From Linear Optimization to Transmission Cost Allocation

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Lagrange Mutliplier

Maximise $f(x_l)$, with equality constraints $g_i(x_l)$ and inequality constraints $h_j(x_l)$

$$\mathcal{L}(x_l, \lambda_i, \mu_j) = f(x_l) - \sum_i \lambda_i \, g_i(x_l) - \sum_j \mu_j \, h_j(x_l) \tag{1}$$

...

Linear Energy Modelling and LMP

We linearly optimize the capacity and dispatch of a simple power system.

$$\max_{d_{n,a,t}, g_{n,s,t}, G_{n,s}} \left(U_{n,a,t}(d_{n,a,t}) - \sum_{n,s} c_{n,s} G_{n,s} - \sum_{n,s,t} o_{n,s} g_{n,s,t} - \sum_{\ell} c_\ell \, F_\ell \right) \tag{2}$$

where

$U_{n,a,t}$ denotes the utility function per bus n, demand type a time step t

$d_{n,a,t}$ denotes the eletric demand

$c_{n,s}$ denotes the capital expenditure (CAPEX) per node n and generator type s

$G_{n,s}$ denotes the generation capacity

$o_{n,s}$ denotes the operational cost (OPEX)

$g_{n,s,t}$ denotes the net generation in MW

c_ℓ denotes the CAPEX per transmission line

F_ℓ denotes the transmission capacity.

Constraints

In the following we neglect the utility $U_{n,a,t}$ of the nodal demand while fixing the demand $d_{n,a,t}$ to a predefined time-series.

The nodal balance constraint ensures that the amount of power that flows into a bus equals the power that flows out of a bus, thus reflects the Kirchhoff Current Law (KCL)

$$\sum_l K_{n,\ell} \, f_{\ell,t} - \sum_s g_{n,s,t} + \sum_a d_{n,a,t} = 0 \quad \perp \quad \lambda_{n,t} \qquad \forall\, n, t \tag{3}$$

Its shadow price mirrors the Locational Marginal Prizes (LMP) $\lambda_{n,t}$ per bus and time step. In a power market this is the €/MWh$_{\text{el}}$-price which a consumer has to pay.

The generation $g_{n,s,t}$ is constraint to its nominal capacity

$$g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s} \leq 0 \quad \perp \quad \bar{\mu}_{n,s,t} \qquad \forall\, n, s, t \tag{4}$$
$$-g_{n,s,t} \leq 0 \quad \perp \quad \underline{\mu}_{n,s,t} \qquad \forall\, n, s, t \tag{5}$$

where $\bar{g}_{n,s,t} \in [0,1]$ is the capacity factor for renewable generators. The constraints yield the KKT variables $\bar{\mu}_{n,s,t}$ and $\underline{\mu}_{n,s,t}$ which due to complementary slackness,

$$\bar{\mu}_{n,s,t} \left(g_{n,s,t} - \bar{g}_{n,s,t}\, G_{n,s} \right) = 0 \qquad \forall\, n, s, t \tag{6}$$
$$\underline{\mu}_{n,s,t}\, g_{n,s,t} = 0 \qquad \forall\, n, s, t \tag{7}$$

are only non-zero if the corresponding constraint is binding.

The transmission capacity F_ℓ limits the flow $f_{\ell,t}$ in both directions, such that

$$f_{\ell,t} - F_\ell \leq 0 \quad \perp \quad \bar{\mu}_{\ell,t} \qquad \forall\, \ell, t \tag{8}$$
$$-f_{\ell,t} - F_\ell \leq 0 \quad \perp \quad \underline{\mu}_{\ell,t} \qquad \forall\, \ell, t \tag{9}$$

The yielding KKT variables $\bar{\mu}_{\ell,t}$ and $\underline{\mu}_{\ell,t}$ are only non-zero if $f_{\ell,t}$ is limited by the trasmission capacity in positive or negative direction, i.e. Constr. (8) or Constr. (9) are binding. The complementary slackness

$$\bar{\mu}_{\ell,t} \left(f_{\ell,t} - F_\ell \right) = 0 \qquad \forall\, \ell, t \tag{10}$$
$$\underline{\mu}_{\ell,t} \left(f_{\ell,t} - F_\ell \right) = 0 \qquad \forall\, \ell, t \tag{11}$$

set the respective KKT for flows staying below the thermal limit to zero.

Flow is a passive variable

We treat the flow $f_{\ell,t}$ an a passive quantity only that is it is only influenced by the nodal injection.

The Power Transfer Distribition Factors (PTDF) $H_{\ell,n}$ determine the changes in the flow on line ℓ for one unit (typically one MW) of net power production at bus n. Thus with a fix demand $d_{n,a,t}$, they direclty link the generation $g_{n,s,t}$ to the flow on each line according to

$$f_{\ell,t}\left(g_{n,s,t}\right) = \sum_n H_{\ell,n} \left(\sum_s g_{n,s,t} - \sum_a d_{n,a,t} \right) \tag{12}$$

The PTDF have a degree of freedom: The slack k_n denotes the contribution of bus n to balancing out total power excess or deficit in the system. It can be dedicated to one bus, a sinlge "slackbus", or to several or all buses. The choice of slack modifies the PTDF accordingly

$$H_{\ell,n} = H^\circ_{\ell,n} - \sum_m H^\circ_{\ell,m}\, k_m \tag{13}$$

where $H^\circ_{\ell,n}$ denote the PTDF with equally distributed slack. When bus n injects excess power, is has to flow to the slack; when bus n extract deficit power, it has to come from the slack. Summing over all ingoing and outgoing flow changes resulting from an positive injection at n yields again the slack

$$\sum_\ell K_{m,\ell}\, H_{\ell,n} = \delta_{m,n} - k_m \tag{14}$$

where $\delta_{m,n}$ on the right hand side represents the positive injection at n.

Breaking Down the Full Lagrangian

$$\mathcal{L}\left(g_{n,s,t}, G_{n,s}, F_\ell, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) = -\sum_{n,s} c_{n,s} G_{n,s} - \sum_{n,s,t} o_{n,s} g_{n,s,t} - \sum_{\ell} c_\ell\, F_\ell \tag{15}$$

$$-\sum_{n,t} \lambda_{n,t} \left(\sum_{\ell} K_{n,\ell}\, f_{\ell,t} - \sum_{s} g_{n,s,t} + \sum_{a} d_{n,a,t} \right) \tag{16}$$

$$-\sum_{n,s,t} \bar{\mu}_{n,s,t} \left(g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s}\right) + \sum_{n,s,t} \underline{\mu}_{n,s,t} g_{n,s,t} \tag{17}$$

$$-\sum_{\ell,t} \bar{\mu}_{\ell,t} \left(f_{\ell,t} - F_\ell\right) + \sum_{\ell,t} \underline{\mu}_{\ell,t} \left(f_{\ell,t} + F_\ell\right) \tag{18}$$

where $\boldsymbol{\lambda} = \{\lambda_{n,t}\}$ and $\boldsymbol{\mu} = \left\{\bar{\mu}_{n,s,t}, \underline{\mu}_{n,s,t}, \bar{\mu}_{\ell,t}, \underline{\mu}_{\ell,t}\right\}$ denote the set of related KKT variables. The global maximum of the Lagrangian requires stationarity with respect to all variables. The stationarity of the generation capacity variable leads to

$$\frac{\partial \mathcal{L}}{\partial G_{n,s}} = 0 \quad \rightarrow \quad c_{n,s} = \sum_{t} \bar{\mu}_{n,s,t}\, \bar{g}_{n,s,t} \qquad \forall\, n, s \tag{19}$$

the stationarity of the transmission capacity to

$$\frac{\partial \mathcal{L}}{\partial F_\ell} = 0 \quad \rightarrow \quad c_\ell = \sum_{t} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) \qquad \forall\, \ell \tag{20}$$

and the stationarity of the generation to

$$\frac{\partial \mathcal{L}}{\partial g_{n,s,t}} = 0 \qquad \forall\, n, s \tag{21}$$

$$\rightarrow \quad o_{n,s} = \lambda_{n,t} - \bar{\mu}_{n,s,t} + \underline{\mu}_{n,s,t} - \sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) H_{\ell,n} - \sum_{m,\ell} \lambda_{m,t}\, K_{m,\ell}\, H_{\ell,n} \qquad \forall\, n, s \tag{22}$$

For the latter, we used Eq. (12) which sets the derivative of the flow with respect to the generation to

$$\frac{\partial f_{\ell,t}}{\partial g_{n,s,t}} = H_{\ell,n} \tag{23}$$

Equations (19), (20) and (22) show how the capital and operational prices translate into dual variables.

Note that Eq. (22) must hold for every choice of slack in the PTDF. According to Eqs. (13) and (14), setting the slack to $k_n = \delta_{m,n}$ results in $H_{\ell,n} = \sum_\ell K_{m,\ell}\, H_{\ell,n} = 0$. This leads to our first represantion for Locational Market Price, which we will refer to as the "Island Solution",

$$\lambda_{n,t} = o_{n,s} + \bar{\mu}_{n,s,t} - \underline{\mu}_{n,s,t} \qquad \forall\, n, s, t \tag{24}$$

Accordingly, the LMP is directly determined by the local operational price and prices for the generation capacity constraint. However, from the Island Solution, we can derive a second representation for the LMP. Feeding Eq. (24) back into Eq. (22) and applying Eq. (14), finally leads to

$$\lambda_{n,t} = -\sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) H_{\ell,n} + \sum_{m} \left(o_{m,s} + \bar{\mu}_{m,s,t} - \underline{\mu}_{m,s,t}\right) k_m \qquad \forall\, n, s, t \tag{25}$$

This equation generalizes the former Island Solution for the LMP which again can be reproduced by setting $k_m = \delta_{m,n}$. It depicts the interdependence of the LMP, that is how $\lambda_{n,t}$ can be decomposed to operational prices and prices for capacity constraints from all generators and transmission lines in the system.

Cost Allocation

The choice of slack k_m in Eq. (25) decides on the generators and lines to which the nodal price at n is allocated to. As we will see, there are various valid options for this choice. Mutliplied by the nodal consumption $d_{n,t} = \sum_a d_{n,a,t}$, the slack translates into the peer-to-peer relation

$$A_{m\to n,t} = d_{n,t}\, k_m \qquad \forall\; n,t \tag{26}$$

that is the power which is produced at bus m at time t and assigned to consumers at bus n. Or the other words, the part of demand $d_{n,t}$ that is covered by generators at bus m. Likewise, multipying the PTDF with the nodal demand will result in the flow induces by the net consumption at n coming from the distributed slack

$$\sum_m A_{m\to n,\ell,t} = -d_{n,t}\, H_{\ell,n} \qquad \forall\; n,\ell,t \tag{27}$$

where the PTDF include the slack k_m according to Eq. (13). The slack k_m in Eqs. (26) and (27) should be chosen such that the sum over all recipients n in $A_{m\to n,t}$ results in the nodal production, thus

$$\sum_n A_{m\to n,t} = \sum_s g_{m,s,t} \qquad \forall\; n,s,t \tag{28}$$

Then, weighted by the nodal production share $\omega_{m,s,t} = g_{m,s,t}/\sum_s g_{m,s,t}$ the allocation distributes all power pruduced by generator (m,s),

$$\sum_n \omega_{m,s,t}\, A_{m\to n,t} = g_{m,s,t} \qquad \forall\; m,s,t \tag{29}$$

and the sum over all producers m and recipients n in $A_{m\to n,\ell,t}$ equals the power flow on line ℓ,

$$\sum_{m,n} A_{m\to n,\ell,t} = f_{\ell,t} \qquad \forall\; \ell,t \tag{30}$$

The quantities $A_{m\to n,t}$ and $A_{m\to n,\ell,t}$ allow a intuitive decomposition of the nodal payments. Multiplying both sides of Equation (25) with the nodal demand $d_{n,t}$ will breakdown the costs payed by consumers at node n to the different expeditures

$$\lambda_{n,t}\, d_{n,t} = \mathcal{C}^F_{n,t} + \mathcal{O}_{n,t} + \mathcal{C}^G_{n,t} \qquad \forall\; n,t \tag{31}$$

with the allocated payments:

$$\mathcal{O}_{n,t} = \sum_{m,s} \mathcal{O}_{n\to(m,s),t} = \sum_{m,s} o_{m,s}\, \omega_{m,s,t}\, A_{m\to n,t} \qquad \text{OPEX for generators} \tag{32}$$

$$\mathcal{C}^G_{n,t} = \sum_{m,s} \mathcal{C}^G_{n\to(m,s),t} = \sum_{m} \bar{\mu}_{m,s,t}\, \omega_{m,s,t}\, A_{m\to n,t} \qquad \text{CAPEX for the generators} \tag{33}$$

$$\mathcal{C}^F_{n,t} = \sum_{\ell} \mathcal{C}^F_{n\to\ell,t} = \sum_{m,\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) A_{m\to n,\ell,t} \qquad \text{CAPEX for the transmission system} \tag{34}$$

Let's have a look at the allocated OPEX first. Consumers at bus n retrieve power from different generators (m,s) and accordingly pay for operational costs. The OPEX allocation can thus be seen as P2P tradings between producers and consumers with fixed production prices. In this way, the generator

(m, s) retrieves the exact amount of money from consumers that it spends on the operation. In other words, all OPEX payments to generator (m,s) sum up to the total OPEX spent at (m,s), thus

$$\sum_n \mathcal{O}_{n\to(m,s),t} = o_{n,s}\, g_{n,s,t} \tag{35}$$

The CAPEX allocation for generators reveil a similar relation. According to the polluter pays principle, it differentiates between consumers who are 'responsible' for investments and those who are not. If $\bar{\mu}_{n,s,t} > 0$, the upper Capacity Constr. (4) is binding. Thus it is these times steps which push investments in $G_{n,s}$. If in contrast $\bar{\mu}_{n,s,t} = 0$, the generation $g_{n,s,t}$ is not bound and investments are not necessary. When summing over all CAPEX payments to generator (m, s) each generator retrieves exactly the cost that were spent to build the capacity $G_{n,s}$,

$$\sum_{n,t} \mathcal{C}^G_{n\to(m,s),t} = c_{n,s}\, G_{n,s} \tag{36}$$

where we used Eqs. (6) and (19). Hence, throughout all time steps each generator (m, s) receives the money it spends for invesments and operation (non-profit rule).

The allocation of CAPEX for the transmission system $\mathcal{C}^F_{n,t}$ builds up on the KKT variables $\bar{\mu}_{\ell,t}$ and $\underline{\mu}_{\ell,t}$. Again the latter translate to the necessity of transmission invesments at ℓ at time t. Consumers which retrieve power flowing on congested lines, yielding a bound Constr. (8) or (9), pay compensations for the resulting investments. Again the sum of all CAPEX payments to line ℓ equal the total CAPEX spent, thus

$$\sum_{n,t} \mathcal{C}^F_{n\to\ell,t} = c_\ell\, F_\ell \tag{37}$$

where we used the complementary slackness Eqs. (10) and (11) and the fact that summing over all sources m and sinks n the allocation equals the actual power flow as stated in Eq. (30).

Adding CO_2 Constraints

Imposing an additional CO_2 constraint limiting the total emission to K,

$$\sum_{n,s,t} e_{n,s}\, g_{n,s,t} \leq \mathrm{K} \quad \perp \quad \mu_{\mathrm{CO2}} \tag{38}$$

with $e_{n,s}$ being the emission factor in tonne-CO_2 per $\mathrm{MWh_{el}}$, returns an effective CO_2 price μ_{CO2} in €/tonne-CO_2. As shown in ... the constraint can be translated in a dual price which shift the operational price per generator

$$o_{n,s} \to o_{n,s} + e_{n,s}\, \mu_{\mathrm{CO2}} \tag{39}$$

This leads to allocated CO_2 cost compensation of node n of

$$\mathcal{E}_{n,t} = \mu_{\mathrm{CO2}} \sum_{m,s} e_{m,s}\, \omega_{m,s,t}\, A_{m\to n,t} \qquad \forall\, n,t \tag{40}$$

which expands the allocation of the electricity cost in Eq. (31) to

$$\lambda_{n,t}\, d_{n,t} = \mathcal{C}^F_{n,t} + \mathcal{O}_{n,t} + \mathcal{C}^G_{n,t} + \mathcal{E}_{n,t} \qquad \forall\, n,t \tag{41}$$

Choices of Slack

The presented cost allocation is sensitive to the choice of slack k_m. The latter shift the shares of $\mathcal{O}_{n,t}$, $\mathcal{C}^G_{n,t}$, $\mathcal{C}^F_{n,t}$ and $\mathcal{E}_{n,t}$ for each node, whereas the total payment per node remains the same. In the following we show the connection of this cost allocation to established flow allocation methods.

...
Notes:
· $A_{m\to n,t}$ defined by Marginal Participation or Equivalent Bilateral Exchanges can be used, these hold all discussed equations.
· One can "invent" other allocation schemes based on other choices of slack, for example $k_m = g_{m,t}$.

Showcase

Network with CO_2 constraint

We illustrate the flow based cost allocation under use of the fictive network shown in Fig. 2a. It consists of nine buses and ten time steps. The solver optimizes the capacity of two generators, wind and gas, per bus. ...

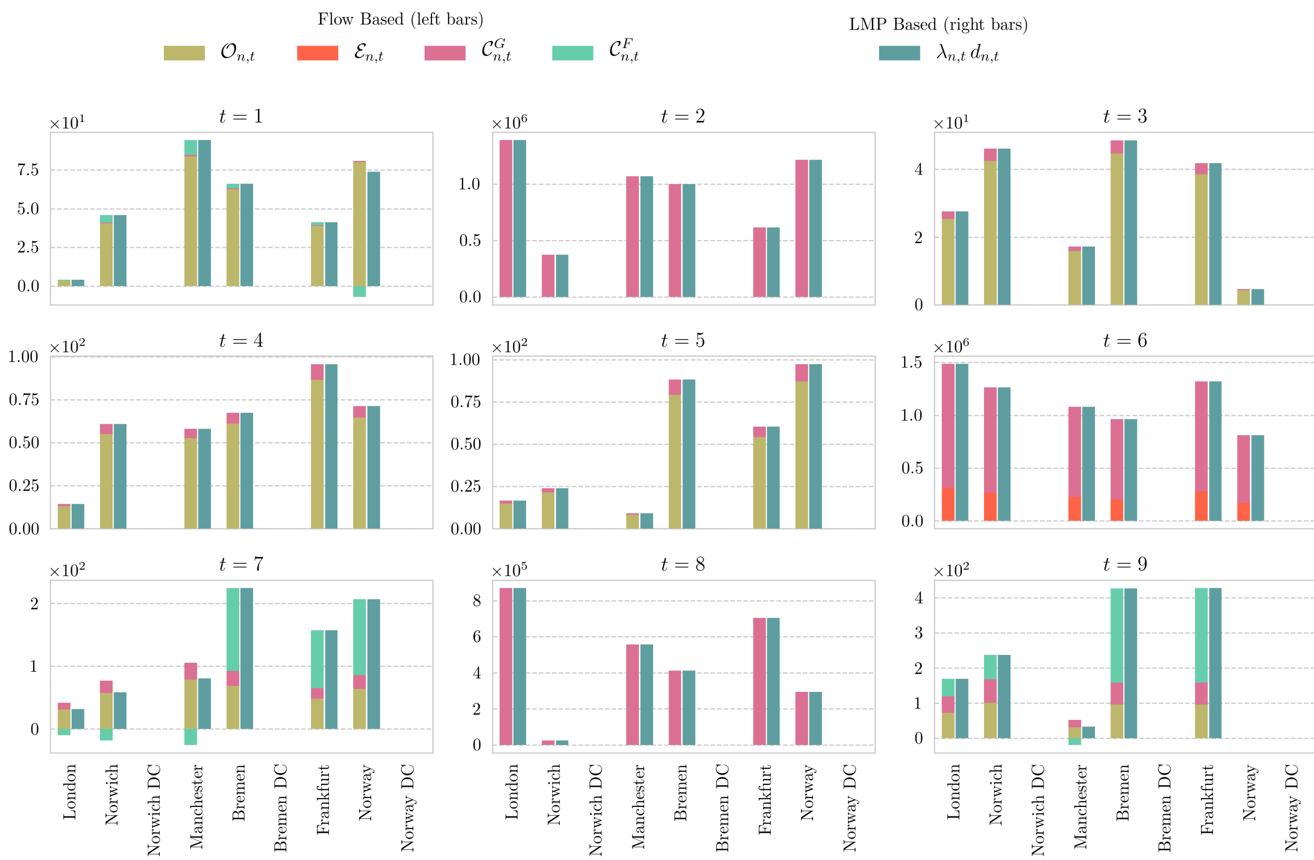

Figure 1: Comparison between the flow based cost allocation and the LMP based cost per consumer. The left bars consist of the allocated OPEX $\mathcal{O}_{n,t}$, the allocated CO_2 cost $\mathcal{E}_{n,t}$, the allocated generator CAPEX $\mathcal{C}^G_{n,t}$ and transmission CAPEX $\mathcal{C}^F_{n,t}$, while the right bars show the of the nodal consumption times the LMP.

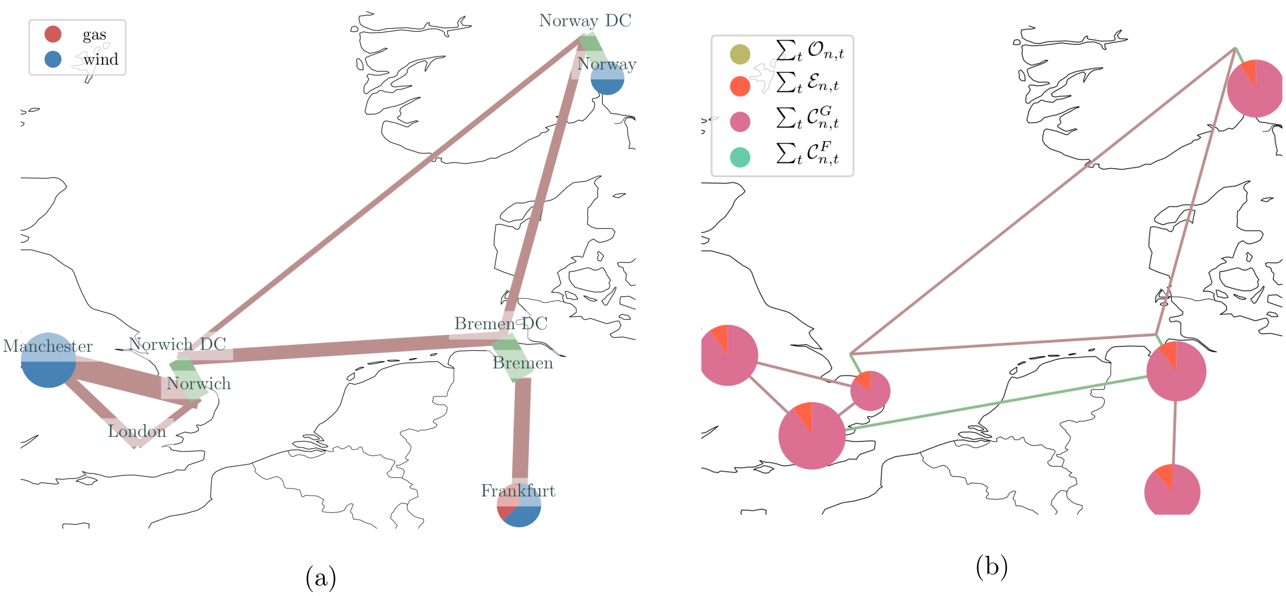

Figure 2: Network used for showcasing. (a) shows the distributing of generation capacities $G_{n,s}$, the widths of the transmission lines are proportional to their thermal limit F_ℓ. (b) shows the total nodal payments according to the cost allocation.