Chapter 1

The Schwinger model

The Schwinger model refers to Quantum Electrodynamics in 1+1 dimensions [1]. It is used as a toy model for Quantum Chromodynamics, because it shows similar properties, such as: confinement, chiral symmetry breaking and topology. Its Lagrangian is given in Minkowski space-time in natural units and for one flavor by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\gamma^{\mu}(i\partial_{\mu} - gA_{\mu})\psi - m\overline{\psi}\psi, \tag{1.1}$$

where $F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$, $A_{\mu}(x)$ is the U(1) gauge field, g is the gauge coupling constant, ψ is the fermion field, $\overline{\psi} = \psi^{\dagger}\gamma^0$ and γ^{μ} are the Dirac matrices, which satisfy $(\gamma^{\mu})^{\dagger} = \gamma^0\gamma^{\mu}\gamma^0$, $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$ with $g_{\mu\nu} = \mathrm{diag}(1, -1)$. A possible representation for γ^{μ} is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.2}$$

With these two matrices, we can define one more matrix

$$\gamma_5 \equiv \gamma^0\gamma^1, \text{ which implies} \quad \{\gamma^{\mu}, \gamma_5\} = 0. \tag{1.3}$$

The equations of motion can be obtained through the Euler-Lagrange equations

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}A_{\mu})}\right) - \frac{\partial\mathcal{L}}{\partial A_{\mu}} = 0 \quad \Rightarrow \quad \partial_{\nu}F^{\nu\mu} = gJ^{\mu}, \quad J^{\mu} \equiv \overline{\psi}\gamma^{\mu}\psi, \tag{1.4}$$

$$\partial_{\mu}\left(\frac{\partial\mathcal{L}}{\partial(\partial_{\mu}\psi)}\right) - \frac{\partial\mathcal{L}}{\partial\psi} = 0 \quad \Rightarrow \quad i\partial_{\mu}\overline{\psi}\gamma^{\mu} + m\overline{\psi} = -gA_{\mu}\overline{\psi}\gamma^{\mu}, \tag{1.5}$$

by taking the complex conjugate of the last equation we have

$$i\gamma^{\mu}\partial_{\mu}\psi - m\psi = g\gamma^{\mu}A_{\mu}\psi. \tag{1.6}$$

Since $F^{\mu\nu}$ is antisymmetric, eq. (1.4) implies that J^{μ} is conserved

$$\partial_{\mu}J^{\mu} = 0. \tag{1.7}$$

If one applies the following global transformations to the fields $\overline{\psi}$ and ψ

$$\psi \to \psi' = e^{i\alpha\gamma_5}\psi, \quad \overline{\psi} \to \overline{\psi}\,' = \overline{\psi}e^{i\alpha\gamma_5}, \quad \alpha \in \mathbb{R}, \tag{1.8}$$

the Lagrangian in eq. (1.1) transforms to

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}e^{i\alpha\gamma_5}\gamma^{\mu}(i\partial_{\mu} - gA_{\mu})e^{i\alpha\gamma_5}\psi - m\overline{\psi}e^{2i\alpha\gamma_5}\psi. \tag{1.9}$$

Since $\{\gamma^\mu, \gamma_5\} = 0$, it follows that

$$e^{-i\alpha\gamma_5}\gamma^\mu = (\mathbb{I} - i\alpha\gamma_5 + \cdots)\,\gamma^\mu = \gamma^\mu\,(\mathbb{I} + i\alpha\gamma_5 + \cdots) = \gamma^\mu e^{i\alpha\gamma_5}. \tag{1.10}$$

Then

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\gamma^\mu(i\partial_\mu - gA_\mu)\psi - m\overline{\psi}e^{2i\alpha\gamma_5}\psi. \tag{1.11}$$

We can see that for $m = 0$, the Lagrangian has a symmetry under the transformation given in eq. (1.8). The Nöther current of this symmetry, known as *axial current*, is

$$J_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi. \tag{1.12}$$

Let us take its derivative by taking into account the mass

$$\begin{aligned}
\partial_\mu J_5^\mu &= \partial_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + \overline{\psi}\gamma^\mu\gamma_5\partial_\mu\psi \\
&= \partial_\mu\overline{\psi}\gamma^\mu\gamma_5\psi - \overline{\psi}\gamma_5\gamma^\mu\partial_\mu\psi \\
&= i(gA_\mu\overline{\psi}\gamma^\mu + m\overline{\psi})\gamma_5\psi + i\overline{\psi}\gamma_5(gA_\mu\gamma^\mu\psi + m\psi) \\
&= igA_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + im\overline{\psi}\gamma_5\psi - igA_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + im\overline{\psi}\gamma_5\psi \\
&= 2im\overline{\psi}\gamma_5\psi,
\end{aligned} \tag{1.13}$$

where we have made use of eqs. (1.3), (1.5) and (1.6). One would expect in the massless model J_5^μ to be conserved. However, it was proved (see ref. [2]) that J_5^μ shows an anomaly at the quantum level. When $m = 0$ one actually has

$$\partial_\mu J_5^\mu = -\frac{g}{\pi}\frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu}. \tag{1.14}$$

In this context it is useful to define

$${}^*F \equiv \frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu} = F^{01} = -F_{01}. \tag{1.15}$$

Now, in 1+1 dimensions the field tensor is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E(x) \\ -E(x) & 0 \end{pmatrix}, \tag{1.16}$$

so ${}^*F = -E$. Furthermore, $F_{\mu\nu} = \epsilon_{\mu\nu}F_{01}$, hence

$$F_{\mu\nu} = -\epsilon_{\mu\nu}{}^*F. \tag{1.17}$$

Let us note that

$$\begin{aligned}
\epsilon^{01}\gamma_1 = -\epsilon_{01}\gamma_1 = -\gamma_1 = \gamma^1 = \gamma^0\gamma^0\gamma^1 = \gamma^0\gamma_5, \\
\epsilon^{10}\gamma_0 = -\epsilon_{10}\gamma_0 = \gamma_0 = \gamma^0 = -\gamma^0\gamma^1\gamma^1 = \gamma^1\gamma^0\gamma^1 = \gamma^1\gamma_5,
\end{aligned} \tag{1.18}$$

therefore $\epsilon^{\mu\nu}\gamma_\nu = \gamma^\mu\gamma_5$. With this expression we can rewrite eq. (1.12) as

$$J_5^\mu = \epsilon^{\mu\nu}J_\nu. \tag{1.19}$$

Since $\epsilon^{\nu\mu}\epsilon_{\mu\sigma} = \delta^\nu_\sigma$, this last equation takes the form

$$J_\mu = \epsilon_{\mu\nu}J_5^\mu. \tag{1.20}$$

Substituting eq. (1.17) in eq. (1.4) reads

$$-\partial_\mu\epsilon^{\mu\nu}\,{}^*F = gJ^\nu \tag{1.21}$$

and by using eq. (1.20) we have

$$-\partial_\mu \epsilon^{\mu\nu}\, {}^*F = g\epsilon^{\nu\mu} J_{5\,\mu}. \tag{1.22}$$

Multiplying by $\epsilon_{\nu\rho}$ yields

$$\partial_\mu\, {}^*F = gJ_{5\,\mu}. \tag{1.23}$$

We can take the derivative in both sides of the equation

$$\partial^\mu \partial_\mu\, {}^*F = g\partial^\mu J_{5\,\mu} = -\frac{g^2}{\pi}\, {}^*F. \tag{1.24}$$

Finally, substituting eq. (1.15) gives

$$\left(\partial^2 + \frac{g^2}{\pi}\right) E = 0, \tag{1.25}$$

which is the equation of a scalar field with mass $\mu^2 = g^2/\pi$. Therefore, in the massless one flavor Schwinger model, a boson of mass μ appears. This result has been generalized to an arbitrary number of massless flavors N [3], where a boson of mass $\mu^2 = Ng^2/\pi$ appears. For massive fermions no general solution exists, although there has been several approaches. We will revise one of those approaches in Chapter 3. A deeper discussion of QED in 1+1 dimensions can be found in refs. [4, 5].

1.1 Confinement

As we mentioned before, the Schwinger model exhibits confinement. We can illustrate this fact by analyzing the classical equations of motion

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{1.26}$$

Let us fix $A_0 = 0$ and suppose that we place a charge q at the origin, then

$$\partial_1 F^{10} = q\delta(x) \Rightarrow \partial_x E = q\delta(x) \Rightarrow E(x) = q\theta(x) + E_0, \tag{1.27}$$

where $\theta(x)$ is the Heaviside function and E_0 is a constant electric field. The latter is fixed according to the value of $E(x)$ at infinity. If one calculates the energy of this configuration, we can see that it diverges

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\, E^2 \to \infty.$$

This means that the finite energy states must have neutral charge. Now, let us consider two charges $\pm q$ at $x = \mp L/2$. The equation of motion reads

$$\partial_x E = q\,\delta\left(x + \frac{L}{2}\right) - q\,\delta\left(x - \frac{L}{2}\right) \Rightarrow E(x) = q\,\theta\left(x + \frac{L}{2}\right) - q\,\theta\left(x - \frac{L}{2}\right) + E_0. \tag{1.28}$$

If we want that $E \to 0$ when $x \to \infty$, we have to fix $E_0 = 0$. Then, the electric field is

$$E(x) = \begin{cases} q, & |x| < \frac{L}{2} \\ 0, & \text{other case} \end{cases}. \tag{1.29}$$

We can calculate the energy of this configuration

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\, E^2 = \frac{1}{2}\int_{-L/2}^{L/2} dx\, q^2 = \frac{q^2 L}{2}. \tag{1.30}$$

We see that the energy grows linearly with the separation of the charges, illustrating confinement. This property holds at the quantum level as well [6].

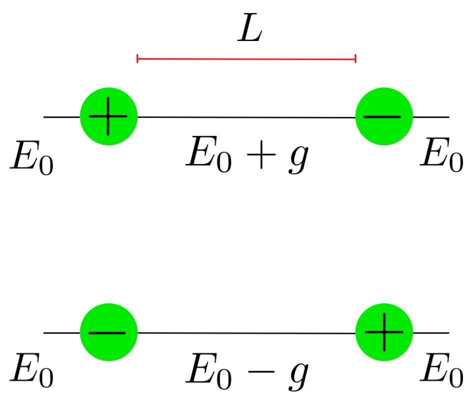

Figure 1.1: Electric field between an electron-positron pair in QED$_2$, considering the background field.

1.2 Vacuum angle

If we do not fix the background field to zero, it is possible to generate electron-positron pairs when the difference of the energy between both particles and the background field is smaller than zero

$$\Delta H = \frac{1}{2}\int_{-L/2}^{L/2} dx\ \left[E(x)^2 - E_0^2\right] < 0. \tag{1.31}$$

The electric field $E(x)$ between the particles is now given by (see figure 1.1)

$$E(x) = E_0 \pm g, \quad -\frac{L}{2} \leq x \leq \frac{L}{2}. \tag{1.32}$$

Pairs can be created when

$$\begin{aligned} \Delta H &= \frac{L}{2}\left(g^2 \pm 2eE_0\right) < 0 \\ &\Leftrightarrow \frac{g}{2} < E_0 \quad \text{or} \quad E_0 < -\frac{g}{2} \\ &\Leftrightarrow \frac{g}{2} < |E_0|. \end{aligned} \tag{1.33}$$

In this context, the *vacuum angle* θ

$$\theta = \frac{2\pi E_0}{g} \tag{1.34}$$

is introduced. Whenever $|\theta| > \pi$, pair production is favorable. $\theta = 0$ refers to confinement. This parameter was introduced to the Schwinger model by Coleman [7]. In QCD a similar parameter appears.

1.3 Chiral symmetry breaking

As we will revise in a more detailed manner in Chapter 5, if one applies the chiral projection operators

$$P_L = \frac{\mathbb{I} - \gamma_5}{2}, \quad P_R = \frac{\mathbb{I} + \gamma_5}{2}, \tag{1.35}$$

to the ψ field, we can transform the Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}_L\gamma^\mu(i\partial_\mu - gA_\mu)\psi_L + \overline{\psi}_R\gamma^\mu(i\partial_\mu - gA_\mu)\psi_R - m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R),$$

$$\psi_R = R\psi, \psi_L = L\psi \tag{1.36}$$

which shows a global symmetry under the transformations

$$\psi_L \to \psi'_L = L\psi_L, \quad \psi_R \to \psi'_R = R\psi_R, \quad L \in \mathrm{SU}(1)_, \quad R \in \mathrm{SU}(1)_R \tag{1.37}$$

when $m = 0$. However, the *chiral condensate*, *i.e.* the vacuum expectation value of $\overline{\psi}\psi$ transforms as

$$\langle \overline{\psi}'\psi' \rangle = \langle \overline{\psi}_R R^\dagger L \psi_L + \overline{\psi}_L L^\dagger R \psi_R \rangle, \tag{1.38}$$

so if $\langle \overline{\psi}\psi \rangle \neq 0$ the symmetry $\mathrm{SU}(1)_L \otimes \mathrm{SU}(1)_R$ breaks spontaneously.

In the N-flavor Schwinger model with degenerate fermion mass m, it has been shown [8] that the chiral condensate has the following dependence on m and θ when $m/\mu \ll 1$

$$\langle \overline{\psi}\psi \rangle = -\frac{\mu}{4\pi}\left(2e^\gamma \cos\frac{\theta}{2}\right)^{\frac{2N}{N+1}} \left(\frac{m}{\mu}\right)^{\frac{N-1}{N+1}}, \quad \mu = \frac{Ng^2}{\pi} \tag{1.39}$$

where γ is the Euler-Mascheroni constant. For the one flavor model we can see that

$$\langle \overline{\psi}\psi \rangle = -\frac{\mu}{2\pi} e^\gamma \cos\frac{\theta}{2}, \tag{1.40}$$

i.e. there is no dependence on the fermion mass. Therefore, the chiral condensate is non vanishing and as a consequence the massless one flavor model shows, indeed, spontaneous chiral symmetry breaking. This happens in four dimensional QCD as well. From eq. (1.39), we make the observation that when $N > 1$ there is no spontaneous symmetry breaking in the Schwinger model.

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