F.M. Faulstich **Math 6590: Homework assignment 4 Due:** Wednesday 04/10/2024.

Exercise 1 *Consider the CP decomposition of* $\mathbf{X} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$*, i.e., the representation of* $\mathbf{X}$ *as a sum of elementary tensors*

$$\mathbf{X} = \sum_{p=1}^{r} \mathbf{v}_{1,p} \otimes \ldots \otimes \mathbf{v}_{d,p} = \sum_{p=1}^{r} \bigotimes_{i=1}^{d} \mathbf{v}_{i,p}$$

for $\mathbf{v}_{i,p} \in \mathbb{R}^{n_i}$*. Define the matrices*

$$\mathbf{V}_i = [\mathbf{v}_{i,1}| \ldots |\mathbf{v}_{i,r}] \tag{1}$$

and show that

$$\mathbf{X}[i_1, \ldots, i_d] = \sum_{p=1}^{r} \mathbf{V}_1[i_1, p]\mathbf{V}_2[i_2, p] \ldots \mathbf{V}_d[i_d, p] \tag{2}$$

Exercise 2 *Consider the sequence*

$$\mathbf{X}_n = n\left(\mathbf{u} + \frac{1}{n}\mathbf{v}\right) \otimes \left(\mathbf{u} + \frac{1}{n}\mathbf{v}\right) \otimes \left(\mathbf{u} + \frac{1}{n}\mathbf{v}\right) \otimes -n\mathbf{u} \otimes \mathbf{u} \otimes \mathbf{u}$$

with $\mathbf{u}$*,* $\mathbf{v} \in \mathbb{R}^m$*,* $\|\mathbf{v}\| = \|\mathbf{u}\| = 1$ *and* $\langle \mathbf{v}, \mathbf{u} \rangle \neq 1$*.*

i) Show that $\mathbf{X}_n \in \mathcal{M}_{\leq r}$ *for all* $n \in \mathbb{N}$

ii) Show that

$$\lim_{n \to \infty} \mathbf{X}_n = \mathbf{v} \otimes \mathbf{u} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{u} + \mathbf{u} \otimes \mathbf{u} \otimes \mathbf{v} \notin \mathcal{M}_{\leq r}$$

Exercise 3 *In this exercise, you will implement and compare the different higher-order SVDs introduced in class.*

i) Implement the Truncated Higher-Order SVD (THOSVD)
Input: Target tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$*, target rank* $r = (r_1, \ldots, r_d)$
Output: Core tensor $\mathbf{C} \in \mathbb{R}^{r_1 \times \ldots \times r_d}$*, basis matrices* $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$ *for* $1 \leq k \leq d$

ii) Implement the Sequential Truncated Higher Order SVD (STHOSVD)
Input: Target tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$*, target rank* $r = (r_1, \ldots, r_d)$
Output: Core tensor $\mathbf{C} \in \mathbb{R}^{r_1 \times \ldots \times r_d}$*, basis matrices* $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$ *for* $1 \leq k \leq d$

iii) Implement the Randomized Sequential Truncated Higher Order SVD (R-STHOSVD)
Input: Target tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$*, target rank* $r = (r_1, \ldots, r_d)$*, oversampling parameter* $p \in \mathbb{N}$
Output: Core tensor $\mathbf{C} \in \mathbb{R}^{r_1 \times \ldots \times r_d}$*, basis matrices* $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$ *for* $1 \leq k \leq d$

iv) Implement the sketeched Sequential Truncated Higher Order SVD (sketched-STHOSVD)
Input: Target tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$*, target rank* $r = (r_1, \ldots, r_d)$*, oversampling parameter* $\ell = (\ell_1, \ldots, \ell_d)$
Output: Core tensor $\mathbf{C} \in \mathbb{R}^{r_1 \times \ldots \times r_d}$*, basis matrices* $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$ *for* $1 \leq k \leq d$

v) Implement the sub-sketch Sequential Truncated Higher Order SVD (sub-sketch-STHOSVD)
Input: Target tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$*, target rank* $r = (r_1, \ldots, r_d)$*, oversampling parameter* $\ell = (\ell_1, \ldots, \ell_d)$*, power* $q \in \mathbb{N}$
Output: Core tensor $\mathbf{C} \in \mathbb{R}^{r_1 \times \ldots \times r_d}$*, basis matrices* $\mathbf{U}_k \in \mathbb{R}^{r_k \times n_k}$ *for* $1 \leq k \leq d$

vi) Run experiment 1 outlined in class:
Consider the Hilbert tensor $\mathbf{A} \in \mathbb{R}^{n_1 \times \ldots \times n_d}$ *defined as*

$$\mathbf{A}[i_1, \ldots, i_d] = \frac{1}{i_1 + \ldots + i_d}, \qquad 1 \leq i_j \leq n_j,\ 1 \leq j \leq d$$

Set the tensor parameters:
- $d = 5$
- $n_j = 25,\ j = 1, 2, \ldots, d$
- *Target rank is* (r, r, r, r, r)*, where* $r \in [\![1, 25]\!]$

Choose the computational parameters:
- *Oversampling* $p = 5$
- $l_i = r_i + 2,\ i = 1, 2, \ldots, d$
- *Power parameter* $q = 1$

Reproduce the plots shown in classes (lecture 17)

vii) Now consider a Hilbert tensor $\mathbf{A} \in \mathbb{R}^{500 \times 500 \times 500}$*. Run experiments targeting the ranks*

$$(i \cdot 10, i \cdot 10, i \cdot 10) \qquad \text{for } i \in [\![10]\!]$$

Report the CPU time in seconds and the relative error. Reproduce the numbers shown in class (lecture 17).

viii) Run experiment 2 outlined in class:
Consider the sparse tensor $\mathbf{A} \in \mathbb{R}^{200 \times 200 \times 200}$*:*

$$\mathbf{A} = \sum_{i=1}^{10} \frac{\gamma}{i^2} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i + \sum_{i=11}^{200} \frac{1}{i^2} \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i,$$

where $\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i \in \mathbb{R}^{200}$ *are sparse vectors with* 5% *nonzeros each (generated using the* `sprand` *command in MATLAB)*
Set the tensor parameters:
- $\gamma = 2, 10, 200$
- $n_j = 25,\ j = 1, 2, \ldots, d$
- *Target rank is* (r, r, r)*, where* $r \in [\![20, 100]\!]$

Choose the computational parameters:
- *Oversampling* $p = 5$
- $l_i = r_i + 2,\ i = 1, 2, \ldots, d$
- *Power parameter* $q = 1$

Reproduce the plots shown in class (lecture 17).