A mean-field controlled gain regulation in echo-state networks

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1 Model Description

1.1 Elements of the Model

The following elements constitute the model:

$\epsilon_{\mu\mathrm{y}}, \epsilon_{\sigma\mathrm{y}}, \epsilon_{\mu\mathrm{x}}, \epsilon_{\sigma\mathrm{x}}, \epsilon_{\mathrm{a}}, \epsilon_{\mathrm{b}} \in \mathbb{R}$	Adaptation Rates
$u \in \mathbb{R}^T,\ e \in \mathbb{R}^{N\times T}$	Input Sequence, External Neural Inputs
$w_{\mathrm{in}} \in \mathbb{R}^N,\ W \in \mathbb{R}^{N\times N}$	External Input Weights, Recurrent Weights
$x, y \in \mathbb{R}^{N\times T}$	Membrane Potentials, Neural Activities
$\mu_{\mathrm{y}}, \sigma^2_{\mathrm{y}}, \mu_{\mathrm{x}}, \sigma^2_{\mathrm{x}} \in \mathbb{R}^{N\times T}$	Trailing Averages
$\sigma^2_{\mathrm{ymf}} \in \mathbb{R}^{N\times T}$	Mean Field of Activity Variances
$a, b \in \mathbb{R}^{N\times T}$	Gains, Biases
$\mu_{\mathrm{yt}} \in \mathbb{R}^N$	Target Activities

1.2 Dynamics

Subscript indices $(\cdot)_i$ or $(\cdot)_{ij}$ refer to the dimensions spanned by N in the respective objects, while $(\cdot)(t)$ refers to the index of the dimension spanned by T.

Network Dynamics

$$e_i(t) = u(t) w_{\mathrm{in},i} \tag{1}$$

$$x_i(t) = \sum_j W_{ij} y_j(t-1) \tag{2}$$

$$y_i(t) = \tanh\left\{a_i(t-1)x_i(t) + e_i(t) - b_i(t-1)\right\} \tag{3}$$

Running Averages

$$\mu_{\mathrm{y},i}(t) = (1-\epsilon_{\mu\mathrm{y}})\mu_{\mathrm{y},i}(t-1) + \epsilon_{\mu\mathrm{y}} y_i(t) \tag{4}$$

$$\sigma^2_{\mathrm{y},i}(t) = (1-\epsilon_{\sigma\mathrm{y}})\sigma^2_{\mathrm{y},i}(t-1) + \epsilon_{\sigma\mathrm{y}} \left(y_i(t) - \mu_{\mathrm{y},i}(t)\right)^2 \tag{5}$$

$$\mu_{\mathrm{x},i}(t) = (1-\epsilon_{\mu\mathrm{x}})\mu_{\mathrm{x},i}(t-1) + \epsilon_{\mu\mathrm{x}} x_i(t) \tag{6}$$

$$\sigma^2_{\mathrm{x},i}(t) = (1-\epsilon_{\sigma\mathrm{x}})\sigma^2_{\mathrm{x},i}(t-1) + \epsilon_{\sigma\mathrm{x}} \left(x_i(t) - \mu_{\mathrm{x},i}(t)\right)^2 \tag{7}$$

$$\sigma^2_{\mathrm{ymf},i}(t) = (1-\alpha)\sigma^2_{\mathrm{y},i}(t) + \alpha \left\langle \sigma^2_{\mathrm{y},j}(t) \right\rangle_j \tag{8}$$

Gain and Bias Update

$$a_i(t) = (1-\epsilon_{\mathrm{a}})\, a_i(t-1) + \epsilon_{\mathrm{a}} R \sqrt{\sigma^2_{\mathrm{ymf},i}(t)/\sigma^2_{\mathrm{x},i}(t)} \tag{9}$$

$$b_i(t) = b_i(t-1) + \epsilon_{\mathrm{b}} \left(y_i(t) - \mu_{\mathrm{yt},i}\right) \tag{10}$$

1.3 Distributions

$$p(W_{ij}) = \begin{cases} \delta(W_{ij}) & : i = j \\ (1-p_{\mathrm{W}})\delta(W_{ij}) + p_{\mathrm{W}} \mathcal{N}\left(W_{ij}, \mu = 0, \sigma = \sigma_{\mathrm{W}}/\sqrt{p_{\mathrm{W}} N}\right) & : i \neq j \end{cases} \tag{11}$$

$$p(w_{\mathrm{in},i}) = (1-p_{\mathrm{win}})\delta(w_{\mathrm{in},i}) + p_{\mathrm{win}} \mathcal{N}\left(w_{\mathrm{in},i}, \mu = 0, \sigma = 1\right) \tag{12}$$

$$p(\mu_{\mathrm{yt},i}) = \mathcal{N}\left(\mu_{\mathrm{yt},i}, \mu = 0, \sigma = \sigma_{\mu\mathrm{yt}}\right) \tag{13}$$

1.4 Parameters

Table 1: Model Parameters

N	T	p_{W}	p_{win}	$\epsilon_{\mu\mathrm{y}}$	$\epsilon_{\sigma\mathrm{y}}$	$\epsilon_{\mu\mathrm{x}}$	$\epsilon_{\sigma\mathrm{x}}$	ϵ_{a}	ϵ_{b}	$\sigma_{\mu\mathrm{yt}}$	α	R
10^3	$15 \cdot 10^3$	0.1	1.0	10^{-3}	$5 \cdot 10^{-3}$	10^{-3}	$5 \cdot 10^{-3}$	10^{-3}	10^{-3}	10^{-2}	1.0	1.0

The values/properties of u,R and σ_{W} were left open to experimentation. Generally, the goal of the mechanism was to adapt a in such a way that too large/too small variances in W are compensated, leading to a spectral radius of $\rho\left(a_i W_{ij}\right)$ of R. We tested with u taken from a Gaussian distribution with $1/4$ standard deviation, $R = 1$ and $\sigma_{\mathrm{W}} = 5$. The latter initially caused the spectral radius to be 5.

Gains were initially set to 1, biases to 0.

2 Results

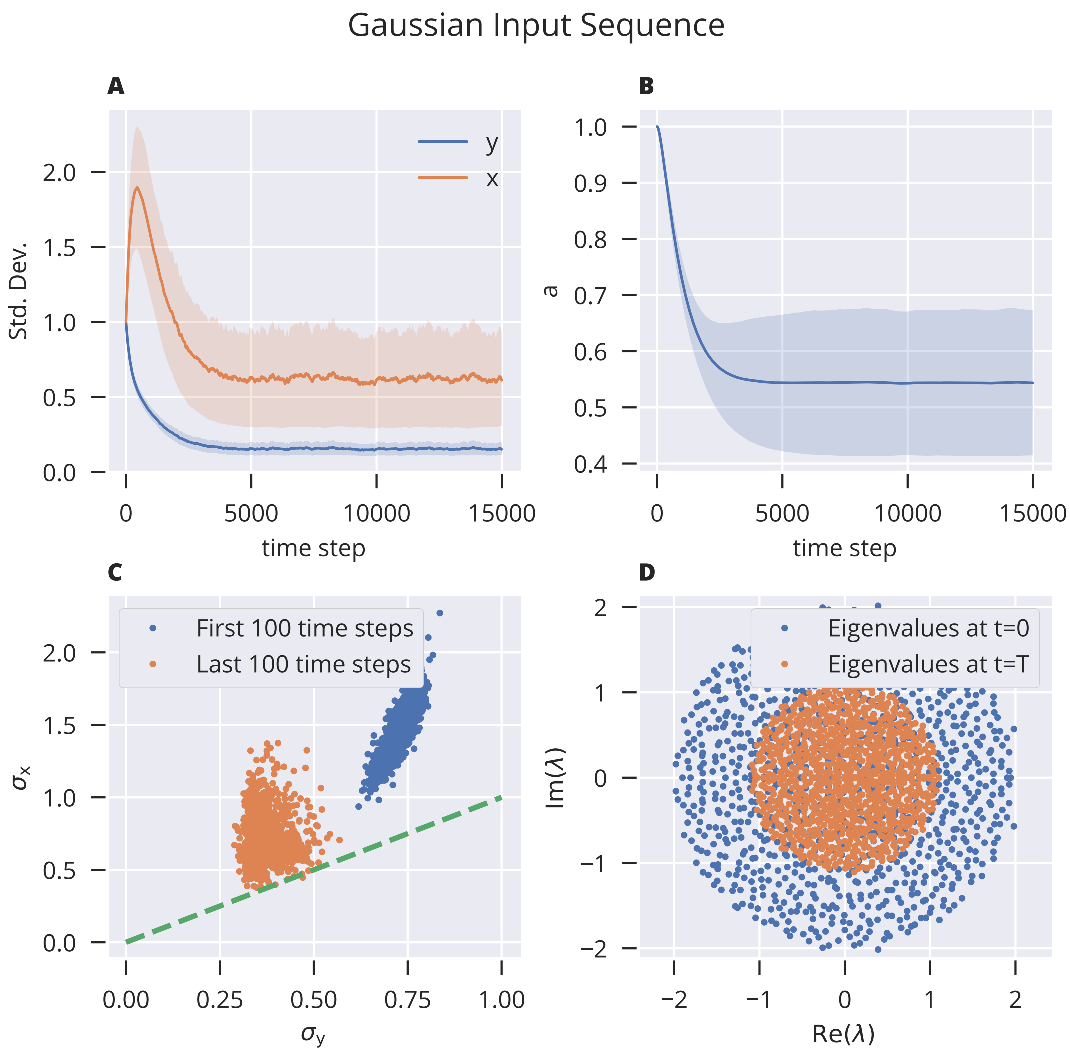

Figure 1: Results for u drawn from a Gaussian distribution with zero mean, $1/4$ standard deviation, $R = 1$ and $\sigma_\mathrm{W} = 2$. Other parameters as in Table 1. **A**: Standard deviations (trailing average over time) of neural activities and membrane potentials. Shaded region denotes standard deviation over the neural population. **B**: Gain mean and standard deviation over the neural population. **C**: Standard deviations of membrane potentials versus standard deviations of activities. Each point represents a neuron. **D**: Eigenvalues of $a_i W_{ij}$.

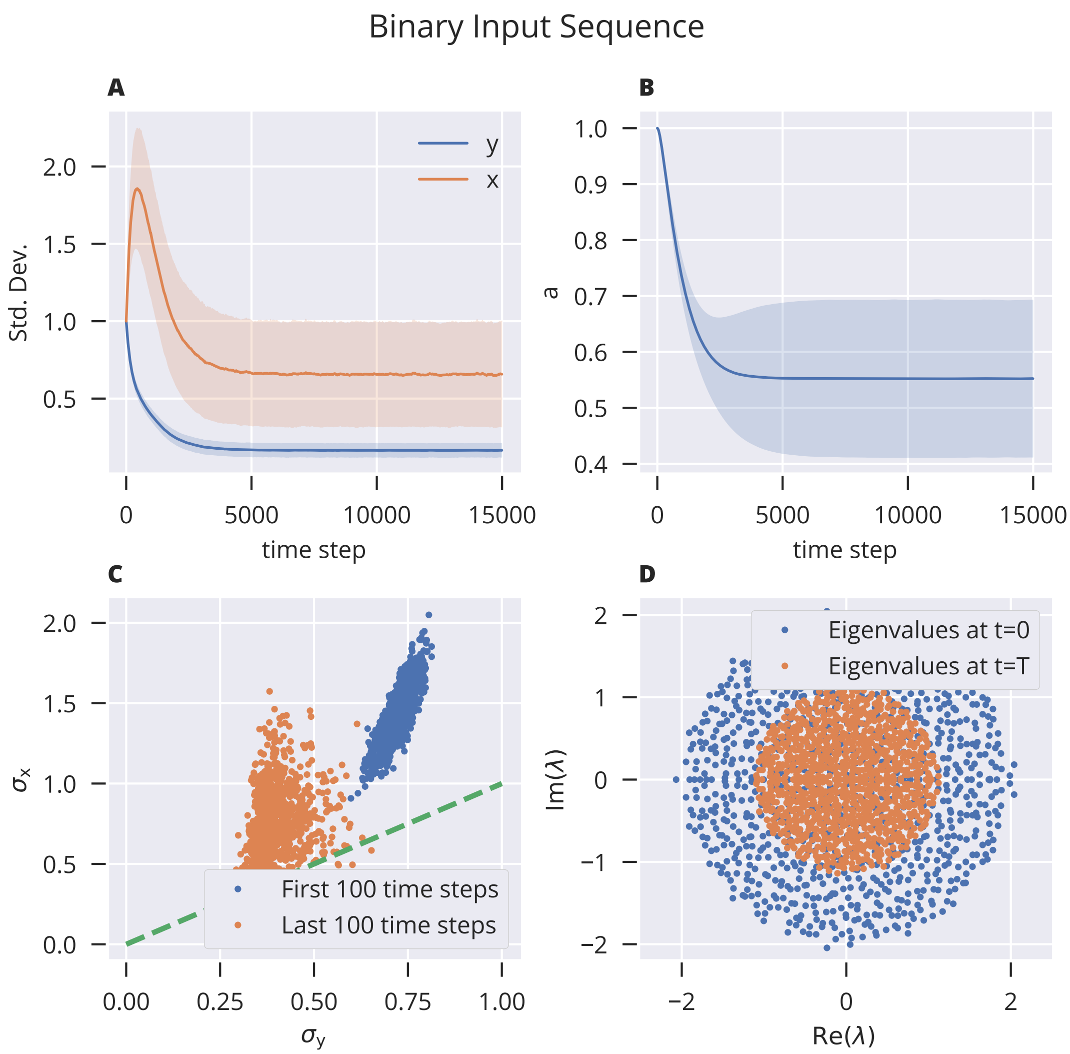

Figure 2: Results as presented in Fig. 1, for u drawn from a symmetric binary distribution, $u(t) \in (-1/4, 1/4)$.

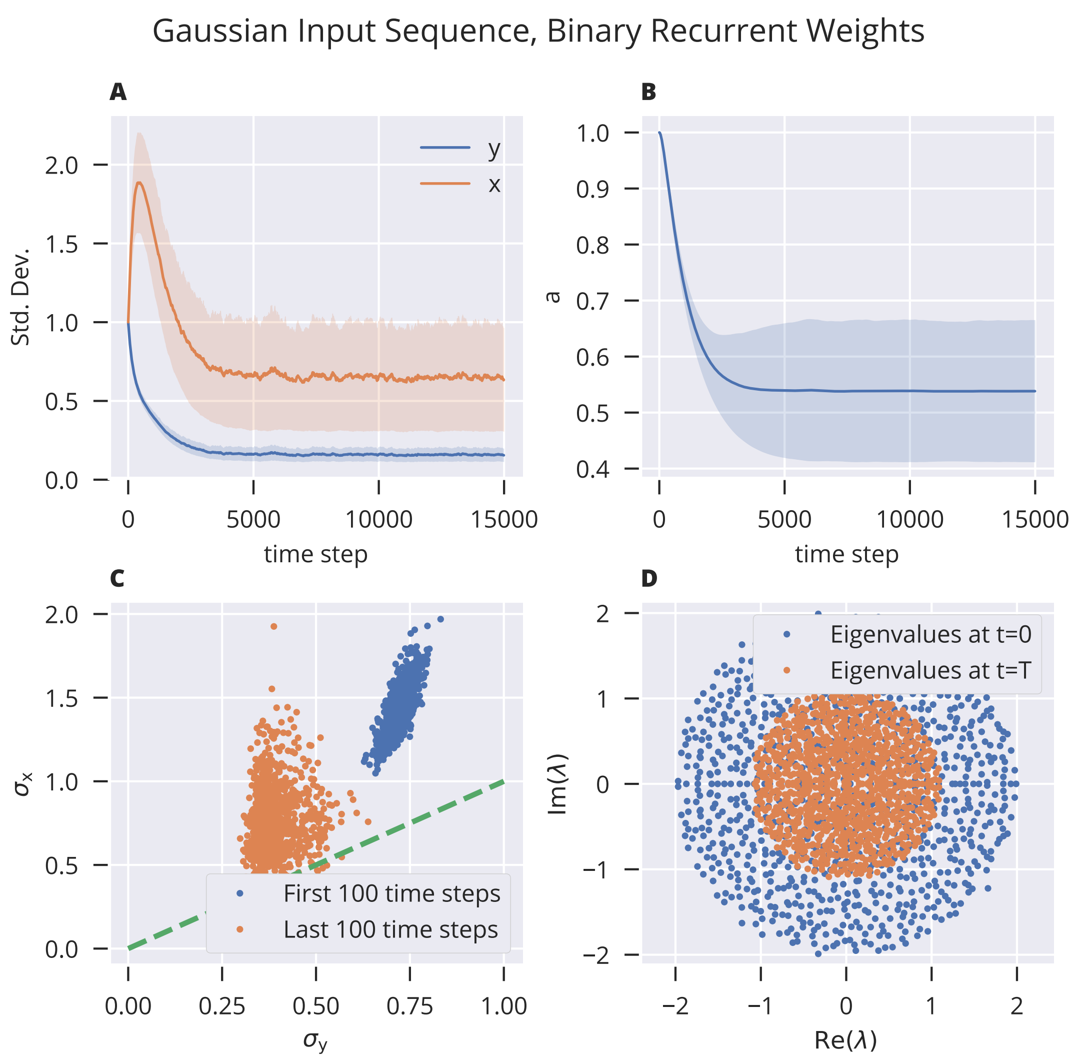

Figure 3: Results as presented in Fig. 1, for W_{ij} drawn from a binary distribution.

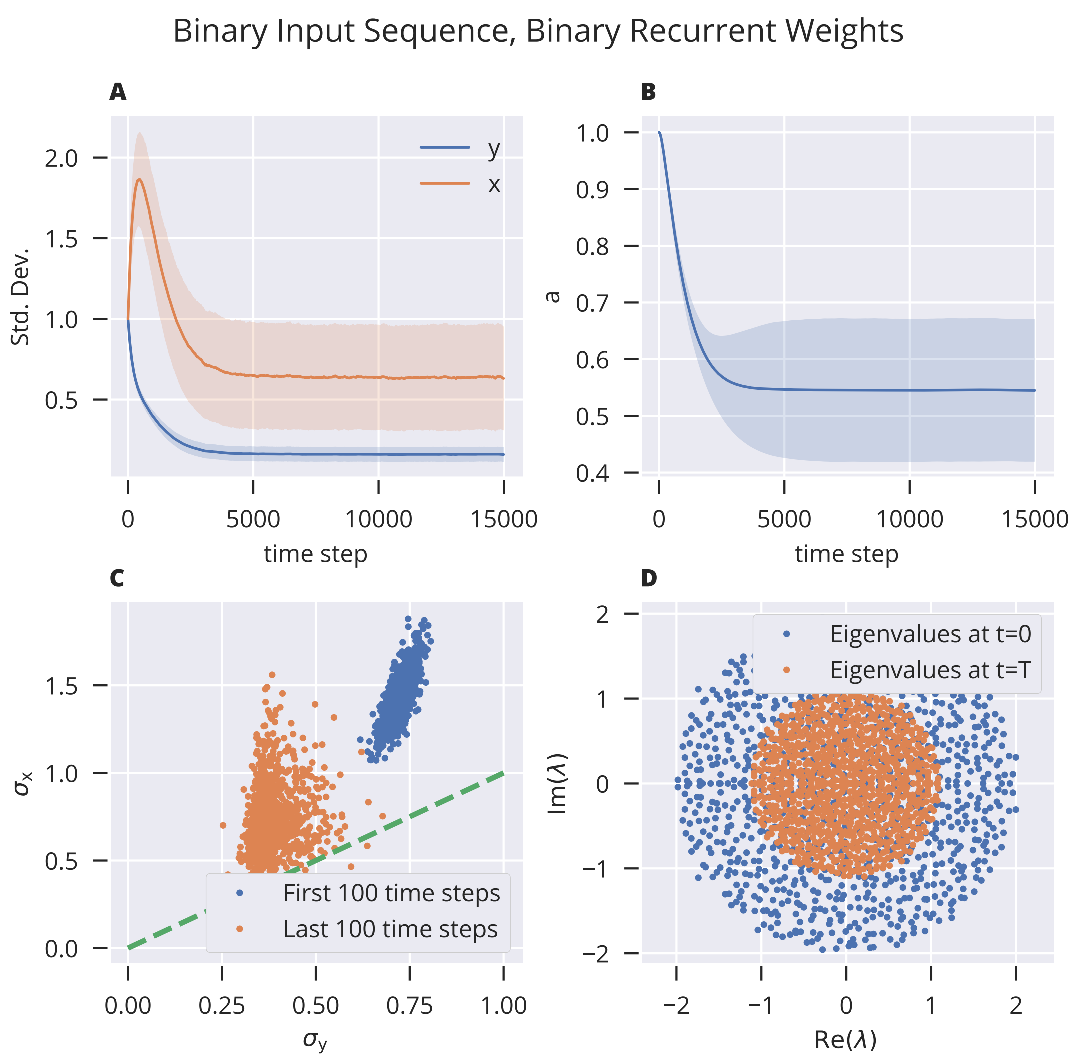

Figure 4: Results as presented in Fig. 1, for W_{ij} drawn from a binary distribution and u drawn from a binary distribution, $u(t) \in (-1/4, 1/4)$.