EJEMPLO (por partes recurrencia)

1) Sea

$I_n = \int_0^1 (3x)^n e^{x/2} dx$ para $n \geq 0$.

a) Verifique que para $n \geq 1$, se cumple:

$$I_n = 2(3)^n e^{1/2} - 6n I_{n-1}$$

$\int e^{kx} dx = \frac{e^{kx}}{k}$

b) Halle I_2

Sol.

Por partes

$u = (3x)^n$

$du = n(3x)^{n-1} \cdot 3dx$

$du = 3n(3x)^{n-1} dx$

$dv = e^{x/2} dx$

$v = \int e^{x/2} dx$

$v = 2e^{x/2}$

$I_n = \int_0^1 (3x)^n e^{x/2} dx$

$I_n = uv - \int v du$

$I_n = (3x)^n \cdot 2e^{x/2} - \int 2e^{x/2} 3n (3x)^{n-1} dx$

$I_n = 2(3x)^n e^{x/2} - 6n \int e^{x/2} (3x)^{n-1} dx$

$I_n = \left[2(3x)^n e^{x/2}\right|_0^1 - 6n \int_0^1 e^{x/2} (3x)^{n-1} dx$

$I_n = 2(3)^n e^{1/2} - 2\left((3(0))^n e^{0/2}\right) - 6n \int_0^1 e^{x/2} (3x)^{n-1} dx$

$I_n = 2(3)^n e^{1/2} - 6n \underbrace{\int_0^1 e^{x/2} (3x)^{n-1} dx}_{I_{n-1}}$

$\boxed{I_n = 2(3)^n e^{1/2} - 6n \cdot I_{n-1}}$ Fórmula recursiva

b) Hallando I_2

i) Si $n = 0$ en $I_n = \int_0^1 (3x)^n e^{x/2} dx$

$\Rightarrow I_0 = \int_0^1 (3x)^0 e^{x/2} dx$

$\Rightarrow I_0 = \int_0^1 e^{x/2}$

$\Rightarrow I_0 = \left[2e^{x/2}\right|_0^1$

$$\Rightarrow \quad I_0 = 2\left[e^{1/2} - e^0\right]$$

$$\Rightarrow \quad \boxed{I_0 = 2\left[e^{1/2} - 1\right]}$$

ii) Si $n=1$ en $\boxed{I_n = 2(3)^n e^{1/2} - 6n I_{n-1}}$

$$\Rightarrow \quad I_1 = 6e^{1/2} - 6I_0$$

$$\Rightarrow \quad I_1 = 6e^{1/2} - 6\left[2e^{1/2} - 2\right]$$

$$\Rightarrow \quad \boxed{I_1 = -6e^{1/2} + 12}$$

iii) si $n=2$ en $\boxed{I_n = 2(3)^n e^{1/2} - 6n I_{n-1}}$

$$I_2 = 2(3)^2 e^{1/2} - 6(2) I_1$$

$$I_2 = 18e^{1/2} - 12\left[-6e^{1/2} + 12\right]$$

$$I_2 = 18e^{1/2} + 72e^{1/2} - 144$$

$$\boxed{I_2 = 90e^{1/2} - 144}$$

2) Sea $I_n = \int_0^2 (2+3x)^n \cdot e^{-4x} dx$ para $n \geq 0$. Halle I_3

Sol.

i) Hallando la fórmula recursiva:

Por partes

$u = (2+3x)^n$

$du = n(2+3x)^{n-1} \cdot 3\,dx$

$du = 3n\,(2+3x)^{n-1} dx$

$dv = e^{-4x} dx$

$v = \int e^{-4x} dx$

$v = -\dfrac{e^{-4x}}{4}$

$$I_n = \int_0^2 (2+3x)^n e^{-4x} dx$$

$$I_n = uv - \int v\,du$$

$$I_n = (2+3x)^n \left(-\frac{e^{-4x}}{4}\right) - \int -\frac{e^{-4x}}{4} \cdot 3n\,(2+3x)^{n-1} dx$$

$$I_n = -\frac{1}{4}(2+3x)^n e^{-4x} + \frac{3n}{4} \int (2+3x)^{n-1} e^{-4x} dx$$

$$I_n = \left[-\frac{1}{4}(2+3x)^n e^{-4x}\Big|_0^2 + \frac{3n}{4}\int_0^2 (2+3x)^{n-1} e^{-4x}dx\right.$$

$$I_n = -\frac{1}{4}(2+3(2))^n e^{-4(2)} - \frac{1}{4}(2+3(0))^n e^{-4(0)} + \frac{3n}{4}\int_0^2 (2+3x)^{n-1} e^{-4x}dx$$

$$I_n = -\frac{1}{4}(8)^n e^{-8} - \frac{1}{4}(2)^n + \frac{3n}{4}\underbrace{\int_0^2 (2+3x)^{n-1} e^{-4x}dx}_{I_{n-1}}$$

$$\boxed{I_n = -\frac{1}{4}(8)^n e^{-8} - \frac{1}{4}(2)^n + \frac{3n}{4}.I_{n-1}}$$ fórmula recursiva para $n \geq 1$

ii) si $n=0$ en $I_n = \int_0^2 (2+3x)^n e^{-4x}dx$

$$\Rightarrow I_0 = \int_0^2 (2+3x)^0 e^{-4x}dx$$

$$\Rightarrow I_0 = \int_0^2 e^{-4x}dx$$

$$\Rightarrow I_0 = \left[-\frac{e^{-4x}}{4}\Big|_0^2\right. \quad \Rightarrow I_0 = -\frac{1}{4}\left[e^{-4(2)} - e^{-4(0)}\right]$$

$$\Rightarrow \boxed{I_0 = -\frac{1}{4}\left[e^{-8} - 1\right]}$$

iii) si $n=1$ en $I_n = -\frac{1}{4}(8)^n e^{-8} - \frac{1}{4}(2)^n + \frac{3n}{4}.I_{n-1}$

$$\Rightarrow I_1 = -\frac{1}{4}(8)^1 e^{-8} - \frac{1}{4}(2)^1 + \frac{3}{4}(1).I_0$$

$$\Rightarrow I_1 = -2e^{-8} - \frac{1}{2} + \frac{3}{4}\left[-\frac{1}{4}(e^{-8}-1)\right]$$

$$\Rightarrow I_1 = -2e^{-8} - \frac{1}{2} - \frac{3}{16}e^{-8} + \frac{3}{16}$$

$$\Rightarrow \boxed{I_1 = -\frac{35}{16}.e^{-8} - \frac{5}{16}}$$

iv) si $n=2$ en $I_n = -\frac{1}{4}(8)^n e^{-8} - \frac{1}{4}(2)^n + \frac{3n}{4}.I_{n-1}$

$$\Rightarrow I_2 = -\frac{1}{4}(8)^2 e^{-8} - \frac{1}{4}(2)^2 + \frac{3}{4}(2).I_1$$

$$\Rightarrow I_2 = -16e^{-8} - 1 + \frac{3}{2}\left[-\frac{35}{16}e^{-8} - \frac{5}{16}\right]$$

$$\Rightarrow I_2 = -16e^{-8} - 1 - \frac{105}{32}e^{-8} - \frac{15}{32}$$

$$\Rightarrow \boxed{I_2 = -\frac{617}{32}e^{-8} - \frac{47}{32}}$$

γ) si $n = 3$ en $I_n = -\frac{1}{4}(8)^n e^{-8} - \frac{1}{4}(2)^n + \frac{3n}{4} \cdot I_{n-1}$

$$\Rightarrow I_3 = -\frac{1}{4}(8)^3 e^{-8} - \frac{1}{4}(2)^3 + \frac{3}{4}(3) \cdot I_2$$

$$\Rightarrow I_3 = -128e^{-8} - 2 + \frac{9}{4}\left[-\frac{617}{32}e^{-8} - \frac{47}{32}\right]$$

$$\Rightarrow I_3 = -128e^{-8} - 2 - \frac{5553}{128} \cdot e^{-8} - \frac{423}{128}$$

$$\Rightarrow \boxed{I_3 = \frac{-21937}{128}e^{-8} - \frac{679}{128}}$$