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Introduction to Algorithms and Data Structures, Exercise Sheet 1

`https://cms.sic.saarland/gralgodat24/`

Total Points: **40** Due: 12:00 Thursday, **October 24**, 2024

You are allowed to collaborate on the exercise sheets in groups of up to 3 students. Each group must submit exactly one solution, that is, exactly one group member should submit the solutions. The first page of your solutions must list all group members and the name of your tutor/tutorial time. Further, cite all external sources that you use (books, websites, research papers, etc.). You need to collect at least 50% of all points on exercise sheets to be admitted to the exam. Upload your solutions in PDF format directly to the CMS, or hand them in physically at the beginning of the lecture.

Exercise 1 (Missing Number) — 5+5 points

Let $A[1,\dots,n]$ be an array such that every number in $\{0,\dots,n\}$ appears exactly once in A, except for one missing number x. In the MissingNumber problem, we are given such an array A and the goal is to compute the missing number x. For example: given the array $A = [4,2,1,0,5]$ (of length $n = 5$), the the missing number is $x = 3$.

Prove correctness of each of the following algorithms, that is, argue that these algorithms correctly solve the MissingNumber problem.

a.

procedure Alg1($A[1\dots n]$)
  Sort(A) ▷ sorts the array such that $A[1] \le A[2] \le \dots \le A[n]$
  for $i = 1,\dots,n$ **do**
    if $A[i] == i$ **then**
      return $i-1$
  return n

b.

procedure Alg2($A[1\dots n]$)
  $y := 0$
  for $i = 1,\dots,n$ **do**
    $y := y + i - A[i]$
  return y

Exercise 2 (Polynomial Evaluation) — 10 points

We are given integer coefficients $a_0,\dots,a_n$ of a polynomial $p(X) = \sum_{i=0}^{n} a_i \cdot X^i$, and an evaluation point z. Prove that the following pseudocode correctly evaluates the polynomial at z, that is, argue that it correctly computes the value $p(z)$. (You can assume that all intermediate values computed in the algorithm fit into a memory cell.)

$y := 0$
for $i = n, n-1, \dots, 0$ **do**
  $y := a_i + z \cdot y$
return y

Exercise 3 (Asymptotic Notation) — 2+2+2+4 points

Prove the following laws that hold for any functions $f_1, f_2, g_1, g_2, f, g, h \geq_{ae} 0$:

a. **Additivity:** If $f_1 = O(g_1)$ and $f_2 = O(g_2)$ then $f_1 + f_2 = O(\max(g_1, g_2))$.

b. **Multiplicativity:** If $f_1 = O(g_1)$ and $f_2 = O(g_2)$ then $f_1 \cdot f_2 = O(g_1 \cdot g_2)$.

c. **Transitivity:** If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

d. $f^c = o(b^f)$ for any function $f = \omega(1)$ and constants $c \geq 0, b > 1$.

Parts a-c also holds after replacing O by o, Ω or ω, but you don't need to prove this.

Hint: It might help to refresh your knowledge of logarithm laws, see e.g. sections 1-4 of `https://en.wikipedia.org/wiki/List_of_logarithmic_identities`

Exercise 4 (Asymptotic Growth) — 10 points

Order the following functions according to their asymptotic growth. Prove your answer.

$$2n^2 - 4n, \quad n^2 \log n, \quad n \log n, \quad 2^n, \quad \log n, \quad n \log\log n.$$

Hint: Use the laws from exercise 3.