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83 PINHOLE CAMERA WITH NON-SQUARE PIXELS

84 SKEWNESS

85 GIVEN ONLY P : CAMERA PROPRIETIES

86 PROVE C IS CAMERA CENTER <u>AND</u> IS NULL-SPACE OF P

87 COMPUTE CAMERA CENTER C <u>SINGULAR</u>

88 COMPUTE CAMERA CENTER C <u>NONSINGULAR</u>

89 RETRIEVE K AND R FROM P

100 FIND F IF WORLD COORDS ARE ALLIGNED TO FIRST CAMERA

101 $[e']_x (k'R'k^{-1}) \Rightarrow (k')^{-T} R' k^T [e]_x$

102 <u>GENERAL CASE</u> OF FINDING F

103 PROPRIETIES OF F $(x'^T F x = 0)$

104 PROPRIETIES OF F (F is unique 3x3 HOMO MATRIX WITH RANK = 2 WITH PROPRIETY $x'^T F x = 0$)

105 OTHER PROPRIETIES OF F:

106 $Fe = 0 = e'^T F$ PROOF

107 DECOMPOSE F MATRIX

108 PROVE e, e' EPIPOLES LIE ON F_S THE STEINER CONIC

109 FIND TANGENT TO F_S STEINER CONIC ON e and e' and X_a (intersection of them)

110 PROOF THAT X_a is POLE OF l_a & l_a IS POLAR OF X_a

111 GEOMETRIC CONSTRUCTION OF EPIPOLAR LINE $l' = Fx$

112 PROJECTIVE INVARIANCE OF F FUNDAMENTAL MATRIX $(\hat{F})$ <u>PROOF</u>

(113) $(P, P') = (PH, P'H)$

F IS INDEPENDENT FROM WORLD COORD.

(114) FIND MATRIX H THAT GIVES CANONICAL FORM OF (P, P')

(115) ASSUMING $P' = [M' | P_4']$ AND WE ARE IN CANONICAL $\Rightarrow F = [p_4']_\times M'$

(116) GIVEN F, FIND (P, P') IN CANONICAL FORM

(117) IF F IS FUNDAMENTAL MATRIX OF (P, P') and $(\tilde{P}, \tilde{P}')$ THEN $\tilde{P} = PH$ AND $\tilde{P}' = P'H$ for some H

(118) GIVEN F AND A SKEW-SYMMETRIC $S \in \mathbb{R}^{3\times3}$ THEN F IS THE FUNDAMENTAL OF (P, P') FOR $P = [I | 0]$, $P' = [SF | e']$

(119) FUNDAMENTAL MATRIX PROPRIETIES RECAP

(120) ESTIMATE FUNDAMENTAL F

(121) 8-POINT ALGORITHM (START)

(122) 8-POINT ALGORITHM (FIND $\underset{\text{SINGULAR}}{F}$)

(123) LEAST SQUARE SOLUTION TO FIND F IN 8-POINTS IF Rank > 8 (OVERDEFINED)

(124) FIND $\tilde{F}$ CLOSEST TO F IN 8 POINT IF F rank = 3

(125) 7-POINT ALGORITHM

(126) 3D RECONSTRUCTION USING F and CANONICAL CAMERA PAIR (FIND X USING x, x')

(127) 3D RECONSTRUCTIONS ($P \Leftrightarrow \tilde{P}$ $\underset{H}{\text{USING}}$)

(128) CORRECT PROJECTION

1 BILINEAR EXERCISE

SCALE FACTOR OF $C=0.25$

BILINEAR INTERPOLATION

for i-j th pixel in target

$$x = (i+0.5)/0.25$$
$$y = (j+0.5)/0.25$$

$$k = \text{round}(x) - 1$$
$$l = \text{round}(y) - 1$$
$$u = x - (k+0.5)$$
$$v = y - (l+0.5)$$

$$a^2_{i,j} = \text{round}\Big((1-v)(1-u)\,a^1_{kl} + (1-v)\,u\,a^1_{k+1,l} + v(1-u)\,a_{k,l+1} + vu\,a_{k+1,l+1}\Big)$$

$$\simeq (1-u)\big((1-v)\,a^1_{kl} + (v)\,a_{k,l+1}\big) + (u)\big((1-v)\,a^1_{k+1,l} + (v)\,a_{k+1,l+1}\big)$$

WHERE FOR U

$$U = x - (K + 0.5) \qquad K = \mathrm{round}(x) - 1$$

$$x = \frac{(i + 0.5)}{0.25}$$

$$U = \frac{(i+0.5)}{0.25} - (K + 0.5)$$

$$= \frac{(i+0.5)}{0.25} - (\mathrm{round}(x) - 1 + 0.5)$$

$$= \frac{(i+0.5)}{0.25} - \left(\mathrm{round}\left(\frac{i+0.5}{0.25}\right) - 1 + 0.5\right)$$

$$= \left(i + \frac{1}{2}\right) \cdot 4 - \mathrm{round}\left(\left(i + \frac{1}{2}\right)4\right) + \frac{1}{2}$$

$$= 4i + 2 + \frac{1}{2} - \mathrm{round}(4i + 2)$$

$$= 4i + \frac{5}{2} - \mathrm{round}(4i + 2)$$

WHERE FOR V

$$V = y - (\ell + 0.5) \qquad \ell = \text{round}(y) - 1$$

$$y = \frac{(j+0.5)}{0.25}$$

$$V = (j+0.5)4 - \ell - 0.5 = (j+0.5)4 - \text{round}(y) + 1 - 0.5$$

$$= (j+0.5)4 - \text{round}\left(\left(j+0.5\right)4\right) + \frac{1}{2}$$

$$= 4j + 2 + \frac{1}{2} - \text{round}(4j+2)$$

$$= 4j + \frac{5}{2} - \text{round}(4j+2)$$

Let's test

$$4i + \frac{5}{2} - \text{round}(4i+2)$$

$i=0$ $\quad \frac{5}{2} - \text{round}(2)$ $\qquad$ $i=1$ $\quad \frac{8+5}{2} - \frac{12}{2} = \frac{1}{2}$

$\frac{5}{2} - \frac{4}{2} = \frac{1}{2}$

So both u & v are $\frac{1}{2}$!

$$\partial^2_{i,j} = (1-u)\left((1-v)\,\partial^1_{kl} + (v)\,\partial^1_{k,l+1}\right) +$$

$$(u)\left((1-v)\,\partial^1_{k+1,l} + (v)\,\partial^1_{k+1,l+1}\right)$$

$$= \left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)\partial^1_{kl} + \left(\frac{1}{2}\right)\partial^1_{k,l+1}\right) + \left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)\partial^1_{k+1,l} + \left(\frac{1}{2}\right)\partial^1_{k+1,l+1}\right)$$

$$= \left(\frac{1}{2}\right)\left(\left(\frac{1}{2}\right)\partial^1_{kl} + \left(\frac{1}{2}\right)\partial^1_{k,l+1} + \left(\frac{1}{2}\right)\partial^1_{k+1,l} + \left(\frac{1}{2}\right)\partial^1_{k+1,l+1}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\partial^1_{kl} + \partial^1_{k,l+1} + \partial^1_{k+1,l} + \partial^1_{k+1,l+1}\right)$$

2 CONVERT MAX SPACE TO PPI

$W = 6cm$

$h = 5cm$

max space in inches

$W = \frac{6}{2.54}$

$h = \frac{5}{2.54}$

max space in pixels

$W = PPI \cdot W_{inches}$

$h \quad PPI \cdot h \cdot inches$

③ SAMPLING & QUANTIZATION

For pixel i,j in image A (sampled)

$$x = (i + 0.5)\,\Delta H \qquad\qquad \Delta H = \frac{H}{M}$$

H — unsampled height
M — sampled height

$$y = (j + 0.5)\,\Delta W \qquad\qquad \Delta W = \frac{W}{N}$$

W — unsampled width
N — sampled width

$$f_{ij} = \underbrace{f(x,y)}_{\text{intensity in original image}}$$

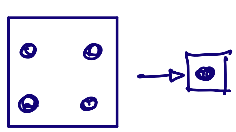
SAMPLING

$$a_{ij} = \lfloor L \cdot f_{ij} \rfloor$$

QUANTIZATION

L = intensity levels
f_{ij} = sampled result

④ SUPERSAMPLING

- 

$$\frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

multiple points mean in one pixel

- Spatial Integration

$$f_{ij} = \frac{1}{\Delta W \cdot \Delta H} \int_{j\Delta W}^{(j+1)\Delta W} \int_{i\Delta H}^{(i+1)\Delta H} f(x,y)\,dx\,dy$$

5 QUANTIZATION ISSUES AND SOLUTIONS

Basic: $a_{ij} = \lfloor L \cdot f_{ij} \rfloor$ whith range $0, \dots, L-1$

>> IF f_{ij} is at max value $\Rightarrow$ $a_{i,j} = L$ $\boxed{\text{OUT OF RANGE}}$

Sol 1: $a_{ij} = \text{round}\left(L f_{ij} - 0.5\right)$

>> Works only if $\text{round}(0.5) = 0$

Sol 2: $a_{ij} = \text{round}\left((L-1) f_{ij}\right)$

>> Now in range, but max bucket $L-1$ is smaller!

Sol 3: $a_{i,j} = \left\lfloor L \cdot \dfrac{f_{ij} - f_*}{f^* - f_*} \right\rfloor$ where

$$f_* = \min_{x,y} f_{x,y}$$

$$f^* = \max_{x,y} f_{x,y}$$

(6) BILINEAR INTERPOLATION

For ∂^2_{ij} intensity:

$$x = (i + 0.5) \cdot [\text{IMAGE MODIFICATION}]$$

$$y = (j + 0.5) \cdot [\text{IMAGE MODIFICATION}]$$

$$k = \text{round}(x) - 1$$

$$\ell = \text{round}(y) - 1$$

$$u = x - (k + 0.5)$$

$$v = y - (\ell + 0.5)$$

$$\partial^2_{ij} = \text{round}\Big((1-v)(1-u)\,\partial^1_{k\ell} + (1-v)(u)\,\partial^1_{k+1,\ell} + (v)(1-u)\,\partial^1_{k,\ell+1} + (v)(u)\,\partial^1_{k+1,\ell+1} \Big)$$

WITH INTEGRAL

$$\partial^2_{ij} = \int_{y-0.5}^{y+0.5} \int_{x-0.5}^{x+0.5} f_1(s,t)\, ds\, td$$

7 GEOMETRIC TRANSFORMATIONS

- SCALING $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} C_x & 0 \\ 0 & C_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = C \begin{pmatrix} x \\ y \end{pmatrix}$

- TRANSLATION $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + t_x \\ y + t_y \end{pmatrix}$

- ROTATION $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}$

- HORIZONTAL SHEAR $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y + S_h x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ S_h & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = S_h \begin{pmatrix} x \\ y \end{pmatrix}$

- VERTICAL SHEAR $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x + S_v y \\ y \end{pmatrix} = \begin{pmatrix} 1 & S_v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = S_v \begin{pmatrix} x \\ y \end{pmatrix}$

⑧ HOMOGENEOUS COORDINATES TRANSFORMATIONS (AFFINE)

- INVARIANT TO AFFINE COMBINATIONS

- SCALING

$$C(c_x, c_y) = \begin{pmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- INVERSE

$$C(c_x, c_y)^{-1} = C\left(\frac{1}{c_x}, \frac{1}{c_y}\right)$$

- TRANSLATION

$$T(t_x, t_y) = \begin{pmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

- INVERSE

$$T(t_x, t_y)^{-1} = T(-t_x, -t_y)$$

- ROTATION

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- INVERSE

$$R(\theta)^{-1} = R(-\theta)$$

- VERTICAL SHEAR

$$S_v(\lambda) = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_v(\lambda)^{-1} = S_v(-\lambda)$$

- HORIZONTAL SHEAR

$$S_h(\lambda) = \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$S_h(\lambda)^{-1} = S_h(-\lambda)$$

(9) FORWARD MAPPING

» For all pixels (x, y) of A_1 use TRANSFORM to compute (x', y') and transfer intensity.

ISSUES: Same pixel A_2 can have an A_1 mapping.
Some pixel A_2 can lack a pre-image in A_1

(10) INVERSE MAPPING

» For all pixels (x', y') in A_2, use TRANSFORM^{-1} to compute (x, y) in A_1 and use any interpolation method to revert back to original

(11) 3-SHEARS ROTATION

$$R(\theta) = S_v(\lambda)\, S_h(\mu)\, S_v(\lambda)$$

$$\lambda = -\tan\frac{\theta}{2} \qquad \mu = \sin\theta$$

(12) CHANGE INTENSITY RESOLUTION

$$r \rightarrow \text{round}\left(\left(r + 0.5\right) \frac{L_2}{L_1} - 0.5\right)$$

(13) TEMPORAL INTENSITY SUPER-SAMPLING

>> Sample source K times

$$\bar{A}_K = \frac{1}{K} \sum_{i=1}^{K} A_i$$

$K \rightarrow \infty$ No noise!

>> On-line (Iterative formulation)

$$\bar{A}_{K+1} = \frac{K \cdot \bar{A}_K + A_{K+1}}{K+1}$$

>> IF UNCORRELATED NOISE

$$\sigma_{\bar{A}_K} = \frac{1}{\sqrt{K}} \sigma_n$$

(14) INTENSITY TRANSFORMATIONS

Discete intensity transformation

$$\overset{1}{T} : \{0, \dots, L_1 - 1\} : \{0, \dots, L_2 - 1\}$$

$$\overset{1}{T}(r) = \text{round}\left(L \cdot T\left(\frac{r + 0.5}{L}\right) - 0.5\right)$$

- NEGATIVE

$$T(r) = 1 - r$$

- LOG TRANSFORM (enhance contrast in dark areas)

$$T(r) = \alpha \log(r/\beta + 1)$$

- $\alpha = 1/\left(\log\left(\frac{1}{\beta} + 1\right)\right)$
- $\beta = 1/\left(\exp\left(\frac{1}{\alpha}\right) - 1\right)$

- EXP TRANSFORM (enhance contrast in bright area)

$$T(r) = \beta\left(\exp(r/\alpha) - 1\right)$$

- $\alpha = 1/\left(\log\left(\frac{1}{\beta} + 1\right)\right)$
- $\beta = 1/\left(\exp\left(\frac{1}{\alpha}\right) - 1\right)$

- GAMMA CORRECTION

$$T(r) = r^{\gamma} \qquad \gamma > 0$$

if $\gamma < 1$ enhance in dark

if $\gamma > 1$ enhance in light

- CONTRAST STRETCHING

Medium intensity contrast enhance

$$T(r) = \frac{1}{2}\left(\frac{\arctan(\delta(2r-1))}{\arctan(\delta)} + 1\right) \qquad \delta > 0$$

MORE

$$T(r) = \frac{1}{2}\left(\lambda \frac{\arctan(\delta(2r-1))}{\arctan(\delta)} + 1\right) \qquad \delta > 0, \ \lambda \geq 1$$

- THREE PIECES

$S_1 = 0$

$S_S = 1$

$r_1 = \text{tresh } 1$

$r_2 = \text{tresh } 2$

$T(r) =$ piecewise
bijective
linear

- INTENSITY LEVEL SLICING

highlight one area

15 HISTOGRAM EQUALIZATION

$$\hat{T}(k) = \text{round}(\tilde{s}_k)$$

where

$$\tilde{s}_k = L \sum_{j=0}^{k-1} p_j + \frac{L}{2} p_k - \frac{1}{2} \quad \text{for } k = 0, \dots, L-1$$

SANITY CHECK:

$h_k = MN/L$ for $k = 0, \dots, L-1$

$p_k = \frac{1}{L} \qquad \tilde{s}_k = k \qquad \hat{T}(k) = k \checkmark$

16 HISTOGRAM MATCHING

Find a T transformation, such that the probability density function of g matches some target $q(s)$

$$T = T_q^{-1} \circ T_p \qquad T_p(r) = \int_0^r p(z)\,dz \qquad T_q(s) = \int_0^s q(z)\,dz$$

IN DISCRETE SETTING

T_p linear with $T_p(k/L) = \bar{p}_k \qquad \bar{p}_k = \sum_{j=0}^{k-1} p_j$

T_q linear with $T_q(k/L) = \bar{q}_k \qquad \bar{q}_k = \sum_{j=0}^{k-1} q_j$

T_q^{-1} linear with $T_q^{-1}(\bar{q}_k) = k/L$

17 LINEAR SPATIAL FILTERING

- box $a = b$, $m = n = 2a+1$, $W_{st} = \frac{1}{m^2}$

- gaussian $a = b$

$$W_{s,t} = \frac{\exp\left(-\frac{s^2+t^2}{2\sigma^2}\right)}{-a\left(\exp\left(-\frac{s^2+t^2}{2\sigma^2}\right)\right)}$$

Padding
- zero
- mirror
- replicate

PERFORMANCE : $O(MNmn)$

18 MEDIAN FILTER

Get median value of kernel window

19 LAPLATIAN FILTER (smoothing)

$OUT = l \ast A$

passing filter

$$l = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ or } l' = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -8 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

20 PREWITT FILTER (Box smoothing)

$$OUT = \sqrt{(g_x \star A)^2 + (g_y \star A)^2}$$

where:

$$g_x = \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{- and -} \quad g_y = \begin{bmatrix} -1 & 0 & 1 \\ -1 & 0 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

21 SOBEL FILTER

$$OUT = \sqrt{(g_x \star A)^2 + (g_y \star A)^2}$$

where:

$$g_x = \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} \quad \text{- and -} \quad g_y = \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

22 BASIC BINAZIZATION

$$\begin{cases} 0 & \text{if } A_{ij} \leq t \\ 1 & \text{otherwise} \end{cases}$$

treshold is selected by fitting a poly OR a gauss. dist

23 IMAGE SEGMENTATION BASICS

conditions:
- $\bigcup_{i=1}^{n} R_i = R$
- R_i connected set $\forall i \in 0,\dots,n$
- $R_i \cap R_j = \emptyset$ for all i,j

- IF SAME VALUES BUT NOT CONNECTED
  ↳ Different regions

24 SIGNAL SMOOTHING & AVG. SMOOTHING

n = kernel size

$$\underbrace{g_i}_{\text{signal out}} = \frac{1}{n}\int_{i-n/2}^{i+n/2} f(x)\,dx$$

$n \times m$ = kernel size

$$B_{ij} = \frac{1}{n \cdot m}\int_{i-n/2}^{i+n/2}\int_{j-\frac{m}{2}}^{j+\frac{m}{2}} A(s,t)\,dt\,ds$$

$$B_{ij} = \sum_{s=-\frac{n}{2}}^{n/2}\sum_{t=-\frac{m}{2}}^{m/2} A_{ij}\, w(i+s, j+t)$$

$$W = \begin{bmatrix} \frac{1}{n+m} & \cdots & \frac{1}{n+m} \\ \vdots & \ddots & \vdots \\ \frac{1}{n+m} & \cdots & \frac{1}{n+m} \end{bmatrix}$$

25 EDGE DETECTION (BASIC CONCEPT)

$$g(x) = \begin{cases} 1 & \text{if } |f'(x)| > t \\ 0 & \text{otherwise} \end{cases}$$

edge detected / not

26 EDGE DETECTION WITH FIRST ORDER DER.

$$W_x = \frac{\partial f(x,y)}{\partial x} = f'(x) \approx \frac{f(x+1,y) - f(x-1,y)}{2}$$

$$W_y = \frac{\partial f(x,y)}{\partial y} = f'(x) \approx \frac{f(x,y+1) - f(x,y-1)}{2}$$

$$W_x = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \qquad W_y = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

27 EDGE PROPRIETIES

$$W_x = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \qquad W_y = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

- Position (x,y)
- Magnitude

$$\left.\begin{aligned} G_x &= W_x \star A \\ G_y &= W_y \star A \end{aligned}\right\} \Rightarrow \text{Magnitude } G = \sqrt{G_x^2 + G_y^2} \cong |G_x| + |G_y|$$

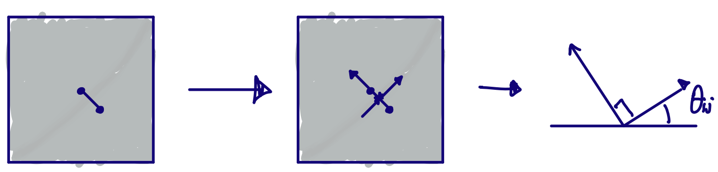
- Orientation $\Theta = \operatorname{atan} \frac{G_x}{G_y}$

(28) CANNY EDGE DETECTION

- INTEGRATION TO SMOOTH IMAGE
- DIFFERENTIATION (27)
- NON-MAXIMA SUPPRESSION: (29)
- HYSTERESIS (DUAL TRESHOLDING) (30)
- CONNECTIVITY ANALYSIS (31)

(29) NON-MAXIMA SUPPRESSION

Transform edge into a single line of pixels



A) Group angles : $\theta' = \begin{cases} 0 & \text{if } -22.5 \leq \theta_{ij} < 22.5 \\ 45 & \text{if } 22.5 \leq \theta_{ij} < 67.5 \\ 90 & \text{if } 67.5 \leq \theta_{ij} < 112.5 \\ 135 & \text{if } 112.5 \leq \theta_{ij} < 157.5 \end{cases}$

B) Compare magnitude + neighbours

if $\theta'_{ij} = 0$:
  if $(G_{ij} < G_{(i-1),j} \,||\, G_{ij} < G_{(i+1),j})$:
    $G'_{ij} = 0$
if $\theta'_{ij} = 45$:
  if $(G_{ij} < G_{(i-1),(j+1)} \,||\, G_{ij} < G_{(i+1),(j-1)})$:
    $G'_{ij} = 0$

if $\theta'_{ij} = 90$:
  if $(G_{ij} < G_{(i-1)j} \,||\, G_{ij} < G_{(i+1)j})$:
    $G'_{ij} = 0$
if $\theta'_{ij} = 135$:
  if $(G_{ij} < G_{(i-1)j} \,||\, G_{ij} < G_{(i+1)j})$:
    $G'_{ij} = 0$
else:
  $G'_{ij} = G_{i,j}$

30 HYSTERESIS TRESHOLDING

Set low & high treshold

pixels above thigh form the image $G'high$

pixels above tlow form the image $G'low$

$G'high$ will contain pixels of large magnitude edges

$G'low$ will contain pixels of low magnitude edges

$$G''_{i,j} = \begin{cases} 2 & \text{if} \quad G'_{ij} \geq thigh \\ 1 & \text{if} \quad thigh > G'_{ij} \geq tlow \\ 0 & \text{otherwise} \end{cases}$$

31 CONNECTIVITY ANALYSIS

Use result of 30 → G''_{ij} to compute an edge image.

- Region growing : • Get a "seed" pixel with G''_{ij} = class 2
  - • Grow seed including neighbours with magnitude ≥ 1
  - • Get new seed with G''_{ij} = class 2

  repeat until all $G''_{ij} = 2$ are included

32 HOUGH TRANSFORM HORIZONTAL LINES

- slope-intercept equation: $y_i = a x_i + b$

  with $a = 0$

  Horizontal!

  $$\boxed{y_i = b}$$

- Create accumulator of discrete voting bins where each bin is a Row in A

A 120 rows $\Rightarrow |H| = 120$

For each (x_i, y_i) in A, we take the position of the x axis and add to $H[x_i]$ the intensity of a_{x_i, y_i}

We then apply non-maxima suppression 29

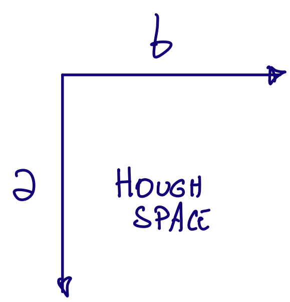
(33) HOUGH TRANSFORM ON LINES ON ANY CHOSEN ANGLE α

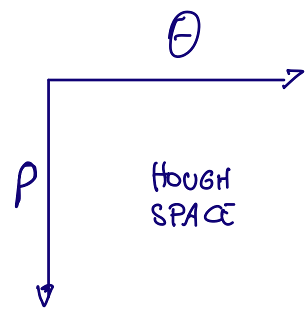
- compute orientation of gradients (27) θ and keep only points with θ close to the angle α to use
- Then put intensities in Hough space following a and b of line equation:

$$y_i = a x_i + b$$

or

$$x_i \cos\theta + y_i \sin\theta = \rho$$



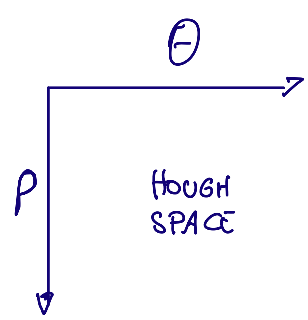


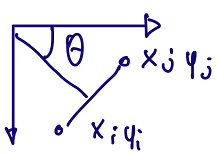
(34) LINE HOUGH TRANSFORM (COMPLETE)

Use <u>normal</u> rappresentation of line using:

$$(x_i \cos\theta + y_i \sin\theta = \rho)$$

and use accumulator:





35 HOUGH CIRCLES

use same procedure but with a,b and a chosen r

$$(x_i - a)^2 + (y_i - b)^2 = r^2$$

36 HARRIS-STEPHENS CORNER DETECTOR

Using Kernel = m x n

- SUM OF SQUARE DIFERENCES AT POSITION (s,t)

$$c(x,y) = \sum_{s=-n/2}^{n/2} \sum_{t=-m/2}^{m/2} \left[A(s+x, t+y) - A(s,t) \right]^2$$

- WEIGHTED SUM OF SQUARE DIFERENCES AT POSITION (s,t)

SMOOTHING KERNEL 24

$$c(x,y) = \sum_{s=-n/2}^{n/2} \sum_{t=-m/2}^{m/2} W(s,t) \left[A(s+x, t+y) - A(s,t) \right]^2$$

– BILINEAR APPROXIMATION

$$C(x,y) = \sum_{s=-n/2}^{n/2} \sum_{t=-m/2}^{m/2} W(s,t) \Big[\big(A(s,t) + x\,G_x(s,t) + y\,G_y(s,t) \big) - A(s,t) \Big]^{\overset{(s,t)}{2}}$$

$$C(x,y) = \begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\text{with } M = \sum_{s} \sum_{t} W(s,t) \begin{bmatrix} G_x(s,t)^2 & G_x(s,t)\,G_y(s,t) \\ G_x(s,t)\,G_y(s,t) & G_y(s,t)^2 \end{bmatrix}$$

Harris matrix

where $G_x = A \star \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \end{bmatrix}$ $\quad G_y = A \star \begin{bmatrix} -0.5 & 0 & 0.5 \end{bmatrix}$

37 HARRIS-STEPHENS WITH EIGENVAL

CALCULATE λ_x & λ_y of M (36) and use that to find $det(M) = \lambda_x + \lambda_y$
$trace(M) = \lambda_x + \lambda_y$

So: $R = det(M) - K \cdot (trace(M))^2$

$det(M) = G_x(s,t)^2 \, G_y(s,t)^2$

$trace(M) = G_x(s,t)^2 + G_y(s,t)^2$

$K = [0.04, 0.06]$

38 COLOR XYZ

$X = \int \bar{x}(\lambda) I(\lambda) \, d\lambda \leftarrow$ L cone

$Y = \int \bar{y}(\lambda) I(\lambda) \, d\lambda \leftarrow$ M cone

$Z \int \bar{z}(\lambda) I(\lambda) \, d\lambda \leftarrow$ S cone

λ : SPD Light

(39) COLOR xyY

$$x = \frac{X}{X+Y+Z} \qquad y = \frac{Y}{X+Y+Z} \qquad z = \frac{Z}{X+Y+Z} = 1-x-y$$

We just use

x, y and Y

Luminance

(40) RGB $\leftrightarrow$ XYZ

$$\begin{pmatrix} R \\ G \\ B \end{pmatrix} = \begin{pmatrix} 3.06 & -1.39 & -0.48 \\ -0.97 & 1.88 & 0.04 \\ 0.07 & -0.23 & 1.07 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$$

(41) RGB $\leftrightarrow$ CMY

$$\begin{pmatrix} C \\ M \\ Y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} R \\ G \\ B \end{pmatrix} \qquad \begin{pmatrix} R \\ G \\ B \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} C \\ M \\ Y \end{pmatrix}$$

(42) CMY $\leftrightarrow$ CMYK

$$(CMY) = \frac{(C-K,\ M-K,\ Y-K)}{(1-K)}$$

43 RGB $\leftrightarrow$ HSI

$$H = \begin{cases} \theta & \text{if } B \le G \\ 360 - \theta & \text{if } B > G \end{cases} \qquad \theta = \arccos\left(\frac{\frac{1}{2}\left[(R-G)+(R-B)\right]}{\sqrt{(R-G)^2+(R-G)(G-B)}} \right)$$

$$S = 1 - \frac{3\min(R,G,B)}{R+G+B} = 1 - \frac{\min(R,G,B)}{I}$$

$$I = \frac{R+G+B}{3}$$

44 HSI $\leftrightarrow$ RGB

if $H \in [0, 120)$

$B = I(1-S)$

$R = I\left(1 - \frac{S\cos H}{\cos(60-H)}\right)$

$G = 3I - (R+B)$

elif $H \in [120, 240)$

$H = H - 120$

$R = I(1-S)$

$G = I\left(1 - \frac{S\cos H}{\cos(60-H)}\right)$

$B = 3I - (R+G)$

elif $H \in [240, 360)$

$H = H - 240$

$G = I(1-S)$

$B = I\left(1 - \frac{S\cos 60}{\cos(60-H)}\right)$

$R = 3I - (G+B)$

(45) PSEUDO - COLORS

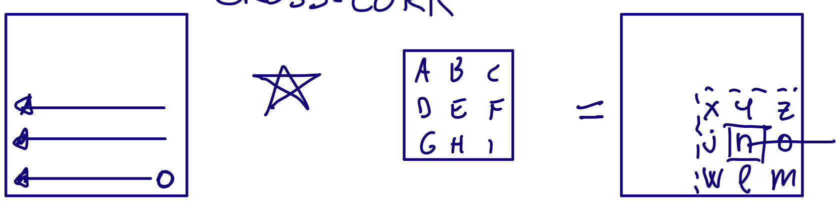
In $f: X \times Y \to I$ $\qquad$ $T_R, T_G, T_B \to I$

out: $g_R, g_G, g_B : X \times Y \to I$

where $g_\star(x,y) = T_\star(f(x,y))$

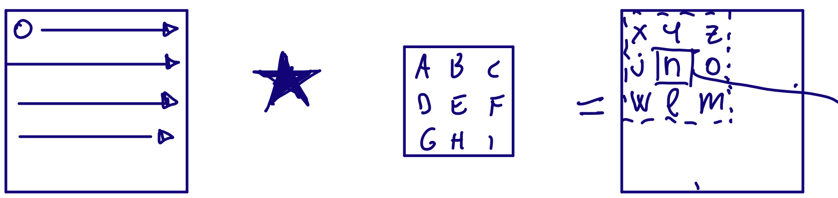
(46) CROSS - CORRELATION vs CONVOLUTION

CROSS-CORR



$= Ax + By + Cz + jD \ldots$

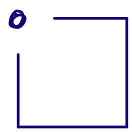
CONVOLUTION

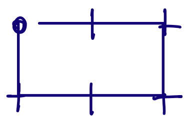


$= I \cdot x + Hy + Gz + \ldots$

47 STRUCTURAL PATTERN MATCHING

- SHAPE NUMBERS





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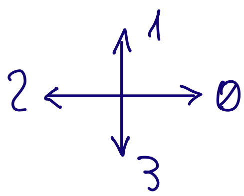

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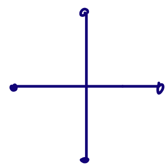
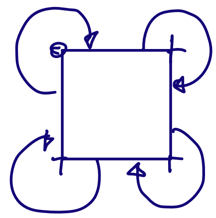
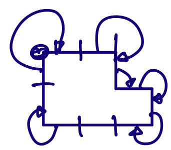
* - SHAPE NUMBERS (Rotation Inv.)





3333



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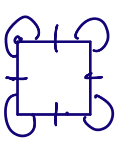
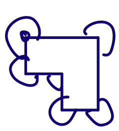
- STRING-BASED PATTERN MATCHING

>> SHIFT THE RESULT OF * FOR BOTH IMAGES TO GET THE MOST 0s on the left And check result 2 at a time

 vs 

30303030 ↓ $\boxed{03}\boxed{03}0303$ (4 under the first 03)

30303313 ↓ $\boxed{03}\boxed{03}3133$

So $\frac{4}{8} = \frac{1}{2}$ DISTANCE $= 1 - \frac{1}{2}$

(48) CONIC SECTION

$$\{(x,y)^T \in \mathbb{R}^2 : ax^2 + bxy + cy^2 + dx + ey + f = 0\}$$

DISCRIMINANT: $\delta = b^2 - 4ac$

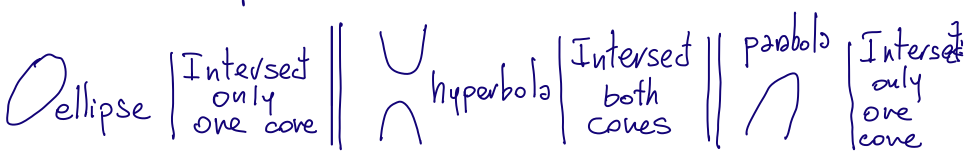
$\delta = -4\Delta$

where $\Delta = \det M$ where $M = \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$

so $\underbrace{\delta}_{\text{DISCRIMINANT}} = -4 \det \begin{pmatrix} a & b/2 \\ b/2 & c \end{pmatrix}$

(49) CONIC TYPES BASED ON DETERMINANT

$$\left.\begin{array}{l} \delta > 0 : \text{hyperbola} \\ \delta < 0 : \text{ellipse} \\ \delta = 0 : \text{parabola} \end{array}\right\} \text{if } a, b, c \text{ are } \underline{\text{not}} \text{ all equal to } \emptyset$$



50 CONIC IN $\mathbb{P}^2$ HOMOGENEOUS

$$ax^2+bxy+cy^2+dxz+eyz+fz^2=0$$

if z is not equal to 1:
To prove this replace $x \to zx$ and $y \to zy$
then

$$ax^2z^2+bxyz^2+cy^2z^2+dxz^2+eyz^2+fz^2=0$$

and if we devide all by z^2 to get to original.

- AS A QUADRATIC FORM: M ⇒ it's determinant will be used for δ!

$$\boxed{X^T C X}=0 \quad , \quad C=\begin{pmatrix} a & b/2 & d/c \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{pmatrix}$$

point in homogeneous
$X=(x,y,z)^T$

$\underbrace{\boxed{X^T C X}}_{\text{QUADRATIC FORM}}=0$ If set to $\varnothing$ we get all the points that lie on the conic in HOMO COORDS

- C IS SYMMETRIC AND $[C]=\{\lambda C\}$ is still the same conic! IN HOMOGENOUS COORDS

(S1) CONICS & DEGREES OF FREEDOM

- Only 5 degrees of Freedom

  -SO-

- CONIC CAN BE DEFINED <u>UNIQUELY</u> by 5 points $x_i = (x_i, y_i, z_i)^T$
  $i = 1, \dots, 5$ in $\underbrace{\text{general position}}_{\text{not 3 of them collinear!}}$

Why? each point x_i gives one constraint on the coefficients

$$\overbrace{(x_i^2, x_i y_i, y_i^2, x_i z_i, y_i z_i, z_i^2)}^{*}(a,b,c,d,e,f)^T = 0$$

* For each of the 5 defining points

>> IF WE ARRANGE THE 5 * AS A MATRIX, WE GET **X**

SO : $\mathbf{X} c = 0$

$$\begin{array}{l} \text{Point 1} \\ \text{Point 2} \\ \vdots \\ \text{Point 5} \end{array}\begin{bmatrix} x_1^2, & x_1 y_1, & y_1^2, & x_1 z_1, & y_1 z_1, & z_1^2 \\ x_2^2, & x_2 y_2, & y_2^2, & x_2 z_2, & y_2 z_2, & z_2^2 \\ & & \vdots & & & \\ x_5^2, & x_5 y_5, & y_5^2, & x_5 z_5, & y_5 z_5, & z_5^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \emptyset$$

"c <u>must</u> be in the null space of **X**"

(52) DEGENERATE CONIC

If 3 of the 5 DEFINING POINTS ARE COLLINEAR $\Rightarrow$ Degenerate

$\det(C) = 0$ will mean a rank-deficient matrix

$\Downarrow$

DEGENERATE CONIC

- If a conic is degenerate: there can be more than one solution! more than 1 conic.

(53) DEGENERATE CONICS IDENTIFICATION

$\delta_{\text{DISCRIM.}} = 0$: if $\text{rank}(C) = 1$ or $(d^2 + e^2 = 4(a+c)f)$: one line

if $(d^2 + e^2 < 4(a+c)f)$: complex conjugate lines [you don't see them] Points with imaginary

else: 2 lines

$\delta_{\text{DISCRIM.}} < 0$: Single point (degenerate ellipse)

So an intersection of 2 non-parallel complex conjugate lines

$\delta_{\text{DISCRIM.}} > 0$: Two intersecting lines (degenerate hyperbola)

54 PROOF OF $C = lm^T + ml^T$ IS THE DEGENERATE CONIC COMPOSED OF LINES l and m

- PROOVE ALL POINTS ON CONIC

A. IF:

x is on l $\Rightarrow$ $x^T l = l^T x = 0$

(x: column vect; l: column vect)

B. IF:

x is on m $\Rightarrow$ $x^T m = m^T x = 0$

(x: column vect; m: column vect)

IN BOTH A or B WE CAN STATE:

$$x^T \underbrace{C}_{lm^T + ml^T} x = x^T\left(lm^T + ml^T\right)x = \left(x^T l\right)\left(m^T x\right) + \left(x^T m\right)\left(l^T x\right)$$

$$= \left(x^T l\right)\left(m^T x\right) + \left(x^T m\right)\left(l^T x\right)$$

IF x is on m $= 0$ IF x is on l $= 0$

SO $x^T C x = 0$! so x is on conic C

So all points on l and all points on m are on the conic C.

- PROOVE DEGENERATE

NOW BECAUSE C is Symmetric $\Rightarrow$ As all Conics

THEN: $C^T = C$

$\Downarrow$

$$\left\{\left(lm^T + ml^T\right)^T = lm^T + ml^T\right.$$

AND

Rank$(C) = 2$ WHY? BECAUSE Null space of C are all the multiples of $\mathbf{x}$

$$\text{null}(C) = \{\lambda \mathbf{x} : \lambda \in \mathbb{R}\} \quad \mathbf{x} = l \times m$$

intersection of the 2 lines

Proof:

$$C(\lambda \mathbf{x}) = \left(lm^T + ml^T\right)(\lambda x) = \lambda\, l \underbrace{m^T x}_{} + \lambda\, m \underbrace{l^T x}_{} = 0$$

x is on line l and also on line m as before!

So AS Rank-Deficient

Rank$(C) = 2$ $\Rightarrow$ Degenerate Conic

(55) TANGENT LINES TO CONIC (PROOF)

$l = C\mathbf{x}$

$\llcorner$ point on the conic

Prove line tangent to conic C at $\mathbf{x}$

Proof:

if l contains x, then $x^T l = \emptyset$

and then as $l = C\mathbf{x}$:

$x^T \underbrace{C x}_{l} = \emptyset \implies l = Cx$ is a line through x

THEN:

- IF l intersects C in 1 point then we have tangent! ✓
- OTHERWISE: we have point $y \neq x$ on l and on C, then also

$y^T C y = \emptyset$ and $l^T y = \emptyset$

We check any points on line l: l^T and $l^T y = \emptyset$

$$\underbrace{(\alpha x + \beta y)^T}_{z \in l} C \underbrace{(\alpha x + \beta y)}_{z \in l} = \underbrace{\alpha^2 x^T C x}_{\emptyset} + \underbrace{2\alpha\beta x^T C y}_{\emptyset} + \underbrace{\beta^2 y^T C y}_{\emptyset} = \emptyset$$

so then C degenerate! WE DON'T CONSIDER IT! ✓

(56) ADJOINT OF A CONIC

$adj(C) =$ if C is full rank then $adj(C) = C^{-1} = C^*$

- l is tangent to C and an element of dual conic $C^* \iff l^T C^* l = 0$

57 Proof that $l^T C^* l = 0$ CAPTURES ALL LINES IN C^*

- if x is on C, ex: $x^T C x = 0$ then $l = Cx$ is a line in C^*
- if C is full rank $\Rightarrow$ $\boxed{x = C^{-1} l}$

$\hookrightarrow l^T C^* l = l^T C^{-1} l \underset{\text{Simmetry}}{=} l^T C^{-T} \underbrace{C C^{-1}}_{I} l$

$= \left(\boxed{C^{-1} l}\right)^T C \left(\boxed{C^{-1} l}\right) = x^T C x = 0$

58 DESCRIBE P^n

- Points and lines in $\mathbb{P}^2$ are represented as vectors in $\mathbb{R}^3 \setminus \{0\}$
- Conics in $\mathbb{P}^2$ are represented as symmetric matrices from $\mathbb{R}^{3\times3} \setminus \{0\}$

59 PROJECTIVE TRANSFORMATIONS IN 2D

Projective trans: mapping $h: \mathbb{P}^2 \to \mathbb{P}$

- h is linear in homo coords
- h is a non-singular matrix $H \in \mathbb{R}^{3\times3}$

So $h(x) = Hx$

- H is homogenious $\Rightarrow$ can be scaled by any λ without changing
- 8 degrees of fredom: 4 original $\Rightarrow$ 4 new points correspondences

(60) PROVE 3 COLLINEAR POINTS STAY COLLINEAR UNDER A PROJECTIVE TRANSFORMATION

>> if x_i points are collinear $\Rightarrow \exists\ l$ line such that $x_i^T l = \emptyset$ for all points i

>> if we take the transformed points Hx_i we claim they lie on the transformed line: $\underbrace{(Hx_i)^T}_{x_i'} \underbrace{(H^{-T} l)}_{l'} = \emptyset$ we claim transformed points are on the line

Line transformed with H^{-T}

$$\underbrace{(Hx_i)^T}_{x'} \underbrace{(H^{-T} l)}_{l'} = x_i^T \underbrace{H^T H^{-T}}_{\substack{(H^{-1}H)^T \\ = \\ I}} l = x_i^T l = \emptyset$$

$x_i' = Hx_i$ lie on l' $\forall i$ ✓

61 FIND MAPPING WITH H 4points ⇒ 4 points

STARTING WITH 4 points: $x_1\ y_1\quad x_3\ y_3$
$x_2\ y_2\quad x_4\ y_4$

CONVERT TO HOMOGENIUS
$Original_1 = (x_1, y_1, 1)$
$Original_2 = (x_2, y_2, 1)$
$Original_3 = (x_3, y_3, 1)$
$Original_4 = (x_4, y_4, 1)$

$$\frac{new X_1}{new Z_1} = \frac{H \cdot Original_1 X}{H \cdot Original_1 Z} \qquad \frac{new y_1}{new Z_1} = \frac{H \cdot Original_1 Y}{H \cdot Original_1 Z}$$

$$\frac{new X_2}{new Z_2} = \frac{H \cdot Original_2 X}{H \cdot Original_2 Z} \qquad \frac{new y_2}{new Z_2} = \frac{H \cdot Original_2 Y}{H \cdot Original_2 Z}$$

$$\frac{new X_3}{new Z_3} = \frac{H \cdot Original_3 X}{H \cdot Original_3 Z} \qquad \frac{new y_3}{new Z_3} = \frac{H \cdot Original_3 Y}{H \cdot Original_3 Z}$$

$$\frac{new X_4}{new Z_4} = \frac{H \cdot Original_4 X}{H \cdot Original_4 Z} \qquad \frac{new y_4}{new Z_4} = \frac{H \cdot Original_4 Y}{H \cdot Original_4 Z}$$

SOLVE H: and get matrix coefficients

H_g is 1! HOMO

(62) TRANSFORM A POINT x TO POINT x'

$$x \mapsto x' = Hx$$

(63) TRANSFORM A LINE l TO l'

$$l \mapsto l' = H^{-T} l$$ PROOF (55)

(64) CONIC C TO CONIC C'

$$C \mapsto C' = H^{-T} \underbrace{C}_{\text{matrix of conic}} H^{-1}$$

x lies on C if only if $x^T C x = 0$

Hx lie on $C' = H^{-T} C H^{-1}$

let's check:

$$(Hx)^T C' (Hx) \overset{?}{=} 0 \Rightarrow (x^T H^T)(H^{-T} C H^{-1})(Hx)$$

$$\Rightarrow x^T C x \qquad \underbrace{H^T H^{-T}}_{I} \qquad \underbrace{H^{-1} H}_{I} \qquad \checkmark$$

(65) DUAL CONIC TRANSFORM PROOF

$$C^* \mapsto C^{*\prime} = HC^*H^T$$

$$l^T C^* l = 0$$

$$(H^{-T}l)^T (HC^*H^T)(H^{-T}l)$$

$$(l^T \underbrace{H^{-1})(H}_{I} C^* \underbrace{H^T)(H^{-T}}_{(H^{-1}H)^T = I} l)$$

$$l^T C^* l = 0 \quad \checkmark$$

(66) SPECIAL CASE: ISOMETRIES

$$H_I = \begin{pmatrix} \epsilon\cos\theta & -\sin\theta & t_x \\ \epsilon\sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

IF $\epsilon = 1, -1$

THEN WE GET

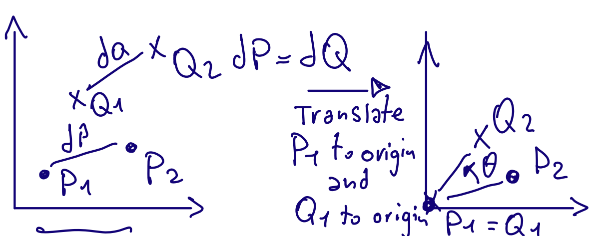
ROTATION + TRANSATION

or

REFLECTION ($\epsilon = -1$)

3 DOF PLUS ϵ

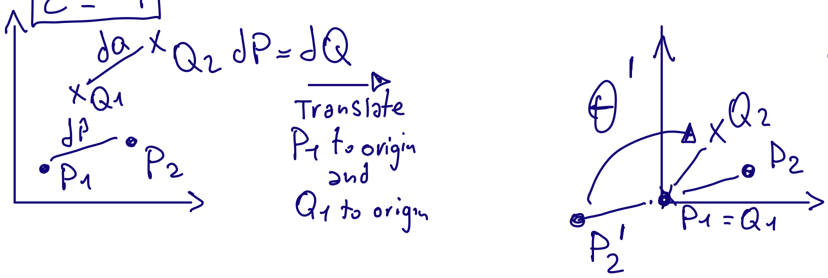
- Determined by 2 points (equidistant pairs)



We have to find θ

$\epsilon = +1$

$\boxed{\epsilon = -1}$



We have to find θ'

67 EUCLEDIAN TRANSFORMATIONS (DISPLACEMENT)

Rotation + Translation

$$H_E = \begin{pmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

R_θ

R_θ is orthogonal: $R_\theta^T R_\theta = R_\theta R_\theta^T = I$

IF $t=0$ pure rotation

IF $R_\theta = I$ pure translation

68 SIMILARITIES (SHAPE-PRESERVING)

$$H_S = \begin{pmatrix} \epsilon S\cos\theta & -S\sin\theta & t_x \\ \epsilon S\sin\theta & S\cos\theta & t_y \\ 0 & 0 & 1 \end{pmatrix}$$

$\epsilon = \pm 1$

S = SCALING

4 DOF PLUS ϵ

with 2 points correspondence

INVARIANT: angles, ratios of length

69 AFFINE TRANSFORMATIONS

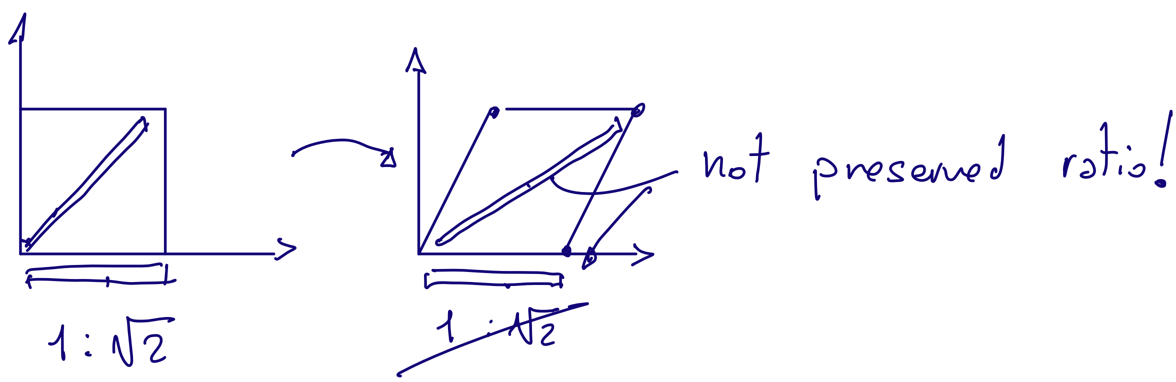
$$H_A = \begin{pmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ non singular}$$

6 degs freedom   3-point correspondence

INVARIANT:

- Parallel lines and ideal points as $l_\infty = H_A^{-T} l_\infty$
- Ratios of lengths of parallel line segments
- Ratio of areas

- example of shear:



70 COMPACT RAPPRESENTATION OF AFFINE TRANS

$$H_A = \begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}$$

IN CARTESIAN

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \mathbf{A}\mathbf{x} + \mathbf{t}$$

How?

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{A} & \mathbf{t} \\ \mathbf{0}^T & 1 \end{pmatrix}\begin{pmatrix} \mathbf{x} \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{A}\mathbf{x} + \mathbf{t} \\ 1 \end{pmatrix} \rightsquigarrow \mathbf{A}\mathbf{x} + \mathbf{t}$$

71 DECOMPOSE LINEAR PART OF AFFINE TRANS IN ROTATION AND SCALINGS

$$A = R_\psi \underbrace{\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}}_{\text{Scaling}} R_\phi = R_\theta R_{-\phi} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R_\phi$$

SING. VALUE DECOMP: $A = \underbrace{U}_{R_\psi} D \underbrace{V^T}_{R_\phi}$ with $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

U, V orthogonal

$$A = \underbrace{UV^T}_{R_\theta} \underbrace{V}_{R_{-\phi}} D \underbrace{V^T}_{R_\phi}$$

72 AFFINE MAPPING, SOHW THAT INFINITY LINE MAPPED TO INFINITY LINE

$$H_A^{-T} \cdot l_\infty \overset{?}{=} l_\infty \qquad \text{with } l_\infty = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$H_A = \begin{pmatrix} A & t \\ 0 & 1 \end{pmatrix}, \quad H_A^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}t \\ 0^T & 1 \end{pmatrix}$$

$\Downarrow$ Proof

$$H_A \cdot H_A^{-1} = \begin{pmatrix} I & -\underbrace{AA^{-1}}_{I}t + t = 0 \\ 0^T & 1 \end{pmatrix} = I_{3\times3}$$

Line at ∞

$$H_A^{-T} = \begin{pmatrix} A^{-T} & 0 \\ -t^T A^{-T} & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Line at ∞ ✓

73 PROJECTIVE TRANSFORMATION

(Like affine but with anything in last row)

$H_P = \begin{pmatrix} A & t \\ v^T & w \end{pmatrix}$ where $v^t = (v_1, v_2)$ } any row vector

which still $\det(H_P) \neq 0$

IN CARTESIAN

$x = \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \dfrac{Ax+t}{v^T x + w}$

<u>Proof</u>

cartesian $\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\begin{pmatrix} A & t \\ v^T & w \end{pmatrix} \begin{pmatrix} \underline{x} \\ 1 \end{pmatrix} = \begin{pmatrix} Ax+t \\ v^T x + w \end{pmatrix} \sim \dfrac{Ax+t}{v^T x + w}$

if $v^T x + w = 0$ map at ∞

ALSO TRANSFORMS COORDINATES AT ∞!

$\begin{pmatrix} A & t \\ v^T & w \end{pmatrix} \begin{pmatrix} \underline{x} \\ 0 \end{pmatrix} = \begin{pmatrix} Ax \\ v^T x \end{pmatrix} \rightsquigarrow \dfrac{Ax}{v^T x}$

point at ∞

if $v^T x = 0$ THEY STAY AT ∞

IF $v = 0$ and $w \neq 0 \rightsquigarrow$ affine trans

- 8 DOF Determined by 4 points - No 3 collinear

INVARIANTS

- lines
- cross ratios $\dfrac{\|P_3 - P_1\| \cdot \|P_4 - P_2\|}{\|P_3 - P_2\| \cdot \|P_4 - P_1\|}$

(74) PROJECTIVE TRANSFORMATION LINE AT ∞

$$l_\infty = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{maps to} \quad l'_\infty = \begin{pmatrix} -A^{-T}v \\ 1 \end{pmatrix}$$

Proof:

$$\begin{pmatrix} A & t \\ v^T & w \end{pmatrix}^{-T} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -A^{-T}v \\ 1 \end{pmatrix}$$

So line at ∞ not necessarly maps to a line at ∞

except if $v = \emptyset$

(75) TRANS PROPRIETIES (WHAT TO CHOOSE?)

- PROJECTIVE $\rightarrow$ INVARIANT:
  - Concurrency
  - Collinearity
  - Order of contact (intersect, tangency, inflection)
  - cross ratio
- AFFINE $\rightarrow$ INVARIANT: - parallelism, ratio of area, ratio of length on collinear, line at infinity
- SIMILARITY $\rightarrow$ INVARIANT: - ratio of lengths, angle, circular point
- EUCLIDIAN $\rightarrow$ INVARIANT: - length, area

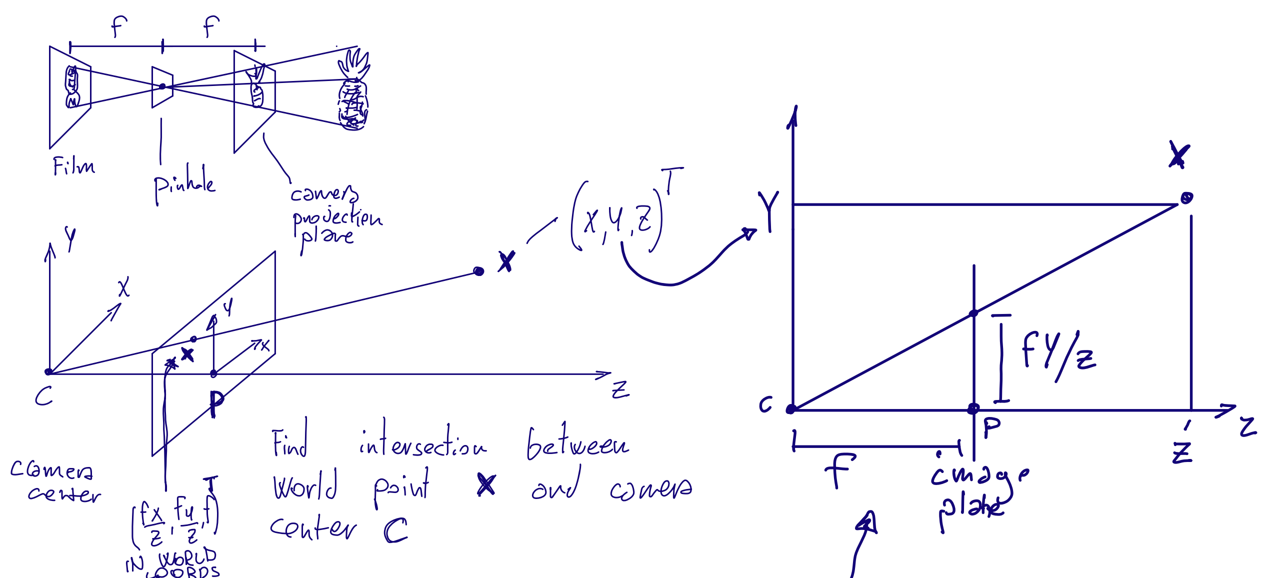
(76) CAMERA BASICS

It's a projection from world point $\mathbb{R}^3$ to an image point $\mathbb{R}^2$.

- $P = \mathbb{R}^{3 \times 4}$

WHERE P has 11 <u>DOF</u>

(77) PINHOLE CAMERA BASIC



Find intersection between World point X and camera center C

ratio of Z coordinate

$$\frac{q}{y} = \frac{f}{Z} = \frac{f \cdot y}{Z}$$

X coordinate is Just f!

F is the same on all points, so we can ignore it

(78) PINHOLE CAMERA MAPPING

$$\mathbf{X} \to x = P\mathbf{X}$$

real

camera mapping

FROM A WORLD POINT $\mathbf{X} = (X, Y, Z, 1)^T$ in $\mathbb{R}^3$

TO AN IMAGE POINT $x = (fx, fy, f)^T$ in $\mathbb{R}^2$

USING P:

$$x = P\mathbf{X} \Rightarrow$$

(79) CENTRAL PROJECTION FOR PINHOLE (with focal length f)

$$P = \begin{pmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{pmatrix}$$

$$K = \begin{pmatrix} f & & \\ & f & \\ & & 1 \end{pmatrix}$$

IN HOMOGENIOUS

Ex: $\begin{pmatrix} f & & & 0 \\ & f & & 0 \\ & & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} fx \\ fy \\ z \end{pmatrix} \sim \begin{pmatrix} \frac{fx}{z} \\ \frac{fy}{z} \\ 1 \end{pmatrix} \simeq \begin{pmatrix} fx \\ fy \\ f \end{pmatrix}$ where $x = \frac{fx}{z}$, $y = \frac{fy}{z}$

- USING OFFSET

$\mathbf{p} = (p_x, p_y)^T$ = OFFSET

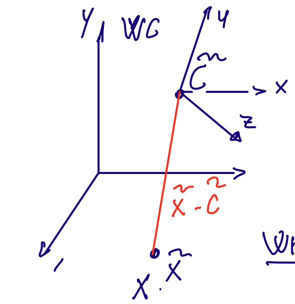
camera param $\mathbf{K} = \begin{pmatrix} f & & p_x \\ & f & p_y \\ & & 1 \end{pmatrix}$

$$P = K[I|0] = \begin{pmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{pmatrix}$$

HOMO

Ex: $\begin{pmatrix} f & & p_x & 0 \\ & f & p_y & 0 \\ & & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \sim \begin{pmatrix} fx + p_x z \\ fy + p_y z \\ z \end{pmatrix} = \begin{pmatrix} \frac{fx}{z} + p_x \\ \frac{fy}{z} + p_y \\ 1 \end{pmatrix}$

(80) NON-CENTRAL ORIENTATION PROJECTION FOR PINHOLE



Express any point from world coord system into camera coordinate system

WHY? So we can do projection into camera image

EXPRESS WORLD POINT IN CAMERA COORDS : $\tilde{\mathbf{X}}$ world

$$\underbrace{\mathbf{X}}_{\text{point coord in camera reference}} = \begin{pmatrix} \underbrace{R}_{\text{rotation matrix of camera orient.}} & -R\underbrace{\tilde{C}}_{\text{world coords of camera center}} \\ 0^T & 1 \end{pmatrix} \tilde{\mathbf{X}}$$

ex: $\begin{pmatrix} \mathbf{X} \\ 1 \end{pmatrix} = \begin{pmatrix} R & -R\tilde{C} \\ \mathbf{0}^T & 1 \end{pmatrix} \begin{pmatrix} \tilde{\mathbf{X}} \\ 1 \end{pmatrix} = \begin{pmatrix} R\tilde{\mathbf{X}} - R\tilde{\mathbf{C}} \\ 1 \end{pmatrix}$

- NO HOMO:

$$\boxed{\mathbf{X} = R\tilde{\mathbf{X}} - R\tilde{\mathbf{C}}} = \underbrace{R}_{\text{Rotation of camera frame to world frame}} \underbrace{(\tilde{\mathbf{X}} - \tilde{\mathbf{C}})}_{\text{Distance between world camera center and world } \tilde{X}}$$

$$R = \text{ortho matrix} = \begin{pmatrix} \hat{x}^T \\ \hat{y}^T \\ \hat{z}^T \end{pmatrix}$$

$\hat{x}, \hat{y}, \hat{z}$ are unit vectors of word coords of camera axes

~ explained in (81) ~

81 GET WORLD COORD OF POINT FROM CAMERA COORD

to get from camera coord $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ to world $\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{X}$, $\begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \tilde{\mathbf{X}}$

$$\tilde{\mathbf{X}} = x \cdot \hat{X} + y \cdot \hat{Y} + z \cdot \hat{Z} + \tilde{C}$$

($\hat{X}, \hat{Y}, \hat{Z}$: world coordinate of camera x, y, z axis)

($\tilde{C}$: CAMERA CENTER IN WORLD COORDINATE)

so

$$\tilde{\mathbf{X}} = \underbrace{\left(\hat{X}, \hat{Y}, \hat{Z}\right)}_{R^{-1}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \tilde{C}$$

$$\tilde{\mathbf{X}} = R^{-1} \mathbf{X} + \tilde{C} \iff R\left(\tilde{\mathbf{X}} - \tilde{C}\right) = \mathbf{X}$$

$\hat{X}, \hat{Y}, \hat{Z}$: unit vectors ortho representing world coords of camera axes

$$R = \text{orthogonal matrix} = \begin{pmatrix} \hat{X}^T \\ \hat{Y}^T \\ \hat{Z}^T \end{pmatrix}$$

$$R^{-1} = R^T$$

(82) GO FROM WORLD TO PIXEL

Rotation matrix : $(\tilde{X}, \tilde{Y}, \tilde{Z})^T$ Camera center in world coordinate

$$\underbrace{X}_{\text{camera coordinates in 3D}} = \underbrace{\begin{pmatrix} R & -R\tilde{C} \\ 0^T & 1 \end{pmatrix}}_{\text{TRANSFORM FROM WORLD TO CAMERA}} \underbrace{\tilde{X}}_{\text{Point in world coord}}$$

From World coordinate to Camera coordinates in homogenious

- FULL PROJECTION -

WE COMBINE IT WITH THE PROJ. TO 2D (THE OLD P!)

World → camera 3D → camera 2D

point in pixels

$$x = P\tilde{X}$$

point in 3D WORLD

where

camera internal parameters

$$P = KR\left[I \mid -\tilde{C}\right] = \left[\underbrace{KR}_{3\times 3} \mid -KR\tilde{C}\right]$$

- FROM CAMERA 3D TO CAMERA 2D

pixels, camera 3D point

$$x = P_{2D} X$$

BUT! X IN CAMERA 3D, X IN WORLD

$$X = \begin{pmatrix} R & -R\tilde{C} \\ 0^T & 1 \end{pmatrix} \tilde{X}$$

pixels

$$x = P_{2D} \begin{pmatrix} R & -R\tilde{C} \\ 0^T & 1 \end{pmatrix} \tilde{X} = K[I \mid 0] \begin{pmatrix} R & -R\tilde{C} \\ 0^T & 1 \end{pmatrix} \tilde{X}$$

$$= K\left[R \mid -R\tilde{C}\right]\tilde{X} = K[R \mid t]\tilde{X}$$

$$t = -R\tilde{C}$$

$$\underbrace{x}_{\text{pixel}} = \underbrace{KR\left[I \mid -\tilde{C}\right]}_{P_{NEW}}\tilde{X}$$

So $P = \underline{9\ DOF}$

(83) PINHOLE CAMERA WITH NON-SQUARE PIXELS

$$K = \begin{pmatrix} a_x & & x_0 \\ & a_y & y_0 \\ & & 1 \end{pmatrix}$$

IF pixels are sized m_x, m_y in image unit distance then:

$$\left.\begin{array}{l} a_x = f m_x \\ a_y = f m_y \end{array}\right\} \text{new focal lengths}$$

$$\left.\begin{array}{l} x_0 = m_x p_x \\ y_0 = m_y p_y \end{array}\right\} \text{new principal point coordinates}$$

So P = 10 DOF in (88)

(84) SKEWNESS

$$K = \begin{pmatrix} a_x & s & x_0 \\ & a_y & y_0 \\ & & 1 \end{pmatrix}$$

$$P = KR[I \mid -\tilde{C}] = K[R \mid t] \qquad t = -R\tilde{C}$$

11 DOF

$\text{rank}(P) = 3 \leftarrow$ finite projective camera

85 GIVEN ONLY P : CAMERA PROPRIETIES

GIVEN $P = [M \mid p_4]$ where $M \in \mathbb{R}^{3\times3}$ and $p_4 \in \mathbb{R}^3$

- WORLD COORDINATES OF CAMERA CENTER $C \in \mathbb{R}^4$ HOMOGENIOUS COORDINATE such that $PC = \emptyset$

- FINITE CAMERA : M is NON singular?

↳ For finite camera non sing. then $C^T = (\tilde{C}^T, 1)$ M - non singular

↳ For camera at infinity then $C^T = (d^T, 0)$ and $Md = 0$ and camera is <u>singular</u>

$P = [M \mid p_4]$, $P \cdot \begin{pmatrix} d \\ 0 \end{pmatrix} = \begin{pmatrix} Md \\ 0 \end{pmatrix} \approx \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ point at infinity in direction d

$Md = \emptyset$

86 PROVE C IS CAMERA CENTER AND IS NULL-SPACE OF P

- Suppose that C is such that $P \cdot C = 0$ (CAMERA CENTER)
- CONSIDER LINE BETWEEN C and some point $A \neq C$

THEN $\underbrace{X(\lambda)}_{\text{any point on line}} = \lambda A + (1-\lambda) C \quad \lambda \in \mathbb{R}$

WE MAP Point in line through A and C

$$P \cdot X(\lambda) = \lambda PA + \underbrace{(1-\lambda) PC}_{\emptyset} = \lambda PA \quad \underline{\forall \lambda \neq 0}$$

THIS MEANS THAT THE WHOLE LINE IS PROJECTED TO SAME IMAGE POINT $\underline{PA}$

WHAT THAT MEANS?

These points must be alligned through the camera center.

Because it works for any line between C and A then Proved that C is camera center ~~or~~ projection center

87) COMPUTE CAMERA CENTER C SINGULAR

$M = (P_1, P_2, P_3)$ $C = (X, Y, Z, W)^T$

then matrix! $P \in \mathbb{R}^3$

$X = \det(P_1, P_3, P_4)$ $Y = -\det(P_1, P_3, P_4)$

$Z = \det(P_1, P_2, P_4)$ $W = -\det(P_1, P_2, P_3)$

88) COMPUTE CAMERA CENTER C NONSINGULAR

$M = (P_1, P_2, P_3)$ $C = (X, Y, Z, W)^T$

$\tilde{C} = -M^{-1} P_4$ $C^T = \underline{(\tilde{C}^T, 1)}$

HOMOGENIOUS!

$P \cdot \begin{pmatrix} \tilde{C} \\ 1 \end{pmatrix} = \begin{pmatrix} -P_4 + P_4 = \emptyset \\ 0 \end{pmatrix} = \emptyset$

89) RETRIEVE K AND R FROM P

$P = [M \mid P_4]$

$\downarrow$

$M = KR$ using RQ decomposition

$-P_4 = -M\tilde{C}$

- RQ DECOMP: $\underline{QR = M^T \Pi}$ where $\Pi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

  COMPUTE QR NORMAL ON THIS

↳ And the result will be: $K = \Pi R^T \Pi$

$R = \Pi Q^T$

(90) PROVE $K \cdot R = M$

$$K \cdot R = \Pi R^T \underbrace{\Pi \cdot \Pi}_{I} \cdot Q^T = \Pi R^T Q^T = \Pi (QR)^T$$

$$= \Pi (M^T \Pi)^T = \underbrace{\Pi \Pi^T}_{I} M$$

$$= M$$

(91) CAMERA AT INFINITY

M IS SINGULAR (HOW IS P ?)

- AFFINE

LAST ROW OF P $= (0,0,0,1)$ MAPS $\infty \to \infty$

(92) PROVE IF 3 POINTS ARE COLLINEAR THEN CONIC IS DEGENERATE

$$\left.\begin{aligned} p^T C p &= 0 \\ q^T C q &= 0 \\ r^T C r &= 0 \end{aligned}\right\} \text{3 points on conic}$$

Since r is collinear with p & q, there

NO-HOMO: $r = \underbrace{(1-\lambda)p + \lambda q}_{\substack{\text{AFFINE COMBO} \\ \text{BARYCENTRIC}}}$ for some $\lambda \in \mathbb{R}$

HOMO: exist $\lambda, \mu \in \mathbb{R}/\{0\}$ then $r = \lambda p + \mu q$

Then $r^T C r = (\lambda p + \mu q)^T C (\lambda p + \mu q)$

$$\underbrace{r^T C r}_{\emptyset !} \overset{!}{=} \lambda^2 \underbrace{p^T C p}_{\emptyset} + 2\lambda\mu \underbrace{p^T C q}_{\emptyset} + \mu^2 \underbrace{q^T C q}_{\emptyset}$$

$$\Rightarrow \underbrace{p^T C q = q^T C p = 0}$$

You can do it for any other point on the line

so we can conclude $r^T C r = 0$ for any r

$\boxed{\text{ANY POINT ON LINE IS ON C}}$

CONT

↓

92 ⇓
CONTINUAT.

- we still need to prove C is degenerate

↳ We get $l = p \times q$ wich is the homogenious coordinate of the line through p and q
l is the vector orthogonal both to p and to q.

$$Cp \times l = \underbrace{Cp \times (p \times q)} = p \cdot \underbrace{(Cp)^T q}_{\text{scalar}} - q \underbrace{(Cp)^T p}_{\text{scalar}}$$

$$A \times B \times C = B \cdot \underbrace{[A^T C]}_{\text{SCAL}} - C \cdot \underbrace{[A^T B]}_{\text{SCAL}}$$

$$= \underbrace{p p^T C q}_{\emptyset} - \underbrace{q p^T C p}_{\emptyset} = \emptyset$$

↓ SAME WAY

$$Cq \times l = \emptyset$$

If cross product of 2 vectors is $\emptyset$

⇓

$Cp \parallel l$ and $Cq \parallel l$

so

$Cp = \lambda l$ & $Cq = \mu l$ for some $\lambda, \mu \in \mathbb{R}$

LET $\underset{\text{POINT}}{s} = \mu p - \lambda q \Rightarrow Cs = C(\mu p - \lambda q) = \mu Cp - \lambda Cq$
$= \mu \lambda l - \lambda \mu l = \emptyset$

We found some point s in nullspace of C and C is degenerate

WITH $s \neq 0$!
Because if $s = 0 \Rightarrow \mu p = \lambda q$
⇓
so p & q would be same point
Because HOMO

(93) COMPUTE THE CAMERA MATRIX

$$P \simeq K[R \mid -R\tilde{C}] \in \mathbb{R}^{3\times 4}$$

camera matrix

11 DOF : 5 internal: in K

6 external: R rotation an position $\tilde{C}$

- GIVEN THE CORRESPONDENCES $X_i \longleftrightarrow x_i$

world image

FIND P SUCH THAT:

$x_i = PX_i$ for all i's

- As P HAS 11 DOF, WE NEED AT LEAST 5 ½ CORRESPONDENCES.

WE CAN USE MORE AND FIND <u>LEAST-SQUARES</u>

94 DLT (DIRECT LINEAR TRANSFORM)

HOMO COORDINATES!

Instead of looking at $\mathbf{x}_i = P\mathbf{X}_i$, we need $\lambda_i \mathbf{x}_i = P\mathbf{X}_i$ for some $\lambda \neq 0$

>> $\underline{\mathbf{x}_i \times P\mathbf{X}_i = 0}$

$$P = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{pmatrix} \begin{matrix} - P_1^T \\ - P_2^T \\ - P_3^T \end{matrix} \qquad P_1 = \begin{pmatrix} p_1 \\ \vdots \\ p_4 \end{pmatrix} \quad P_2 = \begin{pmatrix} p_5 \\ \vdots \\ p_8 \end{pmatrix} \quad P_3 = \begin{pmatrix} p_9 \\ \vdots \\ p_{12} \end{pmatrix}$$

$$P\mathbf{X}_i = \begin{pmatrix} p_1 & p_2 & p_3 & p_4 \\ p_5 & p_6 & p_7 & p_8 \\ p_9 & p_{10} & p_{11} & p_{12} \end{pmatrix} \mathbf{X}_i = \begin{pmatrix} P_1^T X_i \\ P_2^T X_i \\ P_3^T X_i \end{pmatrix}$$

$$x_i \times P\mathbf{X}_i = \begin{pmatrix} y_i P_3^T \mathbf{X}_i - w_i P_2^T \mathbf{X}_i \\ w_i P_1^T \mathbf{X}_i - x_i P_3^T \mathbf{X}_i \\ x_i P_2^T \mathbf{X}_i - y_i P_1^T \mathbf{X}_i \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \overset{4}{\mathbf{0}^T} & \overset{4}{-w_i \mathbf{X}_i^T} & \overset{4}{y_i \mathbf{X}_i^T} \\ w_i \mathbf{X}_i^T & \mathbf{0}^T & -x_i \mathbf{X}_i^T \\ -y_i \mathbf{X}_i^T & x_i \mathbf{X}_i^T & \mathbf{0}^T \end{pmatrix}}_{A_i} \begin{pmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \mathbf{P}_3 \end{pmatrix} = \overset{3\times 12}{A_i} \overset{12}{\mathbf{p}}$$

$$A_i = \begin{pmatrix} a_i^T \\ b_i^T \\ c_i^T \end{pmatrix} \begin{matrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{matrix}$$

So if we take

$$x_i \cdot a_i + y_i \cdot b_i \Rightarrow \begin{pmatrix} w_i y_i \mathbf{X}_i \\ -w_i x_i \mathbf{X}_i \\ x_i y_i \mathbf{X}_i - x_i y_i \mathbf{X}_i \end{pmatrix} = w_i \begin{pmatrix} y_i \mathbf{X}_i \\ -x_i \mathbf{X}_i \\ \mathbf{0} \end{pmatrix} = \boxed{-c_i} \cdot w_i$$

SO WE ONLY NEED:

for each point:

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -w_i(X_r) & -w_i(Y_r) & -w_i(Z_r) & -w_i(1) & y_i(X_r) & y_i(Y_r) & y_i(Z_r) & y_i(1) \\ w_i(X_r) & w_i(Y_r) & w_i(Z_r) & w_i(1) & 0 & 0 & 0 & 0 & x_i(X_r) & x_i(Y_r) & x_i(Z_r) & x_i(1) \end{bmatrix}$$

95 IMAGE POINT x ASSOCIATED TO LINE IN WORLD POINTS

$$X(\lambda) = C + \lambda \begin{pmatrix} (KR)^{-1} x \\ 0 \end{pmatrix}$$

- $KR = M$ $\quad M \rightarrow P = [M \mid p_4]$
- C camera center
- $PC = 0$ $\quad$ • $C^T = \underbrace{(-M^{-1} p_4, 1)}_{\text{HOMO}}{}^T$

WE START:

$$P \cdot X(\lambda) = P\left(C + \lambda \begin{pmatrix} (KR)^{-1} \cdot x \\ 0 \end{pmatrix}\right)$$

$$= [M \mid p_4] \left(\begin{pmatrix} \tilde{C} \\ 1 \end{pmatrix} + \begin{pmatrix} \lambda M^{-1} \cdot x \\ 0 \end{pmatrix} \right)$$

$$= (M\tilde{C} + p_4) + (\lambda x + 0) = \underbrace{\lambda x \approx x}_{\text{HOMO}}$$

$$\tilde{C} = -M^{-1} p_4$$

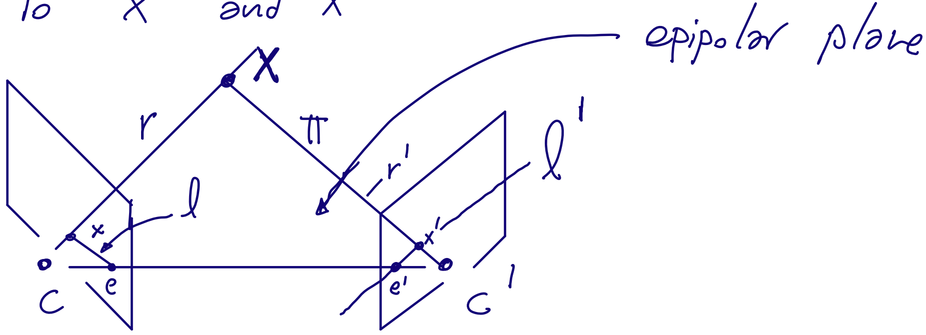
(96) STEREO VISION BASICS

USUALLY: P maps world points $\mathbf{X}$ to image points $x = P\mathbf{X}$.

We use correspondences to map $\mathbf{X}_i \leftrightarrow x_i$

IF 2 CAMERAS:

2 Projection matrices P and P' mapping $\mathbf{X}$ to x and x'

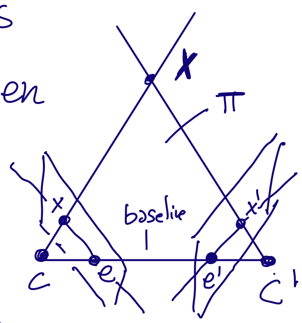


- Given x we know $\mathbf{X}$ must lie on ray $r = \overrightarrow{Cx}$
- We know x' must lie on the ray r' wich is a ray from C' to $\mathbf{X}$ $r' = \overrightarrow{C\mathbf{X}}$

- So x' must lie on the provection of r on the second image whic is l' = epipolar line
- l' = epipolar line is intersection between π and the second image
- e' is provection of C on second image

(97) EPIPOLAR GEOMETRY

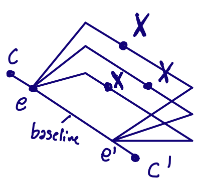
- Π : epipolar plane is uniquely determined by x, C and C' and <u>always</u> contains the baseline between C & C'



- l & l' : epipolar lines are the intersection between the images and Π

- e & e' are the other camera on an image plane

VARYING X : As X changes in height, the plane Π will rotate over baseline



} Pencil of epipolar planes Π

(98) FIND l' EPIPOLAR LINE ON 2^{nd} IMAGE (USING F)

Find mapping from $x \mapsto l'$

- So: $l' = Fx$ F: Fundamental matrix

(98A) PROOF THAT A FUNDAMENTAL MATRIX THAT MAPS x TO l' EXISTS

- $l' = Fx$ is projection of the ray $r = \overrightarrow{Cx}$ into the second image plane

>> we can take 2 points on ray r and project them in 2nd image plane, connect them and have the line.

POINT 1: C, that will project to $\underbrace{e'}_{\text{epipole}} = P'C$

2nd camera projection matrix; first camera world coords

if P is square, $P^{-1} = P^{+}$

original camera point

POINT 2: $\underbrace{P^{+}}_{}x$ ← We project this point to the 2^{nd} image as: $\underbrace{P'P^{+}x}_{}$

Pseudoinverse of P, as P is 3x4 so we can't get inverse } so → P^{+} will be matrix such that $PP^{+} = I \in \mathbb{R}^{3\times3}$ $P^{+} = P^{T}(PP^{T})^{-1}$

HOW TO KNOW THIS IS REALLY A POINT ON RAY $r = \overrightarrow{Cx}$?

Yes!: If we project it in the first image:

$\underbrace{PP^{+}}_{I}x = x$ ✓

>> So now we connect the 2 points to get l'

$$l' = \underbrace{e' \underbrace{\times}_{\text{cross prod}} (P'P^{+}}_{F} x) = \underbrace{[e']_{\times} P'P^{+}}_{\text{NOTATION IN (100)}} x$$

DOESN'T DEPEND ON X! ONLY ON CAMERAS C, C'

99 NOTATION OF X CROSS PRODUCT AS MATRICES

represent $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ AS:

$$[a]_\times = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

THEN THE CROSS PRODUCT $a \times b$ CAN BE REPRESENTED BY:

$$a \times b = \underbrace{[a]_\times}_{\text{skew symmetrix matrix}} b = \left(a^T [b]_\times\right)^T$$

(100) FIND F IF WORLD COORDS ARE ALLIGNED TO FIRST CAMERA

(NO ROTATION AND NO SHIFT!)

$P = K[I|0] = [K|0]$ we only use (79)

WE CAN ASSUME IT! AS WE CHOOSE THE WORLD

$P' = K'[R'|t']$ for second camera

1) LET'S FIND $P^+ = P^T(PP^T)^{-1} = \begin{bmatrix} K^T \\ 0^T \end{bmatrix}\left([K\ 0]\begin{bmatrix} K^T \\ 0^T \end{bmatrix}\right)$

$= \begin{pmatrix} K^T \\ 0^T \end{pmatrix}(KK^T)^{-1} = \underbrace{\begin{pmatrix} K^{-1} \\ 0^T \end{pmatrix}}_{P^+}$

2) LET'S FIND $e' = P'C = \underbrace{[K'R' | K't']}_{P'}\underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{C} = \underbrace{K't'}_{e'}$

3) LET'S FIND $e = PC' = \underbrace{[K|0]}_{P}$ [WE USE (88)] C'

$= [K|0]\begin{bmatrix} -(\underbrace{K'R'}_{M})^{-1}\underbrace{K't'}_{P_4} \\ 1 \end{bmatrix} = K(R')^{-1}\underbrace{(K')^{-1}K'}_{I}t' = \underbrace{K(R')^Tt'}_{e}$

omit -1

(88) $M' = K'R'$, $P_4 = K't'$ $\Rightarrow -(M')^{-1}P_4 = -(K'R')^{-1}K't' = \boxed{-R'^Tt'}$ C'

4) Now with e, e' let's get

WHY WE CAN ? (101)

$F = [e']_\times P'P^T = [e']_\times K'R'K^{-1} = (K')^{-T}R'K^T[e]_\times$

(101) $[e']_\times (K'R'K^{-1}) \Longrightarrow (K')^{-T} R' K^{T} [e]_\times$

rule:

$$\underbrace{[w]_\times}_{\text{matrix vector}} \underbrace{M}_{\text{non singular matrix}} = M^{-T}[M^{-1}w]_\times$$

up to scale factor determinant(M) WITH NO-HOMO

- <u>BUT</u> BECAUSE WE HAVE HOMO:

So if $\lambda \neq 0$ scaling doesn't matter

SO IN OUR CASE:

$$[e']_\times (K'R'K^{-1}) = K'^{-T}[\underbrace{K'^{-1}e'}_{t'}]_\times R'K^{-1} =$$

t' because of $e' = K't'$ in (100)

$$= K'^{-T}[t']_\times R'K^{-1} = K'^{-T}R'[R'^{T}t']_\times K^{-1} = K'^{-T}R'K^{T}[\underbrace{KR'^{T}t'}_{e}]_\times = K'^{-T}R'K^{T}[e]_\times$$

102 GENERAL CASE OF FINDING F

$P=[M|p_4]$ $\quad P'=[M'|p_4']$

then

$e=-MM'^{-1}p_4'+p_4$ $\quad e'=-M'M^{-1}p_4+p_4'$

$F=[e']_\times M'M^{-1}=M'^{-T}M^T[e]_\times$

FIND e and e' (Proof)

$$e=PC'=\underbrace{[M|p_4]}_{P}\underbrace{\begin{bmatrix}-M'^{-1}p_4'\\1\end{bmatrix}}_{c'}$$

Because 88

$= -MM'^{-1}p_4'+p_4$ ✓

$$e'=P'C=\underbrace{[M'|p_4']}_{P'}\begin{bmatrix}-M^{-1}p_4\\1\end{bmatrix}=-M'M^{-1}p_4+p_4'\ ✓$$

1) GET POINTS

FIRST POINT: $C \rightarrow e'$

$$r(\lambda)=C+\lambda\overbrace{\underbrace{\begin{pmatrix}M^{-1}x\\0\end{pmatrix}}_{\text{Point on ray at }\infty}}^{\text{any point on ray}}$$

SECOND POINT:

p_∞ = Point on r at ∞ $\longmapsto P'p_\infty$

$$\overbrace{[M'|p_4']}^{P'}\overbrace{\begin{bmatrix}M^{-1}x\\0\end{bmatrix}}^{p_\infty}=M'M^{-1}x$$

2) GET LINE BETWEEN POINTS

$$l'=e'\underbrace{\times}_{\text{cross product}}(M'M^{-1}x)$$

$$=\underbrace{[e']_\times M'M^{-1}}_{F}x \quad \text{where } F = M'^{-T}M^T[e]_\times$$

$$\overbrace{[e']_\times P'P^+}$$

BUT THIS IS NOT EQUAL $M'M^{-1}$!!!!! For some reason....

(103) PROPRIETIES OF F $\left(x'^T F x = 0\right)$

$x'^T F x = 0$!

- $l' = Fx$ (98), x' lies on l' OBVIOUS

$x'^T F x = x'^T l' = \emptyset$

(104) PROPRIETIES OF F

(F is unique 3x3 HOMO MATRIX WITH RANK = 2 WITH PROPRIETY $x'^T F x = 0$)

IT'S RANK 2 BECAUSE $[e']_x$ IS RANK DEFICIENT

Because: $[e']_x \cdot e' = e' \times e' = \emptyset$

(105) OTHER PROPRIETIES OF F:

- F Fundamental (P, P') <=> F^T Fundamental (P', P)
- $l' = Fx$ is epipolar line of x
- $l = F^T x'$ is epipolar line of x'
- F has 7 DOF
- If $F = [\emptyset]$, both cameras are in the same position

(106) $Fe = 0 = e'^T F$ PROOF

For any $x \neq e$, $l' = Fx$ contains e'

$\Rightarrow \underbrace{e'^T F}_{\text{when can this be } 0?} x = 0 \quad \forall x \neq e$

$\Rightarrow e'^T F$ must be $\emptyset$

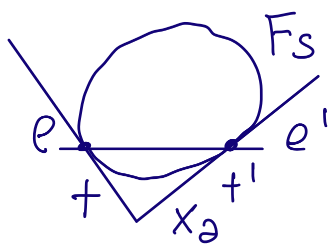
$F^T e' = \emptyset$

(107) DECOMPOSE F MATRIX

$F = F_s + F_a$ where

- F_s is symmetric and represent the STEINER CONIC where the epipoles e, e' lie on.
- F_a is skew-symmetric and can be expressed as $F_a = [x_a]_\times$

where the polar l_a of x_a intersects F_s at e and e'



e is right null space of F

e' is left null space of F

(108) PROVE e, e' EPIPOLES LIE ON F_S THE STEINER CONIC

Since $Fe = \emptyset$ then $\Rightarrow e^T F e = \emptyset$

$\hookrightarrow e^T(F_S + F_a)e \Rightarrow e^T F_S e + \underbrace{e^T F_a e}_{=\emptyset ?} = \emptyset$

PROVE $e^T F_a e = \emptyset$

$\Downarrow$

F_a IT'S A SKEW SYMMETRIC MATRIX

where $F_a = [X_a]_\times$

$\Downarrow$ FROM (99)

$e^T [X_a]_\times e = e^T \underbrace{(X_a \times e)}$

this is Perpendicolar to X_a and e

So $e^T(\text{something} \perp \text{to } e) = \emptyset$

BUT AS WE HAVE $e^T F_S e + e^T F_a e = \emptyset$

$\Downarrow$

$e^T F_S e = \emptyset$

So e lies on conic F_S!

SAME CAN BE DONE FOR e'

109 FIND TANGENT TO F_S STEINER CONIC ON e and e' and X_a (intersection of them)

1) Find tangent lines by:

$$F_S \cdot e = t \qquad F_S \cdot e' = t'$$

2) Find X_a by:

$$\underbrace{t \times t'}_{\text{cross product}}$$

X_a is the POLE of l_a

l_a is the POLAR of X_a

110 PROOF THAT X_a IS POLE OF l_a & l_a IS POLAR OF X_a

We need to proove that e & e' are on $F_S X_a$

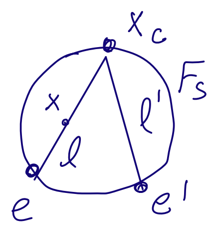
We start by showing $e^T F X_a = 0$ and $e'^T F X_a = 0$ as given in 108

We need now to proove that $F X_a = F_S X_a$

$$F_S X_a = (F - F_a) X_a = F X_a - F_a X_a \quad \text{but } \underbrace{F_a = [X_a]_\times}$$

$$= F X_a - \underbrace{[X_a]_\times X_a}_{0} \Rightarrow F_S X_a = F X_a \ \checkmark$$

(111) GEOMETRIC CONSTRUCTION OF EPIPOLAR LINE $l' = Fx$



1) Find intersection x_c of l and F_s through e and x

2) Construct line $l' = Fx$ through x_c and e'

└ projected on second image

1) FIND X_C

$$l = e + \lambda(\underbrace{x - e}_{\text{vector from } e \text{ to } x})$$

PARAMETRIC

└ parameter (scale factor)

SOLVE FOR λ: $l^T F_S l = 0 \rightarrow$ get 2 values of λ → X_c, e

We care for X_C!

So we use λ corresponding to X_C

in eq: $l = e + \lambda(x - e)$ to get X_C

2) Now that we have X_C we use it to find l' as:

$$X_C \underbrace{\times}_{\text{cross product}} e' = l' = Fx \quad \checkmark$$

(112) PROJECTIVE INVARIANCE OF F FUNDAMENTAL MATRIX ($\hat{F}$) PROOF

$x \mapsto l' = Fx$ and $x'^T F x = 0$ are projective relationships

>> IF WE TRANSFORM IMAGES WITH PROJECTIVE TRANSFORMATIONS $H \in \mathbb{R}^{3\times3}$ AND $H' \in \mathbb{R}^{3\times3}$, THEN THE NEW FUNDAMENTAL IS $\hat{F} = H'^{-T} F H^{-1}$ *

Proof:

$\hat{x} = Hx$

$\hat{x}' = H'x'$

$$\hat{x}'^T \hat{F} \hat{x} = \underbrace{x'^T H'^T}_{\hat{x}'^T} \underbrace{\left(H'^{-T} F H^{-1}\right)}_{\hat{F}} \underbrace{Hx}_{\hat{x}} = 0$$

✓

$$= x'^T \underbrace{H'^T \left(H'^{-T}}_{I} F \underbrace{H^{-1}\right) H}_{I} x = 0$$

$$= x'^T F x = 0$$

Proof 2: $\hat{l}' = \hat{F}\hat{x}$?

$$\hat{l}' = H'^{-T} l' = H'^{-T} F x = H'^{-T} F \underbrace{H^{-1} H}_{I} x = \underbrace{H'^{-T} F H^{-1}}_{\hat{F}\,*} \underbrace{Hx}_{\hat{x}}$$

WE ADD IDENTITY

$$= \hat{F}\hat{x}$$

(113) $(P, P') = (PH, P'H)$

F IS INDEPENDENT FROM WORLD COORD.

Theorem: if $H \in \mathbb{R}^{4\times4}$ represents a spatial projective transformation, the fundamental matrixes for pairs (P, P') and $(PH, P'H)$ are identical

Proof:

For any X, the points $\left.\begin{array}{l} x = PX \\ x' = P'X \end{array}\right|$ correspond as X is same

So: $x'^T F x = \emptyset \implies \underbrace{X^T P'^T}_{x'^T} \underbrace{F}_{F} \underbrace{PX}_{x} = \emptyset$

now X with H^{-1} to $\hat{X} = H^{-1} X$

then $\hat{x} = (PH)\hat{X} = PX = x$

and $\hat{x}' = (P'H)\hat{X} = P'X = x'$

$\implies \hat{x}'^T F \hat{x} = x'^T F x = \emptyset$

SO F IS ALSO FUNDAM FOR $(PH, P'H)$

114 FIND MATRIX H THAT GIVES CANONICAL FORM OF (P, P')

>> CANONICAL FORM $\longrightarrow$ $\underline{P = [I \,|\, 0]}$

Augment P by a 4^{TH} row to a non-singular 4x4 matrix $\rightarrow$ $P^* = \begin{bmatrix} P \\ r^T \end{bmatrix}$

row vector linearly indep. from P

Then $H = P^{*-1}$

Lets prove that we get $P^* \cdot H = \begin{bmatrix} I & 0 \\ 0^T & 1 \end{bmatrix}$

$$= \begin{bmatrix} P \\ r^T \end{bmatrix} \cdot H$$

$$= \begin{bmatrix} P \cdot H \\ r^T H \end{bmatrix} \rightarrow P \cdot H = [I/0]$$

(115) ASSUMING $P' = [M' | P_4']$ AND WE ARE IN CANONICAL $\Rightarrow F = [P_4']_\times M'$

$$P = [M | P_4] \;\&\; P' = [M' | P_4'] \Rightarrow F = [e']_\times M' M^{-1}$$

$$e' = -M' M^{-1} P_4 + P_4'$$

AS (402)

$$\Downarrow$$

$$P = [I | 0] \;\&\; P' = [M' | P_4']$$

$$e' = -M' \underbrace{I}_{M^{-1}} \cdot \underbrace{0}_{P_4} + P_4'$$

$$\boxed{F = [P_4']_\times M'}$$

$$F = [P_4 \times P_1 | P_4 \times P_2 | P_4 \times P_3]$$

116 GIVEN F, FIND (P, P') IN CANONICAL FORM

if $\underbrace{F}_{\text{fundamental matrix}} = \underbrace{[a]_\times A}_{\text{with}} = \underbrace{[\tilde{a}]_\times \tilde{A}}_{\text{with}} \in \mathbb{R}^{3\times 3}$ with rank(F) = 2

with a and $\tilde{a}$ as vectors

> then $\tilde{a} = ka$ with some $k \neq 0$ but very small!

> $\boxed{\tilde{A} = (A + av^T)/k}$ PROVE

Proof of Lemma:

$a^T F = a^T \underbrace{[a]_\times A}_{F} = \underbrace{\vec{0}}_{\text{zero row vector}}$ because $a^T [a]_\times \underbrace{y}_{\text{any vector}}$

$$\Downarrow \qquad\qquad \downarrow$$

$\underbrace{a^T (a}_{\text{ortho}} \underset{\text{cross}}{\times} y) = 0$

LIKEWISE

$\tilde{a}^T F = 0 \Rightarrow$ so a and $\tilde{a}$ are on the left null-space of F

but as rank(F) = 2 then $\tilde{a} = ka$ for some $k \neq 0$

2) $[a]_\times A = [\tilde{a}]_\times \tilde{A} = k[a]_\times \tilde{A} = [a]_\times k\tilde{A}$

$$\Downarrow$$

$[a]_\times \underbrace{(k\tilde{A} - A)}_{\text{zero matrix}} = 0 \Rightarrow [a]_\times \begin{pmatrix}\text{either of} \\ \text{the 3 columns} \\ \text{of}\end{pmatrix} = 0 \Rightarrow k\tilde{A} - A = [a_1 | a_2 | a_3]$

THEN $a \underset{\text{cross}}{\times} a_i = 0$ for $i = 1,2,3$

$a_i = V_i \cdot a$ for some V_i let v be $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \Rightarrow k\tilde{A} - A = [v_1 a | v_2 a | v_3 a] = a \cdot v^T$

$$\tilde{A} = (A + av^T)/k \quad \checkmark$$

117 IF F IS FUNDAMENTAL MATRIX OF (P, P') and $(\tilde{P}, \tilde{P}')$ THEN $\tilde{P} = PH$ AND $\tilde{P}' - P'H$ for some H

» IF (P, P') and $(\tilde{P}, \tilde{P}')$ are in canonical form

ex: $P = [I \,|\, 0] = \tilde{P}$

and $P' = [A \,|\, a]$ and $\tilde{P}' [\tilde{A} \,|\, \tilde{a}]$

then F is fundamental mat so:

$F = [a]_\times A = [\tilde{a}] \times \tilde{A}$ and From 116 LEMMA

$\tilde{A} = (A + a v^T)/k$, $\tilde{a} = k a$ $\quad k \neq 0$

NOW WE LET $H = \begin{bmatrix} I/k & 0 \\ v^T/k & k \end{bmatrix}$

we need to verify that H exists:

- $PH = \underbrace{[I \,|\, 0]}_{P} \underbrace{\begin{bmatrix} I/k & 0 \\ v^T/k & k \end{bmatrix}}_{H} = [I/k \,|\, 0] = \underbrace{\frac{1}{k}}_{\text{HOMO DISAPPEARS}} \tilde{P} = \tilde{P}$ ✓
- $P'H = [A/k + a v^T/k \,|\, a k] = [(A + a v^T)/k \,|\, a k]$

$\overset{!}{=} [\tilde{A} \,|\, \tilde{a}] = \tilde{P}'$ ✓

(118) GIVEN F AND A SKEW-SYMMETRIC $S \in \mathbb{R}^{3\times 3}$ THEN F IS THE FUNDAMENTAL OF (P,P') FOR $P=[I \mid 0]$, $P'=[SF \mid e']$

1. A is skew-symmetric $\Longleftrightarrow A^T=-A \Longleftrightarrow x^TAx=0 \ \forall x$

"$\Rightarrow$" if $A^T=-A$ then $\underbrace{x^TAx}_{\text{we can Transpose all}} = x^TA^Tx = x^T(-A)x = -x^TAx$

So IF $x^TAx=-x^TAx$

THEN $x^TAx=\emptyset$

"$\Leftarrow$" $0 = (\underbrace{x+y}_{\text{any } x+y})^T A (x+y) = \underbrace{x^TAx}_{\emptyset} + x^TAy + y^TAx + \underbrace{y^TAy}_{\emptyset}$

So:

$x^TAy + y^TAx = \emptyset \Rightarrow x^T(-A)y = x^TA^Ty$, $\underbrace{\qquad}_{A^T=-A}$

2. If F is fundamental matrix of $(P,P') \Longleftrightarrow \overset{4\times 4}{\boxed{P'^TFP}}$ is skew-symmet.

$\Longleftrightarrow \underbrace{X^TP'^T}_{x'^T}F\underbrace{PX}_{x}=\emptyset$ for all Xs

$x'^TFx=0$ for $x'=P'X$ and $x=PX$

3. Prove P'^TFP is skew symmetric

$P'^TFP = [SF \mid e']^T F [I \mid 0] = \begin{bmatrix} F^TS^T \\ e'^T \end{bmatrix} [F \mid 0] = \begin{bmatrix} F^TS^TF & 0 \\ \underbrace{e'^TF}_{\substack{\emptyset \\ \text{Left nullspace}}} & 0 \end{bmatrix}$ is skew symmetric

because if S is s-s then is S^T.

(119) FUNDAMENTAL MATRIX PROPRIETIES RECAP

- F depends only on point corresp. not on P or P'
- We can find F without camera calibration
- If cameras don't change, F can be used on other images from same camera
- Given F, (P, P') can be computed up to 3D
- We can reconstruct 3D scene using $\{X_i \leftrightarrow x_i\}$

(120) ESTIMATE FUNDAMENTAL F

- $x = (x, y, 1)^T$ & $x' = (x', y', 1)^T$ corresponding points pair
- $F = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{pmatrix}$

THEN

$$x'^T F x = 0 = (x'x, x'y, x', y'x, y'y, y', x, y, 1) \cdot f$$

where $f = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T$

So for n correspondences $\{x_i' \leftrightarrow x\}$ we get the HOMO LINEAR SYSTEM:

$$Af = 0$$

unknowns for F

$$A = \begin{pmatrix} x_1'x_1 & x_1'y_1 & x_1' & y_1'x_1 & y_1'y_1 & y_1' & x_1 & y_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_n'x_n & x_n'y_n & x_n' & y_n'x_n & y_n'y_n & y_n' & x_n & y_n & 1 \end{pmatrix} \in \mathbb{R}^{n \times 9}$$

SOLVE SYSTEM

Solution if $\text{rnk} \geq 8$
unique if $\text{rnk} = 8$

121 8-POINT ALGORITHM (START)

$Af = 0$ as in 120

It has many solutions if $\text{rank}(A) > 8$

- It has one solution if $\text{rank}(A) = 8$ with $\|f\| = 1$

- If $\text{rank}(A) > 8$ we use least square solution that minimizes $\|Af\|$ with $\|f\| = 1$

unit singular vector of A

f will then be the last column of V in the SVD $A = U\Sigma V^T$

- $$F = \begin{pmatrix} f_1 & f_2 & f_3 \\ f_4 & f_5 & f_6 \\ f_7 & f_8 & f_9 \end{pmatrix}$$ COMPUTED THIS WAY IS <u>NOT</u> SINGULAR!

This is because if $\text{rank}(F) = 3$ then $e'^T F = F e'^T = 0$ IMPLY $e' = 0$

the only e' epipole will be $e' = 0$!

We have to force $\tilde{F}$ to be rank 2, where $\tilde{F}$ is the closest fundamental to F which has rank 3

(122) 8-POINT ALGORITHM (FIND $\tilde{F}$ SINGULAR)

We have to use $\tilde{F}$ as using ~~non-singular~~ F gives this condition:

≫ $\text{rank}(A) = 3$ then $A\mathbf{z} = 0$ and $A^T\mathbf{z} = 0$ implies that $\mathbf{z} = \emptyset$

This will make it so that there will not be an epipole $e' \neq 0$

because $e'^T l' = e'^T F x = \emptyset \quad \forall x$

$$\Updownarrow$$

$$e'^T F = F^T e' = \emptyset \quad \text{so } e' = \emptyset \,!$$

We have to force $\tilde{F}$ to be rank 2, where $\tilde{F}$ is the closest fundamental to F which has rank 3

- CLOSENESS WILL BE CALCULATED WITH FROBENIOUS NORM: $\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}$

- <u>TO FIND THE BEST APPROXIMATION</u>:

We use the Eck-You-Mir algo:

$\underbrace{\text{SVD of } F}_{\text{DECOMPOSE } F} = \underline{U}\, \text{diag}(\sigma_1, \sigma_2, \sigma_3)\, V^T$

zero 3rd element

$$\tilde{F} = U\, \text{diag}(\sigma_1, \sigma_2, 0)\, V^T \quad \text{with } \det = \emptyset$$

THIS WILL MINIMIZE $\|F - \tilde{F}\|_F$ subject to $\det(\tilde{F}) = \emptyset$

123 LEAST SQUARE SOLUTION TO FIND F IN 8-POINTS IF Rank >8 (OVERDEFINED)

1) START BY $SVD(A) = UDV^T$

V = eigenvectors of (A^TA)

2) TAKE LAST COLUMN OF V

This will be your $(f_1 f_2 f_3 f_4 f_5 f_6 f_7 f_8 f_9)^T$
that we use for F

124 FIND $\tilde{F}$ CLOSEST TO F IN 8 POINT IF F rank = 3

1) SVD OF F TO FIND $SVD\ F = U\Sigma V^T$

- U = eigenvectors of FF^T
- $$\Sigma = diag(\sigma_1, \sigma_2, \sigma_3) = \begin{bmatrix} \sqrt{eigen_1} & 0 & 0 \\ 0 & \sqrt{eigen_2} & 0 \\ 0 & 0 & \sqrt{eigen_3} \end{bmatrix}$$

  where $eigen_i$ are eigenvalues of F^TF or FF^T

- V = eigenvectors of F^TF

2) From $diag(\sigma_1, \sigma_2, \sigma_3)$ PUT $\sigma_3 = 0$ and use this new Σ to get

$$\tilde{F} = U \overset{\text{with } 0}{\Sigma} V^T$$

(125) # 7-POINT ALGORITHM

We can use only 7 points so that $rank(A) = 7$

IN THIS CASE $Af = 0$ is 2-DIMENTIONAL SPACE

SO THAT WE GET 2 f results : f_1 & f_2

- f_1 and f will be linearly independent vector that consider all combinations of F_1 and F_2 that would be the 2 resulting fundamental matricies

- We let $\underbrace{F_\alpha}_{\text{General Fundamental}} = \alpha F_1 + (1-\alpha) F_2$ for F_1 & F_2

$$F_1 \downarrow f_1 \qquad F_2 \downarrow f_2$$

and solve $\det(F_\alpha) = 0$

to retrieve $\underline{\alpha}$ THEN USE α in equation $\underbrace{F_\alpha}_{\text{Fund}} = \alpha F_1 + (1-\alpha) F$ to get

<u>One</u> or <u>3</u> Fundamentals F_α ✓

(126) 3D RECONSTRUCTION USING F and CANONICAL CAMERA PAIR (FIND $\mathbf{X}$ USING x, x')

1. Estimate F from points correspondences

2. Compute a canonical camera pair

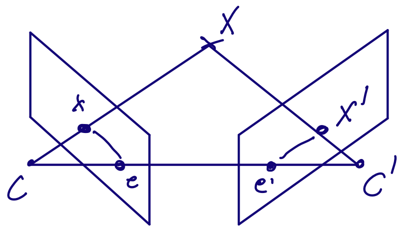
$P = [I \mid 0]$

$P' = \left[[e']_{\times} F \mid e' \right]$ with e' left null space of F.

We know that the cameras might need a COMMON PROJECTIVE TRANSFORMATION

3. We compute world point $\mathbf{X}$ that projects to the corresponding $x = P\mathbf{X}$ and $x' = P'\mathbf{X}$ BY TRIANGULATION

TRIANGULATION:



1) intersect rays $r = \overrightarrow{Cx}$ $r' = \overrightarrow{C'x'}$ and get $\mathbf{X}$ $\left(r \underbrace{\times}_{\text{cross}} r' = \mathbf{X} \right)$

2) Doesn't work for $x = e$ & $x' = e'$

(127) 3D RECONSTRUCTIONS ($P \leftrightarrow \tilde{P}$ USING H) $\boxed{\text{PAGE 266}}$

let $(P, P', \{\mathbf{X}_i\})$ and $(\tilde{P}, \tilde{P}', \{\tilde{\mathbf{X}}_i\})$

be 2 reconstructions from

correspondences $\{x_i \leftrightarrow x_i'\}$ then:

$H \in \mathbb{R}^{4\times 4}$ will be a 3D projective

transformation such that

$\tilde{P} = PH^{-1}$, $\tilde{P}' = P'H^{-1}$ and $\tilde{\mathbf{X}}_i = H\mathbf{X}_i$

For all points where

$Fx_i = x_i'^T F = 0$

128 CORRECT PROJECTION

- Know some real world coords $\hat{X}_j$
- Use canonical (P, P') to reconstruct X_j
- Use H AS IN 127 TO MAP $\hat{X}_j = H X_j$

  SO THAT WE HAVE HOMOGRAPHY THAT RELATES (P, P') to the CORRECT CAMERA PAIR $(\hat{P}, \hat{P}')$

- H has 15 DOF so 5 or more $\hat{X}_j$ will suffice