

## Assignment 8

November 6, 2023

### Exercise 1 [5 points]

Let  $f \in \Pi_3$  be a cubic polynomial and  $g \in \Pi_2$  be the quadratic polynomial that interpolates  $f$  at  $a$ ,  $a+h$ , and  $a+2h$  for some  $a$  and  $h > 0$ , that is,

$$g(a+ih) = f(a+ih), \quad i = 0, 1, 2.$$

Show that

$$\int_a^{a+2h} f(x) dx = \int_a^{a+2h} g(x) dx,$$

which proves that the Simpson's rule has polynomial precision of degree 3.

Hand in your proof.

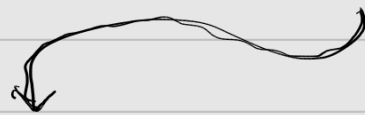
Because  $g$  is quadratic and interpolates  $f$  at  $a, a+h, a+2h$ , we can write  $f$  with in lagrange form:

$$f(x) = g(x) + \frac{f^{(3)}(c)}{3!} \cdot \prod_{i=0}^2 (x - (a+ih)), \quad c \in [a, a+2h]$$

Then:

$$\int_a^{a+2h} f(x) dx = \int_a^{a+2h} g(x) dx + \underbrace{\int_a^{a+2h} \frac{f^{(3)}(c(x))}{3!} \cdot \prod_{i=0}^2 (x - (a+ih)) dx}_{(1)}, \quad c(x) \in [a, a+2h]$$

Let's focus on this term



$$\frac{f^{(3)}(c)}{3!} \int_a^{a+2h} \prod_{i=0}^2 (x - (a+ih)) dx, \quad c \in [a, a+2h] \quad (2)$$

Let's consider the substitution

$$u = x - a$$

$$u = n - a$$

$$du = dn$$

$$n = 0 \Rightarrow u = 0$$

$$n = a + 2h \Rightarrow u = 2h$$

From (2) with substitution  $u$  comes:

$$\frac{f^{(3)}(c)}{3!} \int_0^{2h} u(u-h)(u-2h) du =$$

$$= \frac{f^{(3)}(c)}{3!} \int_0^{2h} (u^3 - 3hu^2 + 2h^2u) du = \frac{f^{(3)}(c)}{3!} \left[ \frac{u^4}{4} - hu^3 + h^2u^2 \right]_0^{2h} =$$

$$= \frac{f^{(3)}(c)}{3!} \left( \frac{2^4 h^4}{2^2} - 2^3 h^4 + 2^2 h^4 \right) = \frac{f^{(3)}(c)}{3!} \underbrace{(2^3 h^4 - 2^3 h^4)}_{=0} =$$

$$= 0$$

Returning to (1) we have that

$$\int_a^{a+2h} f(x) = \int_a^{a+2h} g(x) + 0$$

