Solving Economic Models in Continuous Time using the Pseudo-Transient Method

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This note details how to solve systems of PDEs + algebraic equations associated with economic models in continuous time.

Intuition. Denote Y the solution and denote $F(Y)$ the finite difference scheme corresponding to a model. The goal is to find Y such that $F(Y) = 0$ The existing literature in economics solves for Y using using one of the two methods:

1. Non linear solver for $F(Y) = 0$. A Newton-Raphson update takes the form
$$0 = F(y_t) + J_F(y_t)(y_{t+1} - y_t) \tag{1}$$
The method converges only if the initial guess is sufficiently close to the solution.[1]

2. ODE solver for $F(Y) = \dot{Y}$. The solution of $F(Y) = 0$ is obtained with $T \to +\infty$. [2] With a simple explicit Euler method, updates take the form
$$0 = F(y_t) - \frac{1}{\Delta}(y_{t+1} - y_t) \tag{2}$$
Convergence conditions are given by the Barles-Souganadis theorem. Explicit schemes usually don't satisfy them.

Rather than one of these two methods, I propose to solve for Y using fully implicit Euler method. Updates now take the form

$$\forall t \leq T \qquad 0 = F(y_{t+1}) - \frac{1}{\Delta}(y_{t+1} - y_t)$$

Each time step now requires to solve a non linear equation. I solve this non linear equation using a Newton-Raphson method. These inner iterations take the form

$$\forall i \leq I \qquad 0 = F(y_t^i) - \frac{1}{\Delta}(y_t^i - y_t) + (J_F(y_t^i) - \frac{1}{\Delta})(y_t^{i+1} - y_t^i) \tag{3}$$

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[1] See, for instance, Campbell and Cochrane (1999).

[2] See, for instance, Di Tella (2016), Silva (2015).

We know that the Newton-Raphson method converges if the initial guess is close enough to the solution. Since y_t converges towards y_{t+1} as Δ tends to zero, one can always choose Δ low enough so that the inner steps converge.

I adjust Δ as follows. If the inner iterations do not converge, I decrease Δ. When the inner iteration converges, I increase Δ.

The update Equation (3) can be see as weighted average of the Newton-Raphson step Equation (1) and of the explicit Euler step Equation (2). After a few sucessful implicit time steps, Δ is large and therefore the algorithm becomes like Newton-Rapshon. In particular, the convergence is quadratic around the solution.

I accommodate algebraic equations by setting Δ to $+\infty$ along these coordinates.

Relation with other methods. The method actually corresponds to a method used in he fluid dynamics literature. In this context, it is called the Pseudo-Transient Continuation method, and is denoted Ψtc. Formal conditions for the convergence of the algorithm are given in Kelley and Keyes (1998).

The algorithm with $I = 1$ and Δ constant corresponds to Achdou et al. (2016). They prove the convergence of this algorithm for models in partial equilibrium. Allowing $I > 1$ and adjusting Δ are important to ensure convergence in general equilibrium, which are non linear.

Writing Finite Difference Schemes. It is important to write correctly the finite difference scheme F. A good heuristic is to write it so that the implicit Euler method satisfies the convergence conditions of Barles-Souganadis theorem. In particular,

- Upwind first derivatives to make the scheme monotonous (for instance see Achdou et al. (2016))
- Write each PDE as a no arbitrage condition for a particular asset. Denoting p the price dividend ratio:

$$0 = p(\frac{1}{p} + E[\frac{dD}{D}] + E[\frac{dp}{p}] + \sigma[\frac{dp}{dp}]\sigma[\frac{dD}{dD}] - r - \kappa(\sigma[\frac{dp}{dp}] + \sigma[\frac{dD}{D}]))$$

References

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