Chapter 5

The Witten-Veneziano formula

In this chapter we attempt to compute F_π by means of a different method and we compare the result with the value obtained in the last chapter. The method relies on the Witten-Veneziano formula [1,2], which in 3-flavor QCD relates the mass of the η' meson with m_η, m_π, F_π and the *topological susceptibility* χ_T, defined below. This formula is obtained from the leading order term of a $1/N_c$ expansion, for large N_c (number of colors). The valence quark composition of η and η' is approximately

$$\eta \approx \frac{1}{\sqrt{6}}\left(\overline{u}u + \overline{d}d - 2\overline{s}s\right), \quad \eta' \approx \frac{1}{\sqrt{3}}\left(\overline{u}u + \overline{d}d + \overline{s}s\right). \tag{5.1}$$

In nature the η and η' states are mixed, so the actual composition is given in terms of a *mixing angle* θ_P

$$\begin{pmatrix}\eta \\ \eta'\end{pmatrix} = \begin{pmatrix}\cos\theta_P & -\sin\theta_P \\ \sin\theta_P & \cos\theta_P\end{pmatrix}\begin{pmatrix}\eta_8 \\ \eta_1\end{pmatrix}, \tag{5.2}$$

where

$$\eta_8 = \frac{1}{\sqrt{6}}\left(\overline{u}u + \overline{d}d - 2\overline{s}s\right), \quad \eta_1 = \frac{1}{\sqrt{3}}\left(\overline{u}u + \overline{d}d + \overline{s}s\right). \tag{5.3}$$

η_8 belongs to an octet of states, while η_1 to a singlet. The measured value of θ_P is $-11.3°$ [3].

The Witten-Veneziano formula in QCD reads

$$m_{\eta'}^2 - \frac{1}{2}m_\eta^2 - \frac{1}{2}m_\pi^2 = \frac{6}{F_{\eta'}^2}\chi_T^{\text{que}}. \tag{5.4}$$

where $F_{\eta'}$ is the decay constant of the meson η' and "que" stands for quenched, *i.e.* its value when the fermion mass $m \to \infty$. According to ref. [1], to lowest order in a $1/N_c$ expansion, we have $F_{\eta'} = F_\pi$ in QCD. In the limit of massless fermions, eq. (5.4) is simplified for the Schwinger model[1] [4,5]

$$m_\eta^2 = \frac{2N}{F_\eta^2}\chi_T^{\text{que}}, \tag{5.5}$$

where N is the number of flavors. Nevertheless, in this case the literature is not clear whether the approximation $F_\eta = F_\pi$ is valid. Still, we assume that the relation between the decay constants holds and we attempt to compute F_π in the Schwinger model with eq. (5.5).

[1] According to the literature [4], the most suitable analogy of the heaviest meson of the Schwinger model with QCD would be η'; however, it is not the actual η' particle from QCD and for simplicity we will denote it as η.

χ_T is defined for the Euclidean Schwinger model in the continuum as

$$\chi_T = \int d^2x \, \langle q(x)q(0)\rangle, \tag{5.6}$$

where

$$q(x) = \frac{g}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu}(x) = \frac{g}{2\pi}F_{12}(x) \tag{5.7}$$

is the *topological charge density.* With $q(x)$ we define the *topological charge* as

$$Q = \int d^2x \, q(x). \tag{5.8}$$

We can formulate χ_T in terms of Q as well

$$\chi_T = \frac{\langle Q^2\rangle - \langle Q\rangle^2}{V}, \tag{5.9}$$

where V is the space-time volume. An important property of the topological charge is that it is an integer number. We can see that fact if we rewrite $q(x)$ as a total divergence

$$q(x) = \partial_\mu \Omega_\mu(x), \quad \Omega_\mu(x) = \frac{g}{2\pi}\epsilon_{\mu\nu}A_\nu(x). \tag{5.10}$$

If we consider field configurations of finite action, $F_{\mu\nu}(x)$ has to vanish at infinity, so the gauge field must be gauge equivalent to 0 when $|x| \to \infty$

$$0 = A'_\mu(x) = A_\mu(x) - \frac{1}{g}\partial_\mu\varphi(x). \tag{5.11}$$

Then

$$Q = \int d^2x \, \partial_\mu \left(\frac{g}{2\pi}\epsilon_{\mu\nu}\frac{1}{g}\partial_\nu\varphi(x)\right) = \frac{1}{2\pi}\int_{\partial\mathbb{R}^2} d\sigma_\mu \, \epsilon_{\mu\nu}\partial_\nu\varphi(x), \tag{5.12}$$

where we have used the Gauss theorem. Now, if we consider a circumference of length L, we identify Q with the following integral

$$\lim_{L\to\infty}\frac{1}{2\pi i}\int_0^L dx \, U^*(x)\partial_x U(x), \quad \text{where} \quad U(x) = e^{i\varphi(x)}, \quad U(L) = U(0). \tag{5.13}$$

The last expression is equal to

$$\frac{1}{2\pi}\left[\varphi(L) - \varphi(0)\right] = n \in \mathbb{Z}, \tag{5.14}$$

hence Q is an integer.

As we mentioned in Chapter 3, we can relate m_η with the gauge coupling as follows

$$m_\eta^2 = N\frac{g^2}{\pi}. \tag{5.15}$$

Thus, by determining χ_T we obtain a value for F_π. According to ref. [5], the theoretical expression for χ_T^{que} in infinite volume and in the continuum is

$$\chi_T^{\text{que}} = \frac{g^2}{4\pi^2} \simeq 0.0253\, g^2. \tag{5.16}$$

On the other hand, to measure the topological susceptibility by using lattice simulations we have to discretize the topological charge density. This can be done through the plaquette

Ref.	$\chi_T^{\rm que}/g^2$	F_π
[5]	0.0253	0.3989
[8]	0.023	0.3801
[9]	0.0300(8)	0.4342(58)

Table 5.1: Topological susceptibility in the literature measured for $\beta = 1/g^2 = 4$. The results from refs. [8,9] were obtained by means of lattice simulations, while in ref. [5] $\chi_T^{\rm que}$ corresponds to eq. (5.16).

variables defined in Chapter 2. From eq. (2.96), we know that for a small lattice spacing a, the plaquettes have the following expression

$$U_{\mu\nu}(\vec{n}) = e^{iga^2 F_{\mu\nu}(\vec{n})}. \tag{5.17}$$

Then

$$F_{\mu\nu}(\vec{n}) = -\frac{i}{ga^2}\ln U_{\mu\nu}(\vec{n}). \tag{5.18}$$

That way, we have

$$q(\vec{n}) = -\frac{i}{2\pi a^2}\ln U_{12}(\vec{n}) \tag{5.19}$$

and

$$Q = \sum_{\vec{n}\in L} a^2 q(\vec{n}), \tag{5.20}$$

where $L = \{\vec{n} = (n_1, n_2)|\, n_\mu = 0, 1, \ldots, N_\mu - 1; \mu = 1, 2\}$ is the set of lattice sites.

The lattice configurations generated through Monte Carlo algorithms are sorted in different sectors, where each one is characterized by a topological charge. Furthermore, there is evidence (see e.g. refs. [6, 7]) that the distribution of these configurations corresponds approximately to a Gaussian function. Due to parity symmetry, we also have that

$$\langle Q\rangle = 0. \tag{5.21}$$

Then, one can calculate χ_T on the lattice using the following weighted average

$$\chi_T = \frac{\sum_i Q_i^2\, N_i}{V\sum_i N_i}, \tag{5.22}$$

where i denotes a sector with N_i configurations labeled by a topological charge Q_i.

In the last chapter we showed results of Q obtained with simulations for several lattice sizes, using low statistics (10^3 measurements separated by 10 sweeps). We attempted to compute the topological susceptibility using those results. Unfortunately, even though the topological charge is compatible with $\langle Q\rangle = 0$, χ_T as a function of the fermion mass m does not have a clear behavior (see figure 5.1 for instance). This does not allow us to perform a fit and to extrapolate to the quenched value of χ_T. For that reason, we incremented the number of measurements to 10^4, separated by 100 sweeps, and simulated a 10×64 lattice for $\beta = 4$. This improved the results. In figure 5.2 we show the distribution of the configurations and in figure 5.3 we show the topological susceptibility as a function of the degenerate fermion mass. We used two functions to extrapolate χ_T, from an average we obtain

$$\chi_T^{\rm que} = 0.029(1)g^2. \tag{5.23}$$

Now, we substitute eq. (5.15) in eq. (5.5) and solve for F_π

$$F_\pi^2 = \chi_T^{\rm que}\frac{2\pi}{g^2}. \tag{5.24}$$

Using the result in eq. (5.23) yields

$$F_\pi = 0.4243(76). \tag{5.25}$$

χ_T has been obtained in the Schwinger model before. In table 5.1 we show some values available in the literature together with F_π computed by means of eq. (5.24). We observe that the topological susceptibility value in the literature is in the range 0.023-0.030 for $\beta = 4$, so our result is consistent.

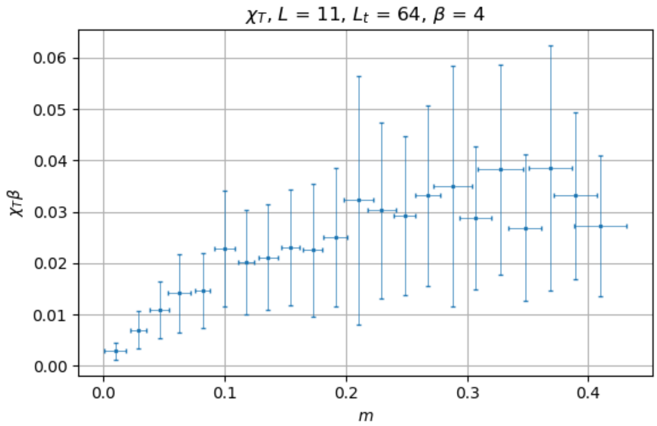
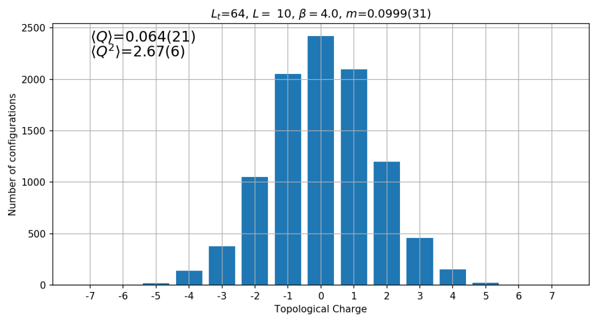
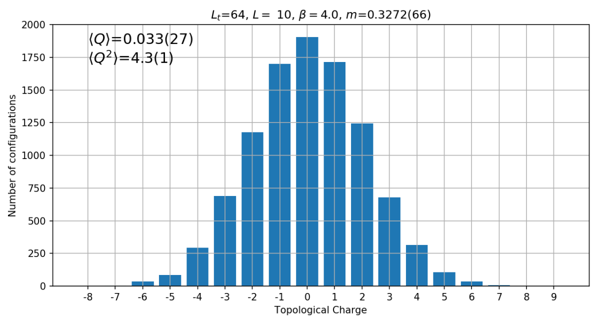


Figure 5.1: Topological susceptibility as a function of the fermion mass m, computed for 10^3 measurements with 10 sweeps between each of them on a 11×64 lattice. χ_T does not have a clear behavior for this amount of measurements.

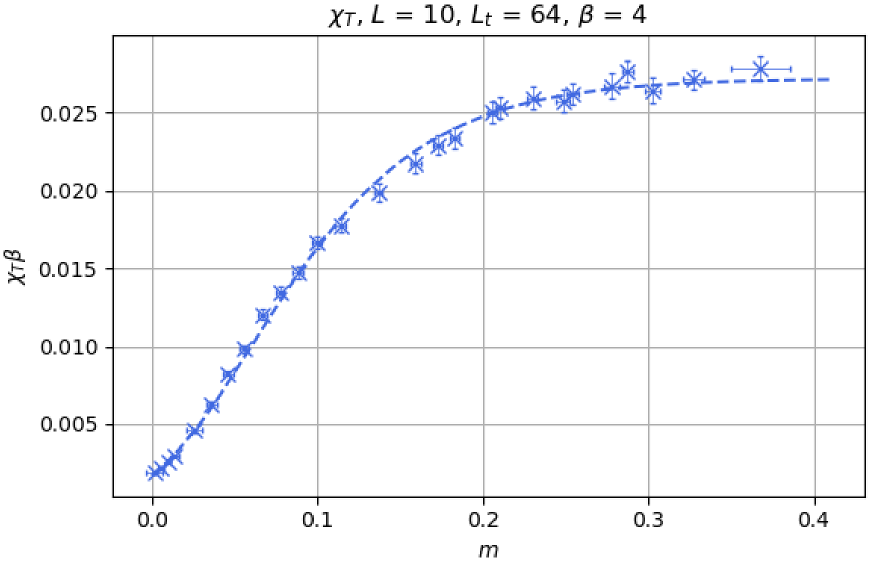


(a) Configurations sorted by their topological charge for $m = 0.0999(31)$.

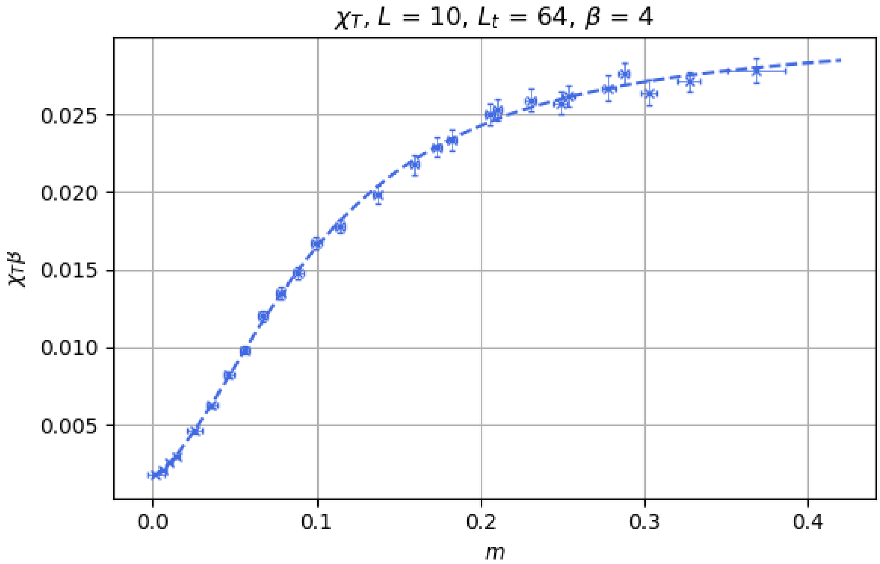


(b) Configurations sorted by their topological charge for $m = 0.3272(66)$.

Figure 5.2: Distribution of the Monte Carlo configurations in different topological sectors for $\beta = 4$. We see approximately a Gaussian distribution. m denotes the degenerate PCAC fermion mass. When the mass is smaller, the configurations occupy less topological sectors.



(a) A function of the form $y = ae^{-be^{-cx}}$ was fitted to the data.



(b) We also fitted a function of the form $y = \frac{a+bx+cx^2}{d+fx+gx^2}$.

Figure 5.3: Topological susceptibility as a function of the degenerate fermion mass obtained with 10^4 measurements. The plots are in lattice unites. We performed two fits in order to extract the value of χ_T when $m \to \infty$. The results yield $\chi_T^{\text{que}}/g^2 = \chi_T^{\text{que}}\beta = 0.029(1)$.

We also tried to verify that this result is independent of β. To do so, we performed more simulations to determine χ_T in the quenched approximation by working with pure gauge theory, *i.e.* by generating Monte Carlo configurations using only the gauge action

$$S_G = \frac{1}{4}\int d^4x\, F_{\mu\nu}F_{\mu\nu}. \tag{5.26}$$

This is more convenient than extrapolating χ_T to infinite m, because the simulations are faster and they yield results for $m \to \infty$. Still, the extrapolation of χ_T to infinite m works as a cross-check with the results of $\beta = 4$ that we obtain with the quenched simulations.

In figure 5.3, we show $\chi_T^{\text{que}}\beta$ for different lattices of dimension $L \times L$ and $\beta = 2, 3, 4$ and 5. We took 10^4 measurements separated by 10 sweeps for $\beta = 2, 3$ and 10^4 measurements separated by 100 sweeps for $\beta = 4$ and 5. In table 5.2 we show $\chi_T^{\text{que}}\beta$ for the different β's that we simulated, together with F_π computed with the Witten-Veneziano formula.

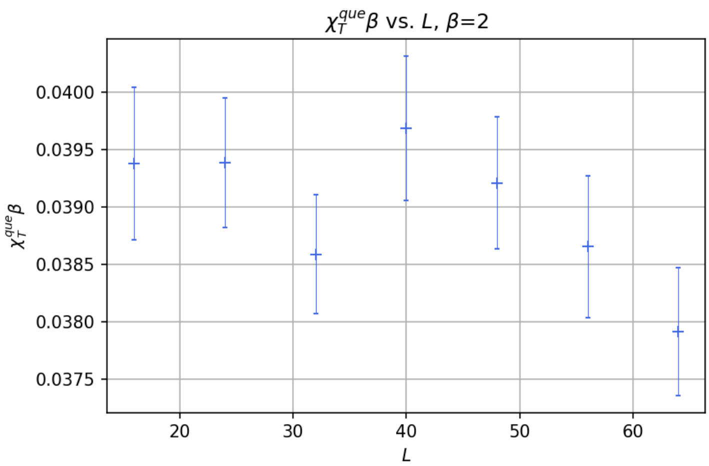
β	$\chi_T^{\text{que}}\beta$	F_π
2	0.0389(2)	0.495(1)
3	0.0335(3)	0.459(2)
4	0.0304(2)	0.437(1)
5	0.0285(4)	0.423(3)

Table 5.2: Results of $\chi_T^{\text{que}}\beta$ and F_π for different β's obtained with pure gauge theory simulations.

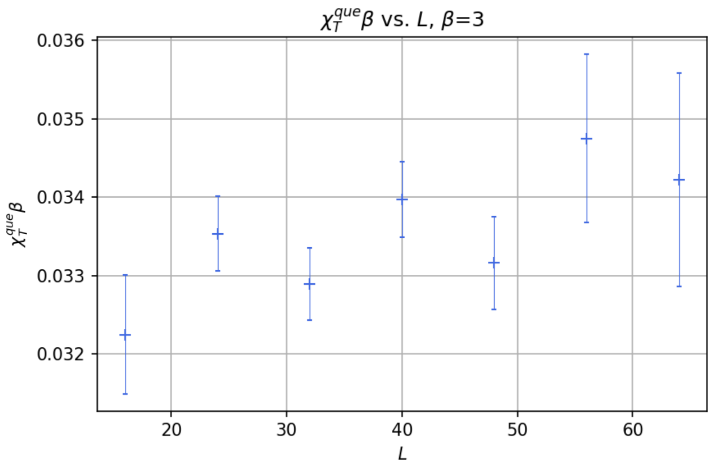
We expected $\chi_T^{\text{que}}\beta$ on the lattice to be independent from β, as in the δ-regime, but we observe that it is not the case. As β increases, $\chi_T^{\text{que}}\beta$ decreases. For $\beta = 4$ the number that we obtained by means of the quenched simulations is compatible, within errors, with the extrapolation that we performed before. In figure 5.4, we show a comparison of our results for $\chi_T^{\text{que}}\beta$ with the values of refs. [5, 9]. We see that the lattice results are above the theoretical prediction, given by eq. (5.16), and they seem to converge to it for large β, which is the continuum limit. Since $\chi_T^{\text{que}}\beta$ is not independent from β, F_π is not independent either. Still, we can perform an extrapolation to the continuum limit by fitting the ansatz $\chi_T^{\text{que}}\beta = a + b/\beta$, where a and b are fit parameters, in order to determine F_π. The extrapolation yields

$$\frac{\chi_T^{\text{que}}}{g^2} = 0.0219(3), \quad F_\pi = 0.3707(23). \tag{5.27}$$

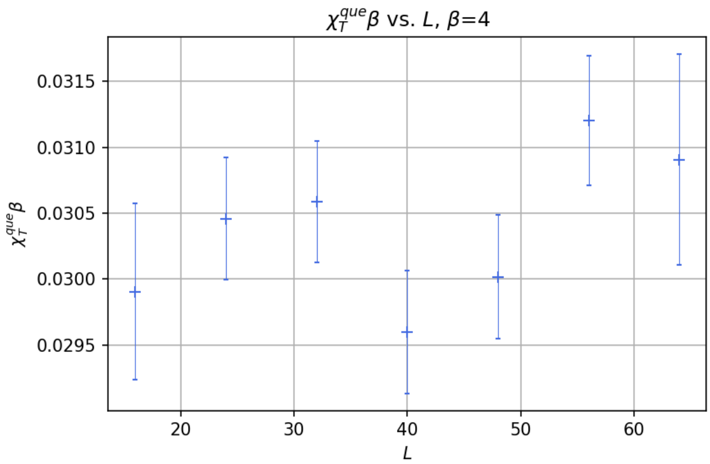
The result of eq. (5.27) is slightly below the theoretical prediction for infinite volume and in the continuum, given by eq. (5.16). Nevertheless, when we compare F_π with the value that we obtained in the δ-regime: $F_\pi = 0.6688(5)$, we see an inconsistency between the two methods used to measure the decay constant. Furthermore, in the δ-regime the result was independent of β, in contrast to the outcome of this chapter. However, let us remember that the actual decay constant involved in the Witten-Veneziano formula is the η. In the Schwinger model, the validity of the approximation $F_\eta = F_\pi$ is not justified in the literature, so we cannot assure that we actually measured F_π in two dimensions with eq. (5.24).



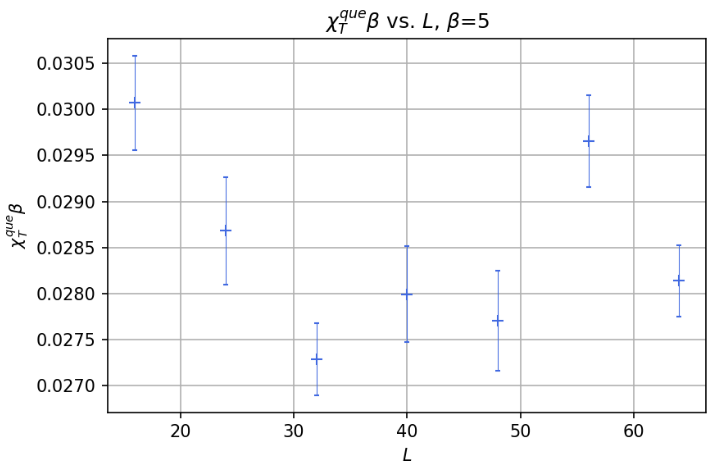
(a) $\chi_T^{\text{que}}\beta$ vs. L for $\beta = 2$. An average yields $\chi_T^{\text{que}} = 0.0389(2)$.



(b) $\chi_T^{\text{que}}\beta$ vs. L for $\beta = 3$. An average yields $\chi_T^{\text{que}} = 0.0335(3)$.



(c) $\chi_T^{\text{que}}\beta$ vs. L for $\beta = 4$. An average yields $\chi_T^{\text{que}} = 0.0304(2)$.



(d) $\chi_T^{\text{que}}\beta$ vs. L for $\beta = 4$. An average yields $\chi_T^{\text{que}} = 0.0285(4)$.

Figure 5.3: $\chi_T\beta$ measured for different β and lattices of dimensions $L \times L$

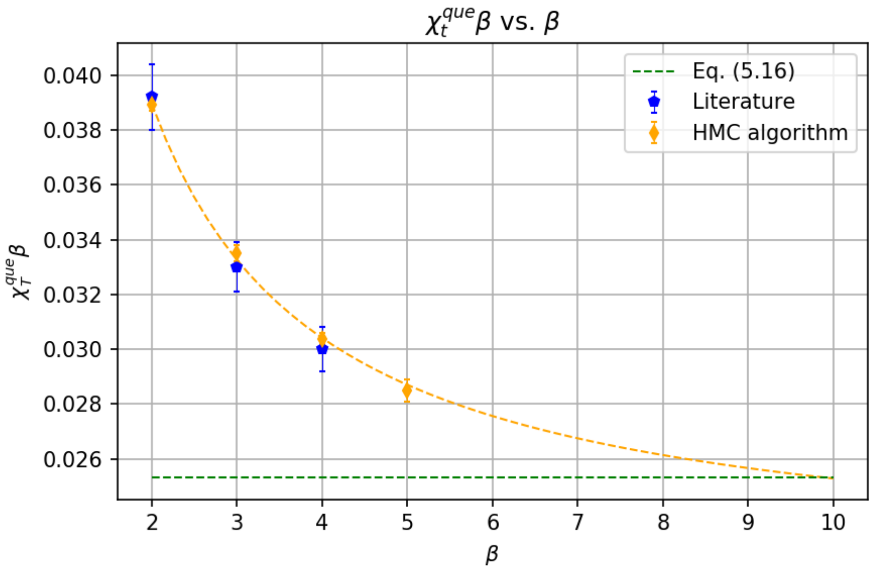


Figure 5.4: $\chi_T^{\text{que}}\beta$ vs. β. The literature values correspond to ref. [9]. HMC algorithm denotes the results that we computed with pure gauge theory simulations. In order to determine $\chi_T^{\text{que}}\beta$ in the continuum we fitted a function of the form $y = a + b/x$, which yielded $\chi_T^{\text{que}}\beta = 0.0219(3)$.

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