

# Flow-Based Cost Allocation in Linear Optimized Power Systems are non-unique but localizable

F. Hofmann

July 19, 2020

## Abstract

### Highlights

- The LMP always breaks down into contributions of different cost terms (OPEX, CAPEX, etc.) of the system.
- Breaking the LMP further down into contributions of components (generator, transmission line, etc.) requires assumptions on power flow assignments.
- Methods like Average Participation or Flow Based Market Coupling allow for localizing the P2P cost assignments and prioritize assignments within one market zone.

### Nomenclature

$\lambda_{n,t}$	Locational Market Price at bus $n$ and time step $t$ in €/MW
$d_{n,t}$	Electric demand per bus $n$ , demand type $a$ , time step $t$ in MW
$g_{s,t}$	Electric generation of generator $s$ , time step $t$ in MW
$f_{\ell,t}$	Active power flow on line $\ell$ , time step $t$ in MW
$o_s$	Operational price in €/MW
$c_s$	Capital Price in €/MW
$c_\ell$	Capital Price in €/MW for transmission capacity on line $\ell$
$G_s$	Generation capacity in MW
$F_\ell$	Transmission capacity in MW
$K_{n,\ell}$	Incidence matrix

## 1 Economic Context

In long-term operation and investment planning models, the total costs  $\mathcal{TC}$  of a power system include all operational expenditures (OPEX)  $\mathcal{O}$  and capital expenditures (CAPEX)  $\mathcal{C}^G$  and  $\mathcal{C}^F$  for production and transmission respectively, thus

$$\mathcal{TC} = \mathcal{O} + \mathcal{E} + \mathcal{C}^G + \mathcal{C}^F + \dots \quad (1)$$

In a network design with minimal  $\mathcal{TC}$ , the Locational Market Price (LMP) describes the marginal price for an incremental increase of electricity demand  $d_{n,t}$  of any consumer  $a$  at bus  $n$ . It is given by the derivative of the total system cost  $\mathcal{TC}$  with respect to the local demand  $d_{n,t}$

$$\lambda_{n,t} = \frac{\partial \mathcal{TC}}{\partial d_{n,t}} \quad (2)$$

This leads to a nodal pricing where over the span of optimized timesteps  $t$ , the system costs are partially or totally paid back by the consumers

$$\mathcal{TC} - \mathcal{R} = \sum_{n,a,t} \lambda_{n,t} d_{n,t} \quad (3)$$

depending on the costs  $\mathcal{R}$  which are independent of the nodal demand

$$\frac{\partial \mathcal{R}}{\partial d_{n,t}} = 0 \quad (4)$$

In most cases, where  $\mathcal{R}$  plays a minor role, the LMP splits into different cost terms assigned to OPEX and CAPEX for production and transmission and eventual other expenditures. We schematically show this behaviour in Fig. 1. Extensive investigations of the LMP as done in [8] already showed this connection however leave the question open how the costs are allocated among the components. In the following we will show how the above presented cost decomposition is extended to full peer-to-peer (P2P) cost allocation including all network components.

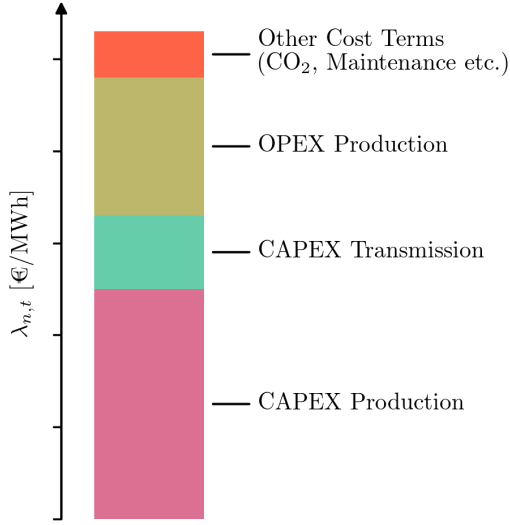


Figure 1: Schematic decomposition of the Locational Market Price  $\lambda_{n,t}$ . In power system model with optimal long-term operation and planning, the total system costs  $\mathcal{TC}$  split into different cost terms, *i.e.* OPEX and CAPEX for production and transmission and possibly other expenditures.

## 2 Nodal Cost Allocation

From inserting Eq. (1) into Eq. (2), we are able to disaggregate the cost terms  $\mathcal{O}$ ,  $\mathcal{C}^G$  and  $\mathcal{C}^F$  into contributions of demand  $d_{n,t}$  at bus  $n$  and time  $t$ . For the CAPEX allocation we will make use of the shadowprices resulting from the cost minimization.

### 2.1 Allocate Operation Expenditures

The total OPEX is given by the operational price  $o_s$  times the generation  $g_{s,t}$  of all generators  $s$ ,

$$\mathcal{O} = \sum_{s,t} o_s g_{s,t} \quad (5)$$

The allocated OPEX  $\mathcal{O}_{n,t}$  which naturally sum up to  $\mathcal{O} = \sum_{n,t} \mathcal{O}_{n,t}$ , are given by

$$\mathcal{O}_{n,t} = \frac{\partial \mathcal{O}}{\partial d_{n,t}} d_{n,t} = \sum_{s,t} o_s \frac{\partial g_{s,t}}{\partial d_{n,t}} d_{n,t} \quad (6)$$

The derivative on the right hand side time the nodal demand can be interpreted as power produced by generator  $s$  and consumed at node  $n$ . Without further explaining how the derivative is calculated, we define this power allocation as

$$A_{s \rightarrow n,t} = \frac{\partial g_{s,t}}{\partial d_{n,t}} d_{n,t} \quad (7)$$

which by feeding back into Eq. (5) has to fulfill

$$g_{s,t} = \sum_n A_{s \rightarrow n,t} \quad (8)$$

The direct payment of node  $n$  to generator  $s$  is then given by

$$\mathcal{O}_{n \rightarrow s,t} = o_s A_{s \rightarrow n,t} \quad (9)$$

Accordingly, consumers at bus  $n$  which retrieve power from a generators  $s$ , directly compensate its operational costs. The cost allocation behaves like P2P tradings between producers and consumers for fixed production prices. In this way, generator  $s$  retrieves the exact amount of money from consumers that it spends on its operation, thus

$$\sum_n \mathcal{O}_{n \rightarrow s,t} = o_s g_{s,t} \quad (10)$$

### 2.2 Allocate Emission Expenditures

Given a fix price for emissions in CO<sub>2</sub> equivalents  $\mu_{\text{CO}_2}$ , the cost term for emission adds up to

$$\mathcal{E} = \mu_{\text{CO}_2} \sum_s e_s g_{s,t} \quad (11)$$

where  $e_s$  denotes the emission factor in tonne-CO<sub>2</sub> per MWh<sub>el</sub> for generator  $s$ . Similar to the allocated OPEX, the emission cost for consumers at bus  $n$  at time  $t$  is allocated through

$$\mathcal{E}_{n,t} = \frac{\partial \mathcal{E}}{\partial d_{n,t}} d_{n,t} = \sum_s \mu_{\text{CO}_2} e_s A_{s \rightarrow n,t} \quad (12)$$

with its peer-to-peer payments

$$\mathcal{E}_{n \rightarrow s,t} = \mu_{\text{CO}_2} e_s A_{s \rightarrow n,t} \quad (13)$$

### 2.3 Allocate Capital Expenditures for Generators

The optimization of the generation capacity  $G_s$  adds the cost term

$$\mathcal{C}^G = \sum_s c_s G_s \quad (14)$$

to  $\mathcal{TC}$ . It combines all capacities times their capital cost  $c_s$ . The nominal capacity constraints the generation  $g_{s,t}$  in the form of

$$g_{s,t} - \bar{g}_{s,t} G_s \leq 0 \quad \perp \quad \bar{\mu}_{s,t} \quad \forall n, s, t \quad (15)$$

$$-g_{s,t} \leq 0 \quad \perp \quad \underline{\mu}_{s,t} \quad \forall n, s, t \quad (16)$$

where  $\bar{g}_{s,t} \in [0, 1]$  is the capacity factor for renewable generators. Using the result of [3] we show in that the capital cost for generator  $s$  translates is payed back by the generation times the shadow price for the upper capacity constraint,

$$c_s G_s = \sum_t \bar{\mu}_{s,t} g_{s,t} \quad (17)$$

With our definition in Eq. (54) the allocated CAPEX can then be defined as

$$C_{n,t}^G = \frac{\partial \mathcal{C}^G}{\partial d_{n,t}} d_{n,t} = \sum_s \bar{\mu}_{s,t} A_{s \rightarrow n,t} \quad (18)$$

with the peer-to-peer allocation

$$C_{n \rightarrow s,t}^G = \bar{\mu}_{s,t} A_{s \rightarrow n,t} \quad (19)$$

So, how does this allocation behave? According to the polluter pays principle, it differentiates between consumers who are ‘responsible’ for investments and those who are not. If  $\bar{\mu}_{s,t}$  (in literature often denoted as the Quality of Supply) is bigger than zero, the upper Capacity Constr. (15) is binding. Thus it is these times steps which push investments in  $G_s$ . If  $\bar{\mu}_{s,t} = 0$ , the generation  $g_{s,t}$  is not bound and investments are not necessary. When summing over all CAPEX payments to generator  $s$  in Eq. (19) and using Eqs. (8) and (17), we see that each generator retrieves exactly the cost that were spent to build the capacity  $G_s$ .

## 2.4 Allocate Capital Expenditures for Transmission Lines

The CAPEX for the transmission system is given by

$$C^F = \sum_{\ell} c_{\ell} F_{\ell} \quad (20)$$

The transmission capacity variable  $F_{\ell}$  limits the flow  $f_{\ell,t}$  in both directions, such that

$$f_{\ell,t} - F_{\ell} \leq 0 \perp \bar{\mu}_{\ell,t} \quad \forall \ell, t \quad (21)$$

$$-f_{\ell,t} - F_{\ell} \leq 0 \perp \underline{\mu}_{\ell,t} \quad \forall \ell, t \quad (22)$$

Again, we use the result of [3] which derives that the investment in line  $\ell$  is payed back by the shadowprices times the flow

$$c_{\ell} F_{\ell} = \sum_t (\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}) f_{\ell,t} \quad (23)$$

We insert this into Eq. (20) and define the individual cost terms as

$$C_{n,t}^F = \frac{\partial \mathcal{C}^F}{\partial d_{n,t}} d_{n,t} = \sum_{\ell,t} (\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}) \frac{\partial f_{\ell,t}}{\partial d_{n,t}} d_{n,t} \quad (24)$$

The marginal change of flow with respect to the nodal demand is again non-unique and requires assumption on where the power is resulting from.

$$\frac{\partial f_{\ell,t}}{\partial d_{n,t}} d_{n,t} = A_{\ell,n,t} \quad (25)$$

fulfilling

$$\sum_n A_{\ell,n,t} = f_{\ell,t} \quad (26)$$

The peer-to-peer cost allocation for the transmission CAPEX is then defined as

$$C_{n \rightarrow \ell,t}^F = (\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}) A_{\ell,n,t} \quad (27)$$

Again we can interpret the shadowprices  $\bar{\mu}_{\ell,t}$  and  $\underline{\mu}_{\ell,t}$  as a measure for necessity of transmission investments at  $\ell$  at time  $t$ . Consumers which retrieve power flowing on congested lines, yielding a bound Constr. (21) or (22), pay compensations for the resulting investments. Again the sum of all CAPEX payments to line  $\ell$  equals the total CAPEX spent. This is seen when summing Eq. (27) over all buses and time steps and using Eqs. (23) and (26)

## 3 Assumptions on Power Allocations

The P2P cost allocation suits for any type of topology and network setup. But so far, the question of how  $A_{s \rightarrow n,t}$  and  $A_{\ell,n,t}$  are defined was left open. We recap that both rely on derivatives of  $g_{s,t}$  and  $f_{\ell,t}$  with respect to the nodal demand  $d_{n,t}$  fulfilling Eqs. (8) and (26).

In fact, it turns out that once the peer-to-peer allocations  $A_{s \rightarrow n,t}$  are defined, the power flow allocations  $A_{\ell,n,t}$  are determined by the linear power flow equations. In order to show this, let  $H_{\ell,n}$  denote Power Transfer Distribution Factors (PTDF) giving the changes in the flow on line  $\ell$  for one unit (typically one MW) of net power production at bus  $n$ . Further let  $K_{n,s}$  be 1 if generator  $s$  is attached to bus

$n$  and zero otherwise. The linear power flow equals the

$$f_{\ell,t} = \sum_m H_{\ell,m} (g_{m,t} - d_{m,t}) \quad (28)$$

where the nodal generation  $g_{m,t} = \sum_s K_{m,s} g_{s,t}$  combines the production of all generators attached to bus  $m$ . Taking the derivative with respect to the demand, leads to

$$A_{\ell,n,t} = \frac{\partial f_{\ell,t}}{\partial d_{n,t}} d_{n,t} = \sum_m H_{\ell,m} (A_{m \rightarrow n,t} - \delta_{n,m} d_{n,t}) \quad (29)$$

where  $A_{m \rightarrow n,t} = \sum_s K_{n,s} A_{s \rightarrow n,t}$  combines all power allocations of the source  $m$ . Equation (29) shows that the peer-to-peer allocation  $A_{s \rightarrow n,t}$  directly determines the flow allocation on line  $\ell$ .

This leaves us with only solving Eq. (8) for  $A_{s \rightarrow n,t}$ . In fact, we have to answer the question of how much power supply does generator  $s$  deliver to bus  $n$ . Unfortunately, the solution is non-unique and requires further assumptions. Established flow allocation schemes approach this problem from different directions. Principally two options exist *what* is allocated

1. gross power injections
2. net power injections

Further it is important *what assumptions* define the allocation, *i.e.* what method is used to define the pairs of sources and sinks. The three suitable approaches we present here are

- a. Equivalent Bilateral Exchanges (EBE) [4] which assumes that every producer supplies every consumer proportional to its share in the total consumption.
- b. Average Participation (AP) [2, 1] which traces the flow from producer to consumer following the law of proportional sharing.
- c. Flow Based Market Coupling (FBMC) which uses zonal PTDF for allocating power within predefined regions. The interregional exchange is only allocating net power deficit or excess of the regions.

We show the mathematical formulation for all combinations **a1** - **c2** in Appendices A.9 to A.12. Principally, type **2** leads to less P2P trades than type **1** as

power from a bus  $n$  with  $g_{n,t} \leq d_{n,t}$  is not assigned to other buses, only to  $n$ . Further, as literature has often pointed out, the EBE principal **a** does not suit for large networks where remote buses would interconnect in the same way as buses in close vicinity [5]. The AP based type **b** tackles this problem by restricting P2P trades to those which are traceable when applying the proportional sharing principal. Therefore  $A_{s \rightarrow n,t}$  denotes that part of power produced by bus  $m$  which, when only following in the direction of  $f_{\ell,t}$ , ends up at bus  $n$ . Type **c** further allows to control the regions or market zones which are netted out in a first calculation. If in a region  $R$  the generation undercuts the demand,  $\sum_{n \in R} g_{s,t} \leq \sum_{n \in R} d_{n,t}$ , none of the inner-regional generation is assigned to other regions.

### 3.1 Numerical Example

In the following we showcase a two bus system with one optimized time step and its resulting allocated payments, illustrated in Fig. 2.

The two buses are connected via one transmission line, each has one generator. Whereas the generator at bus 1 has an operational price of 50 €/MWh<sub>el</sub>, the generator at bus 2 has a higher operational price of 200 €/MWh<sub>el</sub>. For both the CAPEX rate is set to 500 €/MW and the maximal capacity is limited to  $\bar{G}_s = 100$  MW. The transmission line has a CAPEX rate of 100 €/MW and no upper capacity limit. With a demand of 60 MW at bus 1 and 90 MW at bus 2, the optimization expands the cheaper generator at bus 1 to its full limit of 100 MW. The 40 MW excess power not consumed at bus 1, flow to bus 2 where the generator is built with only 50 MW.

#### Allocating Gross Power using EBE (**1a**)

Figure 3 shows the allocated transactions on basis of equivalently traded gross power (type **1a**) for both buses 1 & 2 separately. The resulting P2P payments are given in Fig. 4. The upper graph Fig. 3a shows that  $A_{1,1} = 40$  MW at bus 1 are self-sustained. Consumers at bus 1 consequently pay 2k € OPEX and 22k € CAPEX to the generator at bus 1. The remaining 20 MW come from bus 2 and induce a sub-flow on line 1 of  $A_{2 \rightarrow 1,1} = -20$ . As this flow is in contrary direction to the total flow, it is relieving the transmission system. This translates to a congestion reward for consumers at bus 1 of  $c_{\ell=1} A_{2 \rightarrow 1,1} = 2k$  € which is exactly the cost that had to be spent on the

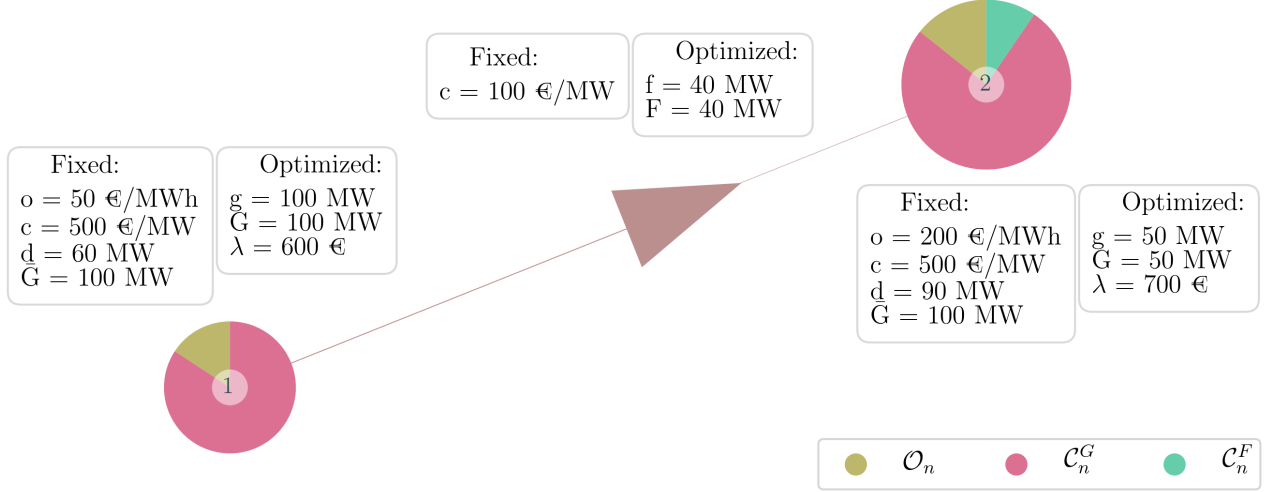


Figure 2: Illustrative example of a 2-bus network with one optimized time step. Fixed prices and constraining values are given in the left box for each bus and the transmission line. Optimized values are given in the right boxes. Bus 1 has a cheaper operational price  $o$ , capital prices are the same for both. As both generator capacities are constraint to 100 MW, the optimization also deploys the generator at bus 2. The resulting electricity prices  $\lambda$  are then a composition of all prices for operation and capital investments.

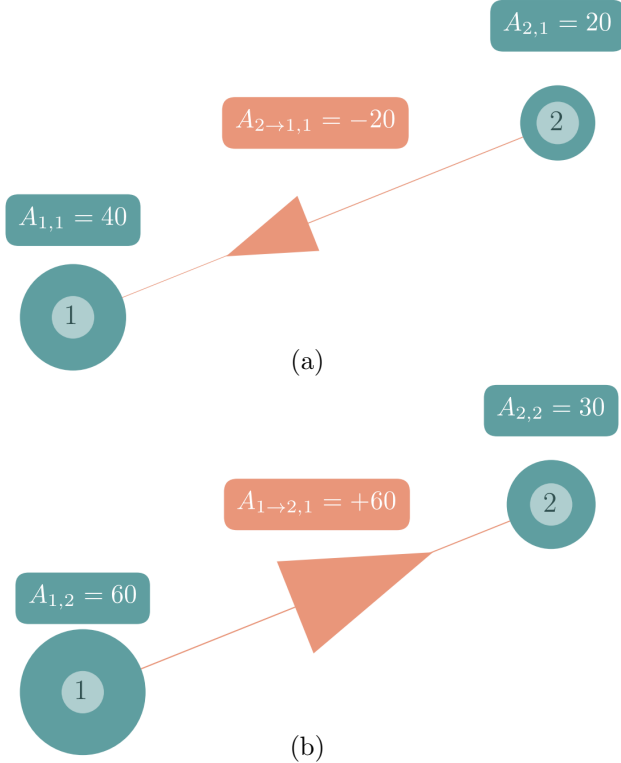


Figure 3: Power allocations for bus 1 (a) and bus 2 (b) of the example network in Fig. 2. Bus 1 retrieves 40 MW from itself and 20 MW from bus 2. The latter in turn retrieves 60 MW from bus 1 and self supplies 30 MW. The sum of both net flows equals the resulting flow of  $f = 40 \text{ MW}$ .

	$O_n$		$C_n^G$		$C_n^F$
1	2k €	4k €	22k €	10k €	-2k €
2	3k €	6k €	33k €	15k €	6k €
	1	2	1	2	1
	m		m		$\ell$

Figure 4: Full P2P cost allocation for the example setup shown in Fig. 2. The payments are derived on the basis of Eqs. (9), (19) and (27). Consumers at bus  $n$  have to pay each generator proportional to their consumption. As we only consider one time step the proportionality applies for OPEX  $O_n$  and CAPEX  $C_n$ . As bus 1 induces a relieving flow on line 1 and therefore “prevents” further transmission expansion, it is rewarded proportional to the relief.

transmission system if bus 1 didn’t induce a relieving flow, see again Fig. 4.

The lower graph Fig. 3b illustrates the impact of consumption at bus 2. As  $d_2$  is higher than  $d_1$  the retrievements from both generators are proportionately increased as well as the OPEX and CAPEX allocations to the generators. But instead of a re-

lieving flow, consumers at bus 2 drive the burdening flow in direction of congestion. Hence the payoff to the transmission system is positive and much higher than for bus 1.

The sum of all rows in the payoff matrix in Fig. 4 yields the revenues of the assets  $m, \ell$ . These values match their overall spendings, *e.g.* the total revenue of the transmission line is 4k € which equals the cost for investments  $c_1 F_1$ . The sum of all columns yields the total payment of consumers at bus  $n$ . For example the sum of payments of consumers at bus 1 is 36k €. This is exactly the electricity price of 600 €/MW times the consumption of 60 MW,  $\lambda_1 d_1$ .

The fact that OPEX and CAPEX allocations are proportional to the total consumption at a bus results from optimizing one time step only. In larger optimization problems with multiple time steps the CAPEX allocation takes effect only for time steps in which one or more of the capacity constraints Constrs. (15), (21) and (22) become binding.

### Allocating Net Power using EBE (2a)

In contrast the to equivalent allocation of gross power production, netting out injections for each bus leads to less P2P payments. The resulting payment given in Fig. 5 builds on the allocated power flow shown Fig. 6 in Appendix A.13. As bus 2 does not produce excess power, none of its power production is assigned to bus 1 and thus no payment of bus 1 to bus 2 allocated. Neither has bus 1 to pay fee to the transmission system as it only exports power. So all its consumers pay to its local generators. Bus 2 in constrast bear all CAPEX for the transmission system as well as CAPEX and OPEX for generators at bus 1.

Again the cummulative payments per bus meet the nodal spendings  $\lambda_{n,t} d_{n,t}$ . The cummulative revenues per generator and transmission line meet the all CAPEX and OPEX.

As the other schemes b1 - c2 apart from the EBE scheme don't distinguishably modify allocations within the presented example, we move on to a more realistic setup where systemic and local effects show up.

## 4 Application Case

		$\mathcal{O}_n$		$\mathcal{C}_n^G$		$\mathcal{C}_n^F$	
$n$	1	3k €	0k €	33k €	0k €	0k €	
	2	2k €	10k €	22k €	25k €	4k €	
		1	2	1	2	1	$\ell$
		$m$		$m$			

Figure 5: Full P2P cost allocation for the example setup shown in Fig. 2 when equivalently allocation net power injection (scheme a2).



## A Appendix

### A.1 LMP from Optimization

The nodal balance constraint ensures that the amount of power that flows into a bus equals the power that flows out of a bus, thus reflects the Kirchhoff Current Law (KCL)

$$\sum_l K_{n,\ell} f_{\ell,t} - g_{n,t} + d_{n,t} = 0 \quad \forall n, t \quad (30)$$

Its shadow price mirrors the Locational Marginal Prizes (LMP)  $\lambda_{n,t}$  per bus and time step. In a power market this is the €/MWh<sub>el</sub>-price which a consumer has to pay.

### A.2 Full Lagrangian

$$\begin{aligned} \mathcal{L}(g_{s,t}, f_{\ell,t}, G_s, F_\ell, \boldsymbol{\lambda}, \boldsymbol{\mu}) = & \sum_{n,s} c_s G_s + \sum_{n,s,t} o_s g_{s,t} + \sum_\ell c_\ell F_\ell \\ & + \sum_{n,t} \lambda_{n,t} \left( \sum_\ell K_{n,\ell} f_{\ell,t} - \sum_s K_{n,s} g_{s,t} + d_{n,t} \right) \\ & + \sum_{\ell,c,t} \lambda_{c,t} C_{\ell,c} x_{\ell,t} f_{\ell,t} \\ & + \sum_{n,s,t} \bar{\mu}_{s,t} (g_{s,t} - \bar{g}_{s,t} G_s) - \sum_{n,s,t} \underline{\mu}_{s,t} g_{s,t} \\ & + \sum_{\ell,t} \bar{\mu}_{\ell,t} (f_{\ell,t} - F_\ell) - \sum_{\ell,t} \underline{\mu}_{\ell,t} (f_{\ell,t} + F_\ell) \end{aligned} \quad (31)$$

where  $\boldsymbol{\lambda} = \{\lambda_{n,t}, \lambda_{c,t}\}$  and  $\boldsymbol{\mu} = \{\bar{\mu}_{s,t}, \underline{\mu}_{s,t}, \bar{\mu}_{\ell,t}, \underline{\mu}_{\ell,t}\}$  denote the set of related KKT variables. The global maximum of the Lagrangian requires stationarity with respect to all variables. The stationarity of the generation capacity variable leads to

$$\frac{\partial \mathcal{L}}{\partial G_s} = 0 \rightarrow c_s = \sum_t \bar{\mu}_{s,t} \bar{g}_{s,t} \quad \forall n, s \quad (32)$$

the stationarity of the transmission capacity to

$$\frac{\partial \mathcal{L}}{\partial F_\ell} = 0 \rightarrow c_\ell = \sum_t (\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}) \quad \forall \ell \quad (33)$$

and the stationarity of the generation to

$$\frac{\partial \mathcal{L}}{\partial g_{s,t}} = 0 \rightarrow o_s = K_{n,s} \lambda_{n,t} - \bar{\mu}_{s,t} + \underline{\mu}_{s,t} \quad \forall n, s \quad (34)$$

$$0 = \frac{\partial \mathcal{L}}{\partial f_{\ell,t}} \quad (35)$$

$$0 = \sum_n K_{n,\ell} \lambda_{n,t} + \lambda_{c,t} C_{\ell,c} x_{\ell,t} - \bar{\mu}_{\ell,t} + \underline{\mu}_{\ell,t} \quad \forall n, s \quad (36)$$

### A.3 Proof Generation Capacity Payback

As stated in [3] the capital cost of a generator  $s$  is equal to the shadowprice  $\bar{\mu}_{s,t}$  times the upper availability  $\bar{g}_{s,t}$  of all time steps

$$c_s = \sum_t \bar{\mu}_{s,t} \bar{g}_{s,t} \quad (37)$$

Constrs. (15) and (16) which yield the KKT variables  $\bar{\mu}_{s,t}$  and  $\underline{\mu}_{s,t}$  imply the complementary slackness,

$$\bar{\mu}_{s,t} (g_{s,t} - \bar{g}_{s,t} G_s) = 0 \quad \forall n, s, t \quad (38)$$

$$\underline{\mu}_{s,t} g_{s,t} = 0 \quad \forall n, s, t \quad (39)$$

Multiplying both sides of Eq. (32) with  $G_s$  and using Eq. (38) leads to

$$c_s G_s = \sum_t \bar{\mu}_{s,t} g_{s,t} \quad (40)$$

The zero-profit rule for generators is obtained by multiplying Eq. (34) with  $g_{s,t}$  and using Eqs. (39) and (40) which results in

$$c_s G_s + \sum_t o_s g_{s,t} = \sum_t \lambda_{n,t} K_{n,s} g_{s,t} \quad (41)$$

It states that over the whole time span, all OPEX and CAPEX for generator  $s$  (left hand side) are payed back by its revenue (right hand side).

### A.4 Proof Flow Capacity Payback

The yielding KKT variables  $\bar{\mu}_{\ell,t}$  and  $\underline{\mu}_{\ell,t}$  are only non-zero if  $f_{\ell,t}$  is limited by the transmission capacity in positive or negative direction, i.e. Constr. (21) or Constr. (22) are binding. For flows below the thermal limit, the complementary slackness

$$\bar{\mu}_{\ell,t} (f_{\ell,t} - F_\ell) = 0 \quad \forall \ell, t \quad (42)$$

$$\underline{\mu}_{\ell,t} (f_{\ell,t} + F_\ell) = 0 \quad \forall \ell, t \quad (43)$$

sets the respective KKT to zero. When multiplying Eq. (33) with  $F_\ell$  and using the complementary slackness Eqs. (42) and (43) we obtain

$$c_\ell F_\ell = \sum_t (\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}) f_{\ell,t} \quad (44)$$

Again we can use this to formulate the zero-profit rule for transmission lines. We multiply Eq. (36) with  $f_{\ell,t}$ , which finally leads us to

$$c_{\ell} F_{\ell} = \sum_n K_{n,\ell} \lambda_{n,t} f_{\ell,t} + \lambda_{c,t} C_{\ell,c} x_{\ell} f_{\ell,t} \quad (45)$$

It states that the congestion revenue of a line (first term right hand side) subtracted by the cost for cycle constraint exactly matches its CAPEX.

### A.5 CO<sub>2</sub> constraint

Imposing an additional CO<sub>2</sub> constraint limiting the total emission to  $K$ ,

$$\sum_{n,s,t} e_s g_{s,t} \leq K \quad \perp \quad \mu_{\text{CO}_2} \quad (46)$$

with  $e_s$  being the emission factor in tonne-CO<sub>2</sub> per MWh<sub>el</sub>, returns an effective CO<sub>2</sub> price  $\mu_{\text{CO}_2}$  in €/tonne-CO<sub>2</sub>. As shown in ... the constraint can be translated in a dual price which shift the operational price per generator

$$o_s \rightarrow o_s + e_s \mu_{\text{CO}_2} \quad (47)$$

### A.6 Power Transfer Distribution Factors and Flow Allocation

In linear power flow models, the Power Transfer Distribution Factors (PTDF)  $H_{\ell,n}$  determine the changes in the flow on line  $\ell$  for one unit (typically one MW) of net power production at bus  $n$ . Thus for a given production  $g_{s,t}$  and demand  $d_{n,t}$ , they directly link to the the resulting flow on each line,

$$f_{\ell,t} = \sum_n H_{\ell,n} (g_{n,t} - d_{n,t}) \quad (48)$$

where  $g_{n,t} = \sum_s g_{s,t}$  and  $d_{n,t} = \sum_a d_{n,t}$  combine all generators  $s$  and all consumers  $a$  attached to  $n$ . The PTDF have a degree of freedom: The slack  $k_n$  denotes the contribution of bus  $n$  to balancing out total power excess or deficit in the system. It can be dedicated to one bus, a single “slackbus“, or to several or all buses. The choice of slack modifies the PTDF according to

$$H_{\ell,n}(k_m) = H_{\ell,n}^{\circ} - \sum_m H_{\ell,m}^{\circ} k_m \quad (49)$$

where  $H_{\ell,n}^{\circ}$  denote the PTDF with equally distributed slack. When bus  $n$  injects excess power, it

has to flow to the slack; when bus  $n$  extract deficit power, it has to come from the slack. Summing over all ingoing and outgoing flow changes resulting from a positive injection at  $n$  yields again the slack

$$\sum_{\ell} K_{m,\ell} H_{\ell,n} = \delta_{m,n} - k_m \quad (50)$$

Note that  $\delta_{m,n}$  on the right hand side represents the positive injection at  $n$ . Established flow allocation schemes [cite] have used this degree of freedom in order to allocate power flows and exchanges to market participants. Under the assumption that consumers account for all power flows in the grid, the slack is set to  $k_n^*$  such that

$$f_{\ell,t} = - \sum_n H_{\ell,n} (k_m^*) d_{n,t} \quad (51)$$

With such a choice the flow can be reproduced from the demand-side of the system only. Now each term in the sum on the right hand side stands for the individual contribution of consumers at node  $n$  to the network flow  $f_{\ell,t}$ . In other words, each nodal demand  $d_{n,t}$  induces a subflow originating from the slack  $k_n^*$  which all together add up to  $f_{\ell,t}$ . These subflows, in turn, can be further broken down to contributions for each bus  $m$  in the slack, such that we get the subflow of individual  $m \rightarrow n$  relations, that is

$$A_{s \rightarrow n, \ell, t} = (H_{\ell,m}^{\circ} - H_{\ell,n}^{\circ}) k_m^* d_{n,t} \quad (52)$$

It indicates the flow on line  $\ell$  coming from generators at  $m$  and supplying the demand  $d_{n,t}$ . When summing over all sources  $m$  it yields the total subflow induced by  $d_{n,t}$ , the same term as in the sum on the right hand side in Eq. (51); when summing over all sources and sinks, it yields again the power flow, thus

$$f_{\ell,t} = \sum_{m,n} A_{s \rightarrow n, \ell, t} \quad (53)$$

As mentioned before, the consumed power  $d_{n,t}$  has to come from the slack  $k_m^*$ . As proven in Appendix A.7, for each P2P relation  $m \rightarrow n$ , the “traded“ power  $A_{s \rightarrow n, t}$  amounts to

$$A_{s \rightarrow n, t} = k_m^* d_{n,t} \quad (54)$$

Finally, when summing over all sinks the P2P trades yield the nodal generation (see Appendix A.8)

$$\sum_n A_{s \rightarrow n, t} = g_{m,t} \quad (55)$$



and summing over all sources yields the nodal demand

$$\sum_m A_{s \rightarrow n, t} = d_{n, t} \quad (56)$$

which follows from the fact that  $\sum_n k_n^* = 1$ . Both allocation quantities  $A_{s \rightarrow n, t}$  and  $A_{s \rightarrow n, \ell, t}$  can be broken down to generators  $s$  or consumers  $a$  by multiplying with the nodal production share  $\omega_{n, s, t} = g_{s, t} / \sum_s g_{s, t}$  and the nodal consumer share  $\omega_{n, a, t} = d_{n, t} / \sum_a d_{n, t}$  respectively.

The solution to  $k_n^*$  follows from combining Eqs. (54) and (55), which sets it to the share of the total production  $k_n^* = c g_{n, t}$  with  $c$  being defined as  $c = 1 / \sum_n g_{n, t}$ . That leads to the demand  $d_{n, t}$  of every single consumers  $a$  being supplied by all generators  $s$  in the network proportional to their gross production  $g_{s, t}$ .

## A.7 Proof Equation (54)

Equation (54) follows from summing  $A_{s \rightarrow n, \ell, t}$  over all incoming flows to  $n$  and taking into account the power that  $n$  provides by itself,  $k_n^* d_{n, t}$ , which leads us to

$$\begin{aligned} A_{s \rightarrow n, t} &= k_n^* d_{n, t} - \sum_{\ell} K_{n, \ell} A_{s \rightarrow n, \ell, t} \\ &= k_n^* d_{n, t} - \sum_{\ell} K_{n, \ell} (H_{\ell, m}^{\circ} - H_{\ell, n}^{\circ}) k_m^* d_{n, t} \\ &= k_n^* d_{n, t} - \left( \delta_{n, m} - \frac{1}{N} - \delta_{n, n} + \frac{1}{N} \right) k_m^* d_{n, t} \\ &= k_n^* d_{n, t} - (\delta_{n, m} - 1) k_m^* d_{n, t} \\ &= k_m^* d_{n, t} \end{aligned}$$

where we used Eq. (50) and the fact that the equally distributed slack amounts to  $1/N$  for all  $N$  nodes in the network.

## A.8 Proof of Equation (55)

The relation follows from multiplying Eq. (51) with  $\sum_m K_{m, \ell}$ , and solving for  $A_{s \rightarrow n, t}$

$$\begin{aligned} \sum_m K_{m, \ell} f_{\ell, t} &= - \sum_{m, n} K_{m, \ell} H_{\ell, n} d_{n, t} \\ g_{m, t} - d_{m, t} &= -\delta_{m, n} d_{n, t} + k_m^* d_{n, t} \\ A_{s \rightarrow n, t} &= g_{m, t} - d_{m, t} + \delta_{m, n} d_{n, t} \\ \sum_n A_{s \rightarrow n, t} &= g_{m, t} \end{aligned}$$

## A.9 Allocating Gross Injections with EBE

Allocating gross injections using the Equivalent Bilateral Exchanges Principal is simplistic and straightforward scheme. Accordingly each generator supplies each bus proportional to its power demand. The allocation is given by

$$A_{s \rightarrow n, t} = \frac{g_{s, t}}{\sum_s g_{s, t}} d_{n, t} \quad (57)$$

## A.10 Allocating Net Injections with EBE

Allocating net power injections using the EBE methods leads to the same result as the Marginal Participation (MP) [7] algorithm when allocating to consumers only, see [6] for further insight. We calculate it by setting

$$A_{m \rightarrow n, t} = \delta_{m, n} p_{m, t}^{\circ} + \gamma_t p_{n, t}^{-} p_{m, t}^{+} \quad (58)$$

where

- $p_{n, t}^{+} = \min(g_{n, t} - d_{n, t}, 0)$  denotes the nodal net production
- $p_{n, t}^{-} = \min(d_{n, t} - g_{n, t}, 0)$  denotes the nodal net consumption
- $p_{n, t}^{\circ} = \min(p_{n, t}^{+}, p_{n, t}^{-})$  the denotes nodal self-consumption. That is the power generated and at the same time consumed at node  $n$  and
- $\gamma_t = (\sum_n p_{n, t}^{+})^{-1} = (\sum_n p_{n, t}^{-})^{-1}$  is the inverse of the total injected/extracted power at time  $t$ .

The allocation  $A_{s \rightarrow n, t}$  from generator  $s$  to  $n$ , is given by multiplying  $A_{m \rightarrow n, t}$  with the nodal share  $g_{s, t} / g_{n, t}$ .

## A.11 Allocating Net Power using AP

Allocating net injections using the AP method is derived from [1]. In a lossless network the downstream and upstream formulations result in the same P2P allocation which is why we restrict ourselves to the downstream formulation only. In a first step we define a time-dependent auxiliary matrix  $\mathcal{J}_t$  which is the inverse of the  $N \times N$  with directed power flow  $m \rightarrow n$  at entry  $(m, n)$  for  $m \neq n$  and the total flow passing node  $m$  at entry  $(m, m)$  at time step  $t$ . Mathematically this translates to

$$\mathcal{J}_t = (\text{diag}(p^+) + \mathcal{K}^- \text{diag}(f) K)_t^{-1} \quad (59)$$

where  $\mathcal{K}^-$  is the negative part of the directed Incidence matrix  $\mathcal{K}_{n,\ell} = \text{sign}(f_{\ell,t}) K_{n,\ell}$ . Then the distributed slack for time step  $t$  is given by

$$A_{m \rightarrow n,t} = \mathcal{J}_{m,n,t} p_{m,t}^+ p_{n,t}^- \quad (60)$$

### A.12 Allocating Gross Power using AP

We use the same allocation as in Appendix A.11 but replace the net nodal production  $p_{n,t}^+$  by the gross nodal production  $g_{n,t}$  which leads to

$$\mathcal{J}_t = (\text{diag}(g) + \mathcal{K}^- \text{diag}(f) K)_t^{-1} \quad (61)$$

The distributed slack is for time step  $t$  is then given by

$$A_{s \rightarrow m,t} = \mathcal{J}_{m,n} g_{s,t} d_{n,t} \quad (62)$$

### A.13 Example: Power Flow Allocations of different Types

### A.14 Brownfield Optimization and Capacity Restrictions

Constraining the capacities  $G_s$  for a subset  $S$  of generators to lower or upper limits in the form of

$$G_s \geq \underline{G}_s \perp \underline{\mu}_s^{\text{nom}} \quad \forall n, s \in S \quad (63)$$

$$G_s \leq \bar{G}_s \perp \bar{\mu}_s^{\text{nom}} \quad \forall n, s \in S \quad (64)$$

alters the objective value as soon as one of those become bounding. The complementary slackness state is

$$\bar{\mu}_s^{\text{nom}} (G_s - \bar{G}_s) = 0 \quad (65)$$

$$\underline{\mu}_s^{\text{nom}} (G_s - \underline{G}_s) = 0 \quad (66)$$

$$(67)$$

The CAPEX paybacks for generators Eqs. (40) and (44) change to

$$\frac{\partial \mathcal{L}}{\partial G_s} = 0 \rightarrow \quad (68)$$

$$c_s = \sum_t \bar{\mu}_{s,t} \bar{g}_{s,t} + \underline{\mu}_s^{\text{nom}} - \bar{\mu}_s^{\text{nom}} \quad \forall n, s \quad (69)$$

for generators. Multiplying Eq. (69) by  $G_s$

$$c_s G_s = \sum_t \bar{\mu}_{s,t} g_{s,t} + \underline{\mu}_s^{\text{nom}} G_s - \bar{\mu}_s^{\text{nom}} \bar{G}_s \quad \forall n, s \quad (70)$$

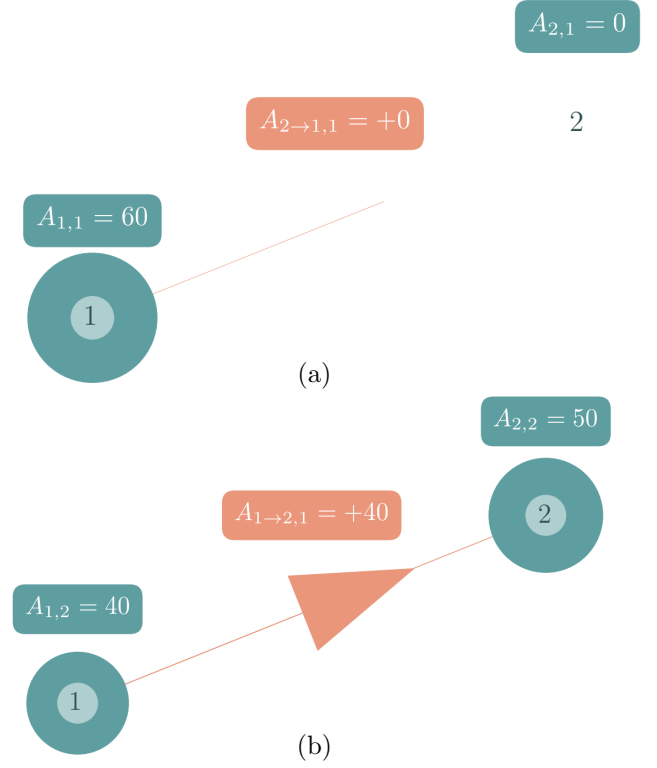


Figure 6: Power allocations for bus 1 (a) and bus 2 (b) of the example network in Fig. 2 using equivalently allcated net power injections (scheme 2a). Bus 1 retrieves 60 MW from itself and nothing from bus 2. The latter in turn retrieves 40 MW from bus 1 and self supplies 50 MW. The P2P trades are less in number and more intuitive then with allocating gross flow.

Multiplying Eq. (69) by the amount of expansion  $(G_s - \underline{G}_s)$

$$c_s G_s^{\text{exp}} = \sum_t \bar{\mu}_{s,t} (g_{s,t} - \bar{g}_{s,t} G_s) + \bar{\mu}_s^{\text{nom}} (\bar{G}_s - G_s) \quad \forall n, s \quad (71)$$

playaround

$$c_s G_s = \sum_t \bar{\mu}_{s,t} g_{s,t} + \underline{\mu}_s^{\text{nom}} G_s - \bar{\mu}_s^{\text{nom}} \bar{G}_s \quad \forall n, s \quad (72)$$

## References

- [1] Chira Achayuthakan et al. “Electricity Tracing in Systems With and Without Circulating Flows: Physical Insights and Mathematical Proofs”. en. In: *IEEE Transactions on Power Systems* 25.2 (May 2010), pp. 1078–1087. ISSN: 0885-8950, 1558-0679. DOI: [10.1109/TPWRS.2009.2037506](#).
- [2] J. Bialek. “Tracing the Flow of Electricity”. en. In: *IEE Proceedings - Generation, Transmission and Distribution* 143.4 (1996), p. 313. ISSN: 13502360. DOI: [10.1049/ip-gtd:19960461](#).
- [3] T. Brown and L. Reichenberg. “Decreasing Market Value of Variable Renewables Is a Result of Policy, Not Variability”. In: *arXiv:2002.05209 [econ, math, q-fin]* (Feb. 2020). arXiv: [2002.05209 \[econ, math, q-fin\]](#).
- [4] F.D. Galiana, A.J. Conejo, and H.A. Gil. “Transmission Network Cost Allocation Based on Equivalent Bilateral Exchanges”. en. In: *IEEE Transactions on Power Systems* 18.4 (Nov. 2003), pp. 1425–1431. ISSN: 0885-8950. DOI: [10.1109/TPWRS.2003.818689](#).
- [5] H.A. Gil, F.D. Galiana, and A.J. Conejo. “Multiarea Transmission Network Cost Allocation”. en. In: *IEEE Transactions on Power Systems* 20.3 (Aug. 2005), pp. 1293–1301. ISSN: 0885-8950. DOI: [10.1109/TPWRS.2005.851951](#).
- [6] Fabian Hofmann et al. “Flow Allocation in Meshed AC-DC Electricity Grids”. en. In: *Energies* 13.5 (Mar. 2020), p. 1233. ISSN: 1996-1073. DOI: [10.3390/en13051233](#).
- [7] H. Rudnick, R. Palma, and J.E. Fernandez. “Marginal Pricing and Supplement Cost Allocation in Transmission Open Access”. In: *IEEE Transactions on Power Systems* 10.2 (May 1995), pp. 1125–1132. ISSN: 08858950. DOI: [10.1109/59.387960](#).
- [8] Fred C. Schweppe et al. *Spot Pricing of Electricity*. en. Boston, MA: Springer US, 1988. ISBN: 978-1-4612-8950-0 978-1-4613-1683-1. DOI: [10.1007/978-1-4613-1683-1](#).