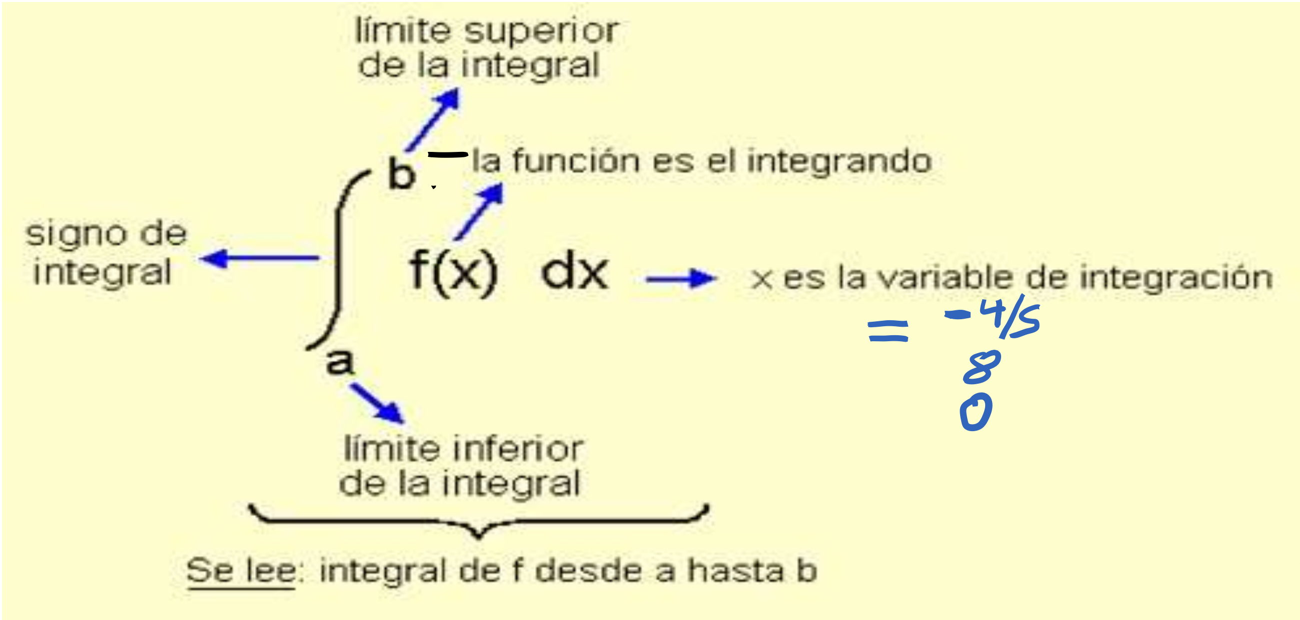
LA INTEGRAL DEFINIDA



$$\int (x^2+3x-2)\,dx$$

$$\int_2^4 (x^2+3x-2)\,dx \rightarrow f(x)=x^2+3x-2$$

está definida en $[2,4]$

$\therefore$ f es continua en $[2,4]$

$$\int_{-3}^{3} \frac{3}{x-2}\,dx$$

la función $f(x)=\frac{3}{x-2}$

no está definida en $x=2\in[-3,3]$

$\therefore$ f no es continua en $[-3,3]$

$\boxed{\frac{5}{0}}$

no está definido

$\frac{0}{0}$

$\frac{0}{8}=0$

$$\int_0^4 \ln(3-x)$$

$$\int_0^2 \frac{4x+2}{x+1}\,dx \;;\; f(x)=\frac{4x+2}{x+1}$$

no está definida en $x=-1\notin[0,2]$

$\therefore$ f es continua en $[0,2]$

PROPIEDADES DE LA INTEGRAL DEFINIDA

Dadas la funciones f y g integrables en $[a,b]$ y $k \in \Re$.

1. $\int_a^b k\ f(x)dx = k\int_a^b f(x)dx$

2. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

3. $\int_{-a}^{a} f(x)dx = 0$, si $f(x)$ es impar

Función impar

$f(-x) = -f(x)$

Por ejemplo:

$$\int_{-2}^{2} (x^3 + 2x)dx = 0$$

Por ejemplo

$f(x) = x^3 + 2x$

si $f(-x) = (-x)^3 + 2(-x)$

$= -x^3 - 2x$

$= -(x^3 + 2x)$

$= -f(x)$

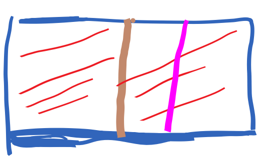
$\therefore$ f es impar

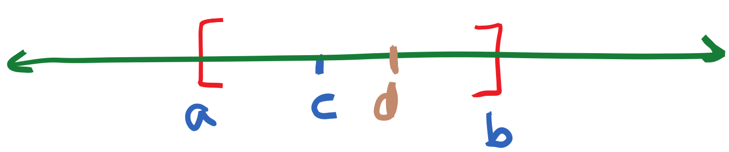
4. $\int_a^b f(x)dx = -\int_b^a f(x)dx$, si $a > b$.

Por ejemplo:

$$\int_3^0 (x^2 - 3x)dx = -\int_0^3 (x^2 - 3x)dx$$

5. Si c es un punto interior de $[a,b]$ entonces: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

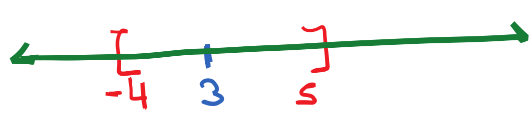




$$f(x) = \begin{cases} x^2+3, & x>2 \\ \text{Sen}x; & x<3 \end{cases}$$

Por ejemplo

Sea $3 \in [-4, 5] \rightarrow$



$$\Rightarrow \int_{-4}^{5} \sqrt{3x^2+2}\,dx = \int_{-4}^{3} \sqrt{3x^2+2}\,dx + \int_3^5 \sqrt{3x^2+2}\,dx$$

TEOREMA FUNDAMENTAL DEL CÁLCULO

Sea f una función continua en el intervalo $[a,b]$. y sea F una función tal que $F'(x) = f(x) \quad \forall \in [a,b]$ entonces

$$\int_a^b f(x)dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$$

Por ejemplo :

$$\int_0^1 (x^2+2)dx = \left[\frac{x^3}{3}+2x\right|_0^1$$

$$= \frac{(1)^3}{3}+2(1) - \left(\frac{(0)^3}{3}+2(0)\right)$$

$$= \frac{7}{3}$$

EJEMPLO (Cambio de variable)

1) Dada la integral $I=\int_{-1}^{4}(5a\,x+1)\sqrt{x+5}\,dx$, determinar el valor de la constante a si: $\frac{1}{2}(I-9)=15$.

Sol

i) $I=\int_{-1}^{4}(5a\,x+1)\sqrt{x+5}\,dx$

Cambio de variable

$z=\sqrt{x+5}$

$z^2=x+5$

$x=z^2-5$

$dx=2z\,dz$

*) Si $x=-1$ en $z=\sqrt{x+5}$

$\Rightarrow z=\sqrt{4}$

$\Rightarrow \boxed{z=2}$

*) Si $x=4$ en $z=\sqrt{x+5}$

$\Rightarrow z=\sqrt{9}$

$\Rightarrow \boxed{z=3}$

$$\Rightarrow I=\int_{2}^{3}\left[5a(z^2-5)+1\right]z\,2z\,dz$$

$$\Rightarrow I=2\int_{2}^{3}\left[5az^2-25a+1\right]z^2\,dz$$

$$I=2\int_{2}^{3}\left[5az^4-25az^2+z^2\right]dz$$

$$I=2\left[\cancel{5}a\,\frac{z^5}{\cancel{5}}-25a\cdot\frac{z^3}{3}+\frac{z^3}{3}\right|_{2}^{3}$$

$$I=2\left[a(243)-25a(9)+9-\left(a(32)-25a\left(\frac{8}{3}\right)+\frac{8}{3}\right)\right]$$

$$I=2\left[a(\sqrt{x+5})^5-\frac{25a}{3}(\sqrt{x+5})^3+\frac{(\sqrt{x+5})^3}{3}\right|_{-1}^{4}$$

$$I=2\left[\frac{158}{3}a+\frac{19}{3}\right]$$

ii) Hallando $a=?$

Si $\frac{1}{2}(I-9)=15$

$\Rightarrow I-9=30$

$\Rightarrow I=39$

$\Rightarrow 2\left[\frac{158}{3}a+\frac{19}{3}\right]=39$

$\Rightarrow \frac{158}{3}a+\frac{19}{3}=\frac{39}{2}$

$\Rightarrow \frac{158}{3}a=\frac{39}{2}-\frac{19}{3}$

$\Rightarrow \frac{158}{3}a=\frac{79}{6}$

$\Rightarrow a=\frac{3}{158}\cdot\frac{79}{6}$

Luego, $\boxed{a=1/4}$

2) Dada la integral $I=\int_1^{17}\left(5x-\frac{3}{2}\cdot z\right)\sqrt{26-x}\,dx$, determinar el valor de la constante z si: $3(I-277)=7063$.

Sol

$$I=\int_1^{17}\left(5x-\frac{3}{2}z\right)\sqrt{26-x}\,dx$$

Cambio variable

$$\boxed{w=\sqrt{26-x}}$$
$$w^2=26-x$$
$$\boxed{x=26-w^2}$$
$$dx=(0-2w)dw$$
$$\boxed{dx=-2w\,dw}$$

*) Si $x=1$ en $w=\sqrt{26-x}$
$$w=\sqrt{26-1}$$
$$\boxed{w=5}$$

*) Si $x=17$ en $w=\sqrt{26-x}$
$$w=\sqrt{26-17}$$
$$w=\sqrt{9}$$
$$\boxed{w=3}$$

$$I=\int_5^3\left[5(26-w^2)-\frac{3}{2}z\right]w(-2w)dw$$

$$I=-2\int_5^3\left[130-5w^2-\frac{3}{2}z\right]w^2dw$$

$$I=-2\int_5^3\left[130w^2-5w^4-\frac{3}{2}zw^2\right]dw$$

$$I=-2\left[130\frac{w^3}{3}-\cancel{5}\frac{w^5}{\cancel{5}}-\frac{\cancel{3}}{2}z\frac{w^3}{\cancel{3}}\right|_5^3$$

$$I=-2\left[\frac{130}{3}w^3-w^5-\frac{z}{2}w^3\right|_5^3$$

$$I=-2\left[\frac{130}{3}(3)^3-(3)^5-\frac{z}{2}(3)^3-\left(\frac{130}{3}(5)^3-(5)^5-\frac{z}{2}(5)^3\right)\right]$$

$$I=-2\left[927-\frac{27}{2}z-\left(\frac{6875}{3}-\frac{125}{2}z\right)\right]$$

$$\boxed{I=-2\left[-\frac{4094}{3}+49z\right]}$$

Por dato: $3(I-277)=7063$

$$\Rightarrow I-277=\frac{7063}{3}$$
$$\Rightarrow I=\frac{7063}{3}+277$$
$$\Rightarrow I=\frac{7894}{3}$$

$$\Rightarrow -\cancel{2}\left[-\frac{4094}{3}+49z\right]=\frac{\cancel{7894}\ 3947}{3}$$
$$\Rightarrow -\left[-\frac{4094}{3}+49z\right]=\frac{3947}{3}$$
$$\Rightarrow \frac{4094}{3}-49z=\frac{3947}{3}$$
$$\Rightarrow \frac{4094}{3}-\frac{3947}{3}=49z$$
$$\Rightarrow 49=49z$$
$$\Rightarrow \boxed{z=1}$$

3) Al realizar el cambio de variable $z^3 = 1 - x^2$ en $I = \int_{-3}^{0} x^5 \cdot \left(\sqrt[3]{1-x^2}\right)^2 dx$

y $A = \frac{880}{169371} I + 3$, se determina:

Sol

$$I = \int_{-3}^{0} x^2 \cdot x^2 \left(\sqrt[3]{1-x^2}\right)^2 x\,dx$$

Cambio de variable

$$\boxed{z^3 = 1 - x^2}$$

$$\boxed{x^2 = 1 - z^3}$$

$$2x\,dx = -3z^2 dz$$

$$\boxed{x\,dx = -\frac{3z^2}{2} dz}$$

*) Si $x = -3$ en $z^3 = 1 - x^2$

$\Rightarrow z = \sqrt[3]{1-(-3)^2}$

$\Rightarrow z = \sqrt[3]{-8}$

$\Rightarrow \boxed{z = -2}$

*) Si $x = 0 \Rightarrow z^3 = 1 - x^2$

$\Rightarrow z = \sqrt[3]{1-0}$

$\boxed{z = 1}$

$$I = \int_{-2}^{1} (1-z^3)(1-z^3)\left(\sqrt[\cancel{3}]{z^{\cancel{3}}}\right)^2 \left(-\frac{3z^2}{2}\right) dz$$

$$I = -\frac{3}{2}\int_{-2}^{1} \Big(1 \underbrace{-z^3 - z^3}_{-2z^3} + z^6\Big) \cdot \underbrace{z^2 \cdot z^2}_{z^4} dz$$

$$I = -\frac{3}{2}\int_{-2}^{1} \left(z^4 - 2z^7 + z^{10}\right) dz$$

$$I = -\frac{3}{2}\left[\frac{z^5}{5} - \frac{\cancel{2}^{1} z^8}{\cancel{8}_{4}} + \frac{z^{11}}{11}\right]_{-2}^{1}$$

$$I = -\frac{3}{2}\left[\frac{1}{5} - \frac{1}{4} + \frac{1}{11} - \left(\frac{(-2)^5}{5} - \frac{1}{4}(-2)^8 + \frac{(-2)^{11}}{11}\right)\right]$$

$$I = -\frac{3}{2}\left[\frac{9}{220} - \left(-\frac{14112}{55}\right)\right]$$

$$I = -\frac{3}{2}\left[\frac{56457}{220}\right]$$

$$\boxed{I = -\frac{169371}{440}}$$

Por dato:

$$A = \frac{880}{169371} \cdot I + 3$$

$$A = \frac{\cancel{880}^{2}}{\cancel{169371}} \cdot \left(-\frac{\cancel{169371}}{\cancel{440}_{1}}\right) + 3$$

$$A = (2)(-1) + 3$$

$$\boxed{A = 1}$$

2) Calcular la siguiente integral $I = \int_{-\sqrt[9]{3}}^{-1} x^{26}\left(\sqrt[7]{x^9+2}\right)^2 dx$ mediante el cambio de variable

$$z^9 = x^9 + 2$$

3) Calcular la siguiente integral $I = \int_{\sqrt[8]{247}}^{\sqrt[8]{6552}} \frac{x^{15}}{\left(\sqrt[8]{x^8+9}\right)^3} dx$ mediante el cambio de variable

$$z^8 = x^8 + 9$$

Sol

$$z^8 = x^8 + 9$$

$$\Rightarrow x^8 = z^8 - 9$$

$$\Rightarrow \cancel{8}x^7 dx = \cancel{8}z^7 dz$$

$$\Rightarrow x^7 dx = z^7 dz$$

*) Si $x = \sqrt[8]{247}$ en $z^8 = x^8 + 9$

$$\Rightarrow z^8 = \left(\sqrt[\cancel{8}]{247}\right)^{\cancel{8}} + 9$$

$$\Rightarrow z^8 = 256$$

$$\Rightarrow z = \sqrt[8]{256}$$

$$\Rightarrow \boxed{z = 2}$$

*) Si $x = \sqrt[8]{6552}$ en $z^8 = x^8 + 9$

$$\Rightarrow z = \sqrt[8]{x^8 + 9}$$

$$\Rightarrow z = \sqrt[8]{\left(\sqrt[\cancel{8}]{6552}\right)^{\cancel{8}} + 9}$$

$$\Rightarrow z = \sqrt[8]{6552 + 9}$$

$$\Rightarrow z = \sqrt[8]{6561}$$

$$\Rightarrow \boxed{z = 3}$$

$$\Rightarrow I = \int_{\sqrt[8]{247}}^{\sqrt[8]{6552}} \frac{x^{15}}{\left(\sqrt[8]{x^8+9}\right)^3} dx$$

$$I = \int \frac{x^8}{\left(\sqrt[8]{x^8+9}\right)^3} \cdot x^7 dx$$

$$I = \int_2^3 \frac{(z^8 - 9)}{\left(\sqrt[\cancel{8}]{z^{\cancel{8}}}\right)^3} \cdot z^7 dz$$

$$I = \int_2^3 \frac{(z^8 - 9)}{\cancel{z^3}} \cdot z^{\cancel{7}} dz$$

$$I = \int_2^3 (z^8 - 9) z^4 dz$$

$$I = \int_2^3 (z^{12} - 9z^4) dz$$

$$I = \left[\frac{z^{13}}{13} - \frac{9z^5}{5}\right]_2^3$$

$$I = \frac{(3)^{13}}{13} - \frac{9}{5}(3)^5 - \left(\frac{(2)^{13}}{13} - \frac{9}{5}(2)^5\right)$$

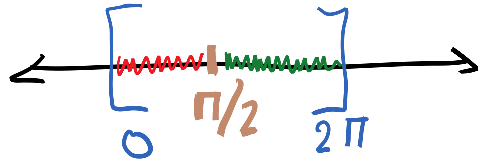
$$I = \frac{1594323}{13} - \frac{2187}{5} - \left(\frac{8192}{13} - \frac{288}{5}\right)$$

$$I = \frac{7943184}{65} - \frac{37216}{65}$$

$$\boxed{I = \frac{7905968}{65}}$$

EJEMPLOS

1) Evalúe la siguiente integral $\int_0^{2\pi} f(x)dx$



Donde $f(x)=\begin{cases}\left(x-\dfrac{\pi}{3}\right)^3+\dfrac{5}{\pi} &, \quad 0\le x\le \pi/2\\ \left(\dfrac{3}{\pi}\right)\cdot\sin^2\left(\dfrac{5x}{2}\right) &, \quad \pi/2<x<2\pi\end{cases}$

Sol

$$I=\int_0^{2\pi} f(x)dx$$

$$I=\int_0^{\pi/2} f(x)\,dx+\int_{\pi/2}^{2\pi} f(x)dx$$

$$I=\int_0^{\pi/2}\left[\left(x-\frac{\pi}{3}\right)^3+\frac{5}{\pi}\right]dx+\int_{\pi/2}^{2\pi}\frac{3}{\pi}\,\underbrace{\operatorname{sen}^2\left(\frac{5x}{2}\right)}\,dx$$

$$\operatorname{sen}^2x=\frac{1-\cos(2x)}{2}$$

$$\int(ax+b)^n dx=\frac{(ax+b)^{n+1}}{n+1}\cdot\frac{1}{a}$$

$$I=\left[\frac{\left(x-\frac{\pi}{3}\right)^4}{4}+\frac{5}{\pi}x\right]_0^{\pi/2}+\frac{3}{\pi}\int_{\pi/2}^{2\pi}\frac{1-\cos\left(\cancel{2}\left(\frac{5x}{\cancel{2}}\right)\right)}{2}\,dx$$

$$I=\frac{1}{4}\left(\frac{\pi}{2}-\frac{\pi}{3}\right)^4+\frac{5}{\cancel{\pi}}\left(\frac{\cancel{\pi}}{2}\right)-\left(\frac{1}{4}\left(0-\frac{\pi}{3}\right)^4+\frac{5}{\pi}(0)\right)+\frac{3}{2\pi}\int_{\pi/2}^{2\pi}(1-\cos(5x))dx$$

$$I=\underline{\frac{1}{4}\left(\frac{\pi}{6}\right)^4}+\frac{5}{2}-\underline{\left(\frac{1}{4}\cdot\frac{\pi^4}{81}\right)}+\frac{3}{2\pi}\left[x-\frac{\operatorname{sen}(5x)}{5}\right]_{\pi/2}^{2\pi}$$

$$I=-\frac{5}{1728}\cdot\pi^4+\frac{5}{2}+\frac{3}{\pi}\left[2\pi-\frac{\cancel{\operatorname{sen}(10\pi)}^{\,0}}{5}-\left(\frac{\pi}{2}-\frac{1}{5}\cdot\cancel{\operatorname{sen}\left(\frac{5\pi}{2}\right)}^{\,1}\right)\right]$$

$$I=-\frac{5}{1728}\cdot\pi^4+\frac{5}{2}+\frac{3}{\pi}\left[2\pi-\left(\frac{\pi}{2}-\frac{1}{5}\right)\right]$$

$$I = -\frac{5}{1728}\cdot \pi^4 + \frac{5}{2} + \frac{3}{\pi}\left[2\pi - \frac{\pi}{2} + \frac{1}{5}\right]$$

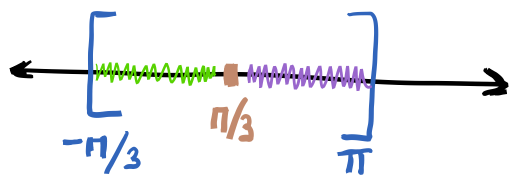
$$I = -\frac{5}{1728}\cdot \pi^4 + \frac{5}{2} + \frac{3}{\pi}\left[\frac{3\pi}{2} + \frac{1}{5}\right]$$

$$I = -\frac{5}{1728}\cdot \pi^4 + \frac{5}{2} + \frac{9}{2} + \frac{3}{5\pi}$$

$$I = -\frac{5}{1728}\cdot \pi^4 + \frac{14}{2} + \frac{3}{2\pi}$$

$$\boxed{I = -\frac{5}{1728}\cdot \pi^4 + \frac{3}{2\pi} + 7}$$

2) Evalúe la siguiente integral $\displaystyle\int_{-\pi/3}^{\pi} f(x)dx$



Donde $f(x) = \begin{cases} \sin^5\left(\dfrac{\pi\, x^{21}}{21}\right) - 3 & , \quad -\pi/3 \le x \le \pi/3 \\ 2 + \cos^2\left(\dfrac{3x}{2}\right) & , \quad \pi/3 < x < \pi \end{cases}$

SOLUCIÓN

$$I = \int_{-\pi/3}^{\pi} f(x)dx = \int_{-\pi/3}^{\pi/3} f(x)dx + \int_{\pi/3}^{\pi} f(x)dx$$

$$I = \int_{-\pi/3}^{\pi/3} \left[\operatorname{sen}^5\left(\frac{\pi x^{21}}{21}\right) - 3\right]dx + \int_{\pi/3}^{\pi} \left[2 + \underbrace{\cos^2\left(\frac{3x}{2}\right)}_{\text{identidad trigonométrica}}\right]dx$$

$$\boxed{\cos^2 x = \frac{1 + \cos(2x)}{2}}$$

$$I = \underbrace{\int_{-\pi/3}^{\pi/3} \operatorname{sen}^5\left(\frac{\pi x^{21}}{21}\right)dx}_{\text{propiedad}} - \int_{-\pi/3}^{\pi/3} 3dx + \int_{\pi/3}^{\pi}\left[2 + \frac{1 + \cos\left(\cancel{2}\left(\frac{3x}{\cancel{2}}\right)\right)}{2}\right]dx$$

propiedad $\displaystyle\int_{-a}^{a} f(x)dx = 0$; si $f(x)$ es impar

$$I = 0 - \left[3x\Big|_{-\pi/3}^{\pi/3}\right] + \int_{\pi/3}^{\pi}\left[\frac{4 + 1 + \cos(3x)}{2}\right]dx$$

$$I = -3\left[\frac{\pi}{3} - \left(\frac{-\pi}{3}\right)\right] + \frac{1}{2}\int_{\pi/3}^{\pi}\left[5 + \cos(3x)\right]dx$$

$$I = \cancel{-3}\left[\frac{2\pi}{\cancel{3}}\right] + \frac{1}{2}\left[5x + \frac{\operatorname{sen}(3x)}{3}\right]\Bigg|_{\pi/3}^{\pi}$$

$$I = -2\pi + \frac{1}{2}\left[5\pi + \cancel{\frac{\operatorname{sen}(3\pi)}{3}}^{\,0} - \left(5\cdot\frac{\pi}{3} + \operatorname{sen}(\cancel{3}(\frac{\pi}{\cancel{3}}))\right)\right]$$

$$I = -2\pi + \frac{1}{2}\left[5\pi - \frac{5\pi}{3} - \cancel{\operatorname{sen}(\pi)}^{\,0}\right]$$

$$I = -2\pi + \frac{1}{2}\left[\frac{10\pi}{3}\right]$$

$$I = -2\pi + \frac{5\pi}{3}$$

$$\boxed{I = -\frac{\pi}{3}}$$