Chapter 1

Chiral Perturbation Theory in QCD

As it was mentioned in chapter 2, the low energy level of QCD cannot be treated with perturbation theory, but one can use lattice simulations. The analytic approach to this regime is the *Chiral Perturbation Theory*, which is an effective field theory. The Lagrangian of this effective theory is built by adding all the terms that are consistent with the symmetries of the underlying theory, that in this case is QCD. This lead us to analize the symmetries of the QCD Lagrangian.

1.1 QCD symmetries

Since we are interested in quark masses $m_f \ll \Lambda_{\text{QCD}} \approx 300$ MeV, we will be working only with two flavors: u, d, whose masses are [1]

$$m_u = 1.7 \text{ - } 3.33 \text{ MeV}, \;\; m_d = 4.1 - 5.8 \text{ MeV}. \tag{1.1}$$

Then, in Minkowski space-time, the Lagrangian is given by

$$\begin{aligned} \mathcal{L} &= \sum_{f=u,d} (\overline{q}_f i \not{D} q_f - m_f \overline{q}_f q_f) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a, \\ \not{D} &= \gamma^\mu(\partial_\mu + igA_\mu), \;\; G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + ig[A^a_\mu, A^a_\nu]. \end{aligned} \tag{1.2}$$

q_f is the quark field of the flavor f, $\overline{q}_f = q_f^\dagger \gamma^0$, $A^a_\mu(x)$ are real valued fields related with $A_\mu(x)$ by $A_\mu(x) = \sum_{a=1}^{8} A^a_\mu(x) T_a$, where T_a are the basis elements of the traceless Hermitian 3×3 matrices. This implies that the gauge field is a matrix as well. This Lagrangian is constructed in the same way we did in section 2.3, so it is gauge invariant under transformations of SU(3). We are interested in the chiral symmetry, then let us apply the chiral operator

$$P_R = \frac{1}{2}(\mathbb{I} + \gamma_5), \;\; P_L = \frac{1}{2}(\mathbb{I} - \gamma_5) \tag{1.3}$$

to the quark fields in order to obtain the right-handed and left-handed fields:

$$q_{R_f} = P_{R_f} q, \;\; q_{L_f} = P_{L_f} q. \tag{1.4}$$

By using $\{\gamma_5, \gamma^\mu\}$ and the definition of the chiral operators one can prove the following properties:

$$P_{L,R} = P^\dagger_{L,R}, \;\; P^2_{L,R} = P_{L,R}, \;\; P_R P_L = P_L P_R = 0, \;\; P_{R,L}\gamma^\mu = \gamma^\mu P_{L,R}. \tag{1.5}$$

This implies that

$$\begin{aligned} q_f &= q_{R_f} + q_{L_f}, \\ \overline{q}_{R_f} &= q^\dagger_{R_f}\gamma^0 = q^\dagger_f P_R \gamma^0 = q^\dagger_f \gamma^0 P_L = \overline{q}_f P_L, \\ \overline{q}_{L_f} &= \overline{q}_f P_R, \\ \overline{q}_f &= \overline{q}_{L_f} + \overline{f}_{L_f}. \end{aligned} \tag{1.6}$$

With (1.5) and (1.6) the QCD Lagrangian can be rewritten as

$$\mathcal{L} = \sum_{f=u,d} \left[\overline{q}_{L_f} i\not{D} q_{L_f} + \overline{q}_{R_f} i\not{D} q_{R_f} - m_f(\overline{q}_{R_f} q_{L_f} + \overline{q}_{L_f} q_{R_f}) \right] - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a. \tag{1.7}$$

Now we can apply global chiral transformations:

$$q_{R,L_f} \to q'_{R,L_f} = e^{i\alpha\gamma_5} q_{R,L_f}, \;\; \overline{q}_{R,L_f} \to \overline{q}'_{R,L_f} = \overline{q}_{R,L_f} e^{i\alpha\gamma_5}, \;\; \alpha \in \mathbb{R}. \tag{1.8}$$

Let us note that the mass tern is not invariant under this transformations, so for the moment we will take $m = 0$. On the other hand, since $\{\gamma^\mu, \gamma_5\} = 0$ it follows that $\gamma^\mu e^{i\alpha\gamma_5} = e^{-i\alpha\gamma_5}\gamma^\mu$ and as a result the Lagrangian transforms as:

$$\mathcal{L} = \sum_{f=u,d} \left(\overline{q}_{L_f} i\not{D} q_{L_f} + \overline{q}_{R_f} i\not{D} q_{R_f} \right) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a = \sum_{f=u,d} \left(\overline{q}'_{L_f} i\not{D} q'_{L_f} + \overline{q}'_{R_f} i\not{D} q'_{R_f} \right) - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu}_a. \tag{1.9}$$

Hence, we can see that it is invariant under the transformations (1.8) only when $m = 0$, for that reason this is called the chiral limit. If we write the quark fields as a vector

$$q_{R,L} = \begin{pmatrix} q_{R,L_u} \\ q_{R,L_d} \end{pmatrix}, \;\; \overline{q}_{R,L} = \begin{pmatrix} \overline{q}_{R,L_u} \\ \overline{q}_{R,L_d} \end{pmatrix}, \tag{1.10}$$

it can be seen that the Lagrangian has a global symmetry under $\mathrm{U}(2)_L \otimes \mathrm{U}(2)_R$. An element of U(2) can be decomposed in an element of SU(2) multiplied by a phase factor, thus, we have the following symmetry group for the massless Lagrangian:

$$\mathrm{U}(2)_L \otimes \mathrm{U}(2)_R \leftrightarrow \mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \otimes \mathrm{U}(1)_B \otimes \mathrm{U}(1)_A. \tag{1.11}$$

$\mathrm{U}(1)_A$ is known as axial symmetry and is broken explicitly under quantization. $\mathrm{U}(1)_B$ is associated with the conservation of the Baryon number. Meanwhile, $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ is the chiral symmetry group. The later breaks spontaneously to SU(2), the order parameter of this broken symmetry is the chiral condensate $\langle 0|\, \overline{q}q\, |0\rangle$, so when it is different from zero the chiral symmetry is indeed spontaneously broken. Because of the Goldstone theorem, to this broken symmetry corresponds $2^2 - 1 = 3$ massless Goldstone Bosons. If now we take into account the masses of m_u and m_d, the symmetry is explicitly broken and the Goldsonte Bosons turn into light massive quasi Golsdtone Bosons, which can be related to the pion triplet π^+, π^-, π^0 [1], [2], [3].

1.2 Effective Lagrangian

1.3 Chiral Perturbation Theory regimes

Bibliography

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