. For the one flavor model we can see that

$$\langle\overline{\psi}\psi\rangle = -\frac{\mu}{2\pi}e^\gamma\cos\frac{\theta}{2}, \tag{1.44}$$

i.e. there is no dependence on the fermion mass. Hence the chiral condensate is non-vanishing even when $m = 0$. We also observe from eq. (1.43) that when $N > 1$, there is no chiral symmetry breaking in the massless Schwinger model, since $\langle\overline{\psi}\psi\rangle = 0$.

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