6.867: Exercises (Week 7)

October 27, 2017

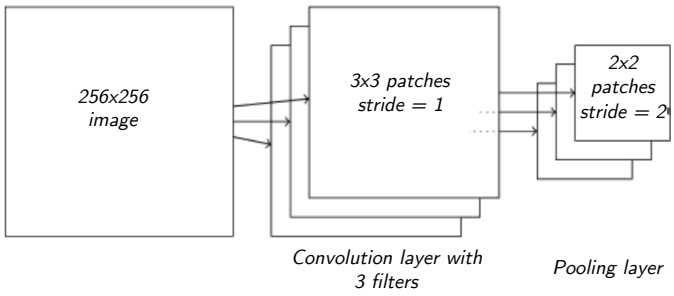
Contents

1 ConvNets

Adapted from UofT CSC321 Final 2017.

Suppose you have a convolutional network with the following architecture:

- The input is an image of size 256×256.
- The first layer is a convolution layer with 3 filters with patches of size 3×3. It uses a stride of 1 and zero padding with 1 pixel border (i.e. adding one row and column to the borders of the image with zero values so the new size is 258 * 258).
- The next layer is a max pooling layer with a stride of 2 and 2×2 filters.



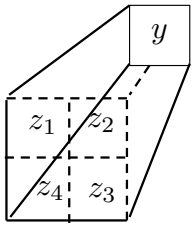
(a) Determine the size of the receptive field for a single unit in the pooling layer. (i.e., determine the size of the region of the input image which influences the activation of that unit.) You may assume the receptive field lies entirely within the image. Hint: you may want to draw a one-dimensional conv net to reason about this problem.

(b) How many units and parameters are there in the convolution layer?

(c) How many units are there in the max pooling layer?

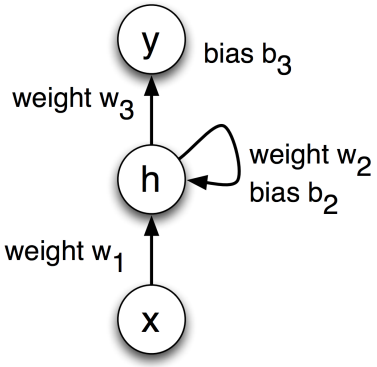
(d) What is the derivative of the output of the max pooling layer with respect to its inputs?

$$y = \max\{z_1, z_2, z_3, z_4\}$$



2 Mystery RNN

Adapted from UofT CSC321 Winter 2015.



You are given an RNN with the architecture shown in the figure above where all inputs and outputs are binary valued (ie. $x, y \in \{0, 1\}$). You observe that it initially outputs 1, but as soon as it gets an input of 0, it switches to outputting 0 for all subsequent time steps. For example, the sequence of inputs 1110101 produced the outputs 1110000. You know that the hidden unit has an initial value of 0 and that it uses the binary threshold function for its non-linearity, which is defined as follows:

$$f(z) = \begin{cases} 1, & z > 0 \\ 0, & otherwise. \end{cases}$$

The output unit is linear. The network can be described by the following set of equations where x_t is the input and y_t is the output at time t and $f()$ is the binary threshold function:

$$h_t = f(w_1 x_t + w_2 h_{t-1} + b_2)$$
$$y_t = w_3 h_t + b_3$$

Provide a set of values for the weights and biases that this network could be using (there is more than one correct answer).

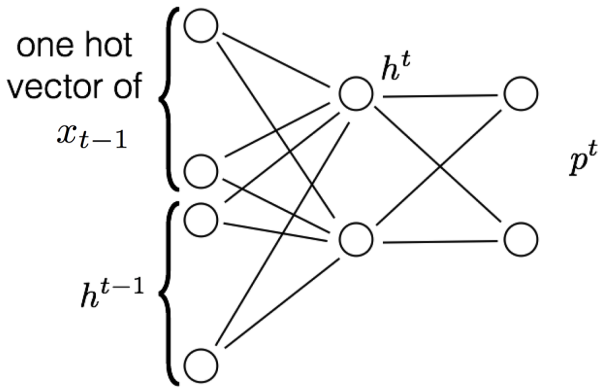
3 RNNs For Language Models

Adapted from MIT 6.864 Fall 2015

Language models often make use of n-gram models, which mean the probability of a given word in a sequence is only dependent on the previous n words. For example, a bi-gram model would model the probability of a given word at position t of a sequence as $p(x_t|x_{t-1})$ whereas a tri-gram model would be $p(x_t|x_{t-1}, x_{t-2})$. In this question we examine the use of RNNs for language models.

<start>	This	is	a	very	very	unimaginative	example	sentence
t=0	t=1	t=2	t=3	t=4	t=5	t=6	t=7	t=8

(a) Consider the example sentence shown above. Is it possible for a bi-gram model to assign a likelihood of 1 to this sentence? Explain.



The figure above shows the architecture for an RNN that we will try to use for language modelling. The hidden state h^t consists of m units with a binary threshold activation function defined as follows:

$$f(z) = \begin{cases} 1, & z > 0 \\ 0, & otherwise. \end{cases}$$

The initial hidden state h^0 is always $h^0_1 = 1$ and $h^0_j = 0$ for $j = 2, \ldots, m$. The output layer is linear followed by a softmax to represent the probabilities over the next word x_t. You can assume all bias terms are -1.

(b) Suppose we eliminate h^{t-1} as input to the network. Is it still possible to set the remaining parameters of the network to assign a likelihood of 1 to the example sentence above? Explain.

(c) Suppose we eliminate x_{t-1} as input to the network. Is it still possible to set the remaining parameters of the network to assign a likelihood of 1 to the example sentence above? Explain.

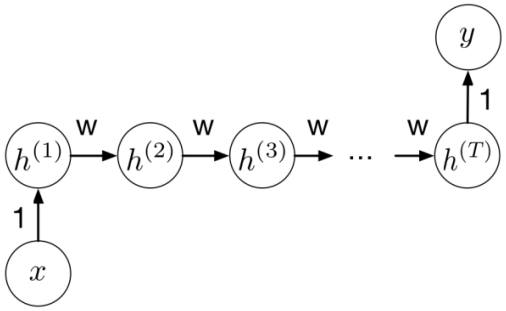
The last few parts will consider the whole network (with both h^{t-1} and x_{t-1} as inputs) and show how it can be used to exactly replicate a tri-gram model.

(d) Why is having h^t be the vector corresponding to the concatenation of one hot vectors for x_{t-1} and x_{t-2} insufficient for producing a tri-gram model?

(e) What is a possible vector representation of h^t that would allow the network to replicate a tri-gram model?

(f) What is a possible parameterization of the network that results in this hidden state representation and the network replicating a tri-gram model?

4 Exploding RNN



Consider the following RNN, which has a scalar input at the first time step, makes a scalar prediction at the last time step, and uses a shifted logistic activation function:

$$\phi(z) = \sigma(z) - 0.5$$

(a) Let z^t denote the input to the activation function at time t. Write the formula for the derivative of the loss $\frac{\delta l}{\delta h^{(t)}}$ in terms of $\frac{\delta l}{\delta h^{(t+1)}}$ for $t < T$.

(b) Suppose the input to the network is $x = 0$. Notice that $h^{(t)} = 0$ for all t. Based on your answer to the previous part, determine the value α such that if $w < \alpha$, the gradient vanishes, while if $w > \alpha$, the gradient explodes. You may use the fact that $\sigma'(0) = 0.25$.

(c) Suppose now that we make this RNN a residual network. Instead of $h^{(t+1)} = \phi(z^t)$, we will connect the input of a recurrent unit to its output, so that we now have

$$h^{(t+1)} = \phi(z^t) + h^t$$

Give a setting of w such that the new gradient does not vanish or explode.

5 More Exploding RNN

Assume we have an RNN with T hidden layers and activation function ϕ. Let h_t and h_T be the hidden unit vectors for hidden layers T and t of the RNN. Assume $t \ll T$. We have the following:

$$z_{t+1} = \mathbf{W}_t h_t + \mathbf{V}_t x_{t+1}$$

$$h_{t+1} = \phi(z_{t+1})$$

(a) Write the derivative of the loss l with respect to h_t in terms of $\delta l/\delta h_T$, $\phi'(z)$ and $\mathbf{W}$ for each layer.

(b) Let's look at the norm of $\delta l/\delta h_t$. We define the norm of a vector to be the usual L2-norm, and the norm of a matrix to be the operator 2-norm. Using the derivative from above, come up with an upper bound for $\|\delta l/\delta h_t\|$. Hint: use the facts that for matrix $\mathbf{A}$ and vector v, $\|\mathbf{A}v\| \leqslant \|\mathbf{A}\|\,\|v\|$.

$$\left\|\frac{\delta l}{\delta h_t}\right\| = \left\|\frac{\delta l}{\delta h_T}\prod_{k=t}^{T-1}\mathbf{D}_{k+1}\mathbf{W}_k^T\right\|$$

$$\leqslant \left\| \frac{\delta l}{\delta h_T} \right\| \prod_{k=t}^{T-1} \left\| \mathbf{D}_{k+1} \mathbf{W}_k^T \right\|$$

(c) If we are looking to prevent exploding or vanishing gradients, what might be a useful property of the weight matrices $\mathbf{W}$? Assuming the weights indeed satisfy this property, simplify the bound given for $\|\delta l/\delta h_t\|$ in the previous part. Hint: if $\mathbf{A}$ is orthogonal, we have $\|\mathbf{AB}\| = \|B\|$. (Recall that an orthogonal matrix is a square matrix whose columns and rows are perpendicular to each other and have norm 1. For orthogonal matrix $\mathbf{A}$, $\mathbf{A}^T\mathbf{A} = \mathbf{I}$.)

(d) Notice that the values of the matrices $\mathbf{D}$ now determine whether or not the gradients explode or vanish. Based on this, why might the ReLU be a good choice for the activation function?

6 Multi-task learning

You come across an exciting data set consisting of a million images of cats. Each image is annotated with (i) the cat's *breed*, (ii) the cat's *gender*, and (iii) the *time* of the day when the picture was taken. You set out to test how predictive these target variables are given the image using deep CNNs. This can be viewed as a *multi-task learning* problem where predicting each target is considered a separate task. You consider the following two approaches:

- Approach 1: Train a separate network for each target variable.
- Approach 2: Train a single network that jointly predicts different target variables based on the last hidden layer. Everything up to the last hidden layer is shared.

(a) You are informed by a cat expert that both (i) and (ii) can be accurately determined based on a common set of morphological features of a cat. Which of the two approaches is preferable for predicting (i) and (ii) together? Why?

(b) Assume the features that were useful for (i) and (ii) contain no predictive signal for (iii) and vice versa. With proper regularization, which approach is preferable for predicting (i) and (iii) together and why? What could happen if we are in the regime of overfitting?

(c) You hypothesize that the first two hidden layers of a network trained for any of the three tasks capture rudimentary image features (e.g., edges, brightness, and contrast) that are useful for any task. Assuming this is indeed the case, find a compromise between Approaches 1 and 2 to design a *single* network for predicting all three target variables together, using the insights from (a) and (b).

(d) You found another data set containing a million images of *dogs*. Each image is annotated with the dog's *breed*, which we would like to predict. How would you modify your model in (c) to simultaneously tackle this fourth task, which is defined over a whole new data set? Briefly describe how the new model would be trained.