From Linear Optimization to Transmission Cost Allocation

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Abstract

Maximizing the welfare of all market participants within a power system is a common approach in energy system modelling. It leads to perfectly scheduled operations of generators and ...

Nomenclature

$\lambda_{n,t}$	Locational Market Price at bus n and time step t in €/MW
$d_{n,a,t}$	Electric demand per bus n, demand type a, time step t in MW
$g_{n,s,t}$	Electric generation per bus n, carrier s, time step t in MW
$f_{\ell,t}$	Active power flow on line ℓ, time step t in MW
$o_{n,s}$	Operational cost (OPEX) in €/MW
$c_{n,s}$	Capital Expenditure (CAPEX) in €/MW
c_{ℓ}	CAPEX per transmission line ℓ in €/MW
$G_{n,s}$	Generation capacity in MW
F_{ℓ}	Transmission capacity in MW
$K_{n,\ell}$	Incidence matrix

1 Economic Context

In long-term operation and investment planning models, the total costs $\mathcal{TC}$ of a power system include all operational expenditures (OPEX) $\mathcal{O}$ and capital expenditures (CAPEX) $\mathcal{C}^G$ and $\mathcal{C}^F$ for production and transmission respectively, thus

$$\mathcal{TC} = \mathcal{O} + \mathcal{C}^G + \mathcal{C}^F + \text{rest} \tag{1}$$

In a network design with minimal $\mathcal{TC}$, the Locational Market Price (LMP) describes the marginal price for an incremental increase of electricity demand $d_{n,a,t}$ of any consumer a at bus n. It is given by the derivative of the total system cost $\mathcal{TC}$ with respect to the local demand $d_{n,a,t}$

$$\lambda_{n,t} = \frac{\partial\, \mathcal{TC}}{\partial\, d_{n,a,t}} \tag{2}$$

This leads to a nodal pricing where over the span of optimized timesteps t, the system costs are totally payed back by the consumers

$$\mathcal{TC} = \sum_{n,a,t} \lambda_{n,t}\, d_{n,a,t} \tag{3}$$

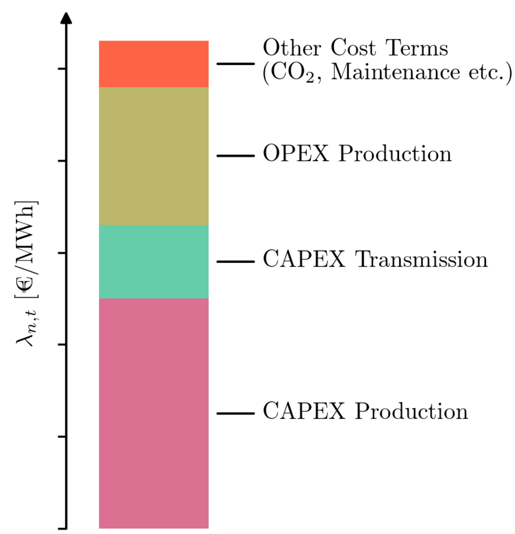

Figure 1: Schematic decomposition of the Locational Market Price $\lambda_{n,t}$. In power system model with optimal long-term operation and planning, the total system costs $\mathcal{TC}$ split into different cost terms, *i.e* OPEX and CAPEX for production and transmission and possibly other expenditures. As $\mathcal{TC}$ is fully payed by the consumers, each LMP $\lambda_{n,t}$ can be decomposed into cost terms compensating the different expenditures at timestep t.

As a direct consequence of Eqs. (1) to (3), the LMP splits into different cost terms assigned to OPEX and

CAPEX for production and transmission and eventual other expenditures. We schematically show this behaviour in Fig. 1. Extensive investigations of the LMP as done in [1] already showed this connection however leave the question open how the costs are allocated among the components. In the following we will show how the above presented cost decomposition is extended to full peer-to-peer cost allocation including all network components.

2 Mathematical Theory

2.1 Power Transfer Distribition Factors and Flow Allocation

In linear power flow models, the Power Transfer Distribition Factors (PTDF) $H_{\ell,n}$ determine the changes in the flow on line ℓ for one unit (typically one MW) of net power production at bus n. Thus for a given production $g_{n,s,t}$ and demand $d_{n,a,t}$, they directly link to the the resulting flow on each line,

$$f_{\ell,t} = \sum_n H_{\ell,n} \left(g_{n,t} - d_{n,t} \right) \tag{4}$$

where $g_{n,t} = \sum_s g_{n,s,t}$ and $d_{n,t} = \sum_a d_{n,a,t}$ combine all generators s and all comsumers a attached to n. The PTDF have a degree of freedom: The slack k_n denotes the contribution of bus n to balancing out total power excess or deficit in the system. It can be dedicated to one bus, a sinlge "slackbus", or to several or all buses. The choice of slack modifies the PTDF according to

$$H_{\ell,n}\left(k_m\right) = H^{\circ}_{\ell,n} - \sum_m H^{\circ}_{\ell,m}\, k_m \tag{5}$$

where $H^{\circ}_{\ell,n}$ denote the PTDF with equally distributed slack. When bus n injects excess power, it has to flow to the slack; when bus n extract deficit power, it has to come from the slack. Summing over all ingoing and outgoing flow changes resulting from a positive injection at n yields again the slack

$$\sum_\ell K_{m,\ell}\, H_{\ell,n} = \delta_{m,n} - k_m \tag{6}$$

Note that $\delta_{m,n}$ on the right hand side represents the positive injection at n. Established flow allocation schemes [cite] haved used this degree of freedom in order to allocate power flows and exchanges to market participants. Under the assumption that consumers account for all power flows in the grid, the slack is set to k^*_n such that

$$f_{\ell,t} = - \sum_n H_{\ell,n}\left(k^*_m\right)\, d_{n,t} \tag{7}$$

With such a choice the flow can be reproduced from the demand-side of the system only. Now each term in the sum on the right hand side stands for the individual contribution of consumers at node n to the network flow $f_{\ell,t}$. In other words, each nodal demand $d_{n,t}$ induces a subflow originating from the slack k^*_n which all together add up to $f_{\ell,t}$. These subflows, in turn, can be further broken down to contributions for each bus m in the slack, such that we get the subflow of individual $m \to n$ relations, that is

$$A_{m\to n,\ell,t} = \left(H^{\circ}_{\ell,m} - H^{\circ}_{\ell,n}\right) k^*_m\, d_{n,t} \tag{8}$$

It indicates the flow on line ℓ coming from generators at m and supplying the demand $d_{n,t}$. When summing over all sources m it yields the total subflow induced by $d_{n,t}$, the same term as in the sum on the right hand side in Eq. (7); when summing over all sources and sinks, it yields again the power flow, thus

$$f_{\ell,t} = \sum_{m,n} A_{m\to n,\ell,t} \tag{9}$$

As mentioned before, the consumed power $d_{n,t}$ has to come from the slack k^*_m. As proofen in Appendix A.1, for each peer-to-peer relation $m \to n$, the "traded" power $A_{m\to n,t}$ amounts to

$$A_{m\to n,t} = k^*_m\, d_{n,t} \tag{10}$$

Finally, when summing over all sinks the peer-to-peer trades yield the nodal generation (see Appendix A.2)

$$\sum_n A_{m\to n,t} = g_{m,t} \tag{11}$$

and summing over all sources yields the nodal demand

$$\sum_m A_{m\to n,t} = d_{n,t} \tag{12}$$

which follows from the fact that $\sum_n k^*_n = 1$. Both allocation quantities $A_{m\to n,t}$ and $A_{m\to n,\ell,t}$ can be broken down to generators s or consumers a by multiplying with the nodal production share $\omega_{n,s,t} = g_{n,s,t} / \sum_s g_{n,s,t}$ and the nodal comsumer share $\omega_{n,a,t} = d_{n,a,t} / \sum_a d_{n,a,t}$ respectively.

The solution to k_n^* follows from combining Eqs. (10) and (11), which sets it to the share of the total production $k_n^* = c\, g_{n,t}$ with c being defined as $c = 1/\sum_n g_{n,t}$. That leads to the demand $d_{n,a,t}$ of every single consumers a being supplied by all generators s in the network proportional to their gross production $g_{n,s,t}$. However as we discuss in Appendix A.3, the solution space can be extended to an individual slack for each node $k_{m,n}^*$.

2.2 Network Optimisation

We linearly cost-optimize the capacity and dispatch of a simple power system.

$$\min_{g_{n,s,t}, G_{n,s}, F_\ell} \left(\sum_{n,s} c_{n,s} G_{n,s} + \sum_{n,s,t} o_{n,s} g_{n,s,t} + \sum_\ell c_\ell F_\ell \right) \tag{13}$$

subject to following physical constraints.

The nodal balance constraint ensures that the amount of power that flows into a bus equals the power that flows out of a bus, thus reflects the Kirchhoff Current Law (KCL)

$$\sum_l K_{n,\ell} f_{\ell,t} - g_{n,t} + d_{n,t} = 0 \perp \lambda_{n,t} \quad \forall\, n, t \tag{14}$$

Its shadow price mirrors the Locational Marginal Prizes (LMP) $\lambda_{n,t}$ per bus and time step. In a power market this is the €/MWh$_{\text{el}}$-price which a consumer has to pay. Note that the flow $f_{\ell,t}$ in Constr. (14) is a passive variable only, given by Eq. (4).

The generation $g_{n,s,t}$ is constraint to its nominal capacity

$$g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s} \leq 0 \perp \bar{\mu}_{n,s,t} \quad \forall\, n, s, t \tag{15}$$

$$-g_{n,s,t} \leq 0 \perp \underline{\mu}_{n,s,t} \quad \forall\, n, s, t \tag{16}$$

where $\bar{g}_{n,s,t} \in [0, 1]$ is the capacity factor for renewable generators. The constraints yield the KKT variables $\bar{\mu}_{n,s,t}$ and $\underline{\mu}_{n,s,t}$ which due to complementary slackness,

$$\bar{\mu}_{n,s,t} \left(g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s} \right) = 0 \quad \forall\, n, s, t \tag{17}$$

$$\underline{\mu}_{n,s,t}\, g_{n,s,t} = 0 \quad \forall\, n, s, t \tag{18}$$

are only non-zero if the corresponding constraint is binding.

The transmission capacity F_ℓ limits the flow $f_{\ell,t}$ in both directions, such that

$$f_{\ell,t} - F_\ell \leq 0 \perp \bar{\mu}_{\ell,t} \quad \forall\, \ell, t \tag{19}$$

$$-f_{\ell,t} - F_\ell \leq 0 \perp \underline{\mu}_{\ell,t} \quad \forall\, \ell, t \tag{20}$$

The yielding KKT variables $\bar{\mu}_{\ell,t}$ and $\underline{\mu}_{\ell,t}$ are only non-zero if $f_{\ell,t}$ is limited by the trasmission capacity in positive or negative direction, i.e. Constr. (19) or Constr. (20) are binding. The complementary slackness

$$\bar{\mu}_{\ell,t} \left(f_{\ell,t} - F_\ell \right) = 0 \quad \forall\, \ell, t \tag{21}$$

$$\underline{\mu}_{\ell,t} \left(f_{\ell,t} - F_\ell \right) = 0 \quad \forall\, \ell, t \tag{22}$$

set the respective KKT for flows staying below the thermal limit to zero.

2.3 Allocating Nodal Payments

$$\begin{aligned}
&\mathcal{L}\left(g_{n,s,t}, f_{\ell,t}, G_{n,s}, F_\ell, \boldsymbol{\lambda}, \boldsymbol{\mu}\right) = \\
&\quad \sum_{n,s} c_{n,s} G_{n,s} + \sum_{n,s,t} o_{n,s} g_{n,s,t} + \sum_\ell c_\ell F_\ell \\
&\quad + \sum_{n,t} \lambda_{n,t} \left(\sum_\ell K_{n,\ell} f_{\ell,t} - \sum_s g_{n,s,t} + \sum_a d_{n,a,t} \right) \\
&\quad + \sum_{\ell,c,t} \lambda_{c,t} C_{\ell,c} x_\ell f_{\ell,t} \\
&\quad + \sum_{n,s,t} \bar{\mu}_{n,s,t} \left(g_{n,s,t} - \bar{g}_{n,s,t} G_{n,s} \right) - \sum_{n,s,t} \underline{\mu}_{n,s,t} g_{n,s,t} \\
&\quad + \sum_{\ell,t} \bar{\mu}_{\ell,t} \left(f_{\ell,t} - F_\ell \right) - \sum_{\ell,t} \underline{\mu}_{\ell,t} \left(f_{\ell,t} + F_\ell \right)
\end{aligned} \tag{23}$$

where $\boldsymbol{\lambda} = \{\lambda_{n,t}, \lambda_{c,t}\}$ and $\boldsymbol{\mu} = \left\{\bar{\mu}_{n,s,t}, \underline{\mu}_{n,s,t}, \bar{\mu}_{\ell,t}, \underline{\mu}_{\ell,t}\right\}$ denote the set of related KKT variables. The global maximum of the Lagrangian requires stationarity with respect to all variables. The stationarity of the generation capacity variable leads to

$$\frac{\partial \mathcal{L}}{\partial G_{n,s}} = 0 \;\rightarrow\; c_{n,s} = \sum_t \bar{\mu}_{n,s,t}\, \bar{g}_{n,s,t} \quad \forall\, n, s \tag{24}$$

the stationarity of the transmission capacity to

$$\frac{\partial \mathcal{L}}{\partial F_\ell} = 0 \;\rightarrow\; c_\ell = \sum_t \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t} \right) \quad \forall\, \ell \tag{25}$$

and the stationarity of the generation to

$$\frac{\partial \mathcal{L}}{\partial g_{n,s,t}} = 0 \;\rightarrow\; o_{n,s} = \lambda_{n,t} - \bar{\mu}_{n,s,t} + \underline{\mu}_{n,s,t} \quad \forall\; n, s \tag{26}$$

Solving Eq. (26) for the $\lambda_{n,t}$, leads to our first representation for Locational Market Price, which we will refer to as the "Island Solution",

$$\lambda_{n,t} = o_{n,s} + \bar{\mu}_{n,s,t} - \underline{\mu}_{n,s,t} \quad \forall\; n, s, t \tag{27}$$

It connects the LMP directly with the local operational price and prices for the generation capacity constraint. However, we can derive a second representation for $\lambda_{n,t}$. Starting from the stationarity of the flow

$$0 = \frac{\partial \mathcal{L}}{\partial f_{\ell,t}} \tag{28}$$

$$\begin{aligned} 0 = &\sum_{m,\ell,t} \lambda_{m,t}\, K_{m,\ell} + \sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) \\ &+ \sum_{\ell,c,t} \lambda_{c,t}\, C_{\ell,c}\, x_\ell \end{aligned} \tag{29}$$

and multipying each term with the Power Transfer Distribution Factor $H_{\ell,n}$ leaves us with

$$0 = \lambda_{n,t} - \sum_{m} \lambda_{m,t}\, k_m + \sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) H_{\ell,n} \tag{30}$$

According to Eq. (6), the first term splits into the LMP at n and the LMP weighted with the slack. The final term disappears as the $C_{\ell,c}\, x_\ell$ is the kernel of the PTDF $H_{\ell,n}$, so $\sum_l C_{\ell,c}\, x_\ell H_{\ell,n} = 0$. Solving Eq. (30) for $\lambda_{n,t}$ and replacing $\lambda_{m,t}$ of the right hand side with the expression of the Island Solution in Eq. (27) leads to

$$\begin{aligned} \lambda_{n,t} = &\sum_{m} o_{m,s}\, k_m + \sum_{m} \left(\bar{\mu}_{m,s,t} - \underline{\mu}_{m,s,t}\right) k_m \\ &- \sum_{\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) H_{\ell,n} \quad \forall\; n, s, t \end{aligned} \tag{31}$$

In contrast to the Island Solution in Eq. (27), this representation connects $\lambda_{n,t}$ with all other prices in the system. Moreover, it must hold for any choice of k_m and correspondingly decomposes the LMP to operational prices $o_{m,s}$ and prices for capacity bounds for generators and transmission lines. When setting the slack to $k^*_m = c\, g_{m,s,t}$, the LMP decomposes to prices of generators proportional to their production plus an extra term for the transmission usage. This directly links to the flow allocation presented above. Multiplied with the nodal demand $d_{n,t}$ the left hand side of Eq. (31) turns into the total payment of n and the right hand side into the different payment allocations,

$$\lambda_{n,t}\, d_{n,t} = \mathcal{O}_{n,t} + \mathcal{C}^G_{n,t} + \mathcal{C}^F_{n,t} \quad \forall\; n, t \tag{32}$$

which we define as

$$\mathcal{O}_{n,t} = \sum_{m,s} o_{m,s}\, \omega_{m,s,t}\, A_{m\rightarrow n,t} \tag{33}$$

$$\mathcal{C}^G_{n,t} = \sum_{m,s} \bar{\mu}_{m,s,t}\, \omega_{m,s,t}\, A_{m\rightarrow n,t} \tag{34}$$

$$\mathcal{C}^F_{n,t} = \sum_{m,\ell} \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) A_{m\rightarrow n,\ell,t} \tag{35}$$

Equation (33) denotes the payments for generators' OPEX, Eq. (34) for generators' CAPEX and Eq. (35) for the transmission system's CAPEX respectively. All those payment allocation build up on the phycally allocated flows presented in Eqs. (8) and (10). Thus with the right choice of slack the power flow allocations directly tranlate into a cost allocation. Further, the allocated payments can be considered on a more detailed level as

$$\mathcal{O}_{n\rightarrow(m,s),t} = o_{m,s}\, \omega_{m,s,t}\, A_{m\rightarrow n,t} \tag{36}$$

$$\mathcal{C}^G_{n\rightarrow(m,s),t} = \bar{\mu}_{m,s,t}\, \omega_{m,s,t}\, A_{m\rightarrow n,t} \tag{37}$$

$$\mathcal{C}^F_{n\rightarrow\ell,t} = \left(\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}\right) \sum_{m} A_{m\rightarrow n,\ell,t} \tag{38}$$

Let's have a look at the allocated OPEX in Eqs. (33) and (36) first. Consumers at bus n retrieve power from different generators (m, s) and accordingly compensate their operational costs. The OPEX allocation behaves like P2P tradings between producers and consumers with fixed production prices. In this way, the generator (m, s) retrieves the exact amount of money from consumers that it spends on the operation. In other words, all OPEX payments to generator (m,s) sum up to the total OPEX spent at (m,s), thus

$$\sum_{n} \mathcal{O}_{n\rightarrow(m,s),t} = o_{n,s}\, g_{n,s,t} \tag{39}$$

The CAPEX allocation for generators $\mathcal{C}^G_{n,t}$ defined in Eqs. (34) and (37), reveils a similar relation. According to the polluter pays principle, it differentiates between consumers who are 'responsible' for investments and those who are not. If $\bar{\mu}_{n,s,t}$ (in literature often denoted as the Quality of Supply) is bigger than zero, the upper Capacity Constr. (15) is binding. Thus it is these times steps which push investments in $G_{n,s}$. If $\bar{\mu}_{n,s,t} = 0$, the generation $g_{n,s,t}$ is not bound and investments are not necessary. When summing over all CAPEX payments to generator (m, s) each generator retrieves exactly the cost that were spent to build the capacity $G_{n,s}$,

$$\sum_{n,t} \mathcal{C}^G_{n\to(m,s),t} = c_{n,s}\, G_{n,s} \tag{40}$$

where we used Eqs. (17) and (24). So in total, throughout all time steps each generator (m, s) receives the money it spends for invesments and operation, reflecting the zero-profit rule.

The allocation of CAPEX for the transmission system $\mathcal{C}^F_{n,t}$, defined in Eqs. (35) and (38), builds up on the KKT variables $\bar{\mu}_{\ell,t}$ and $\underline{\mu}_{\ell,t}$. Again the latter translate to the necessity of transmission invesments at ℓ at time t. Consumers which retrieve power flowing on congested lines, yielding a bound Constr. (19) or (20), pay compensations for the resulting investments. Again the sum of all CAPEX payments to line ℓ equal the total CAPEX spent, thus

$$\sum_{n,t} \mathcal{C}^F_{n\to\ell,t} = c_\ell\, F_\ell \tag{41}$$

where we used the complementary slackness Eqs. (21) and (22) and the fact that summing over all sources m and sinks n the allocation equals the actual power flow as stated in Eq. (9).

2.4 Adding Further Constraints

CO_2 Constraint

Imposing an additional CO_2 constraint limiting the total emission to K,

$$\sum_{n,s,t} e_{n,s}\, g_{n,s,t} \leq \mathrm{K} \;\perp\; \mu_{\mathrm{CO2}} \tag{42}$$

with $e_{n,s}$ being the emission factor in tonne-CO_2 per $\mathrm{MWh_{el}}$, returns an effective CO_2 price μ_{CO2} in €/tonne-CO_2. As shown in ... the constraint can be translated in a dual price which shift the operational price per generator

$$o_{n,s} \to o_{n,s} + e_{n,s}\, \mu_{\mathrm{CO2}} \tag{43}$$

This leads to allocated CO_2 cost compensation of node n of

$$\mathcal{E}_{n,t} = \mu_{\mathrm{CO2}} \sum_{m,s} e_{m,s}\, \omega_{m,s,t}\, A_{m\to n,t} \quad \forall\; n, t \tag{44}$$

which expands the allocation of the electricity cost in Eq. (32) to

$$\lambda_{n,t}\, d_{n,t} = \mathcal{C}^F_{n,t} + \mathcal{O}_{n,t} + \mathcal{C}^G_{n,t} + \mathcal{E}_{n,t} \quad \forall\; n, t \tag{45}$$

Lower and Upper Capacity Limits

Constraining the capacities $G_{n,s}$ for a subset S of generators to lower or upper limits in the form of

$$G_{n,s} \geq \underline{G}_{n,s} \qquad \forall n, s \in S \tag{46}$$

$$G_{n,s} \leq \bar{G}_{n,s} \qquad \forall n, s \in S \tag{47}$$

or doing likewise with a subset L of transmission lines,

$$F_\ell \geq \underline{F}_\ell \qquad \forall \ell \in L \tag{48}$$

$$F_\ell \leq \bar{F}_\ell \qquad \forall \ell \in L \tag{49}$$

does not have any effect on the above presented allocation. The additional cost directly translate t¡o the Quality of Supply $\bar{\mu}_{n,s,t}$ for production and $\bar{\mu}_{\ell,t} - \underline{\mu}_{\ell,t}$ for transmission respectively. This also counts for limiting the overall production capacity $\sum_{n,s} G_{n,s}$ or transmission capacity $\sum_\ell F_\ell$.

3 Numerical Example

The peer-to-peer cost allocation suits for any type of topology or network setup. In the following we showcase a two bus system with one optimized time step, shown in Fig. 2 and its resulting allocated payments. The two buses are connected via a transmission line and both have generators. Whereas the generator at bus 1 has an operational price of 50 €/$\mathrm{MWh_{el}}$, the generator at bus 2 has a higher operational price of 200 €/$\mathrm{MWh_{el}}$. Both generators have the same CAPEX rate of 500 €/MW and both are constraint to a maxmimal capacity of $\bar{G}_{n,s} = 100$ MW. The transmission line has a CAPEX rate of 100 €/MW and no upper capacity limit. With a demand of 60

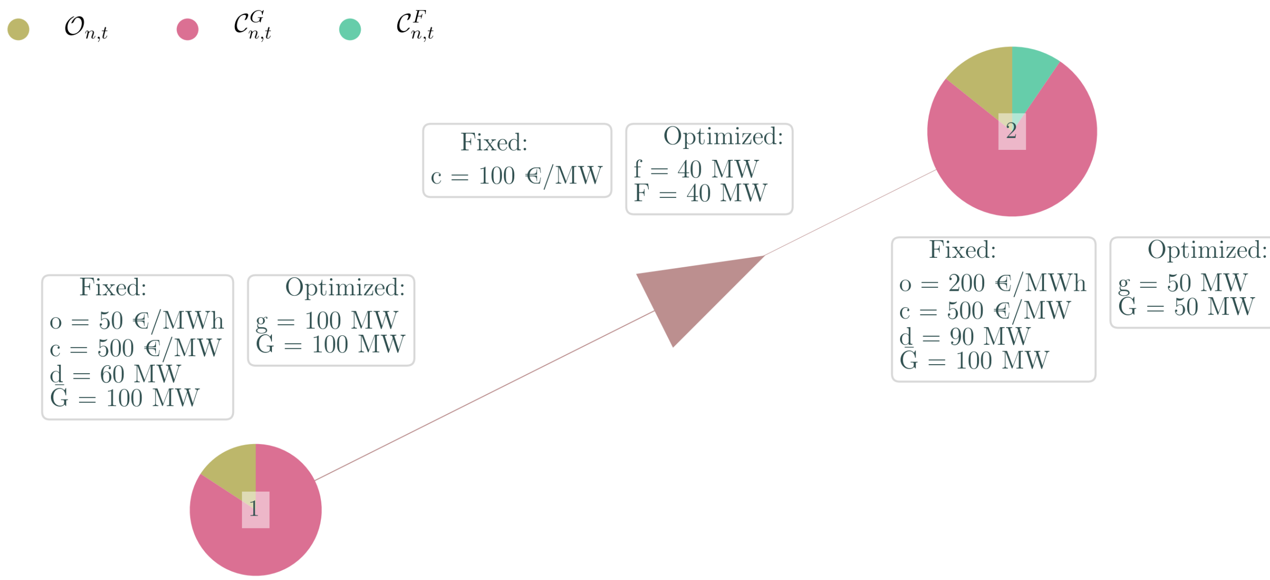

Figure 2: Illustrative Example of a two bus optimized network with one time step.

MW at bus 1 and 90 MW at bus 2, the optimization deploys the generator at bus 1 with its full limit of 100 MW. The 40 MW excess power not consumed at bus 1, flow to bus 2 where the generator is only built with 50 MW.

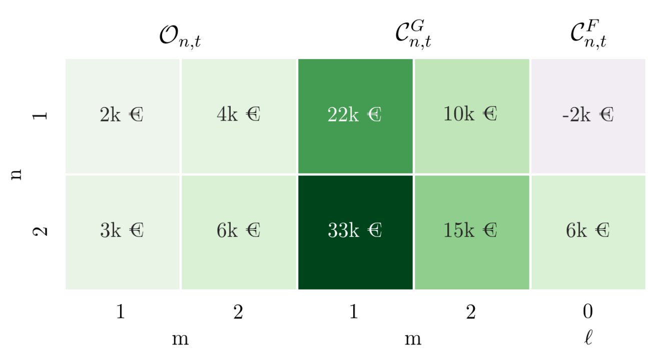

Figure 3: Full cost allocation for the example setup shown in Fig. 2. The payments on the basis of Eqs. (36) to (38)

References

[1] Fred C. Schweppe et al. *Spot Pricing of Electricity*. en. Boston, MA: Springer US, 1988. ISBN: 978-1-4612-8950-0 978-1-4613-1683-1. DOI: 10.1007/978-1-4613-1683-1.

A Appendix

A.1 Proof Equation (10)

Equation (10) follows from summing $A_{m\to n,\ell,t}$ over all incoming flows to n and taking into account the power that n provides by itsself, $k^*_n\, d_{n,t}$, which leads us to

$$
\begin{aligned}
A_{m\to n,t} &= k^*_n\, d_{n,t} - \sum_\ell K_{n,\ell}\, A_{m\to n,\ell,t} && (50)\\
&= k^*_n\, d_{n,t} - \sum_\ell K_{n,\ell}\left(H^\circ_{\ell,m} - H^\circ_{\ell,n}\right) k^*_m\, d_{n,t} && (51)\\
&= k^*_n\, d_{n,t} - \left(\delta_{n,m} - \frac{1}{N} - \delta_{n,n} + \frac{1}{N}\right) k^*_m\, d_{n,t} && (52)\\
&= k^*_n\, d_{n,t} - \left(\delta_{n,m} - 1\right) k^*_m\, d_{n,t} && (53)\\
&= k^*_m\, d_{n,t} && (54)
\end{aligned}
$$

where we used Eq. (6) and the fact that the equally distributed slack amounts to $1/N$ for all N nodes in the network.

A.2 Proof of Equation (11)

The relation follows from multiplying Eq. (7) with $\sum_m K_{m,\ell}$, and solving for $A_{m\to n,t}$

$$
\begin{aligned}
\sum_m K_{m,\ell}\, f_{\ell,t} &= -\sum_{m,n} K_{m,\ell}\, H_{\ell,n}\, d_{n,t} && (55)\\
g_{m,t} - d_{m,t} &= -\delta_{m,n}\, d_{n,t} + k^*_m\, d_{n,t} && (56)\\
A_{m\to n,t} &= g_{m,t} - d_{m,t} + \delta_{m,n} d_{n,t} && (57)\\
\sum_n A_{m\to n,t} &= g_{m,t} && (58)
\end{aligned}
$$

A.3 Solution Space of k^*_m

All choices of k^*_m fulfilling Eq. (7) determine the solution space of k^*_n. In the $N\times N$ nodal space this translates to the constraint given by Eq. (11) which denotes

$$g_{m,t} = \sum_n k^*_m\, d_{n,t} \tag{59}$$

Solving for k^*_m directly leads to

$$k^*_m = c\, g_{m,s,t} \tag{60}$$

where c is the inverse of the total consumption or production $c = 1/\sum_n d_{n,t} = 1/\sum_n g_{n,t}$.

However Eq. (59) has an inherent degree of freedom and can be reformulated as

$$g_{m,t} = \sum_n k^*_{m,n}\, d_{n,t} \tag{61}$$

where $k^*_{m,n}$ denote the individual choice of slack for each bus n. For example the Marginal Participation cite[] algorithm takes only net injections into account. This sets the individual slack to

$$k^*_{m,n} = \frac{\delta_{m,n}\, p^\circ_{m,t} + \gamma_t\, p^-_{n,t}\, p^+_{m,t}}{d_{n,t}} \tag{62}$$

where

- $p^+_{n,t} = \min\left(g_{n,t} - d_{n,t}, 0\right)$ denotes the nodal net production
- $p^-_{n,t} = \min\left(d_{n,t} - g_{n,t}, 0\right)$ denotes the nodal net consumption
- $p^\circ_{n,t} = \min\left(p^+_{n,t}, p^-_{n,t}\right)$ the denotes nodal self-consumption. That is the power generated and at the same time consumed at node n and
- $\gamma_t = \dfrac{1}{\sum_n p^+_{n,t}} = \dfrac{1}{\sum_n p^-_{n,t}}$ is the inverse of the total injected/extracted power at time t.