Available online at www.sciencedirect.com
SCIENCE DIRECT®
Applied Mathematics and Computation 168 (2005) 1169–1174
APPLIED MATHEMATICS AND COMPUTATION
www.elsevier.com/locate/amc

A study of convergence on the Newton-homotopy continuation method

Tzong-Mou Wu

Department of Mechanical Engineering, Far East College, Hsin-Shih, Chung-Hwa Road, Tainan 744, Taiwan, Province of China

Abstract

The traditional Newton method is a well-known and popular method for solving non-linear equation. It has high efficient in the convergence speed. However, we always need to guess the initial value in the iteration process. Good initial guess value can solve the equation quickly. Bad initial guess value usually will yield divergence. Homotopy continuation method is a kind of perturbation method. It can guarantee the answer by a certain path if we choose the auxiliary homotopy function, or call start system, well. This paper presents some useful rules for the choice of the auxiliary homotopy function to avoid the problem of divergence of traditional Newton method.

Keywords: Numerical method; Newton method; Homotopy continuation method

1. Introduction

Homotopy continuation method was known as early as in the 1930s. This method was used by kinematician in the 1960s at US for solving mechanism

E-mail address: tm.wu@cc.fec.edu.tw

0096-3003/$ - see front matter
doi:10.1016/j.amc.2003.10.068

synthesis problems. The latest development was done by Morgan [1,2] at GM. We also have two important literatures: Garcia [3] and Allgower and Georg [4]. Continuation method gives a set of certain answers and some simple iteration process to obtain our solutions more exactly. He [5–7] studied the homotopy method through a series of different non-linear ordinary differential equations. He [8] also applied the Taylor's expansion theorem to algebraic equations and derive another iteration formula. However, the formula has the danger of divergence due to the choice of bad initial guess value. Abbasbandy [9] made the effort to develop the modified Adomian decomposition method for solving algebraic equations, and obtain the well numerical result about computing speed. Similarly, the Abbasbandy's formula still has the chance of divergence. Therefore, this paper attempts to study some robust rules and provide the choice method of the auxiliary homotopy function for homotopy continuation method to avoid the problem of divergence of traditional Newton method when solving the non-linear equations.

2. Newton-homotopy continuation method

Before the study of the auxiliary homotopy function, we consider the following non-linear algebraic equation

$$f(x) = 0. \tag{1}$$

To solve this equation, we have many different methods such as bisection method, fixed-point iteration, secant method, Newton–Raphson method and so on. As we known, the Newton's method is a faster method than others. The numerical iteration formula of Newton's method for solving this equation is given as

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{2}$$

Observing Eq. (2), we find that the x_{n+1} value will become infinite when $f'(x_n) = 0$ or $f'(x_n) \to 0$. Now, we will apply the homotopy continuation method and develop some useful rules for the choice of the auxiliary homotopy function to avoid the divergence of iteration result in Eq. (2).

Given a set of equations in n variables $x_1, x_2, \ldots, x_n$. We modify the equations by omitting some of the terms and adding new ones until we have a new system of equations, the solutions to which may be easily guessed/given/known. We then deform the coefficients of the new system into the coefficients of the original system by a series of small increments, and we follow the solution through the deformation, using methods such as Newton–Raphson. This is called Newton-homotopy continuation original system.

If we wish to find the solution in Eq. (1), we choose a new simple start system or call auxiliary homotopy function

$$g(x) = 0. \tag{3}$$

The auxiliary homotopy function $g(x)$ must be known or controllable and easy to solve. The initial guess value x_0 in traditional Newton method we name it start value in Newton-homotopy continuation method. Then, we define the homotopy continuation as

$$H(x,t) \equiv tf(x) + (1-t)g(x) = 0. \tag{4}$$

Where t is an arbitrary parameter and varies from 0 to 1, i.e. $t \in [0,1]$. Therefore, we have the following two boundary conditions

$$\begin{aligned} H(x,0) &= g(x), \\ H(x,1) &= f(x). \end{aligned} \tag{5}$$

This is the famous homotopy continuation method. It is also called Bootstrap method or Parameter–Perturbation method, but these names did not become popular.

Our goal in this study is to solve the $H(x, t) = 0$ instead of $f(x) = 0$ by varying parameter t from 0 to 1 and avoid the situation of divergence. So we rewrite Eq. (2) as

$$x_{n+1} = x_n - \frac{H(x_n,t)}{H'(x_n,t)}, \tag{6}$$

where the situation of divergence occurs at

$$H'(x,t) = tf'(x) + (1-t)g'(x) = 0 \quad \text{or} \quad \to 0. \tag{7}$$

To avoid divergence, we have to let $H'(x,t) \neq 0$ or not $\to 0$. In other words, we need to guarantee $tf'(x) + (1-t)g'(x) \neq 0$ or not $\to 0$. By this reason, we can assume

$$\begin{aligned} &tf'(x) + (1-t)g'(x) = C \quad \text{or} \quad Ce^x \quad \text{or} \quad C\sin x + D \quad \text{or} \\ &\quad C\cos x + D \cdots. \end{aligned} \tag{8}$$

Where the $C, D, E \ldots$ are constants and not equal to zero. Integrate this equation to yield

$$tf(x) + (1-t)g(x) = Cx + K \quad \text{or} \quad Ce^x + K \cdots. \tag{9}$$

K is an arbitrary number. It can be zero. Here, our task is to determine an appropriate auxiliary homotopy function $g(x)$ to satisfy the following three principles/rules concluded from Eq. (9):

$$(1) \quad \text{For } t = 0 \Rightarrow g(x) = Cx + K \quad \text{or} \quad Ce^x + K \cdots, \tag{10}$$

$$(2) \quad \text{For } t = 1 \Rightarrow f(x) = Cx + K \quad \text{or} \quad Ce^x + K \cdots, \tag{11}$$

(3) For $t \in (0,1) \Rightarrow g(x)$

$$= C_1 f(x) + C_2 x + K \quad \text{or} \quad C_1 f(x) + C_2 \mathrm{e}^x + K \cdots. \tag{12}$$

Similarly the $C_1, C_2 \ldots$ are constants and not equal to zero. For convenience in the general construction and programming of Newton-homotopy continuation method, we can choose $g(x) = Cx + K$, $C\mathrm{e}^x + K$, $C\sin x + Dx + K$ or the form like Eq. (12).

Above are the principles/rules for choosing the appropriate auxiliary homotopy function $g(x)$. In the following section, we will run two examples to compare the improved Newton-homotopy method and traditional Newton method.

Example 1. Consider a polynomial algebraic equation

$$f(x) = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 6x + 1 = 0. \tag{13}$$

We can easy to detect the roots of $f'(x) = 0$ are $x = -2 \vee 3$. If we choose the auxiliary homotopy function $g(x) = x$, $-\mathrm{e}^x + 1$ or $x - 1$, respectively, to run the Newton-homotopy iteration formula (6). The results therefore are obtained in Table 1. Since we already change the original equation $f(x)$ into a new equation $tf(x) + (1 - t)g(x)$, the same start value x_0 thus be possible to converge to distinct solution.

Example 2. Consider a triangle non-linear equation

$$f(x) = \sin x - \frac{1}{2}x = 0. \tag{14}$$

Similarly, we have the roots of $f'(x) = 0$ are $x = -60° \pm n(360°) \vee 60° \pm n(360°)$, $n \in N$. We choose the auxiliary homotopy function $g(x) = 100x$ or $\sin x - x + 1$, respectively in this case, to run the Newton-homotopy method and obtain the results in Table 2.

Clearly, we find the use of traditional Newton method will easily become divergence when the bad initial guess value x_0 been chosen. However, this situation can be avoided in the improved Newton-homotopy continuation

Table 1
The results of traditional Newton and improved Newton-homotopy method in Example 1

x_0	$g(x)$	Newton	Newton-homotopy
−2	x	Diverge	−3.65270475885147
−2	$-\mathrm{e}^x + 1$	Diverge	0.164655385738659
3	$x - 1$	Diverge	4.98804937311281
3.1	$x - 1$	4.98804937311281	4.98804937311281

Table 2
The results of traditional Newton and improved Newton-homotopy method in Example 2

x_0	$g(x)$	Newton	Newton-homotopy
60°	$100x$	Diverge	0°
−60°	$\sin x - x + 1$	Diverge	108.60382157811°
−61°	$\sin x - x + 1$	108.60382157811°	108.60382157811°

method no matter what the start value x_0 been decided if we propose the appropriate auxiliary homotopy function $g(x)$ by the rules in Eqs. (10) or (12).

3. Conclusion

Typically, it is usually a big trouble and disadvantage for us to do the algebraic operation, for example, solving the non-linear equations. Fortunately, by the aid of computer science, the non-linear equations will be solved no more difficulty. With the computer improved and the numerical technique developed, the solutions of non-linear equations become easier. We can use current high-speed processor to determine the solutions quickly.

Although, we already have many different numerical methods to help us to treat these non-linear equations. The traditional Newton–Raphson method is a popular numerical method. However, it still has some well-known critical defects. By means of the development of numerical continuation technique, the defects in traditional numerical methods such as the acquirement of a good initial guess, the problems of convergence and the computing time will be avoided. Homotopy continuation method is one of the famous improved numerical continuation methods. Its convergence speed is fast, also the algorithm is clear and easy.

This paper combines Newton's and homotopy continuation method and proposes some robust rules for the choice of the auxiliary homotopy function for the Newton-homotopy continuation method to avoid the problem of divergence of many traditional numerical methods when solving the non-linear equations. We can get well outcome by this new method. This new interesting method will provide another numerical approach. It is hoped that the work presented here will contribute towards progress in the numerical techniques and other relative fields for scientists or engineers.

References

[1] A.P. Morgan, A Method for Computing all Solutions to Systems of Polynomial Equations, GM Research Publication, GMR 3651, 1981.

[2] A.P. Morgan, Solving Polynomial Systems Using Continuation for Engineering and Scientific Problems, Prentice-Hall Book Company Inc, Englewood Cliffs, New Jersey, 1987.

[3] C.B. Garcia, W.I. Zangwill, Pathways to Solutions, Fixed Points, and Equilibria, Prentice-Hall Book Company Inc, Englewood Cliffs, New Jersey, 1981.
[4] E.L. Allgower, K. Georg, Numerical Continuation Methods, An Introduction, Springer-Verlag, New York, 1990.
[5] J.H. He, Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering 178 (1999) 257–262.
[6] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, International Journal of Non-linear Mechanics 35 (2000) 37–43.
[7] J.H. He, Homotopy perturbation methods: a new nonlinear analytical technique, Applied Mathematics and Computation 135 (2003) 73–79.
[8] J.H. He, A new iteration method for solving algebraic equations, Applied Mathematics and Computation 135 (2003) 81–84.
[9] S. Abbasbandy, Improving Newton–Raphson method for nonlinear equations by modified Adomian decomposition method, Applied Mathematics and Computation 145 (2003) 887–893.