Numerical Algorithms 23/24 Fabian Gobet

Assignment 8 November 6, 2023

Exercise 1 [5 points]

Let $f \in \Pi_3$ be a cubic polynomial and $g \in \Pi_2$ be the quadratic polynomial that interpolates f at a, $a+h$, and $a+2h$ for some a and $h > 0$, that is,

$$g(a+ih) = f(a+ih), \quad i = 0, 1, 2.$$

Show that

$$\int_a^{a+2h} f(x)dx = \int_a^{a+2h} g(x)dx,$$

which proves that the Simpson's rule has polynomial precision of degree 3.

Hand in your proof.

Because g is quadratic and interpolates f at $a, a+h, a+2h$, we can write f with in lagrange form:

$$f(x) = g(x) + \frac{f^{(3)}(c)}{3!} \cdot \prod_{i=0}^{2} (x - (a+ih)), \quad c \in [a, a+2h]$$

Then:

$$\int_a^{a+2h} f(x)\,dx = \int_a^{a+2h} g(x)\,dx + \underbrace{\int_a^{a+2h} \frac{f^{(3)}(c(x))}{3!} \cdot \prod_{i=0}^{2} (x-a-ih)\,dx}, \quad c(x) \in [a, a+2h] \qquad (1)$$

Let's focus on this term

$$\frac{f^{(3)}(c)}{3!} \int_a^{a+2h} \prod_{i=0}^{2} (x-a-ih)\,dx, \quad c \in [a, a+2h] \qquad (2)$$

Let's consider the substitution

$u = x - a$

$u = x - a$

$du = dx$

$x = 0 \Rightarrow u = 0$

$x = a+2h \Rightarrow u = 2h$

From (2) with substitution u comes:

$$\frac{f^{(3)}(c)}{3!} \int_0^{2h} u(u-h)(u-2h)\, du =$$

$$= \frac{f^{(3)}(c)}{3!} \int_0^{2h} u^3 - 3hu^2 + 2h^2 u \, du = \frac{f^{(3)}(c)}{3!} \left[\frac{u^4}{4} - hu^3 + h^2u^2 \right]_0^{2h} =$$

$$= \frac{f^{(3)}(c)}{3!} \left(\frac{2^4 h^4}{2^2} - 2^3h^4 + 2^2h^4 \right) = \frac{f^{(3)}(c)}{3!} \left(\underbrace{2^3h^4 - 2^3h^4}_{0} \right) =$$

$$= 0$$

Returning to (1) we have that

$$\int_a^{a+2h} f(x) = \int_a^{a+2h} g(x) + 0 \qquad \blacksquare$$