Chapter 3

Hosotani's approach to the Schwinger model

In the 1990s, Yukio Hosotani *et al.* published several studies of the massive Schwinger model for an arbitrary number of flavors, where they reduced the N flavor massive model to a quantum mechanical system of $N-1$ degrees of freedom. This allowed them to derive analytic predictions for the boson masses that appear in the model and for the chiral condensate as well. In the massless N flavor model, it is known that $N-1$ massless bosons will result together with one massive boson of mass $\mu = \sqrt{N/\pi}g$ [1]. On the other hand, in the Schwinger model with degenerate non-zero fermion mass the approach by Hosotani gives as a result N bosons, $N-1$ of them with the same mass μ_2 and one with mass $\mu_1 > \mu_2$. If $m \to 0$ then $\mu_2 \to 0$ and $\mu_1 \to \mu$ [2].

We will revise the most important equations of this appproach without the derivation of them, together with a numerical solution that gives predictions of the boson masses and the chiral condensate for the two flavor model at finite temperature. Unfortunately, the reliability of these solutions is limited to $m \ll \mu$.

3.1 Reduction to a quantum mechanical system

The QED Lagrangian for the N flavor Schwinger model is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N} \overline{\psi}_f \left[\gamma^\mu \left(i\partial_\mu - gA_\mu\right) - m_f\right] \psi_f. \tag{3.1}$$

The index f denotes the flavor. We are going to consider degenerate fermion masses, that is $m_f \equiv m$.

The first step is to map the Schwinger model onto a circle of circumference L and to impose the following boundary conditions of the fermion fields and the gauge field

$$\begin{aligned} \psi_f(t, x+L) &= -e^{2\pi i\alpha_f}\psi_f(t,x), \\ A_\mu(t, x+L) &= e^{2\pi i\alpha_f}A_\mu(t,x), \end{aligned} \tag{3.2}$$

where α_f is a phase factor. It is important to note that if one performs a Wick rotation $it \to x$ and then interchanges $x \leftrightarrow \tau$, with τ the Euclidean time, then the boundary condition is

$$\begin{aligned} \psi_f\left(\tau + \frac{1}{T}, x\right) &= -e^{2\pi i\alpha_f}\psi_f(\tau,x), \\ A_\mu\left(\tau + \frac{1}{T}, x\right) &= e^{2\pi i\alpha_f}A_\mu\left(\tau,x\right), \end{aligned} \tag{3.3}$$

where $T = \frac{1}{L}$ is the temperature [3]. That way, by setting $\alpha_f = 0$ it is possible to relate the model on the circle with the finite temperature one, which is of interest.

Next one uses the *bosonization method* to reduce the model to a quantum mechanical system of $N-1$ degrees of freedom. The main idea is to write the fields in terms of bosonic operators that obey certain commutation relations, see e.g. [2], [4]. However, those steps are rather tedious and here we only review the resulting formulation. After bosonization, the following equation

$$\hat{H}\left|\Phi_0\right\rangle = E_0 \left|\Phi_0\right\rangle, \tag{3.4}$$

where $\left|\Phi_0\right\rangle$ is the vacuum state and E_0 is the vacuum energy, is reduced to

$$\begin{aligned}
\left[-\triangle_\varphi + \kappa F_N(\varphi_1, ..., \varphi_N)\right] g(\varphi_1, ..., \varphi_N) &= \epsilon g(\varphi_1, ..., \varphi_N), \\
\kappa_0 = \frac{N}{\pi N-1} m L \overline{B} e^{-\pi/\mu N L}, \quad F_N(\varphi_1, ..., \varphi_N) &= -\sum_{f=1}^{N} \cos\varphi_f, \\
\overline{B} = \left[B(\mu_1 L)\right]^{1/N} \left[B(\mu_2 L)\right]^{1-(1/N)}, \quad \epsilon &= \frac{N L E_0}{2\pi} + \frac{\pi N^2}{12}, \\
B(z) = \frac{z}{4\pi}\exp\left[\gamma + \frac{\pi}{z} - 2\int_1^\infty \frac{du}{(e^{uz}-1)\sqrt{u^2-1}}\right]&.
\end{aligned} \tag{3.5}$$

$\gamma = 0.57721566490...$ is the Euler-Mascheroni constant, ϵ has to be determined together with $g(\varphi_1, ..., \varphi_N)$ and φ_f are angular variables constrained by

$$\varphi_N = \theta - \sum_{f=1}^{N-1} \varphi_f. \tag{3.6}$$

θ is known as the *vacuum angle*, it can be restricted to $(-\pi, \pi)$. $\triangle_\varphi$ is the Laplacian of the system, given by

$$\begin{aligned}
\triangle_\varphi &= \sum_{f=1}^{N-1}\left(\frac{\partial}{\varphi_f} - i\beta_f\right)^2 - \frac{2}{N-1}\sum_{f<f'}^{N-1}\left(\frac{\partial}{\partial\varphi_f} - i\beta_f\right)\left(\frac{\partial}{\partial\varphi_{f'}} - i\beta_{f'}\right) \\
\beta_f &= \alpha_f - \alpha_N.
\end{aligned} \tag{3.7}$$

The first line in eq. (3.5) is an eigenvalue problem for a system of $N-1$ degrees of freedom, due to the restriction (3.6). This equations enable us to find solutions for μ_2, see below. First we simplify eqs. (3.5) for $N = 2$. The Laplacian is now given by:

$$\triangle_\varphi = \left(\frac{d}{d\varphi_1} + i\delta\alpha\right)^2 = \frac{1}{i^2}\left(i\frac{d}{d\varphi} - \delta\alpha\right)^2 = -\left(i\frac{d}{d\varphi} - \delta\alpha\right)^2, \tag{3.8}$$

with $\delta\alpha = \alpha_2 - \alpha_1$. On the other hand, in virtue of the constriction 3.6, the function $F_2(\varphi_1, \varphi_2)$ can be written as

$$F_2(\varphi_1) = -\cos(\varphi_1) - \cos(\varphi_1 - \theta). \tag{3.9}$$

It is possible to rewrite the last expression as $-2\cos\frac{\theta}{2}\cos\left(\varphi_1 - \frac{\theta}{2}\right)$, this can be seen using trigonometric identities

$$\begin{aligned}
-2\cos\frac{\theta}{2}\cos\left(\varphi_1 - \frac{\theta}{2}\right) &= -2\cos^2\frac{\theta}{2}\cos\varphi_1 + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}\sin\varphi_1 \\
&= -2\left(\frac{1+\cos\theta}{2}\right)\cos\varphi_1 + \sin\theta\sin\varphi_1 \\
&= -\cos\varphi_1 - \cos\theta\cos\varphi_1 + \sin\theta\sin\varphi_1 \\
&= -\cos\varphi_1 - \cos(\varphi_1 - \theta).
\end{aligned} \tag{3.10}$$

Substituting the result for $\triangle_\varphi$ and $F_2(\varphi_1)$ in (3.5) yields

$$\left[\left(i\frac{d}{d\varphi_1}-\delta\alpha\right)^2-2\kappa_0\cos\frac{\theta}{2}\cos\left(\varphi_1-\frac{\theta}{2}\right)\right]g(\varphi_1)=\epsilon g(\varphi_1). \tag{3.11}$$

Let us define

$$\kappa\equiv 2\kappa_0\cos\frac{\theta}{2}=\frac{4}{\pi}mL\cos\frac{\theta}{2}\left[B(\mu_1L)B(\mu_2L)\right]^{1/2}e^{-\pi/2\mu L}, \tag{3.12}$$

thus, (3.11) can be written as

$$\left[\left(i\frac{d}{d\varphi_1}-\delta\alpha\right)^2-\kappa\cos\left(\varphi_1-\frac{\theta}{2}\right)\right]g(\varphi_1)=\epsilon g(\varphi_1). \tag{3.13}$$

Finally, we can apply a change of variable $\varphi=\varphi_1-\frac{\theta}{2}$ and define $f(\varphi)=g(\varphi+\frac{\theta}{2})$. Then, equation (3.13) takes the form

$$\left[\left(i\frac{d}{d\varphi}-\delta\alpha\right)^2-\kappa\cos\varphi\right]f(\varphi)=\epsilon f(\varphi). \tag{3.14}$$

According to [1], [4] and [5], the masses μ_1, μ_2 and the chiral condensate $-\langle\overline{\psi}\psi\rangle_\theta$ can be obtained through the following equations when $m\ll\mu$

$$\begin{aligned}\mu_2^2 &= \frac{2\pi^2}{L^2}\kappa\int_{-\pi}^{\pi}d\varphi\cos\varphi|f_0(\varphi)|^2,\ \ \varphi\in(-\pi,\pi),\\ \mu_1^2 &= \mu^2+\mu_2^2,\\ \langle\overline{\psi}\psi\rangle_\theta &= -\frac{\mu_2^2}{4\pi m},\end{aligned} \tag{3.15}$$

$f_0(\varphi)$ denotes the ground state function of (3.14), which obeys $f_0(\varphi+2\pi)=f_0(\varphi)$ and it has to be normalized.

Now we need to find a solution to eq. (3.14) in order to calculate μ_2; however, κ already depends on μ_2. This means that equations (3.12), (3.14) and (3.15) must be solved in a self consistent way. Analytically this is difficult for general values, but it can be done numerically. Still, there is one limit case that is worth analyzing, because it will provide a cross check with the numerical solutions of the next section. Let us suppose that $\delta\alpha=0$ and $\mu L\gg 1$. Since $\kappa\propto L$ then $\kappa\gg 1$, besides, the largest contribution of $\kappa\cos\varphi$ emerges when φ is close to zero, so we can approximate $\kappa\cos\varphi\approx\kappa-\kappa\frac{\varphi^2}{2}$. Hence

$$L\gg 1,\ \delta\alpha=0\Rightarrow-\frac{d^2f}{d\varphi^2}-\kappa\left(1-\frac{\varphi^2}{2}\right)f=\epsilon f. \tag{3.16}$$

With the ansatz $f(\varphi)=e^{-b\varphi^2}$ we obtain

$$-\frac{d^2f}{d\varphi^2}-\kappa\left(1-\frac{\varphi^2}{2}\right)f=\left(-2b-4b^2\varphi^2-\kappa+\kappa\frac{\varphi^2}{2}\right)e^{-b\varphi^2}, \tag{3.17}$$

then

$$\epsilon=-2b-4b^2\varphi^2-\kappa+\kappa\frac{\varphi^2}{2}. \tag{3.18}$$

Let us remember that ϵ is a constant, so it cannot depend on φ, this forces to fix $b = \sqrt{\frac{\kappa}{8}}$. As a result, the normalized solution to (3.14) under the previous assumptions is

$$f(\varphi) = \frac{1}{\int_{-\pi}^{\pi} d\varphi |e^{-\sqrt{\frac{\kappa}{8}}\varphi^2}|^2} e^{-\sqrt{\frac{\kappa}{8}}\varphi^2}. \tag{3.19}$$

We will denote

$$I \equiv \frac{\int_{-\pi}^{\pi} d\varphi \cos\varphi \, e^{-2\sqrt{\frac{\kappa}{8}}\varphi^2}}{\int_{-\pi}^{\pi} d\varphi \, e^{-2\sqrt{\frac{\kappa}{8}}\varphi^2}}, \tag{3.20}$$

in that manner we can rewrite the first equation of (3.15) as

$$\mu_2^2 = \frac{2\pi^2}{L^2}\kappa I = \frac{8}{L^2\pi} mL \cos\frac{\theta}{2} \left[B(\mu_2 L) B(\mu_1 L)\right]^{1/2} e^{-\pi/2\mu L} I. \tag{3.21}$$

Due to $L \gg 1$, we are able to obtain a simpler form of the $B(z)$ function in (3.5). It can be seen directly from its expression that if $z \gg 1$ the exponential term in the denominator of the integrand vanishes, thus the integral is suppressed, together with the factor π/z. Then

$$B(z) \approx \frac{z e^{\gamma}}{4\pi}, \qquad z \gg 1. \tag{3.22}$$

With this result the value for μ_2 is approximately

$$\mu_2^2 \approx \frac{8\pi^2}{L^2\pi} mL \cos\frac{\theta}{2} \left(\frac{\mu_1 L e^{\gamma}}{4\pi}\right)^{1/2} \left(\frac{\mu_2 L e^{\gamma}}{4\pi}\right)^{1/2} e^{-\pi/2\mu L} I = 2m \cos\frac{\theta}{2} e^{\gamma} \mu_1^{1/2} \mu_2^{1/2} e^{-\pi/2\mu L} I. \tag{3.23}$$

Now, let us remember that equation (3.15) is valid when $m \ll \mu$, that way we can approximate the value of μ_1 by $\mu = \sqrt{2}g/\sqrt{\pi}$, since we know that it is the value for μ_1 when the fermions are massless. By taking the limit $L \to \infty$, it follows that $e^{-\pi/2\mu L} \to 1$. Let us analyze the integral I in (3.20) by expanding $\cos\varphi$ in Taylor series

$$I = 1 + \sum_{n=1}^{\infty} \frac{\int_{-\pi}^{\pi} d\varphi \frac{(-1)^n \varphi^{2n}}{(2n)!} e^{-2\sqrt{\frac{\kappa}{8}}\varphi^2}}{\int_{-\pi}^{\pi} d\varphi \, e^{-2\sqrt{\frac{\kappa}{8}}\varphi^2}}. \tag{3.24}$$

If we take the limit $L \to \infty$ then $\kappa \to \infty$ and we have the following relations for the integrands of the second term in the latter expression

$$\begin{aligned} \lim_{k\to\infty} \varphi^{2n} e^{-2\sqrt{\frac{\kappa}{8}}\varphi^2} &= 0, \\ \lim_{k\to\infty} e^{-2\sqrt{\frac{\kappa}{8}}\varphi^2} &= A\delta(\varphi), \end{aligned} \tag{3.25}$$

where A is an irrelevant constant and $\delta(\varphi)$ is the Dirac delta. Therefore, the second term in equation (3.24) vanishes when $L \to \infty$ and we are left with $I = 1$. This can also be seen numerically, as it is shown in figure 3.1. This simplifies equation (3.23) to

$$\mu_2^2 = 2e^{\gamma} m \cos\frac{\theta}{2} \mu^{1/2} \mu_2^{1/2}, \qquad L \to \infty, \; m \ll \mu. \tag{3.26}$$

We are finally left with

$$\begin{aligned} \mu_2 &= \left(4e^{2\gamma} \mu m^2 \cos\frac{\theta}{2}\right)^{1/3} \\ &= \left(4e^{2\gamma} \sqrt{\frac{2}{\pi}} g m^2 \cos\frac{\theta}{2}\right)^{1/3}. \end{aligned} \tag{3.27}$$

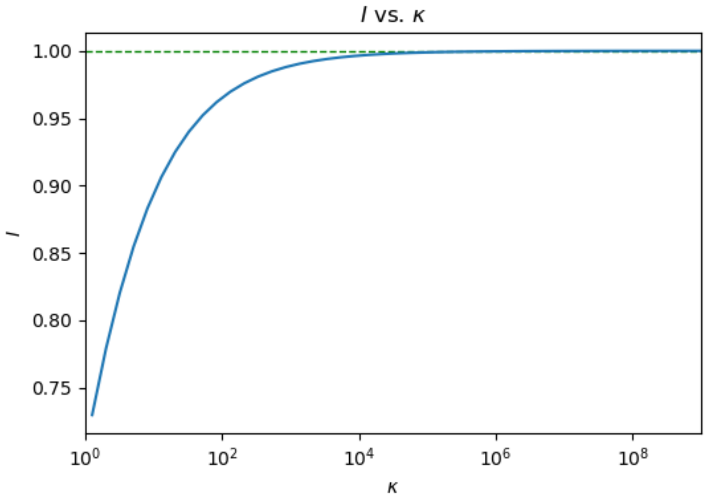

Figure 3.1: Behavior of $I(\kappa)$ defined in (3.24)

The constant terms can be evaluated, giving

$$\left(4e^{2\gamma}\sqrt{\frac{2}{\pi}}\right)^{1/3} = 2.1633... \tag{3.28}$$

Therefore

$$\mu_2 = 2.1633...\cos\frac{\theta}{2}(m^2 g)^{1/3}, \qquad L \to \infty,\ m \ll \mu. \tag{3.29}$$

In the two flavor massive Schwinger model we can make the analogy of the degenerate fermions with a degenerate version of the quarks u and d, that way one can relate μ_1 with the mass of the η boson and μ_2 with the pion mass from QCD. So from now on we will denote $\mu_1 = m_\eta$ and $\mu_2 = m_\pi$. There are two predictions for m_π at infinite volume and small fermion mass m, the first one is a semi-classical prediction [6], that is equal to (3.29) by taking $\theta = 0$. The other prediction was deduced by A. Smilga [7] and it is slightly different from the semi-classical one: $m_\pi = 2.008(m^2 g)^{1/3}$.

It is possible to derive more expressions for limiting cases, however, the rest of the analysis will be done numerically.

3.2 Numerical solution

The first step to solve equations (3.12), (3.14), (3.15) is to find a solution of the differential equation that involves $f(\varphi)$ with the condition $f(\varphi + 2\pi) = f(\varphi)$, $\varphi \in (-\pi, \pi)$. Turns out that if one sets $\delta\alpha = 0$, performs the change of variable $\varphi = 2x$ and defines $a \equiv 4\epsilon$, $q \equiv -2\kappa$, then equation (3.14) is taken to

$$\frac{d^2 f}{dx^2} + (a - 2q\cos 2x)f = 0,\ \ f(x+\pi) = f(\pi),\ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \tag{3.30}$$

which is the quantum pendulum equation or the *Mathieu equation*, whose solutions are known as *Mathieu functions*:

$$\frac{1}{\sqrt{\pi}} ce_n\left(\frac{\varphi}{2}, -2\kappa\right),\ \ \frac{1}{\sqrt{\pi}} se_n\left(\frac{\varphi}{2}, -2\kappa\right),\ \ n \text{ an even number due to the } 2\pi \text{ of } f(\varphi). \tag{3.31}$$

Furthermore, if $\delta\alpha \neq 0$ one can perform the change of variable $f(\varphi) = e^{-i\delta\alpha\varphi} g(\varphi)$ so the derivatives and the boundary condition are

$$\begin{aligned} \frac{df}{d\varphi} &= e^{-i\delta\alpha}\left(\frac{dg}{d\varphi} - i\delta\alpha g\right), \\ \frac{d^2 f}{d\varphi^2} &= e^{-i\delta\alpha\varphi}\left(-2i\delta\alpha\frac{dg}{d\varphi} - \delta\alpha^2 + \frac{d^2 g}{d\varphi^2}\right), \\ g(\varphi + 2\pi) &= e^{i2\pi\delta\alpha} g(\varphi) \end{aligned} \tag{3.32}$$

and substituting in (3.14) yields

$$-\frac{d^2 g}{d\varphi^2} - \kappa\cos\varphi\, g = \epsilon g. \tag{3.33}$$

This is the same equation for $f(\varphi)$ when $\delta\alpha = 0$ but with a different boundary condition given by (3.32)[1]. Its solutions are non periodic solutions to the Mathieu equation, better known as *Floquet solutions*. There are some analytic expressions for the Floquet solutions and for the Mathieu functions ce_n, se_n, however, they are very complex (see for example [9], [10]) and the best way to proceed is by discretizing equation (3.14) in order to get a matrix eigenvalue problem that will allows us to find the ground state function $f_0(\varphi)$.

Let us expand equation (3.14)

$$-\frac{d^2 f}{d\varphi^2} - 2i\delta\alpha\frac{df}{d\varphi} + \delta\alpha^2 f - \kappa\cos\varphi = \epsilon f \tag{3.34}$$

and divide the interval $(-\pi, \pi)$ in $N+1$ sites separated by $\Delta\varphi = 2\pi/N$, such that the derivatives can be interchanged by discrete derivatives of second order:

$$f_j = f(\varphi_j), \quad \frac{df}{d\varphi} \to \frac{f_{j+1} - f_{j-1}}{2\Delta\varphi}, \quad \frac{d^2 f}{d\varphi^2} \to \frac{f_{j+1} - 2f_j + f_{j-1}}{\Delta\varphi^2}, \quad \varphi \in (-\pi, \pi), \quad f_0 = f_N. \tag{3.35}$$

Substituting them in (3.34) gives

$$-\frac{f_{j+1} - 2f_j + f_{j-1}}{\Delta\varphi^2} - 2i\delta\alpha\frac{f_{j+1} - f_{j-1}}{2\Delta\varphi} + \delta\alpha^2 f_j - \kappa\cos\varphi_j\, f_j = \epsilon f_j. \tag{3.36}$$

The index j runs from 0 to $N-1$ (we are assuming $f_{-1} = f_{N-1}$ since f obeys periodic boundary), thus we have N algebraic equations that can be written as matrices if we interchange the following

$$\frac{f_{j+1} - 2f_j + f_{j-1}}{\Delta\varphi^2} \to \frac{1}{\Delta\varphi^2} \underbrace{\begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & \cdots & \cdots & 1 \\ 1 & -2 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & -2 & 1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}}_{\mathbb{A}} \underbrace{\begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix}}_{\vec{f}},$$

[1] For this reason, equation (3.14) is also known as the *Damped Mathieu Equation* [8]

$$\frac{f_{j+1}-f_{j-1}}{2\Delta\varphi} \to \frac{1}{2\Delta\varphi} \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}}_{\mathbb{B}} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \end{pmatrix},$$

$$\delta\alpha^2 f_j - \kappa\cos\varphi_i\, f_j \to \underbrace{\mathrm{diag}(\delta\alpha^2-\kappa\cos\varphi_0, \delta\alpha^2-\kappa\cos\varphi_1, ..., \delta\alpha^2-\kappa\cos\varphi_{N-1})}_{\mathbb{C}}\,\vec{f}. \quad (3.37)$$

In that manner, the N algebraic equations can be expressed as:

$$\left(-\frac{\mathbb{A}}{\Delta\varphi^2} - \frac{i\delta\alpha}{\Delta\varphi}\mathbb{B} + \mathbb{C}\right)\vec{f} = \epsilon\vec{f}, \quad (3.38)$$

where $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ and $\vec{f}$ are defined in (3.37). This is a linear algebra eigenvalue problem that can be solved using subroutines of different programs (e.g. Python), then the eigenvectors $\vec{f}$ can be obtained, however they will not be normalized as $\int_{-\pi}^{\pi} d\varphi|f(\varphi)|^2 = 1$, so one must use a numerical integrator to normalize the resultant vector $\vec{f}$. Reference [1] mentions some limiting case solutions of the ground state of (3.34)

$$f(\varphi) \approx \begin{cases} \frac{1}{\sqrt{2\pi}}\left[1+\frac{\kappa}{1-4\delta\alpha^2}(\cos\varphi - 2i\delta\alpha\,\mathrm{sen}\,\varphi)\right], & \text{for } \frac{\kappa}{1\pm2\delta\alpha} \ll 1 \\ \frac{1}{\sqrt{2\pi}}\left[\frac{1}{\sqrt{2}}\left(1+e^{\mp i\varphi}\right)+\frac{\kappa}{4\sqrt{2}}\left(e^{\pm i\varphi}+e^{\mp 2i\varphi}\right)\right], & \text{for } \delta\alpha = \pm\frac{1}{2}, \kappa \ll 1 \\ \frac{1}{\int_{-\pi}^{\pi}|e^{-i\delta\alpha\varphi-\sqrt{\frac{\kappa}{8}}\varphi^2}|^2 d\varphi} e^{-i\delta\alpha\varphi-\sqrt{\frac{\kappa}{8}}\varphi^2}, & \text{for } \kappa \gg 1 \end{cases} \quad (3.39)$$

One can compare the numerical result with this limit cases in order to verify the outcome of diagonalizing (3.38). In figure 3.2 such comparison is shown.

The next step is to find solutions for m_π and the chiral condensate $\langle\overline{\psi}\psi\rangle = -m_\pi^2/4\pi m$. We had the following system of equations

$$\left[\left(i\frac{d}{d\varphi}-\delta\alpha\right)^2 - \kappa\cos\varphi\right] f(\varphi) = \epsilon f(\varphi),$$

$$(m_\pi L)^2 = 2\pi^2\kappa\int_{-\pi}^{\pi} d\varphi\cos\varphi|f_0(\varphi)|^2,$$

$$\kappa = \frac{4}{\pi} mL\cos\frac{\theta}{2}\left[B(m_\eta L)\right]^{1/2}\left[B(m_\pi L)\right]^{1/2} e^{-\pi/2\mu L}. \quad (3.40)$$

From now on we will write $\beta = 1/g^2$, then $\mu = \sqrt{2/\beta\pi}$.

System (3.40) can be solved as a non linear system of equations or in a self consistent way. The procedure to solve (3.40) as a non linear system can be explained as a recipe:

- First, one has to give a value for κ, $\delta\alpha$, θ, β and m, that way m_π and L are not determined yet.
- Then, the ground state function is calculated numerically using (3.38) for the κ and $\delta\alpha$ we chose. The result is normalized dividing by $\int_{-\pi}^{\pi} d\varphi|f_0(\varphi)|^2$.

- With f_0 already calculated, one computes $m_\pi L$ with the second equation of (3.40), using once again a numerical integrator.
- Note that $m_\pi L$ is already known from the last step, nevertheless, in order to find L and m_π one has to determine the "L-roots" of the last equation in (3.40), this can be done using a root finder (e.g. bisection).

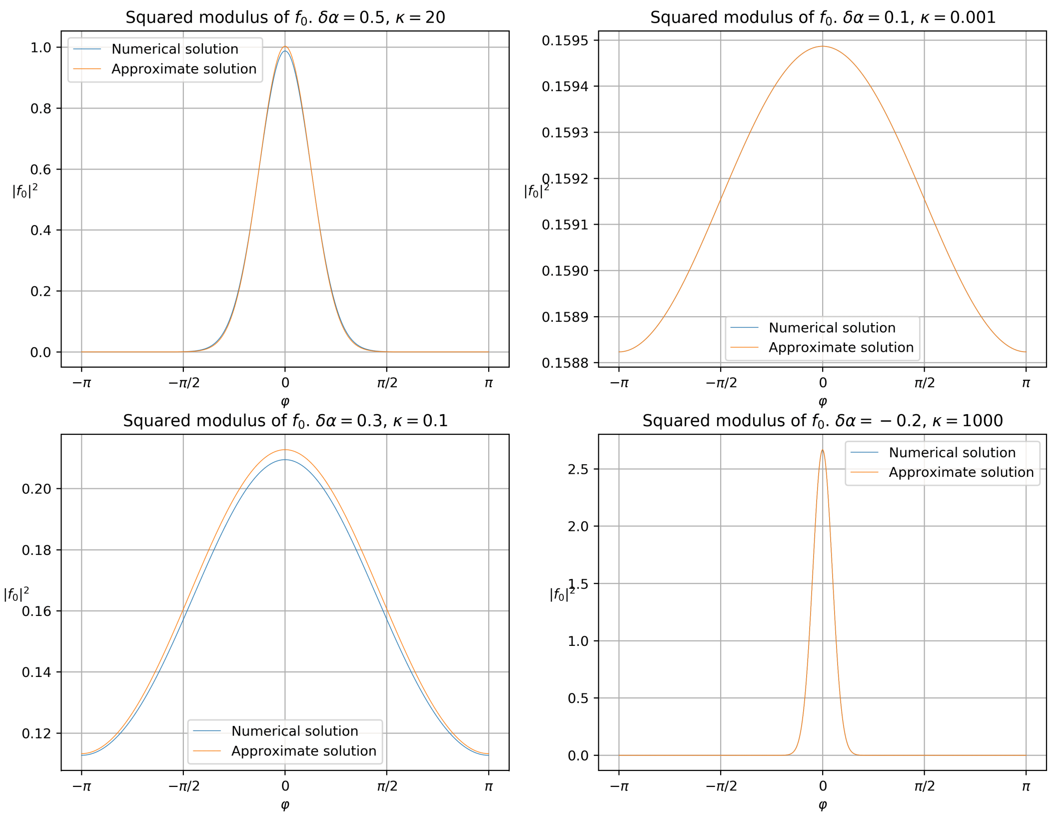

Figure 3.2: Solutions to equation (3.34), the approximate solutions correspond to the expressions in (3.39). In the two right-hand side plots is hard to distinguish between both solutions. $N + 1 = 1,000$ points in the discretization were used.

Following these four steps the system can be solved, however, it is important to note that with this procedure one does not have control over L, but over κ, so if a solution for a specific L is desired, then a scan for several values of κ has to be applied. Still, it is possible to have control over L, but then one can not give an initial value for m and it has to be determined in the same way we computed L in the last four steps, that is, one would have to do the following steps instead:

- Give a value for κ, $\delta\alpha$, θ, β L, and leave m_π and m undetermined.
- Calculate the normalized groundstate $f_0(\varphi)$.
- With f_0 already calculated, one computes m_π with the second equation of (3.40).
- Now one has to determine m in the last equation of (3.40), in this case one can solve for m analytically, there is no need for a root finder.

On the other hand, if one wants to have total control over both variables, L and m, then the system has to be solved self consistently, now the idea is the following:

- Give a value for $\delta\alpha$, θ, β L, m and give an initial guess of the pion mass m_π^{ini}, this last value is the one that will be determined self consistently.

- Calculate κ.
- Calculate a new value of the pion mass, m_π^{new}, using the second equation of (3.40).
- If $|m_\pi^{\text{new}} - m_\pi^{\text{ini}}|$ is smaller than an error that one desires, then m_π^{new} is the result for m_π, otherwise, one has to use m_π^{new} as m_π^{ini} and repeat these four steps until the final value has converged within the error.

The three methods were implemented with Python and all of them give the same results, although the last one can be a little more expensive computationally since one does not know how long is going to take the algorithm to converge and it would depend on the initial guess, so if one is not expecting results for a particular m or L and instead for a wide range of them, finding solutions as a non linear system of equations can be more useful.

The first result to be revised is equation (3.29), because it helps to verify the numerical solution and also allows to check the results with the three different methods explained above. To do that one substitutes m_η for its value in the chiral limit, that is $m_\eta = \mu$ and then solves (3.40). In figure 3.3 the pion mass is shown as a function of $(m^2/\sqrt{\beta})^{1/3}$ for different values of L, we can see that when L grows larger, the values get closer to the semiclassical prediction.

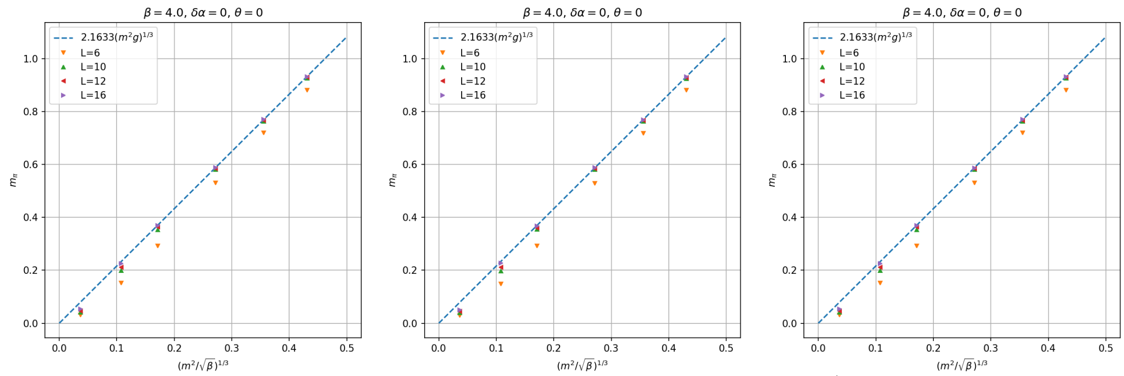

Figure 3.3: Predictions of the pion mass as function of $(m^2\sqrt{\beta})^{1/3}$ when one substitutes $\mu_1 \approx \mu$. Each one of the plots was made with each one of the different methods described above, it can be seen that the result is the same.

If one does not substitute $m_\eta \approx \mu$ and instead writes $m_\eta = \sqrt{m_\pi^2 + \mu^2}$, the result of figure 3.3 is different, since m_π will not converge to equation (3.29). In figure 3.4, the pion mass as a function of $(m^2/\sqrt{\beta})^{1/3}$ and m_η as a function of m are shown, but taking into account the change in m_η.

The chiral condensate can be calculated as well by using the third equation in (3.15), different values of this quantity as function of L and the temperature are shown in figure 3.5.

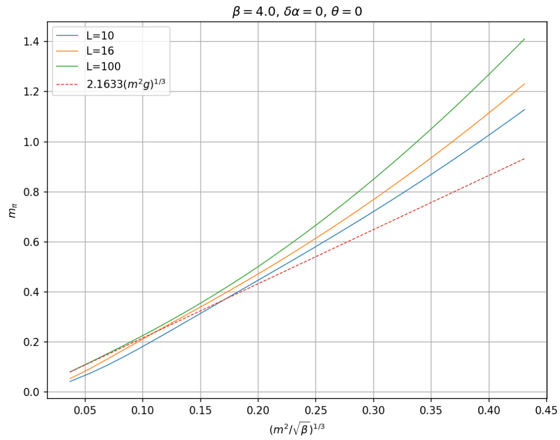

Figure 3.4: Predictions of m_η and m_π for different fermion masses and values of L. It can be seen that as L grows larger, for small m the result gets closer to the semiclassical prediction.

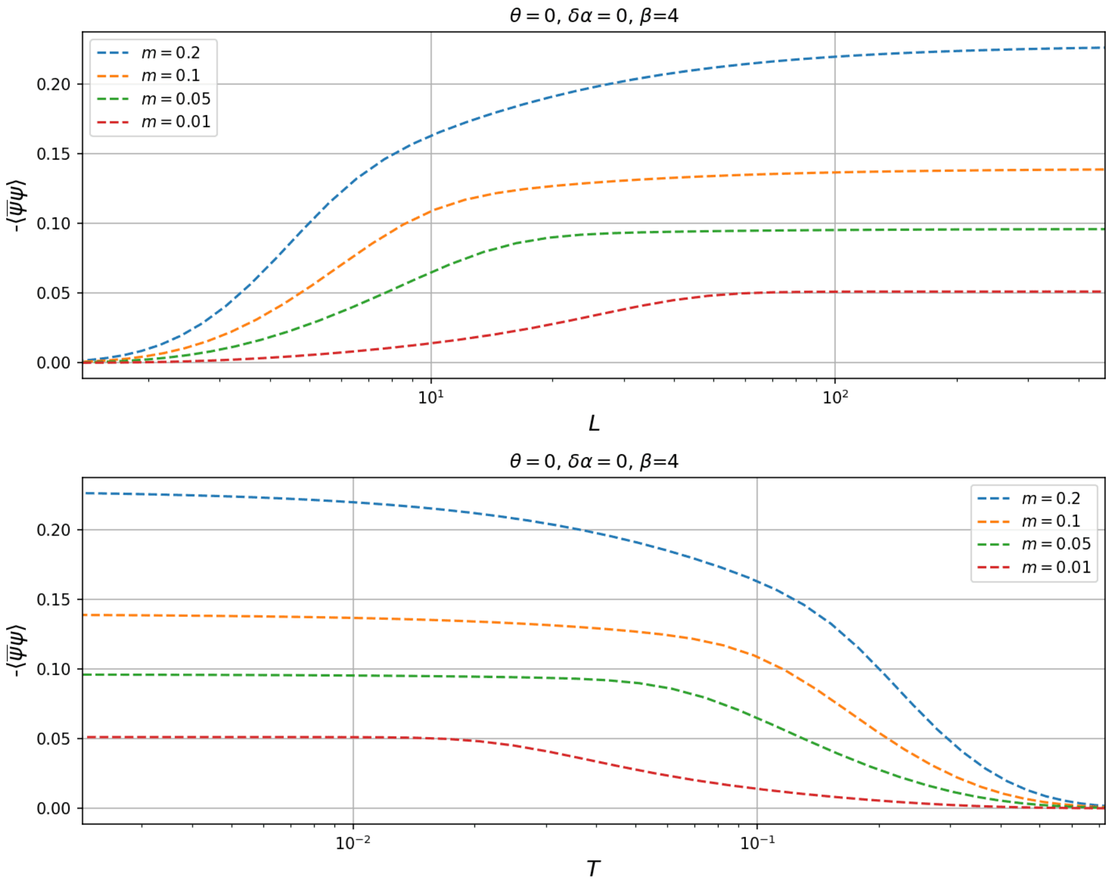

Figure 3.5: Predictions of the chiral condensate -$\langle\overline{\psi}\psi\rangle$ as a function of L and the temperature T. When $m \to 0$, $\langle\overline{\psi}\psi\rangle$ vanishes.

3.3 Lattice simulations results

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