Chapter 4

Chiral Perturbation Theory in QCD

As it was mentioned in chapter 2, the low energy level of QCD cannot be treated with perturbation theory, but one can use lattice simulations. The analytic approach to this regime is the *Chiral Perturbation Theory*, which is an effective field theory. The Lagrangian of this effective theory is built by adding all the terms that are consistent with the symmetries of the underlying theory, that in this case is QCD. This lead us to analize the symmetries of the QCD Lagrangian.

4.1 QCD chiral symmetry

Since we are interested in quark masses $m_f \ll \Lambda_{\text{QCD}} \approx 300$ MeV, we will be working only with two flavors: u, d, whose masses are [1]

$$\begin{aligned} m_u &= 1.7 \text{ - } 3.33 \text{ MeV}, \\ m_d &= 4.1 \text{ - } 5.8 \text{ MeV}. \end{aligned} \tag{4.1}$$

Then, in Minkowski space-time, the Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \sum_{f=u,d} (\overline{q}_f i\gamma^\mu D_\mu q_f - m_f \overline{q}_f q_f) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a, \\ D_\mu &= (\partial_\mu + igA_\mu), \;\; G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ig[A^a_\mu, A^a_\nu]. \end{aligned} \tag{4.2}$$

q_f is the quark field of the flavor f, $\overline{q}_f = q_f^\dagger \gamma^0$, $A^a_\mu(x)$ are real valued fields related with $A_\mu(x)$ by $A_\mu(x) = \sum_{a=1}^{8} A^a_\mu(x) T_a$, where T_a are the basis elements of the traceless Hermitian 3×3 matrices. This implies that the gauge field is a matrix as well. This Lagrangian is constructed in the same way we did in section 2.3, so it is gauge invariant under transformations of SU(3). We are interested in the chiral symmetry, then let us apply the chiral operator

$$P_R = \frac{1}{2}(\mathbb{I} + \gamma_5), \;\; P_L = \frac{1}{2}(\mathbb{I} - \gamma_5) \tag{4.3}$$

to the quark fields in order to obtain the right-handed and left-handed fields:

$$q_{R_f} = P_{R_f} q, \;\; q_{L_f} = P_{L_f} q. \tag{4.4}$$

By using $\{\gamma_5, \gamma^\mu\}$ and the definition of the chiral operators one can prove the following properties:

$$P_{L,R} = P^\dagger_{L,R}, \;\; P^2_{L,R} = P_{L,R}, \;\; P_R P_L = P_L P_R = 0, \;\; P_{R,L}\gamma^\mu = \gamma^\mu P_{L,R}. \tag{4.5}$$

This implies that

$$\begin{aligned} q_f &= q_{R_f} + q_{L_f}, \\ \overline{q}_{R_f} &= q^\dagger_{R_f}\gamma^0 = q^\dagger_f P_R \gamma^0 = q^\dagger_f \gamma^0 P_L = \overline{q}_f P_L, \\ \overline{q}_{L_f} &= \overline{q}_f P_R, \\ \overline{q}_f &= \overline{q}_{L_f} + \overline{f}_{L_f}. \end{aligned} \tag{4.6}$$

With (4.5) and (4.6) the QCD Lagrangian can be rewritten as

$$\mathcal{L} = \sum_{f=u,d} \left[\overline{q}_{L_f} i\gamma^\mu D_\mu q_{L_f} + \overline{q}_{R_f} i\gamma^\mu D_\mu q_{R_f} - m_f(\overline{q}_{R_f} q_{L_f} + \overline{q}_{L_f} q_{R_f}) \right] - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a. \tag{4.7}$$

Now we can apply global chiral transformations:

$$q_{R,L_f} \to q'_{R,L_f} = e^{i\alpha\gamma_5} q_{R,L_f}, \ \ \overline{q}_{R,L_f} \to \overline{q}'_{R,L_f} = \overline{q}_{R,L_f} e^{i\alpha\gamma_5}, \ \ \alpha \in \mathbb{R}. \tag{4.8}$$

Let us note that the mass tern is not invariant under this transformations, so for the moment we will take $m = 0$. On the other hand, since $\{\gamma^\mu, \gamma_5\} = 0$ it follows that $\gamma^\mu e^{i\alpha\gamma_5} = e^{-i\alpha\gamma_5}\gamma^\mu$ and as a result the Lagrangian transforms as:

$$\begin{aligned} \mathcal{L} &= \sum_{f=u,d} \left(\overline{q}_{L_f} i\gamma^\mu D_\mu q_{L_f} + \overline{q}_{R_f} i\gamma^\mu D_\mu q_{R_f} \right) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a \\ &= \sum_{f=u,d} \left(\overline{q}'_{L_f} i\gamma^\mu D_\mu q'_{L_f} + \overline{q}'_{R_f} i\gamma^\mu D_\mu q'_{R_f} \right) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a. \end{aligned}$$

Hence, we can see that it is invariant under the transformations (4.8) only when $m = 0$, for that reason this is called the chiral limit. If we write the quark fields as a vector

$$q_{R,L} = \begin{pmatrix} q_{R,L_u} \\ q_{R,L_d} \end{pmatrix}, \ \ \overline{q}_{R,L} = \begin{pmatrix} \overline{q}_{R,L_u} \\ \overline{q}_{R,L_d} \end{pmatrix}, \ \ q = \begin{pmatrix} q_u \\ q_d \end{pmatrix}, \ \ \overline{q} = \begin{pmatrix} \overline{q}_u \\ \overline{q}_d, \end{pmatrix}, \tag{4.9}$$

it can be seen that the Lagrangian has a global symmetry under $\mathrm{U}(2)_L \otimes \mathrm{U}(2)_R$. An element of U(2) can be decomposed in an element of SU(2) multiplied by a phase factor, thus, we have the following symmetry group for the massless Lagrangian:

$$\mathrm{U}(2)_L \otimes \mathrm{U}(2)_R \leftrightarrow \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \otimes \mathrm{U}(1)_B \otimes \mathrm{U}(1)_A. \tag{4.10}$$

$\mathrm{U}(1)_A$ is known as axial symmetry and is broken explicitly under quantization. $\mathrm{U}(1)_B$ is associated with the conservation of the Baryon number. Meanwhile, $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ is the chiral symmetry group. The later breaks spontaneously to SU(2); the order parameter of this broken symmetry is the chiral condensate $\langle 0| \overline{q}q |0\rangle$, so when it is different from zero the chiral symmetry is indeed spontaneously broken. Because of the Goldstone theorem, to this broken symmetry corresponds $2^2 - 1 = 3$ massless Nambu-Goldstone Bosons (NGB). If now we take into account the masses of m_u and m_d, the symmetry is explicitly broken and the NGB's turn into light massive quasi NGB's, which can be related to the pion triplet π^+, π^-, π^0 since they are the lightest hadrons [1], [2], [3].

4.2 Effective Lagrangian

In order to build the effective Lagrangian one uses the dynamic variables suitable for low-energy, for two quark flavors this means that one replaces the quark and gluon fields by pion fields: $\vec{\pi} = \{\pi_1(x), \pi_2(x), \pi_3(x)\}$. Now one introduces a field $U(x) \in \mathrm{SU}(2)$ defined as

$$U(x) = \exp\left(i\frac{\vec{\pi}\cdot\vec{\tau}}{F_\pi}\right), \quad \vec{\tau} = (\sigma_1, \sigma_2, \sigma_3), \tag{4.11}$$

that transforms under global symmetries of $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ as

$$U(x) \to U'(x) = \Omega_R U(x) \Omega_L^{-1}, \;\; \Omega_R \in \mathrm{SU}(2)_R, \;\; \Omega_L \in \mathrm{SU}(2)_L, \tag{4.12}$$

F_π is known as the pion decay constant and makes the argument of the exponential dimensionless; σ are the Pauli matrices, which are the generators of SU(2), so one can write [1]

$$U(x) = \exp\left(i\frac{\phi(x)}{F_\pi}\right), \quad \phi(x) = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix},$$
$$\pi^0 = \pi_3, \;\; \pi^\pm = \frac{\pi_1 \mp i\pi_2}{\sqrt{2}}. \tag{4.13}$$

The effective Lagrangian $\mathcal{L}_{\text{eff}}$ is written by using $U(x)$ and it must have all the terms that present the symmetries of QCD each one accompanied with a constant denominated low energy constant (LEC); however, those are actually an infinite number of terms, they can be organized in increasing powers of momentum, which is the same as increasing number of derivatives, so one can truncate them. For a massless case, the effective Lagrangian with the least number of derivatives that can be written is

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4}\mathrm{tr}\left(\partial_\mu U \partial^\mu U^\dagger\right). \tag{4.14}$$

The factor of 4 is because if one expands in powers of ϕ one obtains the kinetic term $\frac{1}{2}\mathrm{tr}(\partial_\mu\phi\partial^\mu\phi)$. Let us note that under the transformation of eq. (4.12) the Lagrangian is invariant

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4}\mathrm{tr}\left(\Omega_R\partial_\mu U\Omega_L^{-1}\Omega_L\partial^\mu U^\dagger\Omega_R^{-1}\right) = \frac{F_\pi^2}{4}\mathrm{tr}\left(\Omega_R^{-1}\Omega_R\partial_\mu U\partial^\mu U^\dagger\right) = \frac{F_\pi^2}{4}\mathrm{tr}\left(\partial_\mu U\partial^\mu U^\dagger\right), \tag{4.15}$$

we have used the property $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Thus, $\mathcal{L}_{\text{eff}}$ is indeed chiral invariant.

If now one wants to introduce the masses m_u and m_d to the theory, a term that explicitly breaks the chiral symmetry is added. Such term is

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2}\mathrm{tr}\left(MU^\dagger - UM\right), \quad M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}, \tag{4.16}$$

where s.b. stands for symmetry breaking and F_0 and B_0 are two LEC's. Then, to leading order the massive effective Lagrangian.

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4}\mathrm{tr}\left(\partial_\mu U\partial^\mu U^\dagger\right) + \frac{F_0^2 B_0}{2}\mathrm{tr}\left(MU^\dagger - UM\right). \tag{4.17}$$

For degenerate quark masses $m_u = m_d \equiv m$, the LEC's satisfy the following relations [4]

$$\begin{aligned} F_\pi &= F_0\left[1 + O(m)\right], \\ \langle 0|\,\overline{q}q\,|0\rangle &= -2F_0^2 B_0\left[1 + O(m)\right], \\ m_\pi &= \sqrt{2B_0 m}\left[1 + O(m)\right]. \end{aligned} \tag{4.18}$$

At the chiral limit $F_\pi = F_0$.

4.3 Chiral Perturbation Theory regimes

From eq. (4.17) it can be seen that the leading order LEC's are F_π, B_0 and F_0. These constants can be determined trough Lattice simulations by measuring the chiral condensate, the quark and pion masses and fitting the functions in eqs (4.18); however, since one cannot simulate an infinite volume, three regimes based on the finite volume in Euclidean space $V = L^d$ (note that we are taking the Euclidean time extension to be equal to the spatial volume), with d the dimension, have been studied. For each one of them, eqs. (4.18) are different, but the LEC's are the same. Let us briefly review the regimes:

- The first regime that one can discuss is the *p-regime*, which consists of a large volume compared to the correlation length: $L \gg \xi = m_\pi^{-1}$; besides, the finite volume corrections are suppressed by a factor proportional to $\exp(-m_\pi L)$ [4], [5], so one can expect eqs. (4.18) to be valid.
- If $L \lesssim \xi$ then we refer to the so-called *ϵ-regime*, here the finite volume corrections cannot be neglected. In this regime the chiral condensate has the following dependence on the LEC's, m and the volume when $B_0 m L^2 \ll 1$ [6]

$$\langle 0|\,\overline{q}q\,|0\rangle = -2F_0^2 B_0 \left(\frac{I_1'(F_0^2 B_0 m V)}{I_1(F_0^2 B_0 m V)} - \frac{1}{F_0^2 B_0 m V} \right), \tag{4.19}$$

where I_1 is a modified Bessel function of first kind. When one takes the infinite volume limit, the chiral condensate yields $\langle 0|\,\overline{q}q\,|0\rangle = -2F_0^2 B_0$, recovering the second eq. of (4.18). On the other hand, when $m \to 0$, eq. (4.19) yields [5]

$$\langle 0|\,\overline{q}q\,|0\rangle = -\frac{1}{4} F_0^4 B_0^2 m V. \tag{4.20}$$

We can see that at the chiral limit $\langle 0|\,\overline{q}q\,|0\rangle$ vanishes, this is consistent with the fact that in finite volume there is no spontaneous symmetry breaking.
- Finally, the *δ-regime* is determined by a volume of size $V = L^3 \times L_t$, where the spatial volume is small, but the Euclidean time extension is large, that is $L \lesssim \xi \ll L_t$.

The pion decay constant has already been calculated in the p-regime and the ϵ-regime with 3 flavors several times, giving a result of $F_\pi = 92.1(9)$ MeV [7], [8], [9], [10], meanwhile, the δ-regime is less explored; there F_π has been measured with two flavors obtaining $F_\pi = 78^{+14}_{-10}$ MeV [11].

This last regime is useful from a technical point of view, because the small volume has the advantage of reducing the computing time of the simulations. Besides, the fact that $L \lesssim \xi$ has several consequence. First, it leaves us approximately with only one dimension (a quasi one dimensional field theory), enabling us to perform a quantum mechanical treatment of the system [12]. Another consequence is that at the chiral limit the pion mass does not become massless and instead there is a *residual pion mass* m_π^R. It is possible to obtain an expression of m_π^R as a function of the spatial size L. To do that, one considers the local isomorphism between O(4) and $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ and writes m_π^R in terms of the energy gap of a quantum rotor, which is given by

$$E_j = \frac{j(j+N-2)}{2\Theta}, \tag{4.21}$$

where Θ is the moment of intertia and N stands for the O(N) group. Its value was computed in [13] for general dimension d, yielding

$$\Theta = F_\pi^2 L^{d-1} \left[1 + \frac{N-2}{4\pi F_\pi^2 L^{d-2}} \left(2\frac{d-1}{d-2} + \cdots \right) \right]. \tag{4.22}$$

m_π^R is obtained by substituting $j = 1$ and $N = 4$ in eq. (4.21)

$$m_\pi^R = \frac{3}{2\Theta}. \tag{4.23}$$

In fourth dimensions we have

$$m_\pi^R = \frac{3}{2F_\pi^2 L^3(1+\Delta)}, \quad \Delta = \frac{0.477\cdots}{(F_\pi L)^2} + \ldots, \quad d = 4; \tag{4.24}$$

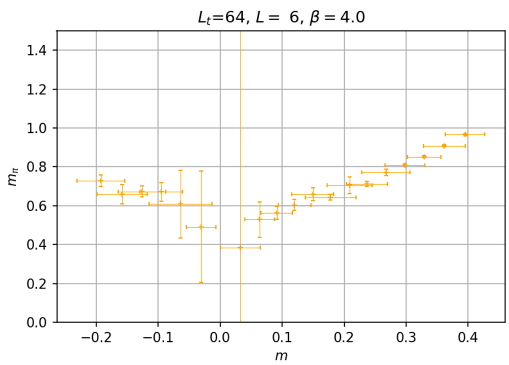
however, let us note that in two dimensions there is a divergence of the next-to-leading term, so instead we just consider the leading term, obtaining

$$m_\pi^R \simeq \frac{3}{2F_\pi^2 L}, \quad d = 2. \tag{4.25}$$

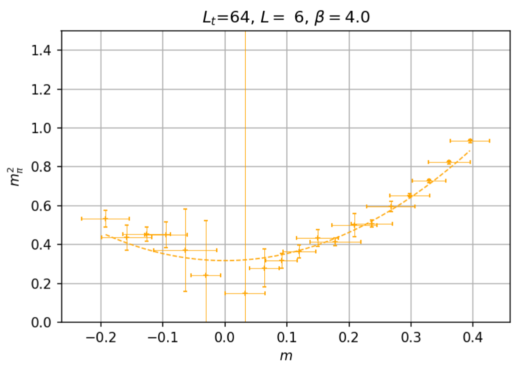
Several simulations were performed in order to obtain the residual pion mass for different volumes L by using the Hybrid Monte Carlo algorithm with the two flavor Schwinger model in order to revise eq. (4.25) and to extract the value of F_π.

4.4 δ-regime results

To determine results for the residual pion mass, first one has to fix a value for the gauge coupling constant g. As in chapter 3, we will denote $\beta = 1/g^2$; most of the work was done for $\beta = 4$, although results for $\beta = 2$ were obtained as well in order to revise the independence of F_π on this parameter. In fig. 4.1 (a), m_π is shown as a function of the degenerate quark mass m for a space size of $L_s = 6$ and a Euclidean time extension $L_t = 64$, we can see that close to the chiral limit the value of m_π becomes very unstable, so one cannot measure directly m_π at $m = 0$ and instead one extrapolates the value to $m = 0$, however, that is easier to do with the plot shown in fig. 4.1 (b), where m_π^2 is shown against m, since there is (qualitatively) a parabolic behavior that enable us to fit a quadratic function and to infer m_π^R. As L_s grows larger, the parabolic behavior is more noticeable.



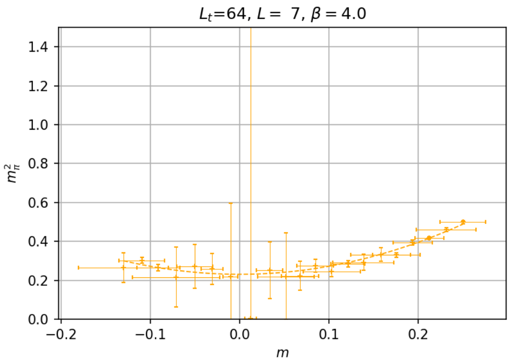
(a) m_π vs. m. It can be seen that near $m = 0$ the pion mass result is not trustable



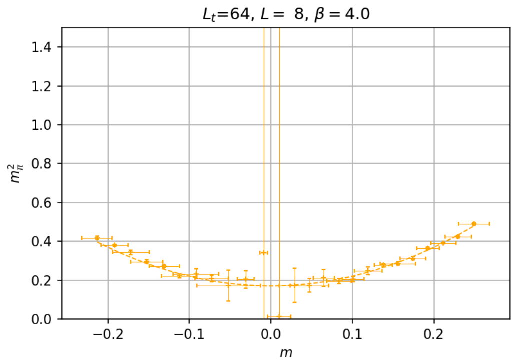
(b) m_π^2 vs. m. One can fit a parabola of the form $y = m_\pi^R + bm^2$ to obtain the value of the residual pion mass: $m_\pi^R = 0.5633(188)$.

Figure 4.1: Results of the measurement of the pion mass as a function of m. Note that there are also values for $m < 0$, they are unphysical, however, in the simulation both masses have to be measured through other parameter of the HMC algorithm which allows negative masses. The errors were computed by using the *Jackknife error*.

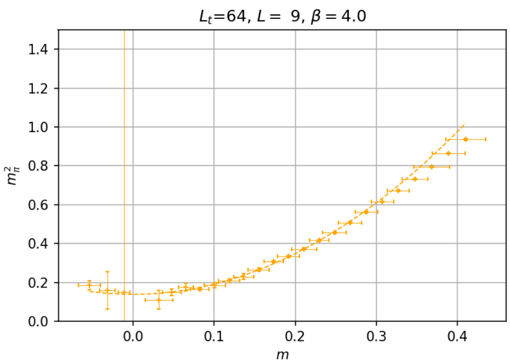
This same procedure was done with $L_s = 7, 8, 9, 10, 11, 12$ and $L_t = 64$. In fig. 4.2 plots of m_π^2 vs. m are shown, together with the extrapolated value of m_π^R.



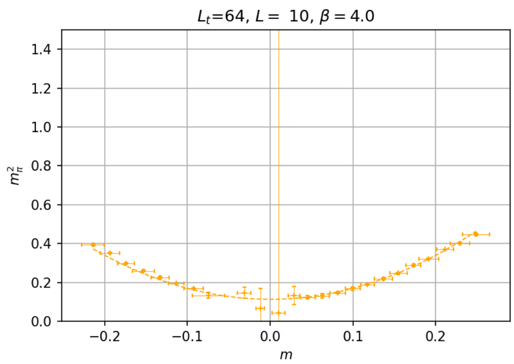
(a) m_π^2 vs. m for $L = 7$, $m_\pi^R = 0.4804(63)$



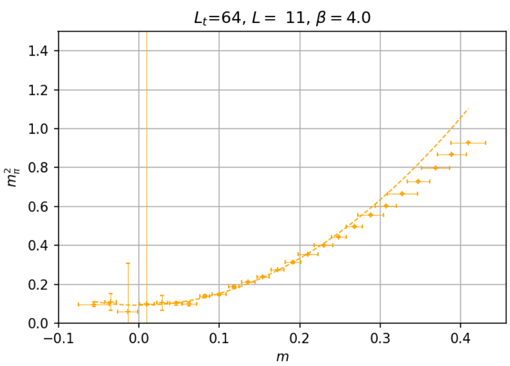
(b) m_π^2 vs. m for $L = 8$, $m_\pi^R = 0.4133(54)$



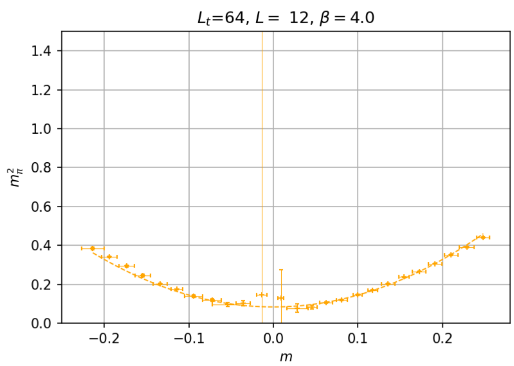
(c) m_π^2 vs. m for $L = 9$, $m_\pi^R = 0.3724(57)$



(d) m_π^2 vs. m for $L = 10$, $m_\pi^R = 0.3361(35)$



(e) m_π^2 vs. m for $L = 11$, $m_\pi^R = 0.3039(54)$



(f) m_π^2 vs. m for $L = 12$, $m_\pi^R = 0.2889(37)$

Figure 4.2: Each result was obtained by averaging 1,000 measurements of different configurations. Between each configuration used, 10 sweeps were performed. All the parabolic fits where made through gnuplot.

Now, one plots m_π^R as a function of N_s and fits a function of the form $3/2LF_\pi^2$, as it is shown in fig. 4.3. The result obtained for the pion decay constant is

$$F_\pi = 0.6677(18), \tag{4.26}$$

for $\beta = 4$, besides, it matches the behavior given by $1/L$ described in eq. (4.25). In fig. 4.4 the same plot is shown, but for $\beta = 2$; the result in that case is

$$F_\pi = 0.6707(84). \tag{4.27}$$

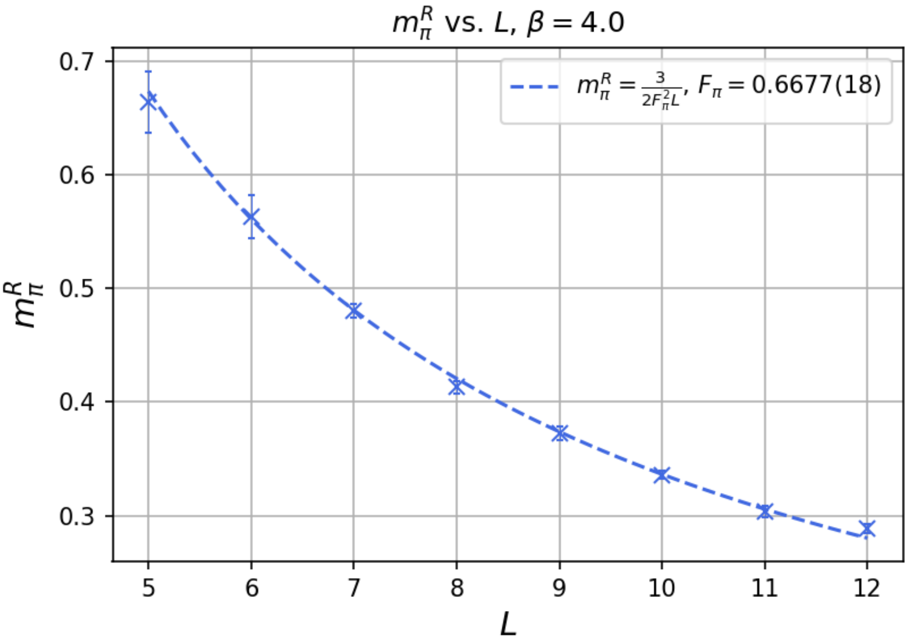


Figure 4.3: Behavior of m_π^R as a function of the volume size L for $\beta = 4$, we can see that as L becomes larger the residual pion mass vanishes, as it should do for an infinite volume.

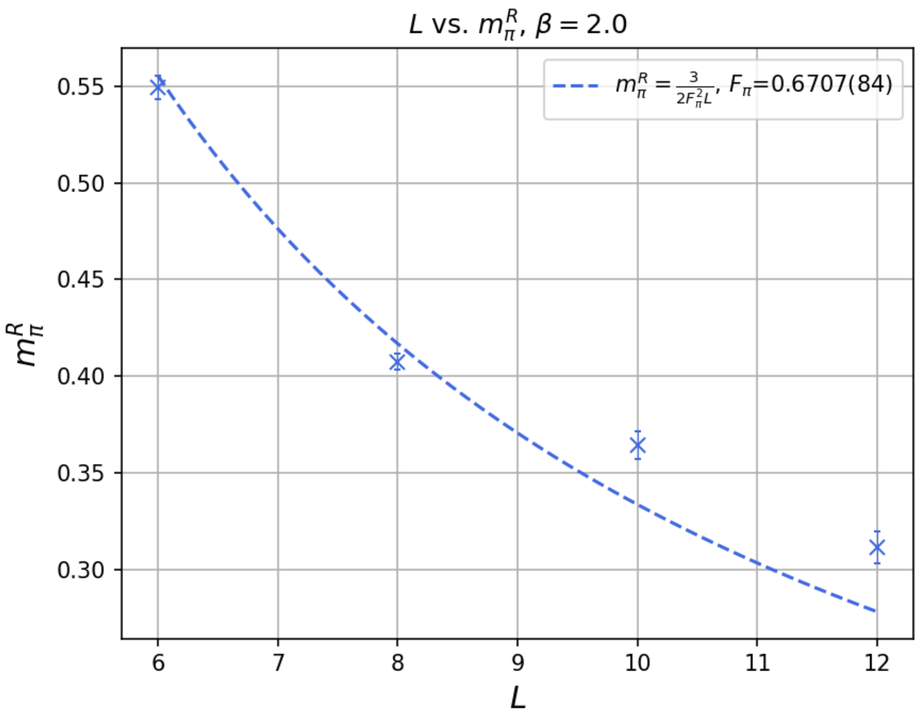


Figure 4.4: m_π^R vs. L for $\beta = 2$

Bibliography

[1] S. Scherer and M. R. Schindler. *A Primer for Chiral Perturbation Theory*. Springer Berlin Heidelberg, 2012.

[2] A. Pich. Chiral perturbation theory. *Rept. Prog. Phys.*, 58:563–610, 1995.

[3] H. Leutwyler. Chiral perturbation theory. *Scholarpedia*, 7(10):8708, 2012.

[4] J. Gasser and H. Leutwyler. Light Quarks at Low Temperatures. *Phys. Lett. B*, 184:83–88, 1987.

[5] M. Golterman. Applications of Chiral perturbation theory to lattice QCD. In *Les Houches Summer School: Session 93: Modern perspectives in lattice QCD: Quantum field theory and high performance computing*, pages 423–515, 12 2009.

[6] J. Gasser and H. Leutwyler. Thermodynamics of chiral symmetry. *Physics Letters B*, 188(4):477–481, 1987.

[7] S. Aoki et al. Review of lattice results concerning low-energy particle physics. *Eur. Phys. J. C*, 77(2):112, 2017.

[8] A. Bazavov et al. Results for light pseudoscalar mesons. *PoS*, LATTICE2010:074, 2010.

[9] R. Arthur, T. Blum, P. A. Boyle, N. H. Christ, N. Garron, R. J. Hudspith, T. Izubuchi, C. Jung, C. Kelly, A. T. Lytle, R. D. Mawhinney, D. Murphy, S. Ohta, C. T. Sachrajda, A. Soni, J. Yu, and J. M. Zanotti. Domain wall QCD with near-physical pions. *Phys. Rev. D*, 87:094514, May 2013.

[10] E. Follana, C. T. H. Davies, G. P. Lepage, and J. Shigemitsu. High-Precision Determination of the π, K, D, and D_s Decay Constants from Lattice QCD. *Phys. Rev. Lett.*, 100:062002, Feb 2008.

[11] W. Bietenholz, M. Göckeler, R. Horsley, Y. Nakamura, D. Pleiter, P.E.L. Rakow, G. Schierholz, and J.M. Zanotti. Pion in a box. *Physics Letters B*, 687(4):410–414, 2010.

[12] H. Leutwyler. Energy Levels of Light Quarks Confined to a Box. *Phys. Lett. B*, 189:197–202, 1987.

[13] P. Hasenfratz and F. Niedermayer. Finite size and temperature effects in the AF Heisenberg model. *Z. Phys. B*, 92:91, 1993.