Driven Random Dynamic Reservoir with Homeostatic Variance Control

1 Model Description

1.1 Dynamics

$$x_i^{t+1} = \tanh\left(g_i^t I_i^{t+1}\right) \tag{1}$$

$$I_i^{t+1} = \sum_{j=1}^{N_{\text{net}}} W_{ij} x_j^t + E_i^{t+1} \tag{2}$$

$$g_i^{t+1} = \mu_g \left[\sigma_{\text{target}}^2 - \left(x_i^t - \langle x_i \rangle\right)^2\right] \tag{3}$$

1.2 Parameters / Settings

W_{ij} is a sparse random matrix with connection probability cf_{net}. Nonzero entries were drawn from a Gaussian distribution $\mathcal{N}(\mu = 0, \sigma = \sigma_{\text{conn}}/\sqrt{N_{\text{net}}\text{cf}_{\text{net}}})$. Diagonal entries were always set to zero.

E_i^t are random vectors of size N_{net} with independent entries drawn from a Gaussian distribution $\mathcal{N}(\mu = 0, \sigma = \sigma_{\text{ext}})$. External input is turned off after $t_{\text{ext.off}}$.

By changing individual gain values g_i, the homeostatic control tries to drive the activity standard deviation of every cell to the value given by σ_{target}. However, this mechanism is also switched off after $t_{\text{ext.off}}$. This is done because we can assume that homeostatic processes would biologically act on much slower timescales than changes in input. Before $t_{\text{ext.off}}$, we can set μ_g to relatively high values to let homeostasis converge under external drive.

See all parameters in Table 1.

Table 1: Model Parameters

Parameter	**Value**
N_{net}	500
cf_{net}	0.1
σ_{conn}	1.0
σ_{ext}	1.0
μ_g	0.0005
σ_{target}	0.33
n_t (Sim. Steps)	200000
$t_{\text{ext.off}}$	100000

2 Results

Exemplary results are shown in Fig. 1.

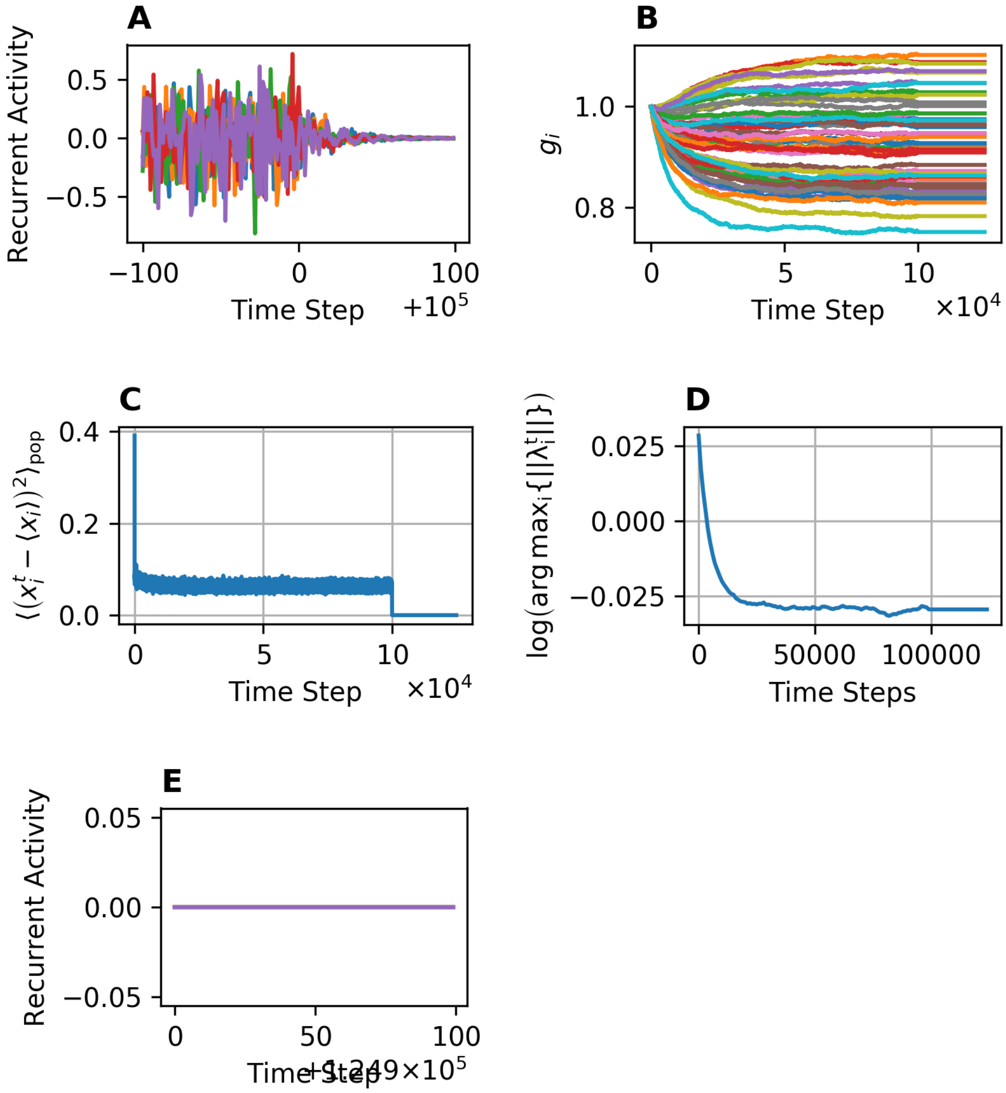

Figure 1: **A**: Sample of activity within $[t_{\mathrm{ext.off}} - 100, t_{\mathrm{ext.off}} + 100]$. **B**: Gain dynamics of $N_{\mathrm{net}}/10$ exemplary neurons. **C**: Population mean of squared activity. **D**: Log. of largest real part of eigenvalues of $g_i^t W_{ij}$.**E**: Sample of population activity for the last 100 steps.

3 Mean Field Approximation

We would like to find an approximate relation between the gain resulting from homeostasis and the input and target variance. In the following, we shall denote by $\langle\cdot\rangle_T$ an average over time and by $\langle\cdot\rangle_P$ over the population. If we linearize the neural activation function and take an average over time, we get

$$\left\langle x_i^2 \right\rangle_T = g_i^2 \left\langle \left(\sum_{j=1}^{N_{\mathrm{net}}} W_{ij} x_j + E_i \right)^2 \right\rangle_T \tag{4}$$

$$= g_i^2 \left\langle \left(\sum_{j=1}^{N_{\mathrm{net}}} W_{ij} x_j \right)^2 \right\rangle_T + g_i^2 E_i^2 \tag{5}$$

$$= g_i^2 \sum_{j,k=1}^{N_{\mathrm{net}}} W_{ij} W_{ik} \left\langle x_j x_k \right\rangle_T + g_i^2 E_i^2 \; . \tag{6}$$

If we assume that the system is in a chaotic state we can set $\left\langle x_j x_k \right\rangle_T = 0$ for $j \neq k$. This leads to

$$\left\langle x_i^2 \right\rangle_T = g_i^2 \left(\sum_{j=1}^{N_{\mathrm{net}}} W_{ij}^2 \left\langle x_j^2 \right\rangle_T + \sigma_{\mathrm{ext}}^2 \right) \tag{7}$$

where we have assumed $\left\langle E_i \right\rangle_T = 0$ for all i.

By design, our homeostatic mechanism fixes all $\left\langle x_i^2 \right\rangle_T$ to $\sigma_{\mathrm{target}}^2$. Thus,

$$\sigma_{\mathrm{target}}^2 = g_i^2 \left(\sigma_{\mathrm{target}}^2 \sum_{j=1}^{N_{\mathrm{net}}} W_{ij}^2 + \sigma_{\mathrm{ext}}^2 \right) \tag{8}$$

$$g_i = \left(\sum_{j=1}^{N_{\mathrm{net}}} W_{ij}^2 + \sigma_{\mathrm{ext}}^2 / \sigma_{\mathrm{target}}^2 \right)^{-1/2} \; . \tag{9}$$

Since W_{ij} is a random Gaussian matrix with variance $\sigma_{\mathrm{conn}}^2 / \left(N_{\mathrm{net}} c f_{\mathrm{net}} \right)$, $\sum_{j=1}^{N_{\mathrm{net}}} W_{ij}^2$ follows a χ^2 - distribution with variance $\frac{2 N_{\mathrm{net}} c f_{\mathrm{net}} \sigma_{\mathrm{conn}}^2}{N_{\mathrm{net}}^2 c f_{\mathrm{net}}^2} = \frac{2 \sigma_{\mathrm{conn}}^2}{N_{\mathrm{net}} c f_{\mathrm{net}}}$. For $N_{\mathrm{net}} \to \infty$, its variance vanishes and consequently, all g_i converge to the same value, namely

$$g = \left(\sigma_{\mathrm{conn}}^2 + \sigma_{\mathrm{ext}}^2 / \sigma_{\mathrm{target}}^2 \right)^{-1/2} \; . \tag{10}$$

This equation predicts that g should not change if the ratio between target and input variance remains constant. We ran a parameter sweep over σ_{ext} and σ_{target} with a network of $N_{\mathrm{net}} = 1000$ neurons and looked at the resulting distribution of gains and the maximal Lyapunov exponent. Importantly, this approximation suggests that the network should tune into a subcritical configuration for any non-vanishing external input. Even though this is not strictly verified in the numerical simulation, see Fig. 2A, it holds for the majority of $\sigma_{\mathrm{ext}} / \sigma_{\mathrm{target}}$ combinations.

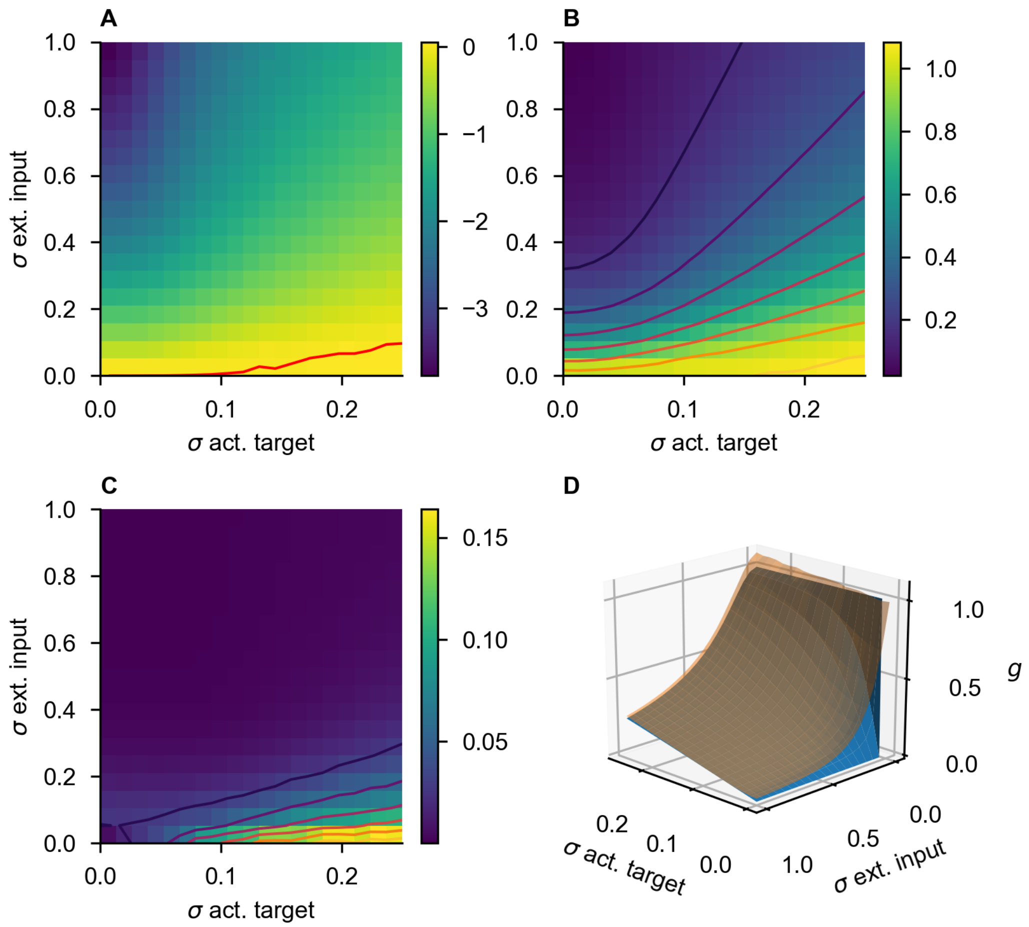

Figure 2: Parameter sweep, run on a network with $N_{\mathrm{net}} = 1000$. **A**: Log of largest absolute value of eigenvalues of $g_i W_{ij}$. Red line marks the zero transition. **B** : $\langle g_i \rangle_P$. **C**: $\left\langle (g_i - \langle g_i \rangle_P)^2 \right\rangle_P$. **D**: Prediction of (10) (blue) vs. numerical result (orange) of $\langle g_i \rangle_P$.