# From Flow Based Cost Allocation to Locational Marginal Prices

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## Linear Energy Modelling and LMP

We linearly optimize the capacity and dispatch of a simple power system.

$$\max_{d_{n,a,t},g_{n,s,t},G_{n,s}} \left( U_{n,a,t}(d_{n,a,t}) - \sum_{n,s} c_{n,s} G_{n,s} - \sum_{n,s,t} o_{n,s} g_{n,s,t} - \sum_{\ell} c_\ell F_\ell \right) \tag{1}$$

where

$U_{n,a,t}$ denotes the utility function per bus $n$, demand type $a$ time step $t$

$d_{n,a,t}$ denotes the eletric demand

$c_{n,s}$ denotes the capital expenditure (CAPEX) per node $n$ and generator type $s$

$G_{n,s}$ denotes the generation capacity

$o_{n,s}$ denotes the operational cost (OPEX)

$g_{n,s,t}$ denotes the net generation in MW

$c_\ell$ denotes the CAPEX per transmission line

$F_\ell$ denotes the transmission capacity.

In the following we neglect the utility $U_{n,a,t}$ of the nodal demand while fixing the demand $d_{n,a,t}$ to a predefined time-series. The generation $g_{n,s,t}$ is constraint to its capacity

$$g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s} \leq 0 \quad \perp \quad \bar{\mu}_{n,s,t} \quad \forall\, n,s,t \tag{2}$$

$$-g_{n,s,t} \leq 0 \quad \perp \quad \underline{\mu}_{n,s,t} \quad \forall\, n,s,t \tag{3}$$

where $\bar{g}_{n,s,t} \in [0,1]$ is the capacity factor for renewable generators. The constraints yield the KKT variables $\bar{\mu}_{n,s,t}$ and $\underline{\mu}_{n,s,t}$ which due to complementary slackness are only non-zero if the corresponding constraint is binding.
The transmission capacity $F_\ell$ limits the flow $f_{\ell,t}$ in both directions, such that

$$f_{\ell,t} - F_\ell \leq 0 \quad \perp \quad \bar{\mu}_{\ell,t} \quad \forall\, \ell,t \tag{4}$$

$$-f_{\ell,t} - F_\ell \leq 0 \quad \perp \quad \underline{\mu}_{\ell,t} \quad \forall\, \ell,t \tag{5}$$

The yielding KKT variables $\bar{\mu}_{\ell,t}$ and $\underline{\mu}_{\ell,t}$ are only non-zero if $f_{\ell,t}$ is limited by the trasmission capacity in positive or negative direction, i.e. Constr. (4) or Constr. (5) are binding.

The nodal balance constraint ensures that the amount of power that flows into a bus equals the power that flows out of a bus, thus reflects the Kirchhoff Current Law (KCL)

$$\sum_{l} K_{n,\ell}\, f_{\ell,t} - \sum_{s} g_{n,s,t} + \sum_{a} d_{n,a,t} = 0 \qquad \perp \quad \lambda_{n,t} \qquad \forall\, n, t \tag{6}$$

Its shadow price mirrors the Locational Marginal Prizes (LMP) $\lambda_{n,t}$ per bus and time step. In a power market this is the €/MWh$_{\mathrm{el}}$-price which a consumer has to pay.

As the shown in [**brown˙decreasing˙2020**], the OPEX and CAPEX of generators follow the non-profit rule which states that their expenses equals their total revenue

$$c_{n,s}\, G_{n,s} + \sum_{t} o_{n,s}\, g_{n,s,t} = \sum_{t} \lambda_{n,t}\, g_{n,s,t} \qquad \forall\, n, s \tag{7}$$

The relation counts for transmission lines where the CAPEX amounts to the total congestion revenue

$$c_{\ell}\, F_{\ell} = -\sum_{n,t} \lambda_{n,t}\, K_{n,\ell}\, f_{\ell,t} \qquad \forall\, \ell \tag{8}$$

Power flow from nodes with low locational prices to nodes with higher prices in the network, thus the nodal price difference in flow direction is negative. Different to Tom's paper, there a minus is missing. Note that we are neglecting the Kirchhoff Voltage Law (KVL) constraints. Finally, the total cost $\mathcal{TC}$, consisting of all CAPEX and OPEX have to be payed by the consumers

$$\mathcal{TC} = \sum_{n,s} c_{n,s}\, G_{n,s} + \sum_{n,s,t} o_{n,s}\, g_{n,s,t} + \sum_{\ell} c_{\ell}\, F_{\ell} = \sum_{n,a,t} \lambda_{n,t}\, d_{n,a,t} \tag{9}$$

## Adding $CO_2$ Constraints

Imposing an additional $CO_2$ constraint limiting the total emission to K,

$$\sum_{n,s,t} e_{n,s}\, g_{n,s,t} \leq \mathrm{K} \qquad \perp \quad \mu_{\mathrm{CO2}} \tag{10}$$

with $e_{n,s}$ being the emission factor in tonne-$CO_2$ per MWh$_{\mathrm{el}}$, returns an effective $CO_2$ price $\mu_{\mathrm{CO2}}$ in €/tonne-$CO_2$. The $CO_2$ price shifts the right hand side of the non-profit relation for generators Eq. (7) to

$$c_{n,s}\, G_{n,s} + \sum_{t} o_{n,s}\, g_{n,s,t} = \sum_{t} \left(\lambda_{n,t} - e_{n,s}\, \mu_{\mathrm{CO2}}\right)\, g_{n,s,t} \qquad \forall\, n, s \tag{11}$$

This shows nicely the duality for exchanging the $CO_2$ Constr. (10) for a shifted OPEX which includes the $CO_2$ costs

$$o_{n,s} \rightarrow o_{n,s} + e_{n,s}\, \mu_{\mathrm{CO2}} \tag{12}$$

## Flow Based Cost Allocation

Besides using LMP, there are others methods for distributing costs in a power system. A common approach is based on Flow Allocation (FA) methods which determine peer-to-peer (P2P) relations between producers (sources) and consumers (sinks) and thus enable to directly assign costs to the consumers. Let $A_{n\to m,t}$ denotes the power that is produced at bus $n$ at time $t$ and assigned to consumers at bus $m$. We will refer to it as a P2P relation between $n$ and $m$. Further let $A_{n\to m,\ell,t}$ denote flow on line $\ell$ at time $t$ caused by the P2P connection $n \to m$.
In the following we show how the LMP can be reproduced on the basis of FA.

### Allocating Operational Expenditures

Consumers at bus $m$ compensate the OPEX proportionally to the assigned production, thus

$$\mathcal{O}_{m,t} = \sum_{n,s} o_{n,s}\, \tilde{g}_{n,s,t}\, A_{n\to m,t} \qquad \forall\ m,t \tag{13}$$

where $\tilde{g}_{n,s,t} = g_{n,s,t} / \sum_s g_{n,s,t}$ denotes the share of generator $s$ of the nodal production.

### Allocating $CO_2$ Cost

Similar to the OPEX (compare Rel. (12)) consumers at bus $m$ compensate the $CO_2$ cost, thus

$$\mathcal{E}_{m,t} = \sum_{n,s} \mu_{\mathrm{CO2}}\, e_{n,s}\, \tilde{g}_{n,s,t}\, A_{n\to m,t} \qquad \forall\ m,t \tag{14}$$

### Allocating Capital Expenditures for Generators

According to the polluter pays principle, the CAPEX allocation should differentiate between consumers who are ‘responsible‘ for investments and those who are not. Thus, it is crucial to know whether a time step $t$ makes investments in $G_{n,s}$ necessary or if it would be feasible without (further) investments. The KKT variable of the upper generator capacity constraint $\bar{\mu}_{n,s,t}$ gives us the needed estimator: If $\bar{\mu}_{n,s,t} > 0$, the upper capacity constraint Constr. (2) is binding and it can be fairly assumed that these times steps cause investments in $G_{n,s}$. If in contrast $\bar{\mu}_{n,s,t} = 0$, the generation $g_{n,s,t}$ is not bound and does not push further investments in $G_{n,s}$. On the basis of the KKT variable $\bar{\mu}_{n,s,t}$ we are able to define a measure $\Phi_{n,s,t}$ for the impact on the investments in $G_{n,s}$

$$\Phi_{n,s,t} = \frac{\bar{\mu}_{n,s,t}}{\sum_t \bar{\mu}_{n,s,t}\, g_{n,s,t}} \tag{15}$$

which weights the contributions of consumers at bus $m$ at time $t$ to the CAPEX

$$\mathcal{C}^G_{m,t} = \sum_{n,s} c_{n,s}\, G_{n,s}\, \Phi_{n,s,t}\, A_{n\to m,t} \qquad \forall\ m,t \tag{16}$$

If consumers at $m$ retrieve power from $n$ when $\Phi_{n,s,t}$ is large, their contribution to investments in $G_{n,s}$ raises. On the other hand, if consumers at $m$ retrieve power from $n$ when $\Phi_{n,s,t} = 0$ , their contribution to $G_{n,s}$ remain unchanged.
*Side note*: A way more simple approach would neglect these thoughts and let consumers at $m$ compensate CAPEX for generator $s$ at bus $n$ proportional to their retrieved power. This would lead us to $\Phi_{n,s,t} = (g_{n,s,t})^{-1}$.

### Allocating Capital Expenditures for Transmission Lines

The allocation of CAPEX per the transmission line $c_\ell \, F_\ell$ follows a similar procedure as for the generators. Again, we build up on KKT variables $\bar{\mu}_{\ell,t}$ and $\underline{\mu}_{\ell,t}$ which translate to the necessity of transmission expansion of $\ell$ at time $t$. The measure $\Phi_{\ell,t}$ respects the need of CAPEX of both flow directions when set to

$$\Phi_{\ell,t} = \frac{\bar{\mu}_{\ell,t} + \underline{\mu}_{\ell,t}}{\sum_t \left(\bar{\mu}_{\ell,t} + \underline{\mu}_{\ell,t}\right) \, f_{\ell,t}} \tag{17}$$

and weights the contributions of consumers at bus $m$ at time $t$ to the CAPEX in line $\ell$,

$$\mathcal{C}^F_{m,t} = \sum_{n,\ell} c_\ell \, F_\ell \, \Phi_{\ell,t}, A_{n\to m,\ell,t} \qquad \forall\, m,t \tag{18}$$

### Reproducing the LMP Based Consumption Costs

The above presented quantities complete an allocation of the total cost $\mathcal{TC}$ which mirrors the cost allocation based on LMP, thus

$$\mathcal{O}_{m,t} + \mathcal{E}_{m,t} + \mathcal{C}^G_{m,t} + \mathcal{C}^F_{m,t} \simeq \lambda_{m,t} \, d_{m,a,t} \qquad \forall\, m,t \tag{19}$$

## Theory

### Lagrange Mutliplier

Maximise $f(x_l)$, with equality constraints $g_i(x_l)$ and inequality constraints $h_j(x_l)$

$$\mathcal{L}(x_l, \lambda_i, \mu_j) = f(x_l) - \sum_i \lambda_i \, g_i(x_l) - \sum_j \mu_j \, h_j(x_l) \tag{20}$$

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### Full Lagrangian

$$\begin{aligned}
\mathcal{L}\left(g_{n,s,t}, G_{n,s}, f_{\ell,t}, F_\ell, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) = &- \sum_{n,s} c_{n,s} G_{n,s} - \sum_{n,s,t} o_{n,s} g_{n,s,t} - \sum_\ell c_\ell \, F_\ell && (21)\\
&- \sum_{n,t} \lambda_{n,t} \left( \sum_\ell K_{n,\ell} \, f_{\ell,t} - \sum_s g_{n,s,t} + \sum_a d_{n,a,t} \right) && (22)\\
&- \sum_{\ell,c,t} \lambda_{c,t} \, C_{\ell,c} \, x_\ell \, f_{\ell,t} && (23)\\
&- \sum_{n,s,t} \bar{\mu}_{n,s,t} \left( g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s} \right) + \sum_{n,s,t} \underline{\mu}_{n,s,t} g_{n,s,t} && (24)\\
&- \sum_{\ell,t} \bar{\mu}_{\ell,t} \left( f_{\ell,t} - F_\ell \right) + \sum_{\ell,t} \underline{\mu}_{\ell,t} \left( f_{\ell,t} + F_\ell \right) && (25)
\end{aligned}$$

### Ansatz

The flow $f_{\ell,t}$ is a passive quantity only, thus

$$\mathcal{L}\left(g_{n,s,t}, G_{n,s}, f_{\ell,t}, F_\ell, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) \to \mathcal{L}\left(g_{n,s,t}, G_{n,s}, F_\ell, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) \tag{26}$$

where flow is given by the product of the Power Transfer Distribibition Factors (PTDF) and the nodal power injection

$$f_{\ell,t}\left(g_{n,s,t}\right) = \sum_n H_{\ell,n} \left( \sum_s g_{n,s,t} - \sum_a d_{n,a,t} \right) \tag{27}$$

Sensitivity of flow $f_{\ell,t}$ against changes of the power production account to the

$$\frac{\partial f_{\ell,t}}{\partial g_{n,s,t}} = H_{\ell,n} \tag{28}$$

$$\frac{\partial \mathcal{L}}{\partial g_{n,s,t}} = 0 \qquad \forall \; n, s \tag{29}$$

$$-o_{n,s} + \lambda_{n,t} - \sum_{m,\ell} \lambda_{m,t} \, K_{m,\ell} \, H_{\ell,n} - \bar{\mu}_{n,s,t} + \underline{\mu}_{n,s,t} - \sum_\ell \bar{\mu}_{\ell,t} H_{\ell,n} + \sum_\ell \underline{\mu}_{\ell,t} \, H_{\ell,n} = 0 \qquad \forall \; n, s \tag{30}$$

Solving for $\lambda_{n,t}$

$$\lambda_{n,t} = o_{n,s} + \bar{\mu}_{n,s,t} - \underline{\mu}_{n,s,t} + \sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) \, H_{\ell,n} + \sum_{m,\ell} \lambda_{m,t} \, K_{m,\ell} \, H_{\ell,n} \qquad \forall \; n, s \tag{31}$$

This equation holds for every choice of slack in the PTDF

$$H_{\ell,n} \to H_{\ell,n} + k_{\ell} \tag{32}$$

Setting the slack to $k_{\ell} = -H_{\ell,n}$ for each node $n$ separately, shows that the Locational Market Price is determined by the local OPEX and prices for the generation capacity constraint

$$\lambda_{n,t} = o_{n,s} + \bar{\mu}_{n,s,t} - \underline{\mu}_{n,s,t} \qquad \forall \; n, s \tag{33}$$

Inserting this into the right hand side of Eq. (31), leads to

$$\lambda_{n,t} = o_{n,s} + \bar{\mu}_{n,s,t} - \underline{\mu}_{n,s,t} + \sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) \, H_{\ell,n} + \sum_{m,\ell} K_{m,\ell} \, H_{\ell,n} \, \left(o_{m,s} + \bar{\mu}_{m,s,t} - \underline{\mu}_{m,s,t}\right) \qquad \forall \; n, s \tag{34}$$

$$\sum_{\ell} K_{m,\ell} \, H_{\ell,n} = \tilde{A}_{n \to m,t} \tag{35}$$

The Marginal Participation measures the sensitivities given in Eq. (28) and multiplies it with nodal injection

$$p_{n,t} \, \frac{\partial f_{\ell,t}}{\partial p_{n,t}} = p_{n,t} \left( H_{\ell,n} + \gamma \, f^{-}_{\ell,t} \right) = \sum_{m} A_{n \to m,\ell,t} \tag{36}$$

$$\frac{\partial f_{\ell,t}}{\partial g_{n,s,t}} = \sum_{m} A_{n \to m,\ell,t} \tag{37}$$