Working Directory

By me (Thanks to for the template)

Contents

Chapter 1

1.1 Indices

Definition 1.1.1: Index Laws

1. $a^m \times a^n = a^{n+m}$
2. $a^m \div a^n = a^{n-m}$
3. $(a^m)^n = a^{m\times n}$
4. $a^{-m} = \frac{1}{a^m}$
5. $a^0 = 1$
6. $a^{\frac{m}{n}} = \sqrt[n]{a^m}$

Example 1.1.1 (Laws in action)

1. $2^3 \times 2^7 = 2^{10}$
2. $\frac{3^6}{3^2} = 3^4$
3. $(5^2)^5 = 5^{10}$
4. $7 \times 2^{-2} = \frac{7}{2^2}$
5. $45^0 = 1$
6. $5^{\frac{-3}{7}} = \frac{1}{\sqrt[7]{5^3}}$

Note:-

Indices are used extremely frequently and there are often multiple laws hidden in each question

1.2 Logarithms (logs)

Definition 1.2.1: Log Laws

1. $log_{(a)}(xy) = \log_{(a)}(x) + \log_{(a)}(y)$
2. $log_{(a)}(\frac{x}{y}) = \log_{(a)}(x) - \log_{a}(y)$
3. $log_{(a)}(x^n) = n\log_{(a)}(x) => \log_{(a)}(\frac{1}{x}) = -\log_{(a)}(x)$
4. $log_{(a)}(a) = 1$
5. $log_{(a)}(1) = 0$

Example 1.2.1 ($p-$Norm)
$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geqslant 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\right)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

Question 1

Prove that triangle inequality is true if $p \geqslant 1$ for $p-$norms. (What goes wrong for $p < 1$?)

Solution: **For Property 3 for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\begin{aligned}
&\sum_i (x_i + y_i)^2 \leqslant \left(\sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2}\right)^2 \\
\Longrightarrow &\sum_i (x_i^2 + 2x_iy_i + y_i^2) \leqslant \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2 \\
\Longrightarrow &\left[\sum_i x_iy_i\right]^2 \leqslant \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]
\end{aligned}$$

So in other words prove $\langle x, y\rangle^2 \leqslant \langle x, x\rangle\langle y, y\rangle$ where

$$\langle x, y\rangle = \sum_i x_iy_i$$

Note:-

- $\|x\|^2 = \langle x, x\rangle$
- $\langle x, y\rangle = \langle y, x\rangle$

- $\langle \cdot, \cdot \rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$

  Here in $\langle x, y \rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leqslant \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}\langle x - \lambda y, x - \lambda y \rangle &= \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle \\ &= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle \\ &= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle\end{aligned}$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x, y \rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x \rangle \geqslant 0$

1.3 Algorithms

Algorithm 1: what

Input: This is some input
Output: This is some output

```
/* This is a comment */

some code here;
x ← 0;
y ← 0;
if x > 5 then
  | x is greater than 5 ;                       // This is also a comment
else
  | x is less than or equal to 5;
end
foreach y in 0..5 do
  | y ← y + 1;
end
for y in 0..5 do
  | y ← y − 1;
end
while x > 5 do
  | x ← x − 1;
end
return Return something here;
```