Abstract

We present a study of the two-flavor Schwinger model by means of lattice simulations, using Wilson fermions and the Hybrid Monte Carlo algorithm. At finite temperature, we measure the masses of the bosons, which are related to m_π and m_η, as a function of the degenerate fermion mass m. We compare the results with the numerical solution of a set of equations obtained by Hosotani et al. based on bosonization, which predict these masses when $m \ll \sqrt{2g^2/\pi}$, where g is the gauge coupling. Furthermore, we measure the pion decay constant F_π in the so-called δ-regime, where finite size effects of the pion mass lead to $F_\pi = 0.6688(5)$. Finally, we measure the quenched topological susceptibility. Applying a two-dimensional version of the Witten-Veneziano formula, we compute the η meson decay constant $F_\eta = 0.374(3)$, which has a lower value than F_π. This is in contrast to large N_c Quantum Chromodynamics, where the two decay constants coincide.

Introduction

Quantum Chromodynamics (QCD) is the theory for the strong interaction. It is described in terms of quark and gluon fields. Phenomenologically, it is known that the quarks and gluons are confined and it is not possible to isolate them. Still, at high energies, the quarks are *asymptotically free* and we can study their interaction by means of perturbative methods. At low energy this is not possible.

A non-perturbative approach to the low energy regime of QCD are lattice simulations, which allow us to derive results from first principles. The first notions of this method were developed by Kenneth Wilson in the 1970s [1]. The general idea is based on the functional integral formalism. Performing a transition to Euclidean time (a Wick rotation), we can interpret the functional integral as a partition function of Statistical Mechanics. Then we discretize the Euclidean space-time and using the partition function we generate field configurations by means of Monte Carlo algorithms. With the configurations we can measure different observables. Numerous results have been obtained with this approach and they agree with experimental measurements, for instance, hadron masses, matrix elements and decay constants[1].

Perhaps, the most noticeable challenge of lattice QCD are the high computational resources that are needed. Therefore, it is more convenient to test numerical techniques in simpler models than QCD. The Schwinger model is a common choice as a toy model for QCD. It represents QED in two dimensions and has similar properties as QCD, such as confinement, topology and chiral symmetry breaking. Also, since the model is two dimensional, simulations do not require that much computer power. The model was introduced by J. Schwinger in 1969 [3, 4]. Later, S. Coleman *et al.* proved the properties that we mentioned before [5, 6].

For one massless fermion, the Schwinger model has an exact solution. However, for $N > 1$ massive flavors there is no precise analytic solution for the chiral condensate or the masses of the bosons that appear in the model. Even so, there have been several analytic approaches. In particular, in Chapter 3 we review the work done by Hosotani *et al.*, which uses *bosonization* to reduce the finite temperature Schwinger model to a quantum mechanical system. Assuming two degenerate flavors of mass m, they arrive at a set of equations, which can be solved numerically in order to compute the chiral condensate and the mass of the bosons that appear, for arbitrary values of m, as long as $m \ll \sqrt{2g^2/\pi}$, where g is the gauge coupling. We compare the solution of Hosotani's equations with lattice simulations of the Schwinger model at finite temperature.

Another non-perturbative approach to the low energy regime of QCD are effective field theories. A particularly successful one is *Chiral Perturbation Theory.* In this theory one considers the spontaneous symmetry breaking of the chiral flavor symmetry of QCD

$$\mathrm{SU}(N)_L \otimes \mathrm{SU}(N)_R \to \mathrm{SU}(N)_{L=R}$$

[1]A review of the most important lattice measurements at low energy can be found in ref. [2].

to write an effective Lagrangian, in terms of a field $U(x)$ in the coset space

$$U(x) \in (\mathrm{SU}(N)_L \otimes \mathrm{SU}(N)_R)/\mathrm{SU}(N)_{L=R} = \mathrm{SU}(N).$$

This field represents the lightest hadrons for N flavors, so at low energy they dominate the theory. Also, the meson field $U(x)$ replaces the quark and gluon fields.

In finite volume, several regimes of Chiral Perturbation Theory are established. We focus on the *δ-regime*, which consists of a small spatial volume but with a large Euclidean time extent. This regime has been little explored in the literature. Although it is unphysical, one can obtain physical results for the low energy constants of QCD. Also, it could be of interest because the small spatial volume enables faster simulations. Based on previous results obtained by Leutwyler, Hasenfratz and Niedermayer [7,8] for dimension $d \geq 3$, we make a conjecture about their two-dimensional version and we verify it by simulating the two-flavor Schwinger model. This allows us to compute the pion decay constant F_π in two dimensions.

Finally, we determine the *quenched topological susceptibility* and, by using the Witten-Veneziano formula [9,10], we obtain the decay constant of the η-meson in two dimensions. In large N_c QCD, it coincides with F_π. We verified whether the relation $F_\eta \simeq F_\pi$ is also valid in the Schwinger model.

Outline

This thesis is composed of the following chapters:

- Chapter 1: We discuss some important features of the Schwinger model and their relation with QCD.
- Chapter 2: We review the path integral and we explain the transition to Euclidean space. We discuss the main ideas about how a lattice simulation is carried out. We present the lattice formulation of Euclidean QED. Finally, we extend the ideas of the lattice formulation from QED to QCD.
- Chapter 3: We review the analytic approach to the Schwinger model by Hosotani *et al.* We compare the results of lattice simulations with a numerical solution to a set of equations, which allow us to compute the masses of the bosons that appear in the theory.
- Chapter 4: We review basic concepts of Chiral Perturbation Theory. We briefly describe three finite volume regimes, which can be used to perform lattice simulations of QCD, with a focus on the δ-regime. We present results of the measurement of the pion decay constant, F_π, in two dimensions.
- Chapter 5: We define the topological charge, the topological susceptibility and we compute them with lattice simulations. In this manner, we determine F_η in two dimensions by using the Witten-Veneziano formula for the Schwinger model.
- Chapter 6: We summarize our results and present the conclusions.
- Appendix: We describe the Hybrid Monte Carlo algorithm for a scalar field, a numerical integral, the jackknife error and the autocorrelation time.

Conclusions

In this work we computed the masses of the bosons that appear in the two-flavor Schwinger model at finite temperature by using lattice simulations. We compared the results with the analytic approach by Hosotani *et al.*, which is only valid for $m \ll \mu = \sqrt{2g^2/\pi}$, so we did not expect a perfect agreement with lattice simulations. Still, we confirm the validity of Hosotani's equations in the low mass regime. We saw that they are not very useful to obtain predictions for m_π and m_η for arbitrary fermion masses. However, they would be applicable to compute m_π and m_η for arbitrary values of the vacuum angle θ, when the degenerate fermion mass is much smaller than μ. This cannot be easily done by lattice simulations, because the Euclidean action becomes complex and $\exp(-S[x])$ does not represent a probability measure for $\theta \neq 0$.

In the δ-regime, we observed that the residual pion mass m_π^R is proportional to $1/L$ for $d = 2$, as we conjectured based on the higher dimensional results previously shown by Leutwyler, Hasenfratz and Niedermayer. Thus, we were able to compute the pion decay constant in two dimensions. Our final result is

$$F_\pi = 0.6688(5). \tag{1}$$

It also turned out that this quantity does not depend on the parameter β and shows mild lattice artifacts.

We determined the η decay constant, F_η, in two dimensions by computing the quenched topological susceptibility and applying the Witten-Veneziano formula to the Schwinger model. We verified that our results for χ_T^{que} are compatible with the literature. From an extrapolation to the continuum limit we obtained

$$\chi_T^{\text{que}}\beta = 0.0223(3). \tag{2}$$

This value is below of the theoretical prediction by Seiler, which states that $\chi_T^{\text{que}}\beta \simeq 0.0253$, but is in agreement with the value determined by Dürr and Hoelbling: $\chi_T^{\text{que}}\beta \simeq 0.023$. With our result of $\chi_T^{\text{que}}\beta$ we obtained

$$F_\eta = 0.374(3). \tag{3}$$

As we mentioned in Chapter 5, in large N_c QCD, to the order $1/N_c$ we can relate $F_{\eta'} = F_\pi$. On the other hand, in the Schwinger model nothing assures that this relation holds. Our results obtained with the Witten-Veneziano formula and in the δ-regime suggest that in Schwinger model the relation $F_\eta = F_\pi$ is not valid. Still, both decay constants are of the same order of magnitude in two dimensions.

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