A Continuous-Time Dynamical System Describing both Rate Encoding and Spiking Neurons

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Introduction

- We investigated a two-dimensional nonlinear system, modeling a wide range of dynamic properties of spiking neurons.
- By altering key parameters of this system, its dynamics become identical to those of a time-continuous rate-encoding model.
- Differences of the dynamical properties of single units as well as of network structures under these two regimes can be treated within the same mathematical framework.

Neuron Model

- The model consists of a two-dimensional non-linear system given by

$$\tau_x \dot{x} = f(x) - y \qquad f(x) = s\sigma\left(a\left(x - s/4\right)\right)$$
$$\tau_y \dot{y} = g(x) - y \qquad g(x) = g_0\sigma\left(a_g\left(x - I\right)\right) - \Delta_y$$
$$u(x,y) := \frac{x+y}{\sqrt{2}} \qquad \sigma(x) = \left(1 + \exp(-x)\right)^{-1}$$

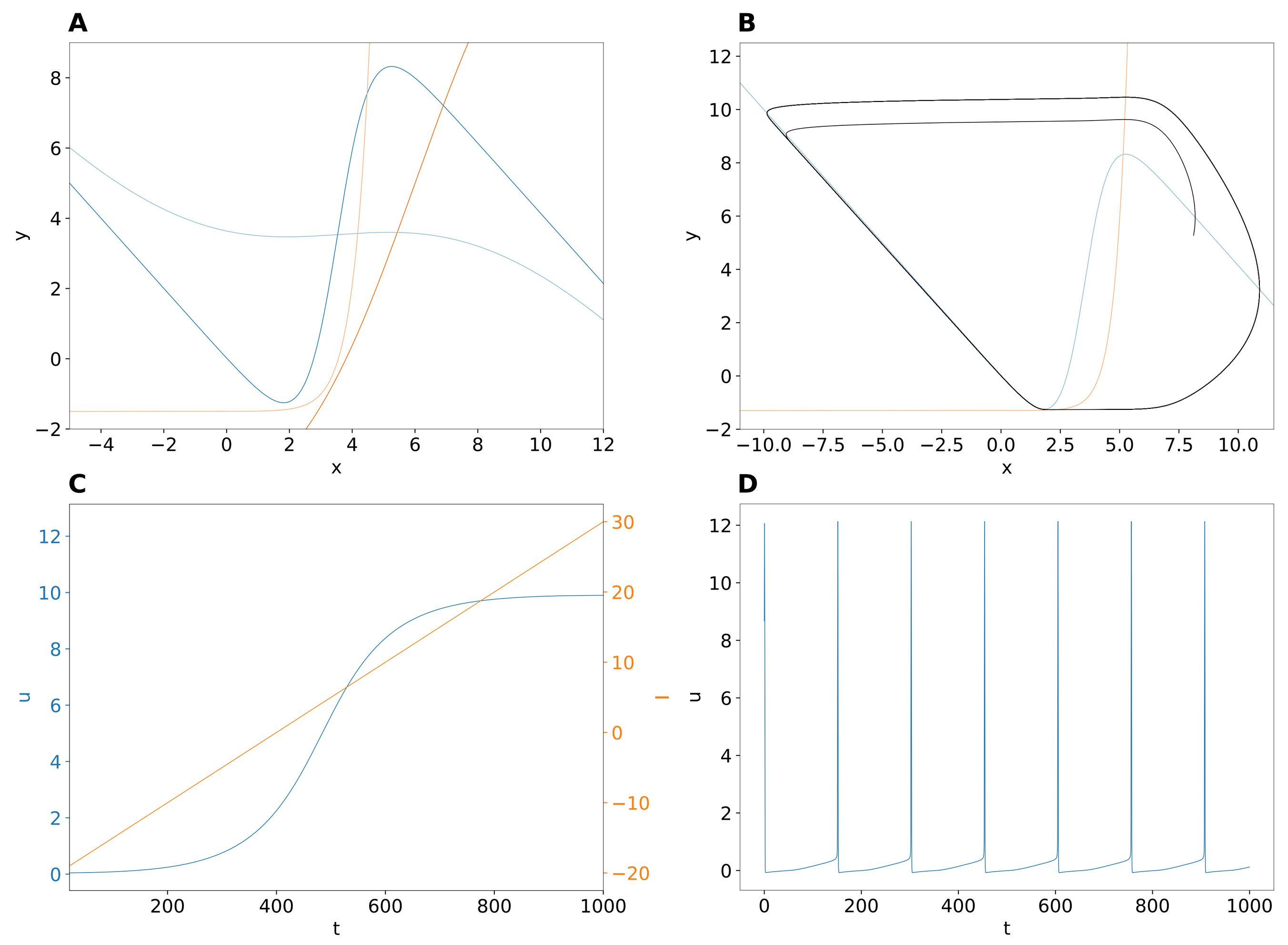

Figure 1: **A**: x/y-nullclines (blue/orange) of the dynamical system for parameter sets generating spiking/non spiking behavior. **B**: Phase plane trajectory for the spiking dynamics. **C**/**D**: Dynamics of readout variable u for the non-spiking/spiking case.

Spiking Neuron Model

- A range of different types of spiking dynamics can be reproduced by this generic system.

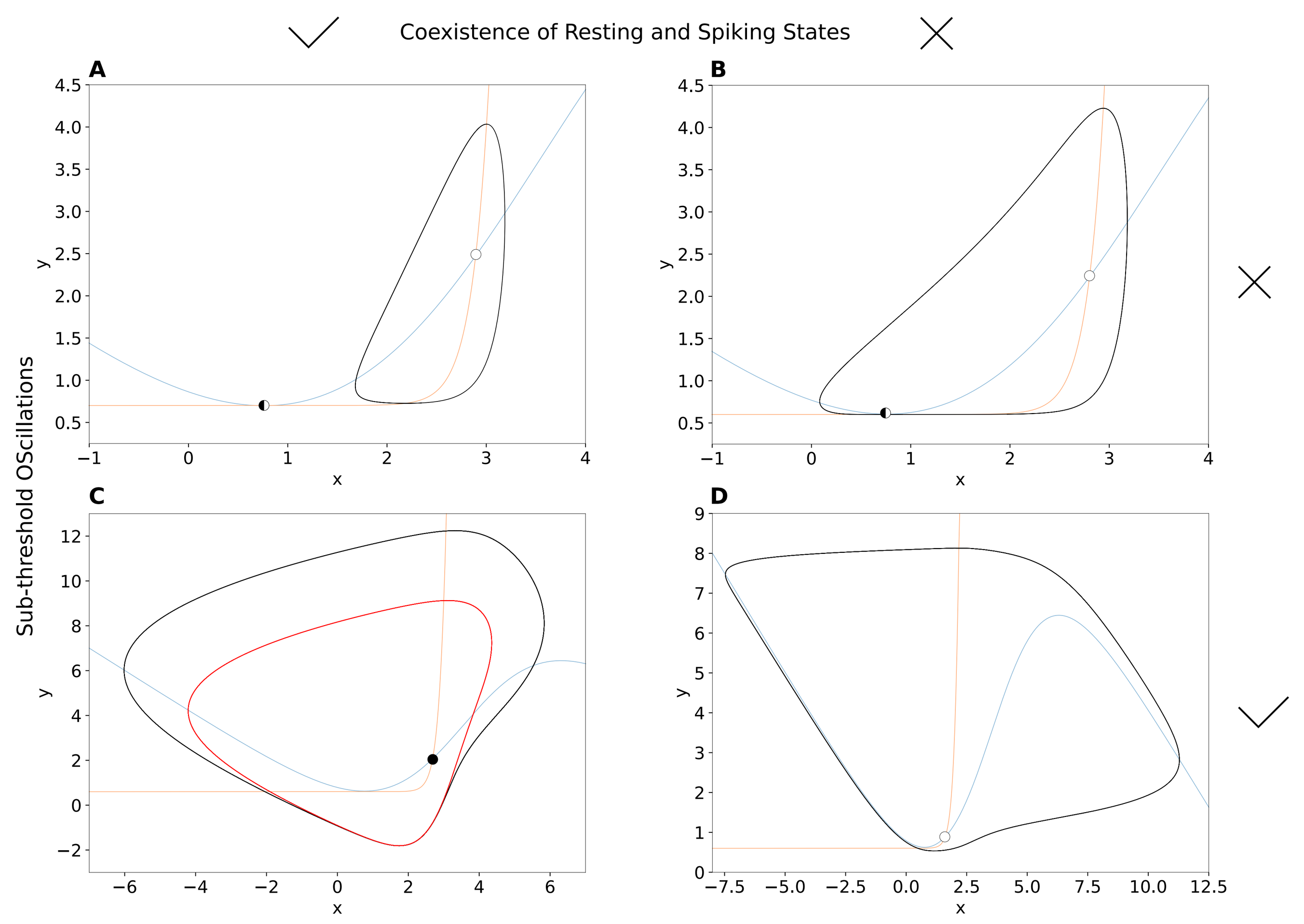

Figure 2: Examples of spiking dynamics/bifurcations observed in the model, ordered by the property of coexistence of resting/spiking states and the possibility of sub-threshold oscillations. **A**: Saddle node bifurcation with coexisting limit cycle. **B**: Saddle node bif. on invariant cycle. **C**: Subcritical Hopf/Fold bif. of limit cycles. **D**: Supercritical Hopf bif.

- Based on the classification by Izhikevich [1], our model was able to generate class 1 (arbitrary small firing rates) as well as class 2 (all-or-none spiking behavior) firing patterns.
- Due to the relatively simple mathematical form of our system, we could perform analytical analyses and predictions about firing behavior closely matching the simulation results, see Fig. 3.

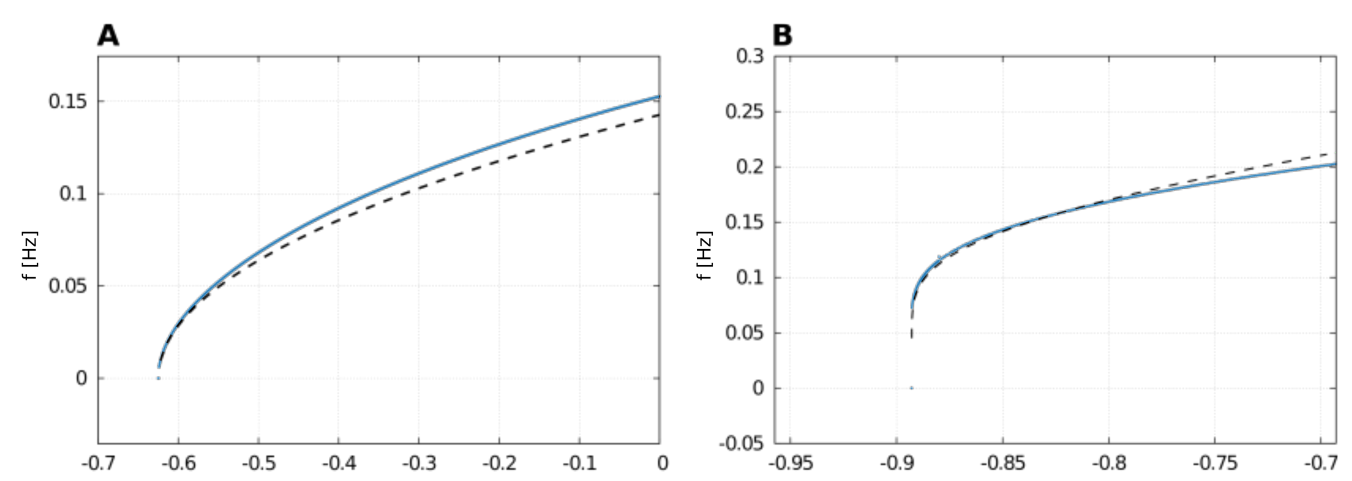

Figure 3: Measured (blue) and predicted firing rates as a function of Δ_y under the configurations shown in Fig. 2A/B.

Fitting Experimental Data

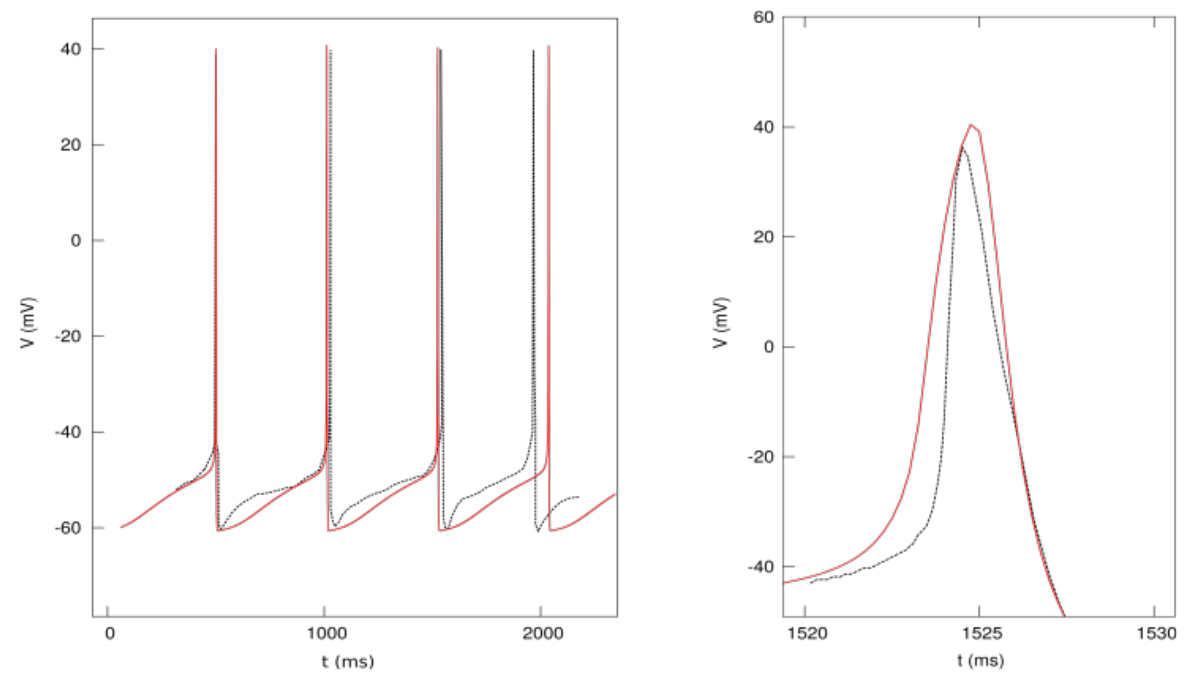

Figure 4: Fit of the dynamical system onto measurements of a dopaminergic neuron in the Substantia Nigra of a mouse brain [2].

- The parameters of the dynamical systems allow to easily control key features of the voltage trace, such as spike height, spike width and threshold potential.

$$u_{\text{spikeheight}} \approx s/\sqrt{2}$$
$$\Delta t_{\text{spike}} \propto g_O^{-1}$$
$$u_{\text{threshold}} \approx 1/\sqrt{2a}$$

Rate Encoding Model

- The system's parameters can be tuned such that dynamics transition from a spiking behavior to a relaxation towards an input-dependent fixed point.
- The projection of this fixed point onto u follows a sigmoidal as input increases, see Fig. 1C.

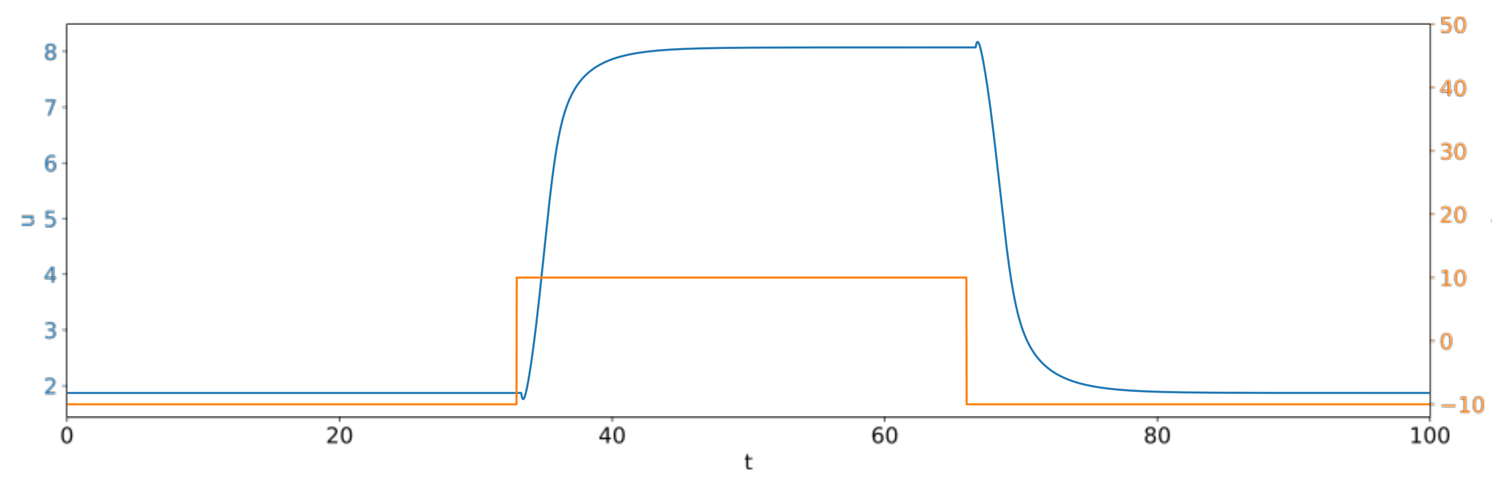

Figure 5: Rate-dynamics response of the readout variable u upon external step input.

- The dynamics of u approximately follow those of 1d-system of the form $\tau\dot{u} = -u + \frac{s}{\sqrt{2}}\sigma\left(c \cdot I\right)$. The time constant τ is then described by $\tau = -\frac{1}{\langle\lambda\rangle}$, where $\langle\lambda\rangle$ is the mean of the eigenvalues of the fixed point's Jacobian.

Usage in Networks

- Combining our model with a simple model of synaptic transmission resulted in stable network dynamics for both spiking and rate encoding dynamics.

$$\tau_g \dot{g} = -g + w \cdot u_{\text{pre}}$$
$$I_{\text{syn}} = g(E_{\text{syn,exc/inh}} - u_{\text{post}})$$

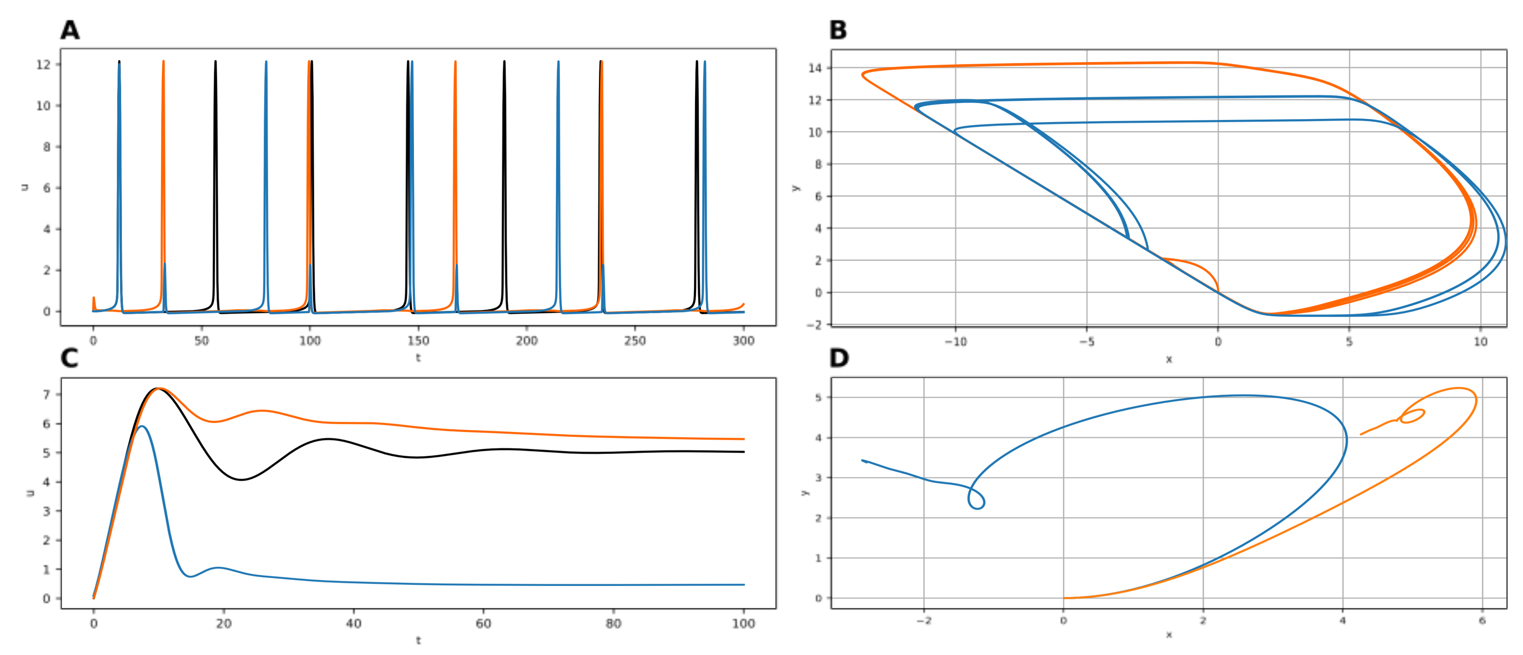

Figure 6: Dynamics of an exc.(blue) and inh.(orange) neuron mutually coupled, including additional input driving the exc. neuron. Black traces in **A**/**C** represent an isolated reference neuron driven by the same external input. **A**/**B**: The network in its spiking state. **C**/**D**: Rate-encoding state, same coupling strengths.

Conclusions

- The model allows for the reproduction of a wide range of dynamical properties of spiking neurons.
- Characteristic properties such as the spike height and width can be controlled very well by certain model parameters.
- Good usability as a rate encoding model with a sigmoid activation function.
- Further investigations should compare how spiking and rate network dynamics map onto each other under the same synaptic coupling.

[1] E. M. Izhikevich. *Dynamical Systems in Neuroscience*. The MIT Press, Cambridge Massachusetts, 2007.

[2] B. P. Bean. The action potential in mammalian central neurons. *Nature Reviews, Neuroscience*, 8, June 2007.