# Chapter 1

## The Schwinger model

The Schwinger model refers to Quantum Electrodynamics in 1+1 dimensions [1]. It used as a toy model for Quantum Chromodynamics, because it shows similar properties, such as: confinement, chiral symmetry breaking and topology. Its Lagrangian is given in Minkowski space-time in natural units by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\gamma^\mu(i\partial_\mu - gA_\mu)\psi - m\overline{\psi}\psi, \tag{1.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $A_\mu(x)$ is the U(1) gauge field, $g$ is the gauge coupling constant, $\psi$ is the fermion field, $\overline{\psi} = \psi^\dagger\gamma^0$ and $\gamma^\mu$ are the Dirac matrices, which satisfy $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with $g_{\mu\nu} = \mathrm{diag}(1, -1)$. A possible representation for $\gamma^\mu$ is

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.2}$$

With these two matrices, we can define

$$\gamma_5 \equiv \gamma^0\gamma^1, \text{ which implies } \quad \{\gamma^\mu, \gamma_5\} = 0. \tag{1.3}$$

The equations of motion can be obtained through the Lagrangian

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\mu)}\right) - \frac{\partial\mathcal{L}}{\partial A_\mu} = 0 \quad \Rightarrow \quad \partial_\nu F^{\nu\mu} = gJ^\mu, \quad J^\mu \equiv \overline{\psi}\gamma^\mu\psi, \tag{1.4}$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu \psi)}\right) - \frac{\partial\mathcal{L}}{\partial \psi} = 0 \quad \Rightarrow \quad i\partial_\mu\overline{\psi}\gamma^\mu + m\overline{\psi} = -gA_\mu\overline{\psi}\gamma^\mu, \tag{1.5}$$

by taking the complex conjugate of the last equation we have

$$i\gamma^\mu\partial_\mu\psi - m\psi = g\gamma^\mu A_\mu\psi. \tag{1.6}$$

Since $F^{\mu\nu}$ is antisymmetric, eq. (1.4) implies that $J^\mu$ is conserved

$$\partial_\mu J^\mu = 0. \tag{1.7}$$

We can introduce another current, named axial current

$$J_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi. \tag{1.8}$$

Let us take its derivative

$$\begin{aligned}
\partial_\mu J_5^\mu &= \partial_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + \overline{\psi}\gamma^\mu\gamma_5\partial_\mu\psi \\
&= \partial_\mu\overline{\psi}\gamma^\mu\gamma_5\psi - \overline{\psi}\gamma_5\gamma^\mu\partial_\mu\psi \\
&= i(gA_\mu\overline{\psi}\gamma^\mu + m\overline{\psi})\gamma_5\psi + i\overline{\psi}\gamma_5(gA_\mu\gamma^\mu\psi + m\psi) \\
&= igA_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + im\overline{\psi}\gamma_5\psi - igA_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + im\overline{\psi}\gamma_5\psi \\
&= 2im\overline{\psi}\gamma_5\psi, \qquad (1.9)
\end{aligned}$$

we have made use of eqs. (1.3), (1.5) and (1.6). Thus, one would expect in the massless model $J_5^\mu$ to be conserved. However, it was proved (see ref. [2]) that $J_5^\mu$ shows an anomaly. When $m = 0$ one actually has

$$\partial_\mu J_5^\mu = -\frac{g}{\pi}\frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu}. \qquad (1.10)$$

In this context it is useful to define

$${}^*F \equiv \frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu} = F^{01} = -F_{01}. \qquad (1.11)$$

In 1+1 dimensions, the field tensor is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E(x) \\ -E(x) & 0 \end{pmatrix}, \qquad (1.12)$$

then ${}^*F = -E$. Furthermore, $F_{\mu\nu} = \epsilon_{\mu\nu}F_{01}$, hence

$$F_{\mu\nu} = -\epsilon_{\mu\nu}{}^*F. \qquad (1.13)$$

Now, let us note that

$$\begin{aligned}
\epsilon^{01}\gamma_1 = -\epsilon_{01}\gamma_1 = -\gamma_1 = \gamma^1 = \gamma^0\gamma^0\gamma^1 = \gamma^0\gamma_5 \\
\epsilon^{10}\gamma_0 = -\epsilon_{10}\gamma_0 = \gamma_0 = \gamma^0 = -\gamma^0\gamma^1\gamma^1 = \gamma^1\gamma^0\gamma^1 = \gamma^1\gamma_5,
\end{aligned} \qquad (1.14)$$

therefore $\epsilon^{\mu\nu}\gamma_\nu = \gamma^\mu\gamma_5$. With this expression we can rewrite eq. (1.8) as

$$J_5^\mu = \epsilon^{\mu\nu}J_\nu. \qquad (1.15)$$

Since $\epsilon^{\nu\mu}\epsilon_{\mu\sigma} = \delta^\nu_\sigma$, this last equation takes the form

$$J_\mu = \epsilon_{\mu\nu}J_5^\mu. \qquad (1.16)$$

Substituting eq. (1.13) in eq. (1.4) reads

$$-\partial_\mu\epsilon^{\mu\nu}\,{}^*F = gJ^\nu \qquad (1.17)$$

and by using eq. (1.16) we have

$$-\partial_\mu\epsilon^{\mu\nu}\,{}^*F = g\epsilon^{\nu\mu}J_{5\,\mu}. \qquad (1.18)$$

Multiplying by $\epsilon_{\nu\rho}$ yields

$$\partial_\mu\,{}^*F = gJ_{5\,\mu}. \qquad (1.19)$$

We can take the derivative in both sides of the equation

$$\partial^\mu\partial_\mu\,{}^*F = g\partial^\mu J_{5\,\mu} = -\frac{g^2}{\pi}\,{}^*F. \qquad (1.20)$$

Finally, substituting eq. (1.11) gives

$$\left(\partial^2 + \frac{g^2}{\pi}\right) E = 0, \tag{1.21}$$

which is the equation of a scalar field with mass $\mu^2 = g^2/\pi$. Therefore, in the massless one flavor Schwinger model, a boson of mass $\mu$ appears. This result has been generalized to an arbitrary number of massless flavors $N$ [3], where a boson of mass $\mu^2 = Ng^2/\pi$ appears. For massive fermions no general solution exists, although there has been several approaches. We will revise one of those approaches in chapter 3. A deeper discussion of QED in 1+1 dimensions can be found in refs. [4, 5].

## 1.1 Confinement

As we mentioned before, the Schwinger model exhibits confinement. We can illustrate this fact by analyzing the classical equations of motion

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{1.22}$$

Let us fix $A_0 = 0$ and suppose that we place a charge $q$ at the origin, then

$$\partial_1 F^{10} = q\delta(x) \Rightarrow \partial_x E = q\delta(x) \Rightarrow E(x) = q\theta(x) + E_0, \tag{1.23}$$

where $\theta(x)$ is the Heaviside function and $E_0$ is a constant electric field. The latter is fixed according to the value of $E(x)$ at infinity. If one calculates the energy of this configuration, we can see that it diverges

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\, E^2 \to \infty.$$

This means that the finite energy states must have neutral charge. Thus, let us consider now two charges $\pm q$ at $x = \mp L/2$. The equation of motion reads

$$\partial_x E = q\,\delta\left(x + \frac{L}{2}\right) - q\,\delta\left(x - \frac{L}{2}\right) \Rightarrow E(x) = q\,\theta\left(x + \frac{L}{2}\right) - q\,\theta\left(x - \frac{L}{2}\right) + E_0. \tag{1.24}$$

If we want that $E \to 0$ when $x \to \infty$, we have to fix $E_0 = 0$. Then, the electric field is

$$E(x) = \begin{cases} q, & |x| < \frac{L}{2} \\ 0, & \text{other case} \end{cases}. \tag{1.25}$$

We can calculate the energy of this configuration

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\, E^2 = \frac{1}{2}\int_{-L/2}^{L/2} dx\, q^2 = \frac{q^2 L}{2}. \tag{1.26}$$

We see that the energy grows linearly with the separation of the charges, illustrating confinement. This property holds in the Schwinger model [6]

## 1.2 Vacuum angle

If we do not fix the background field to zero, it is possible to generate electron-positron pairs when the difference of the energy between both particles and the background field is smaller than zero

$$\Delta H = \frac{1}{2}\int_{-L/2}^{L/2} dx\, \left[E(x)^2 - E_0^2\right] < 0. \tag{1.27}$$

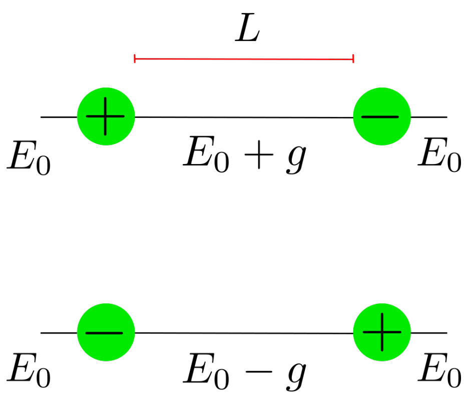


Figure 1.1: Electric field between an electron-positron pair in $\mathrm{QED}_2$, considering the background field.

The electric field $E(x)$ between the particles is now given by (see figure 1.1)

$$E(x) = E_0 \pm g, \quad -\frac{L}{2} \leq x \leq \frac{L}{2}. \tag{1.28}$$

Pairs can be created when

$$\begin{aligned} \Delta H &= \frac{L}{2}\left(g^2 \pm 2eE_0\right) < 0 \\ &\Leftrightarrow \frac{g}{2} < E_0 \quad \text{or} \quad E_0 < -\frac{g}{2} \\ &\Leftrightarrow \frac{g}{2} < |E_0|. \end{aligned} \tag{1.29}$$

In this context, the *vacuum angle* $\theta$

$$\theta = \frac{2\pi E_0}{g} \tag{1.30}$$

is introduced. Whenever $|\theta| > \pi$, pair production is favorable. $\theta = 0$ refers to confinement. This parameter was introduced to the Schwinger model by Coleman [7]. In QCD a similar parameter appears.

## 1.3 Chiral symmetry breaking

If one applies the following global transformations to the fields $\overline{\psi}$ and $\psi$

$$\psi \to \psi' = e^{i\alpha\gamma_5}\psi, \quad \overline{\psi} \to \overline{\psi}' = \overline{\psi} e^{i\alpha\gamma_5}, \quad \alpha \in \mathbb{R}, \tag{1.31}$$

the Lagrangian in eq. (1.1) transforms to

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \overline{\psi} e^{i\alpha\gamma_5}\gamma^\mu (i\partial_\mu - gA_\mu) e^{i\alpha\gamma_5}\psi - m\overline{\psi} e^{2i\alpha\gamma_5}\psi. \tag{1.32}$$

Since $\{\gamma^\mu, \gamma_5\} = 0$, it follows that

$$e^{-i\alpha\gamma_5}\gamma^\mu = (\mathbb{I} - i\alpha\gamma_5 + \cdots)\,\gamma^\mu = \gamma^\mu\,(\mathbb{I} + i\alpha\gamma_5 + \cdots) = \gamma^\mu e^{i\alpha\gamma_5}. \tag{1.33}$$

Then

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\gamma^\mu (i\partial_\mu - gA_\mu)\psi - m\overline{\psi} e^{2i\alpha\gamma_5}\psi. \tag{1.34}$$

We can see that for $m = 0$, the Lagrangian has a symmetry under the transformation given in eq. (1.31). Nevertheless, if one analyzes the vacuum expectation value of $\overline{\psi}\psi$, or the *chiral condensate*, it transforms as

$$\langle \overline{\psi}'_i \psi'_j \rangle \to \left(e^{2i\alpha\gamma_5}\right)_{ij} \langle \psi_i \psi_j \rangle. \tag{1.35}$$

Therefore, if $\langle \psi_i \psi_j \rangle \neq 0$, the symmetry is spontaneously broken when $m = 0$.

In the $N$-flavor Schwinger model with degenerate fermion mass $m$, it has been shown (see ref. [8]) that the chiral condensate has the following dependence on $m$ and $\theta$ when $m/\mu \ll 1$

$$\langle \overline{\psi}\psi \rangle = -\frac{\mu}{4\pi}\left(2e^{\gamma}\cos\frac{\theta}{2}\right)^{\frac{2N}{N+1}}\left(\frac{m}{\mu}\right)^{\frac{N-1}{N+1}}, \tag{1.36}$$

where $\gamma$ is the Euler-Mascheroni constant. For the one flavor model we can see that

$$\langle \overline{\psi}\psi \rangle = -\frac{\mu}{2\pi}e^{\gamma}\cos\frac{\theta}{2}, \tag{1.37}$$

*i.e.* there is no dependence on the fermion mass. Therefore, the chiral condensate is non vanishing and as a consequence the massless one flavor model shows, indeed, spontaneous chiral symmetry breaking. This happens in four dimensional QCD as well. However, let us note from eq. (1.36) that for $N > 1$ there is no spontaneous symmetry breaking in the Schwinger model.

# *Bibliography*


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