Solving Economic Models in Continuous Time using the Pseudo-Transient Method

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This note details how to solve systems of PDEs + algebraic equations associated with economic models in continuous time.

Algorithm. I propose to solve the system corresponding to economic models using the Pseudo-Transient Continuation (denoted Ψtc) method, used in the fluid dynamics literature.

Formal conditions for the convergence of the algorithm are given in Kelley and Keyes (1998), but I now derive an intuition for the algorithm.

Denote Y the solution and denote $F(Y) = 0$ the finite difference scheme corresponding to a model. The existing literature solves for Y using using one of the two methods:

1. Non linear solver for $F(Y) = 0$. Updates take the form

$$0 = F(y_t) + J_F(y_t)(y_{t+1} - y_t)$$

   The method converges only if the initial guess is sufficiently close to the solution. [1]

2. ODE solver for $F(Y) = \dot{Y}$. The solution of $F(Y) = 0$ is obtained with $T \to +\infty$. [2] With a simple explicit Euler method, updates take the form

$$0 = F(y_t) - \frac{1}{\Delta}(y_{t+1} - y_t)$$

   Convergence conditions are given by the Barles-Souganadis theorem. Explicit schemes usually don't satisfy them.

Rather than one of these two methods, I propose to solve for Y using fully implicit Euler method. Updates take the form

$$\forall t \leq T \qquad 0 = F(y_{t+1}) - \frac{1}{\Delta}(y_{t+1} - y_t)$$

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[1] For instance, see Campbell and Cochrane (1999).

[2] For instance, see Di Tella (2016), Silva (2015).

Each time step is a non linear equation, which I solve using a Newton-Raphson method. These inner iterations take the form

$$\forall i \leq I \qquad 0 = F(y_t^i) - \frac{1}{\Delta}(y_t^i - y_t) + (J_F(y_t^i) - \frac{1}{\Delta})(y_t^{i+1} - y_t^i)$$

These inner iterations converge as long as y_t is sufficiently close to y_{t+1}. Therefore I decrease Δ until the inner Newton-Raphson method converges.

How does the method relate to the two algorithms seen above? When $I = 1$ (i.e. with only one inner iteration) the update can be seen as the sum of a Newton-Raphson and an explicit time step

$$\forall t \leq T \qquad 0 = F(y_{t+1}) + (J_F(y_t) - \frac{1}{\Delta})(y_{t+1} - y_t)$$

The step becomes close to a pure Newton-Rapshon step as $\Delta \to +\infty$. Therefore, the convergence is quadratic around the solution.

I accommodate algebraic equations by setting Δ to $+\infty$ for the coordinates of F that correspond to algebraic equations. This ensures that the PDEs are solved backward on a path that always satisfies the algebraic constraints.

Relation with other methods. The algorithm with $I = 1$ and Δ constant corresponds to Achdou et al. (2016), an algorithm for PE models. Allowing $I > 1$ and adjusting Δ are important to ensure convergence in GE models.

Writing Finite Difference Schemes. I now give some heuristics to write correctly the finite difference scheme F given the system of PDEs characterizing some economic model.

The goal is to write the function F so that the implicit Euler method satisfies the convergence conditions of Barles-Souganadis theorem as much as possible. In particular,

- Upwind first derivatives to make the scheme monotonous (for instance see Achdou et al. (2016))
- Write the function F so that $\dot{Y}$ would appear as such (i.e. not multiplied by some parameters). For instance, a typical PDE for the price dividend ratio should be written

$$0 = p(\frac{1}{p} + E[\frac{dD}{D}] + E[\frac{dp}{p}] + \sigma[\frac{dp}{dp}]\sigma[\frac{dD}{dD}] - r - \kappa(\sigma[\frac{dp}{dp}] + \sigma[\frac{dD}{D}]))$$

- When solving a system of PDEs, use the same economic quantities across the different equations. This ensures that the time step is comparable across different equations. For instance use the wealth / consumption of each agent in heterogeneous agent models.

References

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