From Linear Optimization to Transmission Cost Allocation

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Abstract

Maximizing the welfare of all market participants within a power system is a common approach in energy system modelling. It leads to perfectly scheduled operations of generators and

1 Introduction

2 Locational Marginal Price

Nomenclature

$\lambda_{n,t}$	Locational Market Price at bus n and time step t in €/MW
$d_{n,a,t}$	Electric demand per bus n, demand type a, time step t in MW
$g_{n,s,t}$	Electric generation per bus n, carrier s, time step t in MW
$f_{\ell,t}$	Active power flow on line ℓ, time step t in MW
$o_{n,s}$	Operational cost (OPEX) in €/MW
$c_{n,s}$	Capital Expenditure (CAPEX) in €/MW
c_{ℓ}	CAPEX per transmission line ℓ in €/MW
$G_{n,s}$	Generation capacity in MW
F_{ℓ}	Transmission capacity in MW
$K_{n,\ell}$	Incidence matrix

As originally presented in [**schweppe˙spot˙1988**], the Locational Market Price (LMP) describes the marginal price for an addtional unit of electricity demand at bus n, thus it is the derivative of the total system cost $\mathcal{TC}$ with respect to the local demand $d_{n,t}$

$$\lambda_{n,t} = \frac{\partial \mathcal{TC}}{\partial d_{n,t}} \tag{1}$$

Minimizing the total cost $\mathcal{TC}$ of a power system leads to different Locational Marginal Prices (LMP) $\lambda_{n,t}$ for each node n and time stemp t in the system.

A long term investment model of a power system with perfect foresight, The total system cost $\mathcal{TC}$ has to be payed back by the consumers

$$\mathcal{TC} = \sum_{n,a,t} \lambda_{n,t}\, d_{n,a,t} \tag{2}$$

$$\mathcal{TC} = \mathcal{O} + \mathcal{C}^G + \mathcal{C}^F + \text{rest} \tag{3}$$

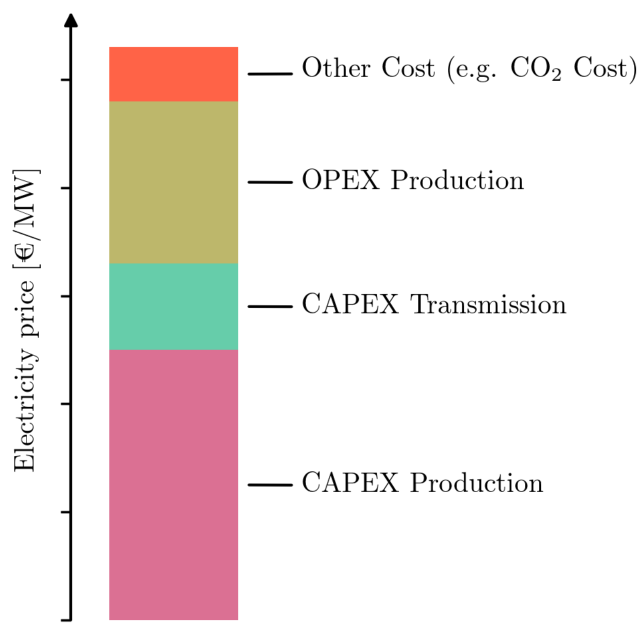

Figure 1: The Locational Market Price

3 Mathematical Theory

Power Transfer Distribition Factors and Flow Allocation

In linear power flow models, the Power Transfer Distribition Factors (PTDF) $H_{\ell,n}$ determine the changes in the flow on line ℓ for one unit (typically one MW) of net power production at bus n. Thus for a given gross production $g_{n,s,t}$ and gross demand $d_{n,a,t}$, they directly link to the the resulting flow on each line,

$$f_{\ell,t} = \sum_n H_{\ell,n} \left(g_{n,t} - d_{n,t}\right) \tag{4}$$

where $g_{n,t} = \sum_s g_{n,s,t}$ and $d_{n,t} = \sum_a d_{n,a,t}$ combine all generators s and all comsumers a attached to n. The PTDF have a degree of freedom: The slack k_n denotes the contribution of bus n to balancing out total power excess or deficit in the system. It can be dedicated to one bus, a sinlge "slackbus", or to several or all buses. The choice of slack modifies the PTDF according to

$$H_{\ell,n} \left(k_m\right) = H^{\circ}_{\ell,n} - \sum_m H^{\circ}_{\ell,m} \, k_m \tag{5}$$

where $H^{\circ}_{\ell,n}$ denote the PTDF with equally distributed slack. When bus n injects excess power, it has to flow to the slack; when bus n extract deficit power, it has to come from the slack. Summing over all ingoing and outgoing flow changes resulting from a positive injection at n yields again the slack

$$\sum_\ell K_{m,\ell} \, H_{\ell,n} = \delta_{m,n} - k_m \tag{6}$$

Note that $\delta_{m,n}$ on the right hand side represents the positive injection at n. Established flow allocation schemes [cite] haved used this degree of freedom in order to allocate power flows and exchanges to market participants. Under the assumption that consumers account for all power flows in the grid, the slack is set to k^*_n such that

$$f_{\ell,t} = - \sum_n H_{\ell,n} \left(k^*_m\right) \, d_{n,t} \tag{7}$$

Therefore, the flow can be reproduced from the demand-side of the system only. Now each term in the sum on the right hand side stands for the individual contribution of consumers at node n to the network flow $f_{\ell,t}$. In other words, each nodal demand $d_{n,t}$ induces a subflow originating from the slack k^*_n which all together add up to $f_{\ell,t}$. These subflows, in turn, can be further broken down to contributions for each bus m in the slack, such that we get the subflow of individual $m \to n$ relations, that is

$$A_{m\to n,\ell,t} = \left(H^{\circ}_{\ell,m} - H^{\circ}_{\ell,n}\right) k^*_m \, d_{n,t} \tag{8}$$

It indicates the flow on line ℓ provided generators at m and supplying the demand $d_{n,t}$. When summing over all sources m it yields the total subflow induced by $d_{n,t}$, the same term as in the sum on the right hand side in **??**; when summing over all sources and sinks, it yields again the power flow, thus

$$f_{\ell,t} = \sum_{m,n} A_{m\to n,\ell,t} \tag{9}$$

As mentioned, the consumed power $d_{n,t}$ has to come from the slack k^*_m. As proofen in **??**, for each peer-to-peer relation $m \to n$, the "traded" power $A_{m\to n,t}$ amounts to

$$A_{m\to n,t} = k^*_m \, d_{n,t} \tag{10}$$

Finally, when summing over all sinks the peer-to-peer trades yield the nodal generation (see **??**)

$$\sum_n A_{m\to n,t} = g_{m,t} \tag{11}$$

and summing over all sources yields the nodal demand

$$\sum_m A_{m\to n,t} = d_{n,t} \tag{12}$$

which straightforwardly follows from the fact that $\sum_n k^*_n = 1$. Both allocation quantities $A_{m\to n,t}$ and $A_{m\to n,\ell,t}$ can be further broken down to generators s or consumers a by multiplying with the nodal production share $\omega_{n,s,t} = g_{n,s,t} / \sum_s g_{n,s,t}$ and the nodal comsumer share $\omega_{n,a,t} = d_{n,a,t} / \sum_a d_{n,a,t}$ respectively.

The solution to k^*_n follows directly from combining Eq. (4) and **??**, which sets it to the share of the total production $k^*_n = c \, g_{n,t}$ with c being defined as $c = 1/ \sum_n g_{n,t}$. That leads to the demand $d_{n,a,t}$ of every single consumers a being supplied by all generators s in the network proportional to their gross production $g_{n,s,t}$. However solution space can be extended to an individual slack for each node $k^*_{m,n}$, which we discuss in **??**.

Network Optimisation

We linearly cost-optimize the capacity and dispatch of a simple power system.

$$\max_{d_{n,a,t}, g_{n,s,t}, G_{n,s}} \left(\sum_{n,s} c_{n,s} G_{n,s} - \sum_{n,s,t} o_{n,s} g_{n,s,t} - \sum_{\ell} c_\ell F_\ell \right) \tag{13}$$

subject to following physical constraints.

The nodal balance constraint ensures that the amount of power that flows into a bus equals the power that flows out of a bus, thus reflects the Kirchhoff Current Law (KCL)

$$\sum_{l} K_{n,\ell} f_{\ell,t} - g_{n,t} + d_{n,t} = 0 \perp \lambda_{n,t} \quad \forall\, n, t \tag{14}$$

Its shadow price mirrors the Locational Marginal Prizes (LMP) $\lambda_{n,t}$ per bus and time step. In a power market this is the €/MWh$_{\mathrm{el}}$-price which a consumer has to pay. Note that the flow $f_{\ell,t}$ in Constr. (10) is a passive variable only, given by Eq. (1).

The generation $g_{n,s,t}$ is constraint to its nominal capacity

$$g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s} \leq 0 \perp \bar{\mu}_{n,s,t} \quad \forall\, n, s, t \tag{15}$$

$$-g_{n,s,t} \leq 0 \perp \underline{\mu}_{n,s,t} \quad \forall\, n, s, t \tag{16}$$

where $\bar{g}_{n,s,t} \in [0, 1]$ is the capacity factor for renewable generators. The constraints yield the KKT variables $\bar{\mu}_{n,s,t}$ and $\underline{\mu}_{n,s,t}$ which due to complementary slackness,

$$\bar{\mu}_{n,s,t} \left(g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s} \right) = 0 \quad \forall\, n, s, t \tag{17}$$

$$\underline{\mu}_{n,s,t}\, g_{n,s,t} = 0 \quad \forall\, n, s, t \tag{18}$$

are only non-zero if the corresponding constraint is binding.

The transmission capacity F_ℓ limits the flow $f_{\ell,t}$ in both directions, such that

$$f_{\ell,t} - F_\ell \leq 0 \perp \bar{\mu}_{\ell,t} \quad \forall\, \ell, t \tag{19}$$

$$-f_{\ell,t} - F_\ell \leq 0 \perp \underline{\mu}_{\ell,t} \quad \forall\, \ell, t \tag{20}$$

The yielding KKT variables $\bar{\mu}_{\ell,t}$ and $\underline{\mu}_{\ell,t}$ are only non-zero if $f_{\ell,t}$ is limited by the trasmission capacity in positive or negative direction, i.e. Constr. (15) or Constr. (16) are binding. The complementary slackness

$$\bar{\mu}_{\ell,t} \left(f_{\ell,t} - F_\ell \right) = 0 \quad \forall\, \ell, t \tag{21}$$

$$\underline{\mu}_{\ell,t} \left(f_{\ell,t} - F_\ell \right) = 0 \quad \forall\, \ell, t \tag{22}$$

set the respective KKT for flows staying below the thermal limit to zero.

Decomposing the Locational Marginal Prices and Allocating the Electricity Costs

$$\begin{aligned}
&\mathcal{L} \left(g_{n,s,t}, f_{\ell,t}, G_{n,s}, F_\ell, \boldsymbol{\lambda}, \boldsymbol{\mu} \right) = \\
&- \sum_{n,s} c_{n,s} G_{n,s} - \sum_{n,s,t} o_{n,s} g_{n,s,t} - \sum_{\ell} c_\ell F_\ell \\
&- \sum_{n,t} \lambda_{n,t} \left(\sum_{\ell} K_{n,\ell} f_{\ell,t} - \sum_{s} g_{n,s,t} + \sum_{a} d_{n,a,t} \right) \\
&- \sum_{\ell,c,t} \lambda_{c,t} C_{\ell,c} x_\ell f_{\ell,t} \\
&- \sum_{n,s,t} \bar{\mu}_{n,s,t} \left(g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s} \right) + \sum_{n,s,t} \underline{\mu}_{n,s,t} g_{n,s,t} \\
&- \sum_{\ell,t} \bar{\mu}_{\ell,t} \left(f_{\ell,t} - F_\ell \right) + \sum_{\ell,t} \underline{\mu}_{\ell,t} \left(f_{\ell,t} + F_\ell \right)
\end{aligned} \tag{23}$$

where $\boldsymbol{\lambda} = \{\lambda_{n,t}, \lambda_{c,t}\}$ and $\boldsymbol{\mu} = \{\bar{\mu}_{n,s,t}, \underline{\mu}_{n,s,t}, \bar{\mu}_{\ell,t}, \underline{\mu}_{\ell,t}\}$ denote the set of related KKT variables. The global maximum of the Lagrangian requires stationarity with respect to all variables. The stationarity of the generation capacity variable leads to

$$\frac{\partial \mathcal{L}}{\partial G_{n,s}} = 0 \;\rightarrow\; c_{n,s} = \sum_{t} \bar{\mu}_{n,s,t}\, \bar{g}_{n,s,t} \quad \forall\, n, s \tag{24}$$

the stationarity of the transmission capacity to

$$\frac{\partial \mathcal{L}}{\partial F_\ell} = 0 \;\rightarrow\; c_\ell = \sum_{t} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t} \right) \quad \forall\, \ell \tag{25}$$

and the stationarity of the generation to

$$\frac{\partial \mathcal{L}}{\partial g_{n,s,t}} = 0 \;\rightarrow\; o_{n,s} = \lambda_{n,t} - \bar{\mu}_{n,s,t} + \underline{\mu}_{n,s,t} \quad \forall\, n, s \tag{26}$$

Solving Eq. (26) for the $\lambda_{n,t}$, leads to our first representation for Locational Market Price, which we will refer to as the "Island Solution",

$$\lambda_{n,t} = o_{n,s} + \bar{\mu}_{n,s,t} - \underline{\mu}_{n,s,t} \quad \forall\, n, s, t \tag{27}$$

It connects the LMP directly with the local operational price and prices for the generation capacity constraint. However, we can derive a second representation for $\lambda_{n,t}$. Starting from the stationarity of the flow

$$0 = \frac{\partial \mathcal{L}}{\partial f_{\ell,t}} \tag{28}$$

$$0 = -\sum_{m,\ell,t} \lambda_{m,t} K_{m,\ell} - \sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) - \sum_{\ell,c,t} \lambda_{c,t} C_{\ell,c} x_{\ell} \tag{29}$$

and multipying each term with the Power Transfer Distribution Factor $H_{\ell,n}$ leaves us with

$$0 = -\lambda_{n,t} + \sum_{m} \lambda_{m,t} k_m - \sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) H_{\ell,n} \tag{30}$$

According to Eq. (3), the first term splits into the LMP at n and the LMP weighted with the slack. The final term disappears as the $C_{\ell,c}\, x_{\ell}$ is the kernel of the PTDF $H_{\ell,n}$, so $\sum_l C_{\ell,c}\, x_{\ell} H_{\ell,n} = 0$. Solving Eq. (30) for $\lambda_{n,t}$ and replacing $\lambda_{m,t}$ of the right hand side with the expression of the Island Solution in Eq. (27) leads to

$$\lambda_{n,t} = \sum_{m} o_{m,s}\, k_m + \sum_{m} \left(\bar{\mu}_{m,s,t} - \underline{\mu}_{m,s,t}\right) k_m - \sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) H_{\ell,n} \quad \forall\; n, s, t \tag{31}$$

In contrast to the Island Solution in Eq. (27), this representation connects $\lambda_{n,t}$ with all other prices in the system. Moreover, it must hold for any choice of k_m and correspondingly decomposes the LMP to operational prices $o_{m,s}$ and prices for capacity bounds for generators and transmission lines. When setting the slack to $k^*_m = c\, g_{m,s,t}$, the LMP decomposes to prices of generators proportional to their production plus an extra term for the transmission usage. This directly links to the flow allocation presented above. Multiplied with the nodal demand $d_{n,t}$ the left hand side of Eq. (31) turns into the total payment of n and the right hand side into the different payment allocations,

$$\lambda_{n,t}\, d_{n,t} = \mathcal{O}_{n,t} + \mathcal{C}^G_{n,t} + \mathcal{C}^F_{n,t} \quad \forall\; n, t \tag{32}$$

which we define as

$$\mathcal{O}_{n,t} = \sum_{m,s} o_{m,s}\, \omega_{m,s,t}\, A_{m\to n,t} \tag{33}$$

$$\mathcal{C}^G_{n,t} = \sum_{m,s} \bar{\mu}_{m,s,t}\, \omega_{m,s,t}\, A_{m\to n,t} \tag{34}$$

$$\mathcal{C}^F_{n,t} = \sum_{m,\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) A_{m\to n,\ell,t} \tag{35}$$

Equation (33) denotes the payments for generators' OPEX, Eq. (34) for generators' CAPEX and Eq. (35) for the transmission system's CAPEX respectively. All those payment allocation build up on the phycally allocated flows presented in Eqs. (4) and (5). Thus with the right choice of slack the power flow allocations directly tranlate into a cost allocation. Further, the allocated payments can be considered on a more detailed level as

$$\mathcal{O}_{n\to(m,s),t} = o_{m,s}\, \omega_{m,s,t}\, A_{m\to n,t} \tag{36}$$

$$\mathcal{C}^G_{n\to(m,s),t} = \bar{\mu}_{m,s,t}\, \omega_{m,s,t}\, A_{m\to n,t} \tag{37}$$

$$\mathcal{C}^F_{n\to\ell,t} = \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) \sum_{m} A_{m\to n,\ell,t} \tag{38}$$

Let's have a look at the allocated OPEX in Eq. (33) and **??** first. Consumers at bus n retrieve power from different generators (m, s) and accordingly compensate their operational costs. The OPEX allocation behaves like P2P tradings between producers and consumers with fixed production prices. In this way, the generator (m, s) retrieves the exact amount of money from consumers that it spends on the operation. In other words, all OPEX payments to generator (m,s) sum up to the total OPEX spent at (m,s), thus

$$\sum_{n} \mathcal{O}_{n\to(m,s),t} = o_{n,s}\, g_{n,s,t} \tag{39}$$

The CAPEX allocation for generators $\mathcal{C}^G_{n,t}$ defined in Eq. (34) and **??**, reveils a similar relation. According to the polluter pays principle, it differentiates between consumers who are 'responsible' for investments and those who are not. If $\bar{\mu}_{n,s,t} > 0$, the upper Capacity Constr. (11) is binding. Thus it is these times steps which push investments in $G_{n,s}$. If $\bar{\mu}_{n,s,t} = 0$, the generation $g_{n,s,t}$ is not bound and investments are not necessary. When summing over all CAPEX payments to generator (m, s) each generator retrieves exactly the cost that were spent to build the capacity $G_{n,s}$,

$$\sum_{n,t} \mathcal{C}^G_{n\to(m,s),t} = c_{n,s}\, G_{n,s} \tag{40}$$

where we used Eqs. (13) and (24). So in total, throughout all time steps each generator (m,s) receives the money it spends for invesments and operation, reflecting the zero-profit rule.

The allocation of CAPEX for the transmission system $\mathcal{C}^F_{n,t}$, defined in Eq. (35) and **??**, builds up on the KKT variables $\bar{\mu}_{\ell,t}$ and $\underline{\mu}_{\ell,t}$. Again the latter translate to the necessity of transmission invesments at ℓ at time t. Consumers which retrieve power flowing on congested lines, yielding a bound Constr. (15) or (16), pay compensations for the resulting investments. Again the sum of all CAPEX payments to line ℓ equal the total CAPEX spent, thus

$$\sum_{n,t} \mathcal{C}^F_{n\to\ell,t} = c_\ell\, F_\ell \tag{41}$$

where we used the complementary slackness Eqs. (17) and (18) and the fact that summing over all sources m and sinks n the allocation equals the actual power flow as stated in Eq. (8).

Adding CO_2 Constraints

Imposing an additional CO_2 constraint limiting the total emission to K,

$$\sum_{n,s,t} e_{n,s}\, g_{n,s,t} \leq \mathrm{K} \perp \mu_{\mathrm{CO2}} \tag{42}$$

with $e_{n,s}$ being the emission factor in tonne-CO_2 per $\mathrm{MWh_{el}}$, returns an effective CO_2 price μ_{CO2} in €/tonne-CO_2. As shown in ... the constraint can be translated in a dual price which shift the operational price per generator

$$o_{n,s} \to o_{n,s} + e_{n,s}\, \mu_{\mathrm{CO2}} \tag{43}$$

This leads to allocated CO_2 cost compensation of node n of

$$\mathcal{E}_{n,t} = \mu_{\mathrm{CO2}} \sum_{m,s} e_{m,s}\, \omega_{m,s,t}\, A_{m\to n,t} \quad \forall\, n,t \tag{44}$$

which expands the allocation of the electricity cost in Eq. (32) to

$$\lambda_{n,t}\, d_{n,t} = \mathcal{C}^F_{n,t} + \mathcal{O}_{n,t} + \mathcal{C}^G_{n,t} + \mathcal{E}_{n,t} \quad \forall\, n,t \tag{45}$$

A Appendix

A.1 Proof Equation (4)

Equation (4) follows from summing $A_{m\to n,\ell,t}$ over all incoming flows to n and taking into account the power that n provides by itsself, $k^*_n\, d_{n,t}$, which leads us to

$$A_{m\to n,t} = k^*_n\, d_{n,t} - \sum_\ell K_{n,\ell}\, A_{m\to n,\ell,t} \tag{46}$$

$$= k^*_n\, d_{n,t} - \sum_\ell K_{n,\ell}\left(H^\circ_{\ell,m} - H^\circ_{\ell,n}\right) k^*_m\, d_{n,t} \tag{47}$$

$$= k^*_n\, d_{n,t} - \left(\delta_{n,m} - \frac{1}{N} - \delta_{n,n} + \frac{1}{N}\right) k^*_m\, d_{n,t} \tag{48}$$

$$= k^*_n\, d_{n,t} - (\delta_{n,m} - 1)\, k^*_m\, d_{n,t} \tag{49}$$

$$= k^*_m\, d_{n,t} \tag{50}$$

where we used Eq. (3) and the fact that the equally distributed slack amounts to $1/N$ for all N nodes in the network.

A.2 Proof of ??

The relation follows from multiplying **??** with $\sum_m K_{m,\ell}$, and solving for $A_{m\to n,t}$

$$\sum_m K_{m,\ell}\, f_{\ell,t} = -\sum_{m,n} K_{m,\ell}\, H_{\ell,n}\, d_{n,t} \tag{51}$$

$$g_{m,t} - d_{m,t} = -\delta_{m,n}\, d_{n,t} + k^*_m\, d_{n,t} \tag{52}$$

$$A_{m\to n,t} = g_{m,t} - d_{m,t} + \delta_{m,n} d_{n,t} \tag{53}$$

$$\sum_n A_{m\to n,t} = g_{m,t} \tag{54}$$

A.3 Solution Space of k^*_m

All choices of k^*_m fulfilling **??** determine the solution space of k^*_n. In the $N\times N$ nodal space this translates to the constraint given by **??** which denotes

$$g_{m,t} = \sum_n k^*_m\, d_{n,t} \tag{55}$$

Solving for k^*_m directly leads to

$$k^*_m = c\, g_{m,s,t} \tag{56}$$

where c is the inverse of the total consumption or production $c = 1/\sum_n d_{n,t} = 1/\sum_n g_{n,t}$.

However **??** has an inherent degree of freedom and can be reformulated as

$$g_{m,t} = \sum_n k^*_{m,n}\, d_{n,t} \tag{57}$$

where $k^*_{m,n}$ denote the individual choice of slack for each bus n. For example the Marginal Participation cite[] algorithm takes only net injections into account. This sets the individual slack to

$$k^*_{m,n} = \frac{\delta_{m,n}\, p^\circ_{m,t} + \gamma_t\, p^-_{n,t}\, p^+_{m,t}}{d_{n,t}} \tag{58}$$

where

- $p^+_{n,t} = \min\left(g_{n,t} - d_{n,t}, 0\right)$ denotes the nodal net production
- $p^-_{n,t} = \min\left(d_{n,t} - g_{n,t}, 0\right)$ denotes the nodal net consumption
- $p^\circ_{n,t} = \min\left(p^+_{n,t}, p^-_{n,t}\right)$ the denotes nodal self-consumption. That is the power generated and at the same time consumed at node n and
- $\gamma_t = \frac{1}{\sum_n p^+_{n,t}} = \frac{1}{\sum_n p^-_{n,t}}$ is the inverse of the total injected/extracted power at time t.