Chapter 1

The Schwinger model

The Schwinger model represents Quantum Electrodynamics in 1+1 dimensions [1]. It is used as a toy model for Quantum Chromodynamics, because it has similar properties, such as: confinement, chiral symmetry breaking and topology. In contrast to QCD, this model does not have a running coupling constant. Its Lagrangian in Minkowski space-time in natural units for one flavor is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \overline{\psi} \gamma^{\mu} (i\partial_{\mu} - gA_{\mu})\psi - m\overline{\psi}\psi, \tag{1.1}$$

where $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$, $A_{\mu}(x)$ is the U(1) gauge field, g is the gauge coupling constant, ψ and $\overline{\psi}$ are independent Grassmann fields in the functional integral formulation (see Chapter 2) and γ^{μ} are the Dirac matrices

$$\gamma^0 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^1 = i\sigma_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \tag{1.2}$$

which satisfy $(\gamma^{\mu})^{\dagger} = \gamma^0 \gamma^{\mu} \gamma^0$, $\{\gamma^{\mu}, \gamma^{\nu}\} = 2g^{\mu\nu}$ with $g_{\mu\nu} = \mathrm{diag}(1, -1)$. We assume $A_{\mu}(x)$ to be dimensionless and g to have mass dimension. With the gamma matrices, we can define one more matrix

$$\gamma_5 \equiv \gamma^0 \gamma^1, \text{ which implies } \quad \{\gamma^{\mu}, \gamma_5\} = 0, \quad \gamma_5^2 = \mathbb{I}, \quad \gamma_5^{\dagger} = \gamma_5. \tag{1.3}$$

The equations of motion can be obtained through the Euler-Lagrange equations

$$\partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\nu})} \right) - \frac{\partial \mathcal{L}}{\partial A_{\nu}} = 0 \quad \Rightarrow \quad \partial_{\nu} F^{\nu\mu} = gJ^{\mu}, \quad J^{\mu} \equiv \overline{\psi} \gamma^{\mu} \psi, \tag{1.4}$$

$$\partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \quad \Rightarrow \quad i\partial_{\mu} \overline{\psi} \gamma^{\mu} + m\overline{\psi} = -g\gamma^{\mu} A_{\mu} \overline{\psi}, \tag{1.5}$$

$$\partial_{\mu} \left(\frac{\partial \mathcal{L}}{\partial(\partial_{\mu} \overline{\psi})} \right) - \frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0 \quad \Rightarrow \quad i\gamma^{\mu} \partial_{\mu} \psi - m\psi = g\gamma^{\mu} A_{\mu} \psi. \tag{1.6}$$

Since $F^{\mu\nu}$ is antisymmetric, eq. (1.4) implies that J^{μ} is conserved

$$\partial_{\mu} J^{\mu} = 0. \tag{1.7}$$

If one applies a global axial transformation to the fields $\overline{\psi}$ and ψ

$$\psi \rightarrow \psi' = e^{i\alpha\gamma_5} \psi, \quad \overline{\psi} \rightarrow \overline{\psi}' = \overline{\psi} e^{i\alpha\gamma_5}, \quad \alpha \in \mathbb{R}, \tag{1.8}$$

the Lagrangian in eq. (1.1) transforms to

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}e^{i\alpha\gamma_5}\gamma^\mu(i\partial_\mu - gA_\mu)e^{i\alpha\gamma_5}\psi - m\overline{\psi}e^{2i\alpha\gamma_5}\psi. \tag{1.9}$$

Since $\{\gamma^\mu, \gamma_5\} = 0$, it follows that

$$e^{-i\alpha\gamma_5}\gamma^\mu = (\mathbb{I} - i\alpha\gamma_5 + \cdots)\,\gamma^\mu = \gamma^\mu\,(\mathbb{I} + i\alpha\gamma_5 + \cdots) = \gamma^\mu e^{i\alpha\gamma_5}. \tag{1.10}$$

Then

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\gamma^\mu(i\partial_\mu - gA_\mu)\psi - m\overline{\psi}e^{2i\alpha\gamma_5}\psi. \tag{1.11}$$

We see that for $m = 0$, the Lagrangian has a symmetry under the transformation given in eq. (1.8). The Noether current of this symmetry, known as *axial current*, is

$$J_5^\mu = \overline{\psi}\gamma^\mu\gamma_5\psi. \tag{1.12}$$

Let us take its derivative by taking into account the mass, using eq. (1.3) and relying on the equations of motion (1.5) and (1.6)

$$\begin{aligned}
\partial_\mu J_5^\mu &= \partial_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + \overline{\psi}\gamma^\mu\gamma_5\partial_\mu\psi \\
&= \partial_\mu\overline{\psi}\gamma^\mu\gamma_5\psi - \overline{\psi}\gamma_5\gamma^\mu\partial_\mu\psi \\
&= i(gA_\mu\overline{\psi}\gamma^\mu + m\overline{\psi})\gamma_5\psi + i\overline{\psi}\gamma_5(gA_\mu\gamma^\mu\psi + m\psi) \\
&= igA_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + im\overline{\psi}\gamma_5\psi - igA_\mu\overline{\psi}\gamma^\mu\gamma_5\psi + im\overline{\psi}\gamma_5\psi \\
&= 2im\overline{\psi}\gamma_5\psi.
\end{aligned} \tag{1.13}$$

Hence, one would expect in the massless model J_5^μ to be conserved. However, it was proved (see refs. [2, 3]) that J_5^μ shows an anomaly at the quantum level. When $m = 0$ one actually has

$$\partial_\mu J_5^\mu = -\frac{g}{\pi}\frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu}. \tag{1.14}$$

This equation is known as the *axial anomaly*. In order to show that the theory is sensitive to this expression, we define

$$^*F \equiv \frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu} = F^{01} = -F_{01} = -E. \tag{1.15}$$

In 1+1 dimensions the Abelian strength field tensor is given by

$$F_{\mu\nu} = \begin{pmatrix} 0 & E(x) \\ -E(x) & 0 \end{pmatrix}, \tag{1.16}$$

which confirms $^*F = -E$. Furthermore, $F_{\mu\nu} = \epsilon_{\mu\nu}F_{01} = \epsilon_{\mu\nu}E$, hence

$$F_{\mu\nu} = -\epsilon_{\mu\nu}{}^*F. \tag{1.17}$$

Let us note that

$$\begin{gathered}
\epsilon^{01}\gamma_1 = -\epsilon_{01}\gamma_1 = -\gamma_1 = \gamma^1 = \gamma^0\gamma^0\gamma^1 = \gamma^0\gamma_5, \\
\epsilon^{10}\gamma_0 = -\epsilon_{10}\gamma_0 = \gamma_0 = \gamma^0 = -\gamma^0\gamma^1\gamma^1 = \gamma^1\gamma^0\gamma^1 = \gamma^1\gamma_5,
\end{gathered} \tag{1.18}$$

therefore $\epsilon^{\mu\nu}\gamma_\nu = \gamma^\mu\gamma_5$. With this expression we can rewrite eq. (1.12) as

$$J_5^\mu = \epsilon^{\mu\nu}J_\nu. \tag{1.19}$$

If we multiply by $\epsilon_{\sigma\mu}$ and use the property $\epsilon^{\nu\mu}\epsilon_{\mu\sigma} = \delta^{\nu}_{\sigma}$, eq. (1.19) takes the form

$$J_\sigma = \epsilon_{\sigma\mu} J_5^\mu, \quad \text{also} \quad J^\sigma = \epsilon^{\sigma\mu} J_{5\,\mu}. \tag{1.20}$$

Substituting eq. (1.17) in eq. (1.4) leads to

$$-\partial_\mu \epsilon^{\mu\nu}\, {}^*F = gJ^\nu \tag{1.21}$$

and by using eq. (1.20) we have

$$-\partial_\mu \epsilon^{\mu\nu}\, {}^*F = g\epsilon^{\nu\mu} J_{5\,\mu}. \tag{1.22}$$

Multiplying by $\epsilon_{\nu\rho}$ yields

$$\partial_\rho\, {}^*F = gJ_{5\,\rho}. \tag{1.23}$$

We can take the derivative on both sides of the equation and rename the dummy index

$$\partial^\mu \partial_\mu\, {}^*F = g\partial^\mu J_{5\,\mu} = -\frac{g^2}{\pi}\, {}^*F. \tag{1.24}$$

Finally, substituting eq. (1.15) gives

$$\left(\partial^2 + \frac{g^2}{\pi}\right) E = 0, \tag{1.25}$$

which is the Klein-Gordon equation of a scalar field with mass $\mu^2 = g^2/\pi$. Therefore, in the massless one flavor Schwinger model, a boson of mass μ appears. This result has been generalized to an arbitrary number of massless flavors N [4], where a boson of mass $\mu^2 = Ng^2/\pi$ appears, along with massless bosons. For massive fermions no general solution exists, although there are several approaches. We will revise one of those approaches in Chapter 3. Deeper discussions of QED in 1+1 dimensions can be found in refs. [5, 6].

1.1 Confinement

As we mentioned before, the Schwinger model exhibits confinement. We can illustrate this fact by analyzing the classical equations of motion

$$\partial_\mu F^{\mu\nu} = J^\nu. \tag{1.26}$$

Let us fix the gauge by setting $A_0 = 0$ and suppose that we place a charge g at the origin, then

$$\partial_1 F^{10}(x) = g\delta(x) \Rightarrow \partial_x E(x) = g\delta(x) \Rightarrow E(x) = g\theta(x) + E_0, \tag{1.27}$$

where $\theta(x)$ is the Heaviside function and E_0 is a constant electric field. If we calculate the energy of this configuration, we see that it diverges

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\, E^2 \to \infty. \tag{1.28}$$

This means that the finite energy states must be charge neutral. Now, let us consider two charges $\pm g$ at $x = \mp L/2$. The equation of motion reads

$$\partial_x E(x) = g\,\delta\left(x + \frac{L}{2}\right) - g\,\delta\left(x - \frac{L}{2}\right) \Rightarrow E(x) = g\,\theta\left(x + \frac{L}{2}\right) - g\,\theta\left(x - \frac{L}{2}\right) + E_0. \tag{1.29}$$

If we set $E_0 = 0$, then the electric field is

$$E(x) = \begin{cases} g & |x| < \frac{L}{2} \\ 0 & \text{otherwise.} \end{cases} \tag{1.30}$$

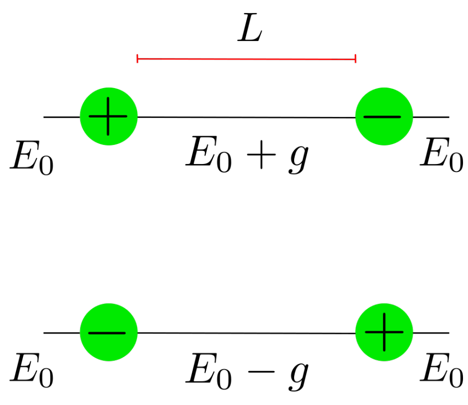

Figure 1.1: Electric field between an electron-positron pair in QED$_2$, considering the background field.

We can calculate the energy of this configuration

$$\frac{1}{2}\int_{-\infty}^{\infty} dx\, E^2 = \frac{1}{2}\int_{-L/2}^{L/2} dx\, g^2 = \frac{g^2 L}{2}. \tag{1.31}$$

We see that the energy grows linearly with the separation of the charges, illustrating confinement. This property holds at the quantum level as well [7].

1.2 Vacuum angle

If we do not fix the background field to zero, it is possible to generate electron-positron pairs when the difference of the energy between both particles together and the background field is smaller than zero

$$\Delta H = \frac{1}{2}\int_{-L/2}^{L/2} dx\, \left[E(x)^2 - E_0^2\right] < 0. \tag{1.32}$$

The electric field $E(x)$ between the particles is now given by (see figure 1.1)

$$E(x) = E_0 \pm g, \quad -\frac{L}{2} \leq x \leq \frac{L}{2}. \tag{1.33}$$

Pairs can be created when

$$\begin{aligned} \Delta H &= \frac{L}{2}\left(g^2 \pm 2gE_0\right) < 0 \\ &\Leftrightarrow \begin{cases} \frac{g}{2} < E_0 & \text{for } E(x) = E_0 - g \\ E_0 < -\frac{g}{2} & \text{for } E(x) = E_0 + g \end{cases} \\ &\Leftrightarrow \frac{g}{2} < |E_0|. \end{aligned} \tag{1.34}$$

In this context, the *vacuum angle* θ

$$\theta = \frac{2\pi E_0}{g} \tag{1.35}$$

is defined. Whenever $|\theta| > \pi$, pair production is favorable. $\theta = 0$ refers to confinement. This parameter was introduced to the Schwinger model by Coleman [8] and it adds the following term to the Lagrangian

$$\mathcal{L}_\theta = \frac{g\theta}{4\pi}\epsilon^{\mu\nu}F_{\mu\nu}. \tag{1.36}$$

We can rewrite $\epsilon^{\mu\nu}F_{\mu\nu}$ as

$$\epsilon^{\mu\nu}F_{\mu\nu} = \partial_\mu(2\epsilon^{\mu\nu}A_\nu), \tag{1.37}$$

which is a divergence. Therefore, $\mathcal{L}_\theta$ does not affect the equations of motion. In QCD a similar parameter appears.

1.3 Chiral symmetry breaking

As we will revise in a more detailed manner in Chapter 4, if one applies the chiral projection operators

$$P_L = \frac{\mathbb{I}-\gamma_5}{2}, \quad P_R = \frac{\mathbb{I}+\gamma_5}{2}, \tag{1.38}$$

to the ψ and $\overline{\psi}$ fields, we can transform the Lagrangian into

$$\begin{gathered}\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}_L\gamma^\mu(i\partial_\mu - gA_\mu)\psi_L + \overline{\psi}_R\gamma^\mu(i\partial_\mu - gA_\mu)\psi_R - m(\overline{\psi}_R\psi_L + \overline{\psi}_L\psi_R), \\ \psi_R = P_R\psi, \quad \psi_L = P_L\psi, \quad \overline{\psi}_R = \overline{\psi}P_L, \quad \overline{\psi}_L = \overline{\psi}P_R, \end{gathered} \tag{1.39}$$

which shows a global symmetry under the transformations

$$\psi_L \to \psi'_L = e^{i\varphi_L}\psi_L, \quad \overline{\psi}_L \to \overline{\psi}'_L = \overline{\psi}_L e^{-i\varphi_L}, \quad e^{i\varphi_L} \in \mathrm{U}(1)_L, \tag{1.40}$$

$$\psi_R \to \psi'_R = e^{i\varphi_R}\psi_R, \quad \overline{\psi}_R \to \overline{\psi}'_R = \overline{\psi}_R e^{-i\varphi_R}, \quad e^{i\varphi_R} \in \mathrm{U}(1)_R \tag{1.41}$$

when $m = 0$. However, the *chiral condensate*, *i.e.* the vacuum expectation value of $\overline{\psi}\psi$ transforms as

$$\langle\overline{\psi}'\psi'\rangle = \langle\overline{\psi}_R e^{i(\varphi_L-\varphi_R)}\psi_L + \overline{\psi}_L e^{i(\varphi_R-\varphi_L)}\psi_R\rangle. \tag{1.42}$$

We see that it is invariant only when $\varphi_L = \varphi_R$, so $\mathrm{U}(1)_L \otimes \mathrm{U}(1)_R$ breaks to $\mathrm{U}(1)_{L=R}$.

In the N-flavor Schwinger model with degenerate fermion mass m, it has been shown [9] that the chiral condensate has the following dependence on m and θ when $m/\mu \ll 1$

$$\langle\overline{\psi}\psi\rangle = -\frac{\mu}{4\pi}\left(2e^\gamma\cos\frac{\theta}{2}\right)^{\frac{2N}{N+1}}\left(\frac{m}{\mu}\right)^{\frac{N-1}{N+1}}, \quad \mu = \frac{Ng^2}{\pi} \tag{1.43}$$

where γ is the Euler-Mascheroni constant. For the one flavor model we can see that

$$\langle\overline{\psi}\psi\rangle = -\frac{\mu}{2\pi}e^\gamma\cos\frac{\theta}{2}, \tag{1.44}$$

i.e. there is no dependence on the fermion mass. Hence the chiral condensate is non vanishing when $m = 0$. We also observe from eq. (1.43) that when $N > 1$, there is no symmetry breaking in the massless Schwinger model, since $\langle\overline{\psi}\psi\rangle = 0$.

Bibliography

[1] J. Schwinger. Gauge Invariance and Mass. II. *Phys. Rev.*, 128:2425–2429, 1962.

[2] S. L. Adler. Axial vector vertex in spinor electrodynamics. *Phys. Rev.*, 177:2426–2438, 1969.

[3] J. S. Bell and R. Jackiw. A PCAC puzzle: $\pi^0 \to \gamma\gamma$ in the σ model. *Nuovo Cim. A*, 60:47–61, 1969.

[4] L. V. Belvedere, K. D. Rothe, B. Schroer, and J. A. Swieca. Generalized Two-dimensional Abelian Gauge Theories and Confinement. *Nucl. Phys. B*, 153:112–140, 1979.

[5] W. Dittrich and M. Reuter. *Selected Topics in Gauge Theories*. Springer, 1986.

[6] D. Tong. Lectures on Gauge Theory. Cambridge. `http://www.damtp.cam.ac.uk/user/tong/gaugetheory/gt.pdf`, 2018. Accessed: 2021-03.

[7] S. R. Coleman, R. Jackiw, and L. Susskind. Charge Shielding and Quark Confinement in the Massive Schwinger Model. *Ann. Phys.*, 93:267, 1975.

[8] S. R. Coleman. More About the Massive Schwinger Model. *Ann. Phys.*, 101:239, 1976.

[9] J. E. Hetrick, Y. Hosotani, and S. Iso. The Interplay between mass, volume, vacuum angle and chiral condensate in N flavor QED in two-dimensions. *Phys. Rev. D*, 53:7255–7259, 1996.