A New Algorithm to Solve PDE Models

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October 13, 2016

This note details a new algorithm to solve PDEs associated with economic models in continuous time.

1 Solving Finite Difference Schemes

1.1 How Ψtc works

Denote Y the solution and denote $F(Y) = 0$ the finite difference scheme corresponding to a model. On can solve for Y using one of the two methods:

1. Non linear solver for $F(Y) = 0$. Updates take the form

$$0 = F(y_t) + J_F(y_t)(y_{t+1} - y_t)$$

   The method converges only if the initial guess is sufficiently close to the solution.

2. ODE solver for $F(Y) = \dot{Y}$. The solution of $F(Y) = 0$ is obtained with $T \to +\infty$ With a simple explicit Euler method, updates take the form

$$0 = F(y_t) - \frac{1}{\Delta}(y_{t+1} - y_t)$$

   Convergence conditions are given by the Barles-Souganadis theorem. Explicit schemes usually don't satisfy them.

I propose to use a fully implicit Euler method, which has better convergence properties than explicit schemes. Updates take the form

$$\forall t \leq T \qquad 0 = F(y_{t+1}) - \frac{1}{\Delta}(y_{t+1} - y_t)$$

Each time step is a non linear equation, which I solve using a Newton-Raphson method. These inner iterations take the form

$$\forall i \leq I \qquad 0 = F(y_t^i) - \frac{1}{\Delta}(y_t^i - y_t) + (J_F(y_t^i) - \frac{1}{\Delta})(y_t^{i+1} - y_t^i)$$

*I thank Valentin Haddad and Ben Moll for useful discussions.

As pointed above, the Newton-Raphson method converges when y_t is sufficiently close to y_{t+1}. Therefore I decrease Δ until the inner Newton-Raphson method converges.[1].
I accomodate algebraic equations by setting $\Delta = 0$ for these equations. This ensures that the PDEs are solved backward on a path that always satisfies the algebraic constraints.

How does the method relate to the two algorithms seen above? When $I = 1$ (i.e. with only one inner iteration) the update can be seen as the sum of a Newton-Raphson and an explicit time step

$$\forall t \leq T \qquad 0 = F(y_{t+1})(J_F(y_t) - \frac{1}{\Delta})(y_{t+1} - y_t)$$

Alternatively, the method can be also be seen as a dampened Newton-Raphson algorithm. As in the Levenberg-Marquardt method, the diagonal of the Jacobian is modified until the algorithm gets close to the solution.

The algorithm usally converges quickly: the convergence is quadratic around the solution, since the algorithm follows the Newton-Rapshon method around the solution.

1.2 Related Methods

- With $I = 1$ and constant Δ, we obtain the method in Achdou, Han, Lasry, Lions (2016) for a partial equilibrium consumption / saving problem with separable preference. I've found that allowing Δ to change and a $I > 1$ are important for the robustness of the algorithm. Moreover, Ψtc handles systems with implicit jacobians and algebraic equations.
- A similar algorithm is used in Fluid Dynamics. In this context, it is called Pseudo-Transient Continuation (denoted Ψtc).

2 Writing Finite Difference Schemes

- Write the finite difference scheme so that the implicit Euler method satisfies the convergence conditions of Barles-Souganadis theorem (as much as possible). In particular,
  - Upwind first derivatives to make the scheme monotonous (for instance see Achdou, Han, Lasry, Lions (2016))
  - Write the function F so that $\dot{Y}$ would appear as such (i.e. not multiplied by some parameters). For instance, a typical PDE for the price dividend ratio should be written

$$0 = p(\frac{1}{p} + E[\frac{dD}{D}] + E[\frac{dp}{p}] + \sigma[\frac{dp}{dp}]\sigma[\frac{dD}{dD}] - r - \kappa(\sigma[\frac{dp}{dp}] + \sigma[\frac{dD}{D}]))$$

[1]However, I cannot prove the convergence of the overall scheme when Δ depends on the step. The Barles-Souganadis theorem ensures that the implicit Euler scheme converges only with Δ fixed.

- When solving for multiple functions, use the same economic quantities across the different equations. This ensures that the time step is comparable across different equations. For instance use the wealth / consumption of each agent in heterogeneous agent models.