

Chapter 4

Chiral Perturbation Theory in QCD

As it was mentioned in chapter 2, the low energy level of QCD cannot be treated with perturbation theory, but one can use lattice simulations. The analytic approach to this regime is the *Chiral Perturbation Theory*, which is an effective field theory. The Lagrangian of this effective theory is built by adding all the terms that are consistent with the symmetries of the underlying theory, that in this case is QCD. This lead us to analyze the chiral symmetry of the QCD Lagrangian.

4.1 QCD chiral symmetry

Since we are interested in low energy, we will work with the flavors whose masses satisfy $m_f \ll \Lambda_{\text{QCD}} \approx 300$ MeV. This means that we only take into account the u and d quarks, with masses in the following range [1]

$$\begin{aligned} m_u &= 1.7 - 3.33 \text{ MeV}, \\ m_d &= 4.1 - 5.8 \text{ MeV}. \end{aligned} \quad (4.1)$$

Then, in Minkowski space-time, the Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \sum_{f=u,d} (\bar{q}_f i \gamma^\mu D_\mu q_f - m_f \bar{q}_f q_f) - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}, \\ D_\mu &= (\partial_\mu + ig A_\mu), \quad G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + ig [A_\mu^a, A_\nu^a]. \end{aligned} \quad (4.2)$$

q_f is the quark field of the f flavor, $\bar{q}_f = q_f^\dagger \gamma^0$, $A_\mu^a(x)$ are real valued fields related to $A_\mu(x)$ by $A_\mu(x) = \sum_{a=1}^8 A_\mu^a(x) T_a$, where T_a are the basis elements of the traceless Hermitian 3×3 matrices. This implies that the gauge field is a matrix as well. This Lagrangian is constructed in the same way we did in section 2.3, so it is gauge invariant under transformations of SU(3). We are interested in analyzing the chiral symmetry, therefore we apply the chiral operators

$$P_R = \frac{1}{2}(\mathbb{I} + \gamma_5), \quad P_L = \frac{1}{2}(\mathbb{I} - \gamma_5) \quad (4.3)$$

to the quark fields in order to obtain the right-handed and left-handed fields

$$q_{Rf} = P_R q_f, \quad q_{Lf} = P_L q_f. \quad (4.4)$$

By using $\{\gamma_5, \gamma^\mu\} = 0$ and the definition of the chiral operators one can prove the following properties

$$P_{L,R} = P_{L,R}^\dagger, \quad P_{L,R}^2 = P_{L,R}, \quad P_R P_L = P_L P_R = 0, \quad P_{R,L} \gamma^\mu = \gamma^\mu P_{L,R}. \quad (4.5)$$

This implies

$$\begin{aligned}
q_f &= q_{R_f} + q_{L_f}, \\
\bar{q}_{R_f} &= q_{R_f}^\dagger \gamma^0 = q_f^\dagger P_R \gamma^0 = q_f^\dagger \gamma^0 P_L = \bar{q}_f P_L, \\
\bar{q}_{L_f} &= \bar{q}_f P_R, \\
\bar{q}_f &= \bar{q}_{L_f} + \bar{f}_{L_f}.
\end{aligned} \tag{4.6}$$

With eqs. (4.5) and eqs. (4.6) the QCD Lagrangian takes the form

$$\mathcal{L} = \sum_{f=u,d} \left[\bar{q}_{L_f} i \gamma^\mu D_\mu q_{L_f} + \bar{q}_{R_f} i \gamma^\mu D_\mu q_{R_f} - m_f (\bar{q}_{R_f} q_{L_f} + \bar{q}_{L_f} q_{R_f}) \right] - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}. \tag{4.7}$$

Now, we use global chiral transformations defined by

$$q_{R,L_f} \rightarrow q'_{R,L_f} = e^{i\alpha\gamma_5} q_{R,L_f}, \quad \bar{q}_{R,L_f} \rightarrow \bar{q}'_{R,L_f} = \bar{q}_{R,L_f} e^{i\alpha\gamma_5}, \quad \alpha \in \mathbb{R}. \tag{4.8}$$

Let us note that the mass term is not invariant under this transformations, so for the moment we will take $m = 0$. On the other hand, since $\{\gamma^\mu, \gamma_5\} = 0$ it follows that $\gamma^\mu e^{i\alpha\gamma_5} = e^{-i\alpha\gamma_5} \gamma^\mu$ and as a result the Lagrangian transforms as

$$\begin{aligned}
\mathcal{L} &= \sum_{f=u,d} \left(\bar{q}_{L_f} i \gamma^\mu D_\mu q_{L_f} + \bar{q}_{R_f} i \gamma^\mu D_\mu q_{R_f} \right) - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu} \\
&= \sum_{f=u,d} \left(\bar{q}'_{L_f} i \gamma^\mu D_\mu q'_{L_f} + \bar{q}'_{R_f} i \gamma^\mu D_\mu q'_{R_f} \right) - \frac{1}{4} G_{\mu\nu}^a G_a^{\mu\nu}.
\end{aligned}$$

Hence, we can see that it is invariant under the transformations (4.8) only when $m = 0$, for that reason this is called the chiral limit. If we write the quark fields as vectors

$$q_{R,L} = \begin{pmatrix} q_{R,L_u} \\ q_{R,L_d} \end{pmatrix}, \quad \bar{q}_{R,L} = \begin{pmatrix} \bar{q}_{R,L_u} \\ \bar{q}_{R,L_d} \end{pmatrix}, \quad q = \begin{pmatrix} q_u \\ q_d \end{pmatrix}, \quad \bar{q} = \begin{pmatrix} \bar{q}_u \\ \bar{q}_d \end{pmatrix}, \tag{4.9}$$

it can be seen that the Lagrangian has a global symmetry $U(2)_L \otimes U(2)_R$. An element of $U(2)$ can be decomposed in an element of $SU(2)$ multiplied by a phase factor, thus, we have the following symmetry group for the massless Lagrangian

$$U(2)_L \otimes U(2)_R \leftrightarrow SU(2)_L \otimes SU(2)_R \otimes U(1)_B \otimes U(1)_A. \tag{4.10}$$

$U(1)_A$ is known as axial symmetry and is broken explicitly under quantization. $U(1)_B$ is associated with the conservation of the Baryon number. Meanwhile, $SU(2)_L \otimes SU(2)_R$ is the chiral symmetry group. The latter breaks spontaneously to $SU(2)$; the order parameter of this broken symmetry is the chiral condensate $\langle 0 | \bar{q} q | 0 \rangle$, so when it is different from zero the chiral symmetry is indeed spontaneously broken. Because of the Goldstone theorem, to this broken symmetry corresponds $2^2 - 1 = 3$ massless Nambu-Goldstone Bosons (NGB). If now we take into account the masses m_u and m_d , the symmetry is explicitly broken and the NGBs turn into light massive quasi NGBs, which can be related to the pion triplet π^+, π^-, π^0 , because they are the lightest hadrons [1, 2, 3].

4.2 Effective Lagrangian

In order to build the effective Lagrangian one uses the dynamic variables suitable for low-energy, for two quark flavors this means that one replaces the quark and gluon fields by pion fields: $\vec{\pi} = \{\pi_1(x), \pi_2(x), \pi_3(x)\}$. Now one introduces a field $U(x) \in \text{SU}(2)$

$$U(x) = \exp\left(i \frac{\vec{\pi} \cdot \vec{\tau}}{F_\pi}\right), \quad \vec{\tau} = (\sigma_1, \sigma_2, \sigma_3), \quad (4.11)$$

that changes under global transformations of $\text{SU}(2)_L \otimes \text{SU}(2)_R$ as

$$U(x) \rightarrow U'(x) = \Omega_R U(x) \Omega_L^{-1}, \quad \Omega_R \in \text{SU}(2)_R, \quad \Omega_L \in \text{SU}(2)_L, \quad (4.12)$$

F_π is known as the pion decay constant and makes the argument of the exponential dimensionless; σ are the Pauli matrices, which are the generators of $\text{SU}(2)$, so one can write [1]

$$U(x) = \exp\left(i \frac{\phi(x)}{F_\pi}\right), \quad \phi(x) = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix},$$

$$\pi^0 = \pi_3, \quad \pi^\pm = \frac{\pi_1 \mp i\pi_2}{\sqrt{2}}. \quad (4.13)$$

The effective Lagrangian \mathcal{L}_{eff} is constructed by using $U(x)$ and it must have all the terms that present the symmetries of QCD. Each term is accompanied by a number denominated low energy constant (LEC), that has to be determined from the underlying theory. However, there are actually an infinite number of terms consistent with the symmetries, they can be organized in increasing powers of momentum, which is the same as increasing number of derivatives, so one can truncate them. For a massless case, the effective Lagrangian with the least number of derivatives that can be written is

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4} \text{tr} \left(\partial_\mu U \partial^\mu U^\dagger \right). \quad (4.14)$$

The factor of 4 is because if one expands in powers of ϕ one obtains the kinetic term $\frac{1}{2} \text{tr}(\partial_\mu \phi \partial^\mu \phi)$. Let us note that under the transformation of eq. (4.12) the Lagrangian is invariant

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4} \text{tr} \left(\Omega_R \partial_\mu U \Omega_L^{-1} \Omega_L \partial^\mu U^\dagger \Omega_R^{-1} \right) = \frac{F_\pi^2}{4} \text{tr} \left(\Omega_R^{-1} \Omega_R \partial_\mu U \partial^\mu U^\dagger \right) = \frac{F_\pi^2}{4} \text{tr} \left(\partial_\mu U \partial^\mu U^\dagger \right), \quad (4.15)$$

we have used the property $\text{tr}(AB) = \text{tr}(BA)$. Thus, \mathcal{L}_{eff} is indeed chiral invariant.

If one introduces the masses m_u and m_d to the theory, a term that explicitly breaks the chiral symmetry is added

$$\mathcal{L}_{\text{s.b.}} = \frac{F_0^2 B_0}{2} \text{tr} \left(M U^\dagger - U M \right), \quad M = \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix}, \quad (4.16)$$

where s.b. stands for symmetry breaking and F_0 and B_0 are two LECs. Then, to leading order the effective Lagrangian with mass reads

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{4} \text{tr} \left(\partial_\mu U \partial^\mu U^\dagger \right) + \frac{F_0^2 B_0}{2} \text{tr} \left(M U^\dagger - U M \right). \quad (4.17)$$

For degenerate quark masses $m_u = m_d \equiv m$, the LECs satisfy the following relations [4]

$$F_\pi = F_0 [1 + O(m)],$$

$$\langle 0 | \bar{q} q | 0 \rangle = -2F_0^2 B_0 [1 + O(m)],$$

$$m_\pi = \sqrt{2B_0 m} [1 + O(m)]. \quad (4.18)$$

At the chiral limit $F_\pi = F_0$.

The effective Lagrangian can be formulated in terms of a normalized field $\vec{S}(x) \in \text{O}(4)$ as well, since there is a local isomorphism between $\text{O}(4)$ and $\text{SU}(2)_L \otimes \text{SU}(2)_R$. The symmetry breaking pattern also occurs

$$\text{O}(4) \rightarrow \text{O}(3) \leftrightarrow \text{SU}(2)_L \otimes \text{SU}(2)_R. \quad (4.19)$$

The \mathcal{L}_{eff} with less number of derivatives reads

$$\mathcal{L}_{\text{eff}} = \frac{F_\pi^2}{2} \partial_\mu \vec{S} \cdot \partial_\mu \vec{S}. \quad (4.20)$$

Now, we introduce an external field \vec{H} that explicitly breaks the symmetry by adding the term

$$\mathcal{L}_{\text{s.b.}} = -\Sigma \vec{H} \cdot \vec{S}, \quad (4.21)$$

where $\Sigma \equiv -\langle 0 | \bar{q}q | 0 \rangle$. \vec{H} plays the same role as the quark mass. This is known as the non-linear σ model [5].

4.3 Chiral Perturbation Theory regimes

From eq. (4.17) it can be seen that the leading order LECs are F_π , B_0 and F_0 . These constants can be determined through lattice simulations by measuring the chiral condensate, the pion and degenerate quark masses and fitting the functions in eqs. (4.18). Nevertheless, one cannot simulate an infinite volume, so three regimes where there is a finite volume in Euclidean space $V = L^d$ (note that we are taking the Euclidean time extension to be equal to the spatial volume), with d the dimension, have been established. For each one of them, eqs. (4.18) can be different, but the LEC's are the same. Let us briefly review the regimes:

- The first regime that one can discuss is the *p-regime*, which consists of a large volume compared to the correlation length: $L \gg \xi = m_\pi^{-1}$; besides, the finite volume corrections are suppressed by a factor proportional to $\exp(-m_\pi L)$ [4, 6], so one can expect eqs. (4.18) to be valid.
- If $L \lesssim \xi$ then we refer to the so-called *ϵ -regime*. Here the finite volume corrections cannot be neglected. In this regime the chiral condensate has the following dependence on the LECs, m and the volume when $B_0 m L^2 \ll 1$ [7]

$$\langle 0 | \bar{q}q | 0 \rangle = -2F_0^2 B_0 \left(\frac{I_1'(F_0^2 B_0 m V)}{I_1(F_0^2 B_0 m V)} - \frac{1}{F_0^2 B_0 m V} \right), \quad (4.22)$$

where I_1 is a modified Bessel function of first kind. When one takes the infinite volume limit, the chiral condensate yields $\langle 0 | \bar{q}q | 0 \rangle = -2F_0^2 B_0$, recovering the second eq. of (4.18). On the other hand, when $m \rightarrow 0$, eq. (4.22) yields [6]

$$\langle 0 | \bar{q}q | 0 \rangle = -\frac{1}{2} F_0^4 B_0^2 m V. \quad (4.23)$$

We can see that at the chiral limit $\langle 0 | \bar{q}q | 0 \rangle$ vanishes, this is consistent with the fact that in finite volume there is no spontaneous symmetry breaking.

- Finally, the *δ -regime* is determined by a volume of size $V = L^3 \times L_t$, where the spatial volume is small, but the Euclidean time extension is large, that is $L \lesssim \xi \ll L_t$.

The pion decay constant has already been calculated in the p -regime and the ϵ -regime with 3 flavors several times, giving a result of $F_\pi = 92.1(9)$ MeV [8, 9, 10, 11], meanwhile, the δ -regime is less explored; there F_π has been measured with two flavors obtaining $F_\pi = 78_{-10}^{+14}$ MeV [12].

This last regime is useful from a technical point of view, because the small volume reduces the computing time of the simulations. Besides, the fact that $L \lesssim \xi$ has several effects. First, at the chiral limit the pion mass does not become massless and instead there is a *residual mass* $m_\pi^R \neq 0$. Another consequence is that there is approximately only one dimension, that enables us to treat the system as a quasi one dimensional field theory, *i.e.* quantum mechanics [13]. Then, $\vec{S}(x)$ describes approximately a particle moving in a unit sphere and one can express m_π^R as the energy gap of a quantum rotor, which is given by

$$E_j = \frac{j(j + N - 2)}{2\Theta}, \quad (4.24)$$

where Θ is the moment of inertia and N stands for the $O(N)$ group. The value of Θ was computed in ref. [14] for a general dimension d and N

$$\Theta = F_\pi^2 L^{d-1} \left[1 + \frac{N-2}{4\pi F_\pi^2 L^{d-2}} \left(2 \frac{d-1}{d-2} + \dots \right) \right]. \quad (4.25)$$

m_π^R is obtained by substituting $j = 1$ and $N = 4$ in eq. (4.24)

$$m_\pi^R = \frac{3}{2\Theta}. \quad (4.26)$$

In four dimensions we have

$$m_\pi^R = \frac{3}{2F_\pi^2 L^3 (1 + \Delta)}, \quad \Delta = \frac{0.477 \dots}{(F_\pi L)^2} + \dots, \quad d = 4; \quad (4.27)$$

however, let us note that in two dimensions there is a divergence of the next-to-leading term, so instead we just consider the leading term. This yields

$$m_\pi^R \simeq \frac{3}{2F_\pi^2 L}, \quad d = 2. \quad (4.28)$$

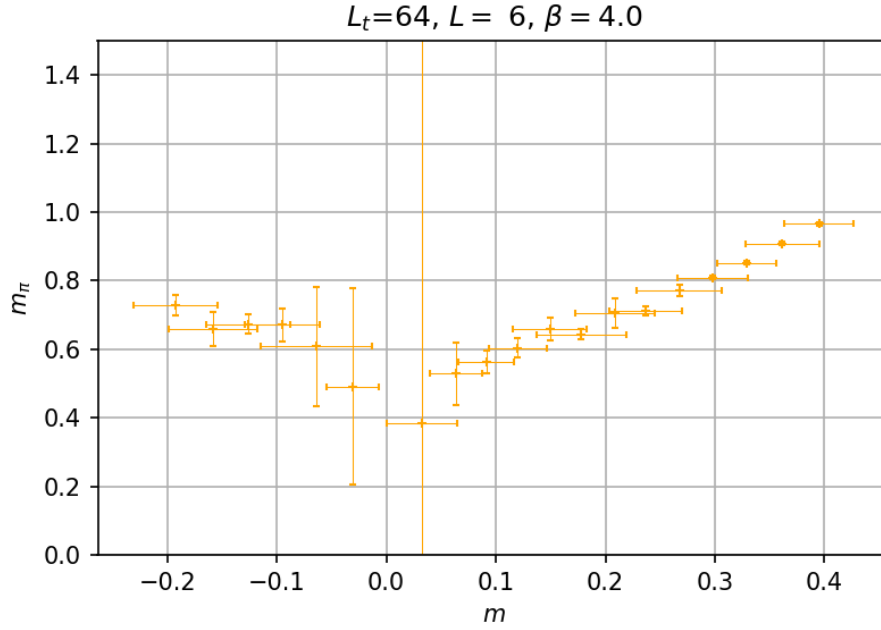
This dimension has not been considered in this context, because there are no NGBs [15]. Still, at finite fermion mass the lightest particles are similar to quasi NGBs and we will refer to them as pions. Note that when $d = 2$, F_π does not have units.

Several simulations of the two flavor Schwinger model were performed in the δ -regime with the HMC algorithm. This allows to obtain m_π^R as a function of L , in order to revise eq. (4.28) and extract the value of F_π .

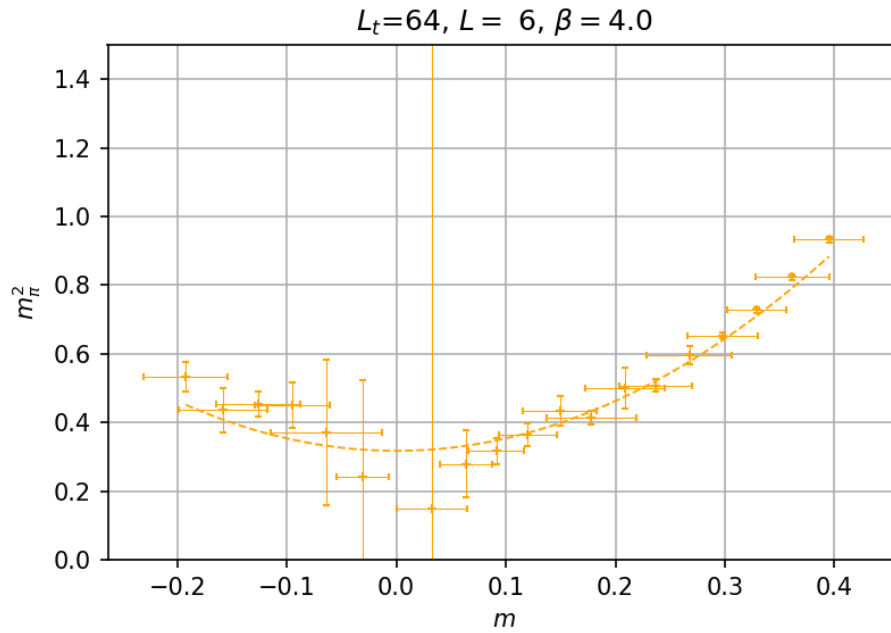
4.4 δ -regime results

To acquire results for the residual pion mass, first one has to fix a value for the gauge coupling constant g . As in chapter 3, we will denote $\beta = 1/g^2$; most of the work was done for $\beta = 4$, although results for $\beta = 2$ were obtained as well to revise the independence of F_π on this parameter. In fig. 4.1 (a), m_π is shown as a function of the degenerate quark mass m for $L = 6$ and a Euclidean time extension $L_t = 64$. We can see that close to the chiral limit the value of m_π becomes very unstable, so one cannot measure directly m_π at $m = 0$ and instead one extrapolates the value to $m = 0$. However, that is easier to do with the plot shown in fig. 4.1 (b), where m_π^2 is plotted against m . It can be seen that

there is (qualitatively) a parabolic behavior that enable us to fit a quadratic function and infer m_π^R . As L becomes large, the parabolic behavior is more noticeable.



(a) m_π vs. m . It can be seen that near $m = 0$ the pion mass result is not trustable



(b) m_π^2 vs. m . One can fit a parabola of the form $y = m_\pi^R + bm^2$ to obtain the value of the residual pion mass: $m_\pi^R = 0.5633(188)$.

Figure 4.1: Results of m_π^2 as a function of m . Note that there are also values for $m < 0$, they are unphysical; however, in the simulation both masses have to be measured through other parameter of the HMC algorithm which gives negative masses. The errors were computed by using the *Jackknife method*.

This same procedure was done with $L = 7, 8, 9, 10, 11, 12$ and $L_t = 64$. In fig. 4.2 plots of m_π^2 vs. m are shown for $\beta = 4.0$ and in fig. 4.3 for $\beta = 2.0$.

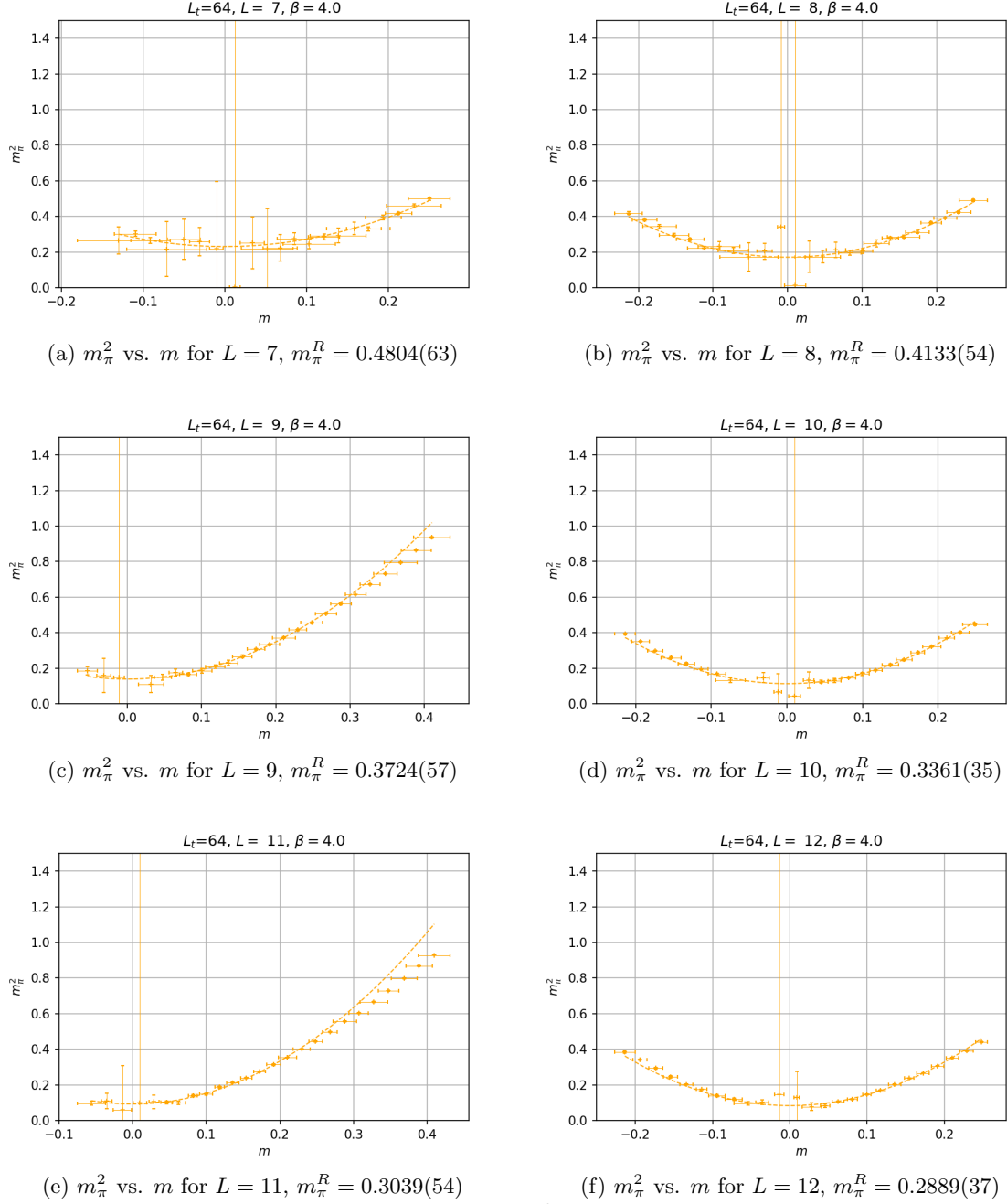


Figure 4.2: Results for $\beta = 4.0$. Each value of m_π^2 and m was obtained by averaging 1,000 measurements of different configurations. Between each configuration used, 10 sweeps were performed. All the parabolic fits were made with gnuplot.

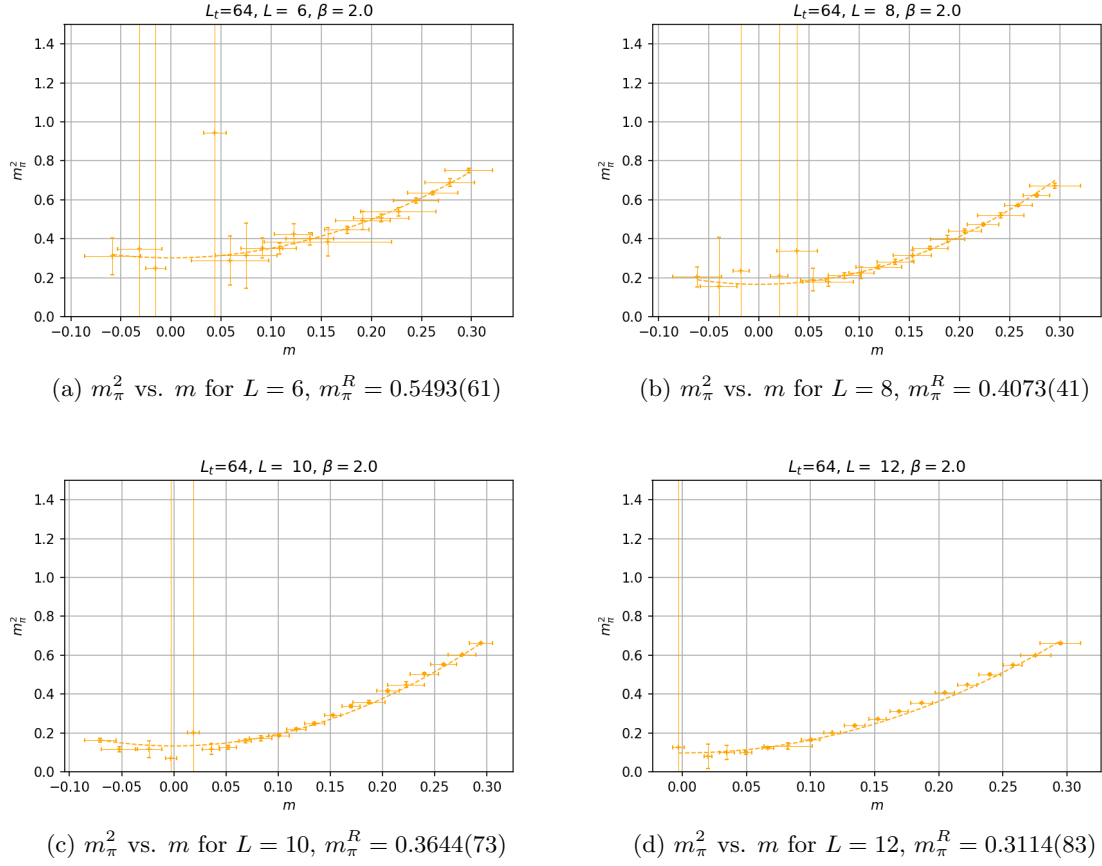


Figure 4.3: Results for $\beta = 2.0$. Each value was obtained by averaging 1,000 measurements of different configurations. Between each configuration used, 10 sweeps were performed.

Now, one plots m_π^R as a function of L and fits a function of the form $3/2LF_\pi^2$, as it is shown in fig. 4.4 and 4.5. The result obtained for the pion decay constant when $\beta = 4.0$ is

$$F_\pi = 0.6677(18), \quad (4.29)$$

while for $\beta = 2$

$$F_\pi = 0.6707(84). \quad (4.30)$$

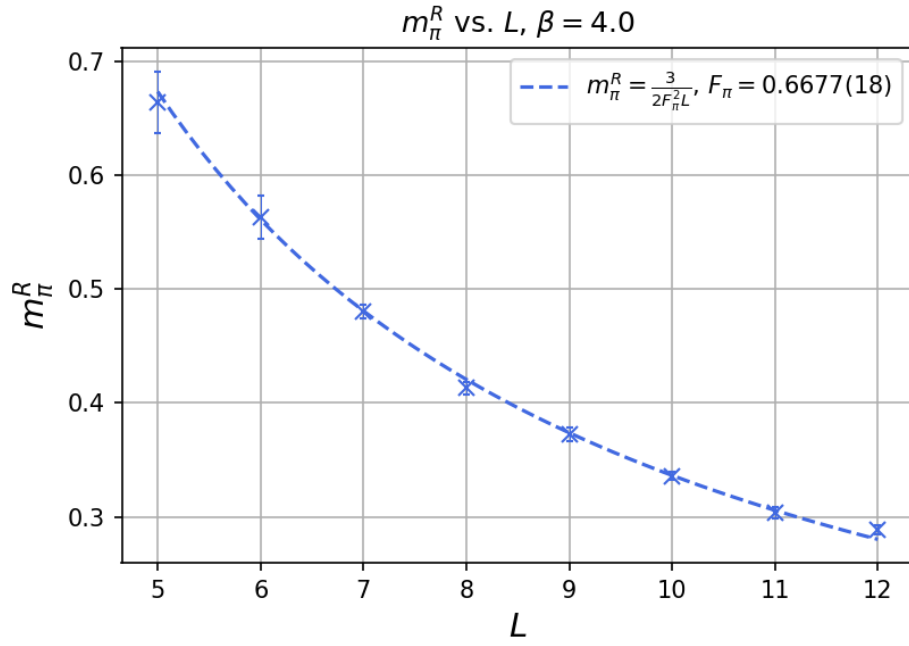


Figure 4.4: Behavior of m_π^R as a function of L for $\beta = 4$. We can see that it matches the behavior of $1/L$ described in eq. (4.28). When L becomes large the residual pion mass vanishes, as it should do for an infinite volume.

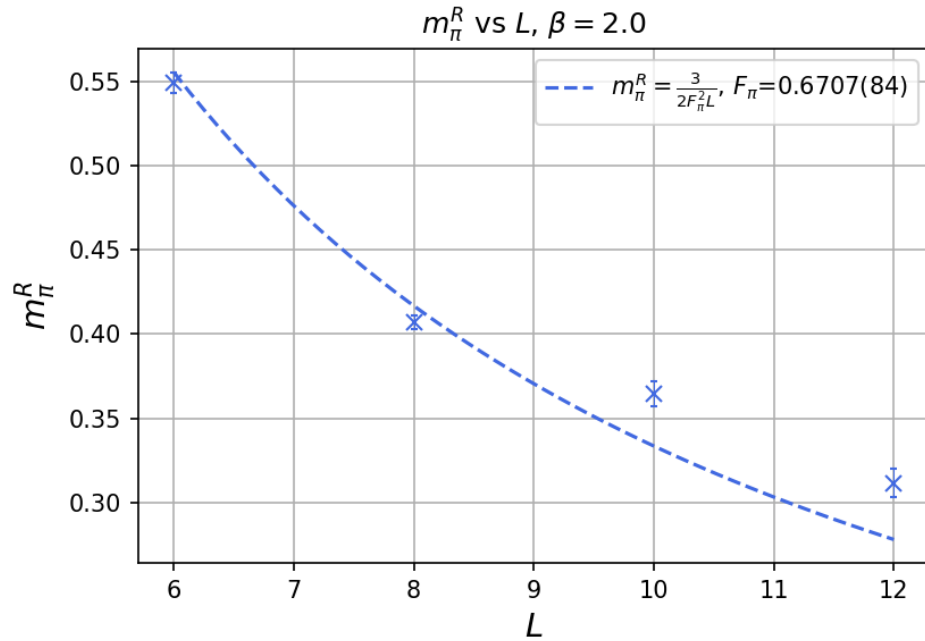


Figure 4.5: m_π^R vs. L for $\beta = 2$

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