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NON-COOPERATIVE GAMES

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Introduction

Von Neumann and Morgenstern have developed a very fruitful theory of two-person zero-sum games in their book *Theory of Games and Economic Behavior*. This book also contains a theory of n-person games of a type which we would call cooperative. This theory is based on an analysis of the interrelationships of the various coalitions which can be formed by the players of the game.

Our theory, in contradistinction, is based on the *absence* of coalitions in that it is assumed that each participant acts independently, without collaboration or communication with any of the others.

The notion of an *equilibrium point* is the basic ingredient in our theory. This notion yields a generalization of the concept of the solution of a two-person zero-sum game. It turns out that the set of equilibrium points of a two-person zero-sum game is simply the set of all pairs of opposing "good strategies."

In the immediately following sections we shall define equilibrium points and prove that a finite non-cooperative game always has at least one equilibrium point. We shall also introduce the notions of solvability and strong solvability of a non-cooperative game and prove a theorem on the geometrical structure of the set of equilibrium points of a solvable game.

As an example of the application of our theory we include a solution of a simplified three person poker game.

Formal Definitions and Terminology

In this section we define the basic concepts of this paper and set up standard terminology and notation. Important definitions will be preceded by a subtitle indicating the concept defined. The non-cooperative idea will be implicit, rather than explicit, below.

Finite Game:

For us an *n-person game* will be a set of n *players*, or *positions*, each with an associated finite set of *pure strategies*; and corresponding to each player, i, a *payoff function*, p_i, which maps the set of all n-tuples of pure strategies into the real numbers. When we use the term *n-tuple* we shall always mean a set of n items, with each item associated with a different player.

Mixed Strategy, s_i :

A *mixed strategy* of player i will be a collection of non-negative numbers which have unit sum and are in one to one correspondence with his pure strategies.

We write $s_i = \sum_\alpha c_{i\alpha} \pi_{i\alpha}$ with $c_{i\alpha} \geqq 0$ and $\sum_\alpha c_{i\alpha} = 1$ to represent such a mixed strategy, where the $\pi_{i\alpha}$'s are the pure strategies of player i. We regard the s_i's as points in a simplex whose vertices are the $\pi_{i\alpha}$'s. This simplex may be re-

garded as a convex subset of a real vector space, giving us a natural process of linear combination for the mixed strategies.

We shall use the suffixes i, j, k for players and α, β, γ to indicate various pure strategies of a player. The symbols s_i, t_i, and r_i, etc. will indicate mixed strategies; $\pi_{i\alpha}$ will indicate the ith player's α^{th} pure strategy, etc.

Payoff function, p_i:

The payoff function, p_i, used in the definition of a finite game above, has a unique extension to the n-tuples of mixed strategies which is linear in the mixed strategy of each player [n-linear]. This extension we shall also denote by p_i, writing $p_i(s_1, s_2, \cdots, s_n)$.

We shall write $\mathfrak{s}$ or $\mathfrak{t}$ to denote an n-tuple of mixed strategies and if $\mathfrak{s} = (s_1, s_2, \cdots, s_n)$ then $p_i(\mathfrak{s})$ shall mean $p_i(s_1, s_2, \cdots, s_n)$. Such an n-tuple, $\mathfrak{s}$, will also be regarded as a point in a vector space, the product space of the vector spaces containing the mixed strategies. And the set of all such n-tuples forms, of course, a convex polytope, the product of the simplices representing the mixed strategies.

For convenience we introduce the substitution notation $(\mathfrak{s}; t_i)$ to stand for $(s_1, s_2, \cdots, s_{i-1}, t_i, s_{i+1}, \cdots, s_n)$ where $\mathfrak{s} = (s_1, s_2, \cdots, s_n)$. The effect of successive substitutions $((\mathfrak{s}; t_i); r_j)$ we indicate by $(\mathfrak{s}; t_i; r_j)$, etc.

Equilibrium Point:

An n-tuple $\mathfrak{s}$ is an *equilibrium point* if and only if for every i

$$(1) \qquad\qquad p_i(\mathfrak{s}) = \max_{\text{all } r_i\text{'s}} [p_i(\mathfrak{s}; r_i)].$$

Thus an equilibrium point is an n-tuple $\mathfrak{s}$ such that each player's mixed strategy maximizes his payoff if the strategies of the others are held fixed. Thus each player's strategy is optimal against those of the others. We shall occasionally abbreviate equilibrium point by eq. pt.

We say that a mixed strategy s_i *uses* a pure strategy $\pi_{i\alpha}$ if $s_i = \sum_\beta c_{i\beta}\pi_{i\beta}$ and $c_{i\alpha} > 0$. If $\mathfrak{s} = (s_1, s_2, \cdots, s_n)$ and s_i uses $\pi_{i\alpha}$ we also say that $\mathfrak{s}$ uses $\pi_{i\alpha}$.

From the linearity of $p_i(s_1, \cdots, s_n)$ in s_i,

$$(2) \qquad\qquad \max_{\text{all } r_i\text{'s}} [p_i(\mathfrak{s}; r_i)] = \max_\alpha [p_i(\mathfrak{s}; \pi_{i\alpha})].$$

We define $p_{i\alpha}(\mathfrak{s}) = p_i(\mathfrak{s}; \pi_{i\alpha})$. Then we obtain the following trivial necessary and sufficient condition for $\mathfrak{s}$ to be an equilibrium point:

$$(3) \qquad\qquad p_i(\mathfrak{s}) = \max_\alpha p_{i\alpha}(\mathfrak{s}).$$

If $\mathfrak{s} = (s_1, s_2, \cdots, s_n)$ and $s_i = \sum_\alpha c_{i\alpha}\pi_{i\alpha}$ then $p_i(\mathfrak{s}) = \sum_\alpha c_{i\alpha}p_{i\alpha}(\mathfrak{s})$, consequently for (3) to hold we must have $c_{i\alpha} = 0$ whenever $p_{i\alpha}(\mathfrak{s}) < \max_\beta p_{i\beta}(\mathfrak{s})$, which is to say that $\mathfrak{s}$ does not use $\pi_{i\alpha}$ unless it is an optimal pure strategy for player i. So we write

$$(4) \qquad\qquad \text{if } \pi_{i\alpha} \text{ is used in } \mathfrak{s} \text{ then } p_{i\alpha}(\mathfrak{s}) = \max_\beta p_{i\beta}(\mathfrak{s})$$

as another necessary and sufficient condition for an equilibrium point.

Since a criterion (3) for an eq. pt. can be expressed by the equating of n pairs of continuous functions on the space of n-tuples $\mathfrak{s}$ the eq. pts. obviously form a closed subset of this space. Actually, this subset is formed from a number of pieces of algebraic varieties, cut out by other algebraic varieties.

Existence of Equilibrium Points

A proof of this existence theorem based on Kakutani's generalized fixed point theorem was published in Proc. Nat. Acad. Sci. U. S. A., 36, pp. 48–49. The proof given here is a considerable improvement over that earlier version and is based directly on the Brouwer theorem. We proceed by constructing a continuous transformation T of the space of n-tuples such that the fixed points of T are the equilibrium points of the game.

THEOREM 1. *Every finite game has an equilibrium point.*

PROOF. Let $\mathfrak{s}$ be an n-tuple of mixed strategies, $p_i(\mathfrak{s})$ the corresponding pay-off to player i, and $p_{i\alpha}(\mathfrak{s})$ the pay-off to player i if he changes to his α^{th} pure strategy $\pi_{i\alpha}$ and the others continue to use their respective mixed strategies from $\mathfrak{s}$. We now define a set of continuous functions of $\mathfrak{s}$ by

$$\varphi_{i\alpha}(\mathfrak{s}) = \max\,(0,\, p_{i\alpha}(\mathfrak{s}) - p_i(\mathfrak{s}))$$

and for each component s_i of $\mathfrak{s}$ we define a modification s_i' by

$$s_i' = \frac{s_i + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})\pi_{i\alpha}}{1 + \sum_\alpha \varphi_{i\alpha}(\mathfrak{s})}\,,$$

calling $\mathfrak{s}'$ the n-tuple $(s_1',\, s_2',\, s_3' \cdots s_n')$.

We must now show that the fixed points of the mapping $T\colon \mathfrak{s} \to \mathfrak{s}'$ are the equilibrium points.

First consider any n-tuple $\mathfrak{s}$. In $\mathfrak{s}$ the ith player's mixed strategy s_i will use certain of his pure strategies. Some one of these strategies, say $\pi_{i\alpha}$, must be "least profitable" so that $p_{i\alpha}(\mathfrak{s}) \leqq p_i(\mathfrak{s})$. This will make $\varphi_{i\alpha}(\mathfrak{s}) = 0$.

Now if this n-tuple $\mathfrak{s}$ happens to be fixed under T the proportion of $\pi_{i\alpha}$ used in s_i must not be decreased by T. Hence, for all β's, $\varphi_{i\beta}(\mathfrak{s})$ must be zero to prevent the denominator of the expression defining s_i' from exceeding 1.

Thus, if $\mathfrak{s}$ is fixed under T, for any i and β $\varphi_{i\beta}(\mathfrak{s}) = 0$. This means no player can improve his pay-off by moving to a pure strategy $\pi_{i\beta}$. But this is just a criterion for an eq. pt. [see (2)].

Conversely, if $\mathfrak{s}$ is an eq. pt. it is immediate that all φ's vanish, making $\mathfrak{s}$ a fixed point under T.

Since the space of n-tuples is a cell the Brouwer fixed point theorem requires that T must have at least one fixed point $\mathfrak{s}$, which must be an equilibrium point.

Symmetries of Games

An *automorphism*, or *symmetry*, of a game will be a permutation of its pure strategies which satisfies certain conditions, given below.

If two strategies belong to a single player they must go into two strategies belonging to a single player. Thus if ϕ is the permutation of the pure strategies it induces a permutation ψ of the players.

Each n-tuple of pure strategies is therefore permuted into another n-tuple of pure strategies. We may call χ the induced permutation of these n-tuples. Let ξ denote an n-tuple of pure strategies and $p_i(\xi)$ the payoff to player i when the n-tuple ξ is employed. We require that if

$$j = i^\psi \qquad\qquad \text{then } p_j(\xi^\chi) = p_i(\xi)$$

which completes the definition of a symmetry.

The permutation ϕ has a unique linear extension to the mixed strategies. If

$$s_i = \sum_\alpha c_{i\alpha} \pi_{i\alpha} \quad \text{we define} \quad (s_i)^\phi = \sum_\alpha c_{i\alpha}(\pi_{i\alpha})^\phi.$$

The extension of ϕ to the mixed strategies clearly generates an extension of χ to the n-tuples of mixed strategies. We shall also denote this by χ.

We define a *symmetric n-tuple* $\mathfrak{s}$ of a game by $\mathfrak{s}^\chi = \mathfrak{s}$ for all χ's

THEOREM 2. *Any finite game has a symmetric equilibrium point.*

PROOF. First we note that $s_{i0} = \sum_\alpha \pi_{i\alpha} / \sum_\alpha 1$ has the property $(s_{i0})^\phi = s_{j0}$ where $j = i^\psi$, so that the n-tuple $\mathfrak{s}_0 = (s_{10}, s_{20}, \cdots, s_{n0})$ is fixed under any χ; hence any game has at least one symmetric n-tuple.

If $\mathfrak{s} = (s_1, \cdots, s_n)$ and $\mathfrak{t} = (t_1, \cdots, t_n)$ are symmetric then

$$\frac{\mathfrak{s} + \mathfrak{t}}{2} = \left(\frac{s_1 + t_1}{2}, \frac{s_2 + t_2}{2}, \cdots, \frac{s_n + t_n}{2} \right)$$

is also symmetric because $\mathfrak{s}^\chi = \mathfrak{s} \longleftrightarrow s_j = (s_i)^\phi$, where $j = i^\psi$, hence

$$\frac{s_j + t_j}{2} = \frac{(s_i)^\phi + (t_i)^\phi}{2} = \left(\frac{s_i + t_i}{2} \right)^\phi,$$

hence

$$\left(\frac{\mathfrak{s} + \mathfrak{t}}{2} \right)^\chi = \frac{\mathfrak{s} + \mathfrak{t}}{2}.$$

This shows that the set of symmetric n-tuples is a convex subset of the space of n-tubles since it is obviously closed.

Now observe that the mapping $T \colon \mathfrak{s} \to \mathfrak{s}'$ used in the proof of the existence theorem was intrinsically defined. Therefore, if $\mathfrak{s}_2 = T\mathfrak{s}_1$ and χ is derived from an automorphism of the game we will have $\mathfrak{s}_2^\chi = T\mathfrak{s}_1^\chi$. If $\mathfrak{s}_1$ is symmetric $\mathfrak{s}_1^\chi = \mathfrak{s}_1$ and therefore $\mathfrak{s}_2^\chi = T\mathfrak{s}_1 = \mathfrak{s}_2$. Consequently this mapping maps the set of symmetric n-tuples into itself.

Since this set is a cell there must be a symmetric fixed point $\mathfrak{s}$ which must be a symmetric equilibrium point.

Solutions

We define here solutions, strong solutions, and sub-solutions. A non-cooperative game does not always have a solution, but when it does the solution is unique. Strong solutions are solutions with special properties. Sub-solutions always exist and have many of the properties of solutions, but lack uniqueness.

S_1 will denote a set of mixed strategies of player i and $\mathfrak{S}$ a set of n-tuples of mixed strategies.

Solvability:

A game is *solvable* if its set, $\mathfrak{S}$, of equilibrium points satisfies the condition

$$(5) \qquad\qquad (\mathfrak{t};\ r_i) \ \epsilon\ \mathfrak{S} \quad \text{and} \quad \mathfrak{s} \ \epsilon\ \mathfrak{S} \ \rightarrow\ (\mathfrak{s};\ r_i) \ \epsilon\ \mathfrak{S} \qquad\qquad \text{for all } i\text{'s.}$$

This is called the *interchangeability* condition. The *solution* of a solvable game is its set, $\mathfrak{S}$, of equilibrium points.

Strong Solvability:

A game is *strongly solvable* if it has a solution, $\mathfrak{S}$, such that for all i's

$$\mathfrak{s} \ \epsilon\ \mathfrak{S} \quad \text{and} \quad p_i(\mathfrak{s};\ r_i) \ =\ p_i(\mathfrak{s}) \ \rightarrow\ (\mathfrak{s};\ r_i) \ \epsilon\ \mathfrak{S}$$

and then $\mathfrak{S}$ is called a *strong solution*.

Equilibrium Strategies:

In a solvable game let S_i be the set of all mixed strategies s_i such that for some $\mathfrak{t}$ the n-tuple $(\mathfrak{t};\ s_i)$ is an equilibrium point. [s_i is the ith component of some equilibrium point.] We call S_i the set of *equilibrium strategies* of player i.

Sub-solutions:

If $\mathfrak{S}$ is a subset of the set of equilibrium points of a game and satisfies condition (1); and if $\mathfrak{S}$ is maximal relative to this property then we call $\mathfrak{S}$ a *sub-solution*.

For any sub-solution $\mathfrak{S}$ we define the ith *factor set*, S_i, as the set of all s_i's such that $\mathfrak{S}$ contains $(\mathfrak{t};\ s_i)$ for some $\mathfrak{t}$.

Note that a sub-solution, when unique, is a solution; and its factor sets are the sets of equilibrium strategies.

THEOREM 3. *A sub-solution, $\mathfrak{S}$, is the set of all n-tuples $(s_1,\ s_2,\ \cdots,\ s_n)$ such that each $s_i \ \epsilon\ S_i$ where S_i is the ith factor set of $\mathfrak{S}$. Geometrically, $\mathfrak{S}$ is the product of its factor sets.*

PROOF. Consider such an n-tuple $(s_1,\ s_2,\ \cdots,\ s_n)$. By definition $\exists\ \mathfrak{t}_1,\ \mathfrak{t}_2,\ \cdots,\ \mathfrak{t}_n$ such that for each i $(\mathfrak{t}_i\ ;\ s_i) \ \epsilon\ \mathfrak{S}$. Using the condition (5) $n-1$ times we obtain successively $(\mathfrak{t}_1\ ;\ s_1) \ \epsilon\ \mathfrak{S}$, $(\mathfrak{t}_1\ ;\ s_1\ ;\ s_2) \ \epsilon\ \mathfrak{S}$, $\cdots$, $(\mathfrak{t}_1\ ;\ s_1\ ;\ s_2\ ;\ \cdots\ ;\ s_n) \ \epsilon\ \mathfrak{S}$ and the last is simply $(s_1,\ s_2,\ \cdots,\ s_n) \ \epsilon\ \mathfrak{S}$, which we needed to show.

THEOREM 4. *The factor sets $S_1,\ S_2,\ \cdots,\ S_n$ of a sub-solution are closed and convex as subsets of the mixed strategy spaces.*

PROOF. It suffices to show two things: (a) if s_i and $s_i' \ \epsilon\ S_i$ then $s_i^* = (s_i + s_i')/2 \ \epsilon\ S_i$; (b) if s_i^* is a limit point of S_i then $s_i^* \ \epsilon\ S_i$.

Let $\mathfrak{t} \ \epsilon\ \mathfrak{S}$. Then we have $p_j(\mathfrak{t};\ s_i) \geqq p_j(\mathfrak{t};\ s_i\ ;\ r_j)$ and $p_j(\mathfrak{t};\ s_i') \geqq p_j(\mathfrak{t};\ s_i'\ ;\ r_j)$ for any r_j, by using the criterion of (1) for an eq. pt. Adding these inequalities, using the linearity of $p_j(s_1,\ \cdots,\ s_n)$ in s_i, and dividing by 2, we get $p_j(\mathfrak{t};\ s_i^*) \geqq$

$p_j(\mathbf{t}; s_i^* ; r_j)$ since $s_i^* = (s_i + s_i')/2$. From this we know that $(\mathbf{t}; s_i)$ is an eq. pt. for any $\mathbf{t} \in \mathcal{S}$. If the set of all such eq. pts. $(\mathbf{t}; s_i^*)$ is added to $\mathcal{S}$ the augmented set clearly satisfies condition (5), and since $\mathcal{S}$ was to be maximal it follows that $s_i^* \in S_i$.

To attack (b) note that the n-tuple $(\mathbf{t}; s_i^*)$, where $\mathbf{t} \in \mathcal{S}$, will be a limit point of the set of n-tuples of the form $(\mathbf{t}; s_i)$ where $s_i \in S_i$, since s_i^* is a limit point of S_i. But this set is a set of eq. pts. and hence any point in its closure is an eq. pt., since the set of all eq. pts. is closed. Therefore $(\mathbf{t}; s_i^*)$ is an eq. pt. and hence $s_i^* \in S_i$ from the same argument as for s_i^*.

Values:

Let $\mathcal{S}$ be the set of equilibrium points of a game. We define

$$v_i^+ = \max_{\mathbf{s} \in \mathcal{S}} [p_i(\mathbf{s})], \qquad v_i^- = \min_{\mathbf{s} \in \mathcal{S}} [p_i(\mathbf{s})].$$

If $v_i^+ = v_i^-$ we write $v_i = v_i^+ = v_i^-$. v_i^+ is the *upper value* to player i of the game; v_i^- the *lower value*; and v_i the *value*, if it exists.

Values will obviously have to exist if there is but one equilibrium point.

One can define *associated values* for a sub-solution by restricting $\mathcal{S}$ to the eq. pts. in the sub-solution and then using the same defining equations as above.

A two-person zero-sum game is always solvable in the sense defined above. The sets of equilibrium strategies S_1 and S_2 are simply the sets of "good" strategies. Such a game is not generally strongly solvable; strong solutions exist only when there is a "saddle point" in *pure* strategies.

Simple Examples

These are intended to illustrate the concepts defined in the paper and display special phenomena which occur in these games.

The first player has the roman letter strategies and the payoff to the left, etc.

Ex. 1				
	5	$a\alpha$	-3	Solution $\left(\dfrac{9}{16}\, a + \dfrac{7}{16}\, b, \dfrac{7}{17}\, \alpha + \dfrac{10}{17}\, \beta\right)$
	-4	$a\beta$	4	
	-5	$b\alpha$	5	$v_1 = \dfrac{-5}{17}, v_2 = +\dfrac{1}{2}$
	3	$b\beta$	-4	
Ex. 2	1	$a\alpha$	1	Strong Solution (b, β)
	-10	$a\beta$	10	
	10	$b\alpha$	-10	$v_1 = v_2 = -1$
	-1	$b\beta$	-1	
Ex. 3	1	$a\alpha$	1	Unsolvable; equilibrium points (a, α), (b, β),
	-10	$a\beta$	-10	and $\left(\dfrac{a}{2} + \dfrac{b}{2}, \dfrac{\alpha}{2} + \dfrac{\beta}{2}\right)$. The strategies in the last case
	-10	$b\alpha$	-10	
	1	$b\beta$	1	have maxi-min and mini-max properties.
Ex. 4	1	$a\alpha$	1	Strong Solution: all pairs of mixed strategies.
	0	$a\beta$	1	
	1	$b\alpha$	0	$v_1^+ = v_2^+ = 1, v_1^- = v_2^- = 0.$
	0	$b\beta$	0	

Ex. 5   1 $a\alpha$ 2   Unsolvable; eq. pts. (a, α), (b, β) and
         -1 $a\beta$ -4 $\left(\dfrac{1}{4}\, a + \dfrac{3}{4}\, b, \dfrac{3}{8}\, \alpha + \dfrac{5}{8}\, \beta\right)$. However, empirical tests
         -4 $b\alpha$ -1
          2    $b\beta$ 1   show a tendency toward (a, α).

Ex. 6   1 $a\alpha$ 1   Eq. pts.: (a, α) and (b, β), with (b, β) an example of
          0    $a\beta$ 0   instability.
          0    $b\alpha$ 0
          0    $b\beta$ 0

Geometrical Form of Solutions

In the two-person zero-sum case it has been shown that the set of "good" strategies of a player is a convex polyhedral subset of his strategy space. We shall obtain the same result for a player's set of equilibrium strategies in any solvable game.

THEOREM 5. *The sets S_1, S_2, $\cdots$, S_n of equilibrium strategies in a solvable game are polyhedral convex subsets of the respective mixed strategy spaces.*

PROOF. An n-tuple $\mathfrak{s}$ will be an equilibrium point if and only if for every i

(6) $$p_i(\mathfrak{s}) = \max_{\alpha} p_{i\alpha}(\mathfrak{s})$$

which is condition (3). An equivalent condition is for every i and α

(7) $$p_i(\mathfrak{s}) - p_{i\alpha}(\mathfrak{s}) \geqq 0.$$

Let us now consider the form of the set S_j of equilibrium strategies, s_j, of player j. Let $\mathbf{t}$ be any equilibrium point, then $(\mathbf{t}; s_j)$ will be an equilibrium point if and only if $s_j \,\epsilon\, S_j$, from Theorem 2. We now apply conditions (2) to $(\mathbf{t}; s_j)$, obtaining

(8) $$s_j \,\epsilon\, S_j \longleftrightarrow \text{for all } i, \alpha \qquad p_i(\mathbf{t}; s_j) - p_i\alpha(\mathbf{t}; s_j) \geqq 0.$$

Since p_i is n-linear and $\mathbf{t}$ is constant these are a set of linear inequalities of the form $F_{i\alpha}(s_j) \geqq 0$. Each such inequality is either satisfied for all s_j or for those lying on and to one side of some hyperplane passing through the strategy simplex. Therefore, the complete set [which is finite] of conditions will all be satisfied simultaneously on some convex polyhedral subset of player j's strategy simplex. [Intersection of half-spaces.]

As a corollary we may conclude that S_j is the convex closure of a finite set of mixed strategies [vertices].

Dominance and Contradiction Methods

We say that s_i' dominates s_i if $p_i(\mathbf{t}; s_i') > p_i(\mathbf{t}; s_i)$ for every $\mathbf{t}$.

This amounts to saying that s_i' gives player i a higher payoff than s_i no matter what the strategies of the other players are. To see whether a strategy s_i' dominates s_i it suffices to consider only pure strategies for the other players because of the n-linearity of p_i.

It is obvious from the definitions that *no equilibrium point can involve a dominated strategy s_i*.

The domination of one mixed strategy by another will always entail other dominations. For suppose s_i' dominates s_i and t_i uses all of the pure strategies which have a higher coefficient in s_i than in s_i'. Then for a small enough ρ

$$t_i' = t_i + \rho(s_i' - s_i)$$

is a mixed strategy; and t_i dominates t_i' by linearity.

One can prove a few properties of the set of undominated strategies. It is simply connected and is formed by the union of some collection of faces of the strategy simplex.

The information obtained by discovering dominances for one player may be of relevance to the others, insofar as the elimination of classes of mixed strategies as possible components of an equilibrium point is concerned. For the t's whose components are all undominated are all that need be considered and thus eliminating some of the strategies of one player may make possible the elimination of a new class of strategies for another player.

Another procedure which may be used in locating equilibrium points is the contradiction-type analysis. Here one assumes that an equilibrium point exists having component strategies lying within certain regions of the strategy spaces and proceeds to deduce further conditions which must be satisfied if the hypothesis is true. This sort of reasoning may be carried through several stages to eventually obtain a contradiction indicating that there is no equilibrium point satisfying the initial hypothesis.

A Three-Man Poker Game

As an example of the application of our theory to a more or less realistic case we include the simplified poker game given below. The rules are as follows:

(a) The deck is large, with equally many *high* and *low* cards, and a hand consists of one card.

(b) Two chips are used to ante, open, or call.

(c) The players play in rotation and the game ends after all have passed or after one player has opened and the others have had a chance to call.

(d) If no one bets the antes are retrieved.

(e) Otherwise the pot is divided equally among the highest hands which have bet.

We find it more satisfactory to treat the game in terms of quantities we call "behavior parameters" than in the normal form of *Theory of Games and Economic Behavior*. In the normal form representation two mixed strategies of a player may be equivalent in the sense that each makes the individual choose each available course of action in each particular situation requiring action on his part with the same frequency. That is, they represent the same behavior pattern on the part of the individual.

Behavior parameters give the probabilities of taking each of the various possible actions in each of the various possible situations which may arise. Thus they describe behavior patterns.

In terms of behavior parameters the strategies of the players may be repre-

sented as follows, assuming that since there is no point in passing with a *high* card at one's last opportunity to bet that this will not be done. The greek letters are the probabilities of the various acts.

	First Moves	Second Moves
I	α Open on *high* β Open on *low*	κ Call III on *low* λ Call II on *low* μ Call II and III on *low*
II	γ Call I on *low* δ Open on *high* ε Open on *low*	ν Call III on *low* ξ Call III and I on *low*
III	ζ Call I and II on *low* η Open on *low* θ Call I on *low* ι Call II on *low*	Player III never gets a second move

We locate all possible equilibrium points by first showing that most of the greek parameters must vanish. By dominance mainly with a little contradiction-type analysis β is eliminated and with it go γ, ζ, and θ by dominance. Then contradictions eliminate μ, ξ, ι, λ, κ, and ν in that order. This leaves us with α, δ, ε, and η. Contradiction analysis shows that none of these can be zero or one and thus we obtain a system of simultaneous algebraic equations. The equations happen to have but one solution with the variables in the range $(0, 1)$. We get

$$\alpha = \frac{21 - \sqrt{321}}{10}, \qquad \eta = \frac{5\alpha + 1}{4}, \qquad \delta = \frac{5 - 2\alpha}{5 + \alpha}, \qquad \varepsilon = \frac{4\alpha - 1}{\alpha + 5}.$$

These yield $\alpha = .308$, $\eta = .635$, $\delta = .826$, and $\varepsilon = .044$. Since there is only one equilibrium point the game has values; these are

$$v_1 = -.147 = -\frac{(1 + 17\alpha)}{8(5 + \alpha)}, \qquad v_2 = -.096 = -\frac{1 - 2\alpha}{4},$$

and

$$v_3 = .243 = \frac{79}{40}\left(\frac{1 - \alpha}{5 + \alpha}\right).$$

A more complete investigation of this poker game is published in Annals of Mathematics Study No. 24, *Contributions to the Theory of Games*. There the solution is studied as the ratio of ante to bet varies, and the potentialities of coalitions are investigated.

Applications

The study of *n*-person games for which the accepted ethics of fair play imply non-cooperative playing is, of course, an obvious direction in which to apply this

theory. And poker is the most obvious target. The analysis of a more realistic poker game than our very simple model should be quite an interesting affair.

The complexity of the mathematical work needed for a complete investigation increases rather rapidly, however, with increasing complexity of the game; so that analysis of a game much more complex than the example given here might only be feasible using approximate computational methods.

A less obvious type of application is to the study of cooperative games. By a cooperative game we mean a situation involving a set of players, pure strategies, and payoffs as usual; but with the assumption that the players can and will collaborate as they do in the von Neumann and Morgenstern theory. This means the players may communicate and form coalitions which will be enforced by an umpire. It is unnecessarily restrictive, however, to assume any transferability or even comparability of the payoffs [which should be in utility units] to different players. Any desired transferability can be put into the game itself instead of assuming it possible in the extra-game collaboration.

The writer has developed a "dynamical" approach to the study of cooperative games based upon reduction to non-cooperative form. One proceeds by constructing a model of the pre-play negotiation so that the steps of negotiation become moves in a larger non-cooperative game [which will have an infinity of pure strategies] describing the total situation.

This larger game is then treated in terms of the theory of this paper [extended to infinite games] and if values are obtained they are taken as the values of the cooperative game. Thus the problem of analyzing a cooperative game becomes the problem of obtaining a suitable, and convincing, non-cooperative model for the negotiation.

The writer has, by such a treatment, obtained values for all finite two person cooperative games, and some special n-person games.

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BIBLIOGRAPHY

(1) VON NEUMANN, MORGENSTERN, Theory of Games and Economic Behavior, Princeton University Press, 1944.
(2) J. F. NASH, JR., *Equilibrium Points in N-Person Games*, Proc. Nat. Acad. Sci. U. S. A. 36 (1950) 48–49.
(3) J. F. NASH, L. S. SHAPLEY, A Simple Three-Person Poker Game, Annals of Mathematics Study No. 24, Princeton University Press, 1950.
(4) JOHN NASH, *Two Person Cooperative Games*, to appear in Econometrica.
(5) H. W. KUHN, *Extensive Games*, Proc. Nat. Acad. Sci. U. S. A., 36 (1950) 570–576.