Abstract

We present a study of the two-flavor Schwinger model by means of lattice simulations, using Wilson fermions and the Hybrid Monte Carlo algorithm. At finite temperature, we measure the mass of the bosons, which are related to m_π and m_η, as a function of the degenerate fermion mass m. We compare the results with the numerical solution of a set of equations obtained by Hosotani *et al.* based on bosonization, which predict these masses when $m \ll \sqrt{2g^2/\pi}$, where g is the gauge coupling. Furthermore, we measure the pion decay constant F_π in the so-called δ*-regime*, where finite size effects of the pion mass lead to $F_\pi = 0.6688(5)$. Finally, we measure the quenched topological susceptibility and applying a 2d version of the Witten-Veneziano formula, we compute the eta meson decay constant F_η. This yields a lower value of $F_\eta = 0.374(3)$.

Introduction

Quantum Chromodynamics (QCD) is the theory for the strong interaction. It is described in terms of quark and gluon fields. Phenomenologically, it is known that the quarks are confined and it is not possible to separate them. Still, at high energies, compared to $\Lambda_{\mathrm{QCD}} \approx 300$ MeV, the quarks are *asymptotically free* and we can study their interaction by means of perturbative methods. At low energy this is not possible.

A non-perturbative approach to the low energy regime of QCD are lattice simulations, which allow to derive results from first principles. The first notions of this tool were developed by Kenneth Wilson in the 70's. The general idea is based on the functional integral formalism. Performing a transition to Euclidean time (a Wick rotation), we can interpret the functional integral as a partition function of Statistical Mechanics. Later we discretize the Euclidean action of the system and using the partition function we generate field configurations by means of Monte Carlo algorithms. With the configurations we can measure different observables. Several results have been obtained with this approach and they agree with experimental measurements. For instance, the mass of the hadrons, the chiral condensate or decay constants[1].

Perhaps, the most noticeable disadvantage of lattice QCD are the high computational resources that are needed. Due to this fact, it is more convenient to test numerical techniques in simpler models than QCD. The Schwinger model is a very common choice as a toy model for QCD. It represents QED in two dimensions and has similar properties with QCD, such as confinement, topology and chiral symmetry breaking. Also, since the model is two dimensional, simulations do not require too much computer power. The model was introduced by J. Schwinger in 1969 [2]. Later, S. Coleman *et al.* proved the properties that we mentioned before [3, 4].

For one massless fermion, the Schwinger model has an exact solution. However, for N massive flavors there is not a precise analytic solution that allow to compute the chiral condensate or the mass of the bosons that appear in the theory. Even so, there have been several analytic approaches. In particular, in Chapter 3 we revise the work done by Hosotani *et al.*, where they use *bosonization* to reduce the finite temperature Schwinger model to a quantum mechanical system. Assuming two degenerate flavors of mass m, they arrive at a set of equations, which can be solved numerically in order to compute the chiral condensate and the mass of the bosons that appear, for arbitrary values of m, as long as $m \ll \sqrt{2g^2/\pi}$, where g is the coupling constant. We compare the solution of Hosotani equations with lattice simulations of the Schwinger model at finite temperature.

Another non-perturbative approach to the low energy regime of QCD are effective field theories. The most succesful one is *Chiral Perturbation Theory.* In this theory one considers the spontaneous symmetry breaking of the chiral symmetry of QCD

$$\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R \to \mathrm{SU}(N)_{L=R}$$

[1] A review of the most important lattice measurements at low energy can be found in ref. [1].

to write an effective Lagrangian, in terms of a $U(x)$ field in the coset space of the symmetry breaking group $(\mathrm{SU}(N)_L \times \mathrm{SU}(N)_R)/\mathrm{SU}(N)_{L=R}$. This field represents the lightest hadrons suitable for N flavors, since at low energy they dominate the theory. Also, the field $U(x)$ replaces the quark and gluon fields.

In finite volume, several regimes to study Chiral Perturbation Theory are established. We focus on the *δ-regime*, which consists of a small spatial volume but with a large Euclidean time extent. This regime has been little explored in the literature, but it could be of interest because the small spatial volume enables to perform faster simulations. It is also possible to determine physical results of low energy QCD from this regime. Based on previous results obtained by Hasenfratz, Leutwyler and Niedermayer for a dimension $d \geq 4$, we make some conjectures about their two-dimensional version and we verify them by simulating the two-flavor Schwinger model. This allows us to compute the pion decay constant F_π in two dimensions.

Finally, we determine the *quenched topological susceptibility* and by using the Witten-Veneziano formula we obtain the decay constant of the eta meson in two dimensions.

Outline

The thesis is composed of the following chapters:

- Chapter 1: We discuss some important features of the Schwinger model and their relation with QCD.
- Chapter 2: We make a deduction of the path integral and we explain the transition to Euclidean space. We discuss some main ideas about how a lattice simulation is carried out. We also discretize the QED Euclidean action by introducing the *link variables*. Finally, we make an extension of the ideas of the discretization of the QED action to QCD.
- Chapter 3: We revise the analytic approach to the Schwinger model made by Hosotani *et al.* We compare the results of lattice simulations with a numerical solution to a set of equations, which allow to compute the masses of the bosons that appear in the theory.
- Chapter 4: We revise basic concepts of Chiral Perturbation Theory. We briefly describe three finite volume regimes that are used to perform lattice simulations of QCD, with focus on the δ-regime. We present results of the measurement of the pion decay constant, F_π in two dimensions.
- Chapter 5: We define the topological charge, the topological susceptibility and we compute them with lattice simulations. Later, we determine F_η in two dimensions by using the Witten-Veneziano formula for the Schwinger model.
- Chapter 6: Conclusions.
- Appendix: A revision of the Hybrid Monte Carlo algorithm for a scalar field, a numerical integral, the jackknife error and the autocorrelation time.

Conclusions

In this work we computed the masses of the bosons that appear in the Schwinger model at finite temperature by using lattice simulations. We compared the results with the analytic approach made by Hosotani *et al.* We mentioned that this approach is only valid for $m \ll \mu = \sqrt{2g^2/\pi}$, so we did not expect a perfect agreement with lattice simulations. Still, we revised the scope of Hosotani equations in the low mass regime. We saw that they are not very useful to obtain predictions for m_π and m_η for arbitrary fermion masses. However, they would be applicable to compute m_π and m_η for arbitrary values of the vacuum angle θ when the degenerate fermion mass is much smaller than μ.

In the δ-regime, we observed that the residual pion mass m_π^R is proportional to $1/L$ for $d = 2$, as we conjectured based on the higher dimensional results previously shown by Hasenfratz, Leutwyler and Niedermayer. With this, we were able to compute the pion decay constant in two dimensions. Our final result is

$$F_\pi = 0.6688(5). \tag{1}$$

It also turned out that this quantity is not dependent on the parameter β and shows few lattice artifacts.

We determined the η decay constant in two dimensions by computing the quenched topological susceptibility and applying the Witten-Veneziano formula to the Schwinger model. We verified that our results of χ_T^{que} were compatible with the literature. From an extrapolation to the continuum limit we obtained

$$\chi_T^{\text{que}}\beta = 0.0223(3). \tag{2}$$

This value is below of the theoretical prediction by E. Seiler [5], which states that $\chi_T^{\text{que}}\beta \simeq 0.0253$. With our result of $\chi_T^{\text{que}}\beta$ we obtained

$$F_\eta = 0.374(3). \tag{3}$$

As we mentioned in Chapter 5, in a $1/N_c$, with large N_c, expansion we can relate $F_{\eta'} = F_\pi$ in QCD. Nevertheless, in the Schwinger model the literature is not clear about this relation. Our results obtained with the Witten-Veneziano formula and in the δ-regime suggest that in QED_2 the relation $F_\eta = F_\pi$ is not valid. Still, both decay constants are of the same order of magnitude in 2d.

Bibliography

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