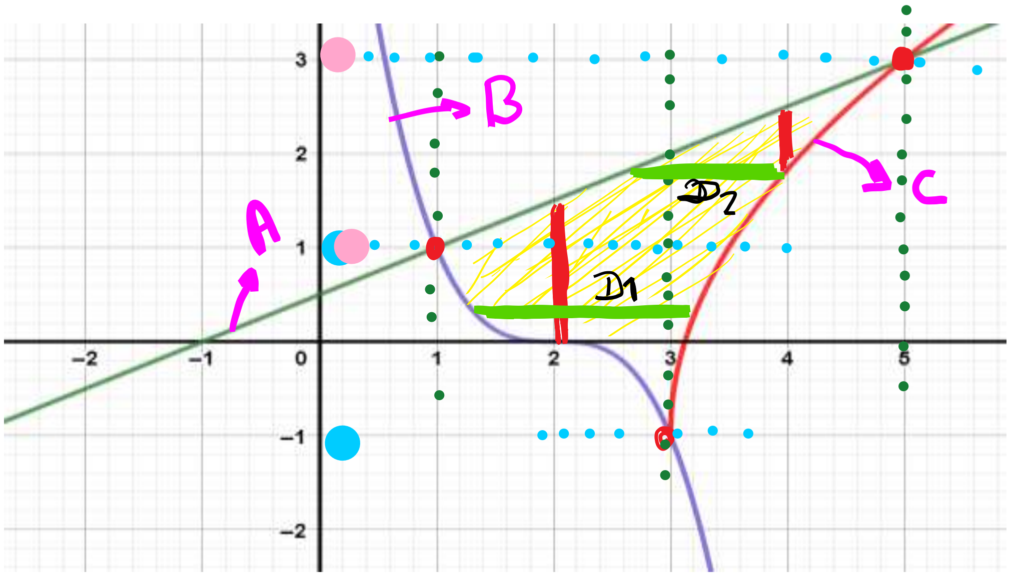
AREA DE UNA REGIÓN PLANA

EJEMPLO
Calcular el área de la región D limitada por las curvas:
1) $x-2y=-1,\quad y=-(x-2)^3,\quad y=-1+\sqrt{8x-24}$



Tipo II

A: $x-2y=-1 \Rightarrow \boxed{x=2y-1}$

B: $y=-(x-2)^3 \Rightarrow -y=(x-2)^3$

$\Rightarrow \sqrt[3]{-y}=x-2$

$\Rightarrow \boxed{x=\sqrt[3]{-y}+2} \Rightarrow \boxed{x=-\sqrt[3]{y}+2}$

C: $y=-1+\sqrt{8x-24}$

$\Rightarrow y+1=\sqrt{8x-24}$

$\Rightarrow (y+1)^2+24=8x \Rightarrow \boxed{x=\frac{1}{8}(y+1)^2+3}$

i) Región D1

$$A(D_1)=\int_{-1}^{1}\Big[\underbrace{\tfrac{1}{8}(y+1)^2+3}_{C}-\underbrace{\left(-y^{1/3}+2\right)}_{B}\Big]dy$$

$$= \int_{-1}^{1} \left[\frac{1}{8}(y+1)^2 + y^{1/3} + 1 \right] dy$$

$$= \left[\frac{1}{8}\cdot\frac{(y+1)^3}{3} + \frac{3}{4}y^{4/3} + y \right|_{-1}^{1}$$

$$= \frac{1}{3} + \frac{3}{4} + 1 - \left(0 + \frac{3}{4} - 1\right)$$

$$\boxed{A(D_1) = \frac{7}{3}}$$

ii) Región D2

$$A(D_2) = \int_{1}^{3} \Big[\underbrace{\tfrac{1}{8}(y+1)^2 + 3}_{C} - \underbrace{(2y-1)}_{A} \Big] dy$$

$$= \int_{1}^{3} \left[\frac{1}{8}(y+1)^2 - 2y + 4 \right] dy$$

$$= \left[\frac{1}{8}\cdot\frac{(y+1)^3}{3} - \frac{\cancel{2}y^2}{\cancel{2}} + 4y \right|_{1}^{3}$$

$$= \frac{8}{3} - 9 + 12 - \left(\frac{1}{3} - 1 + 4\right)$$

$$\boxed{A(D_2) = \frac{7}{3}}$$

Luego, $A(D) = A(D_1) + A(D_2)$

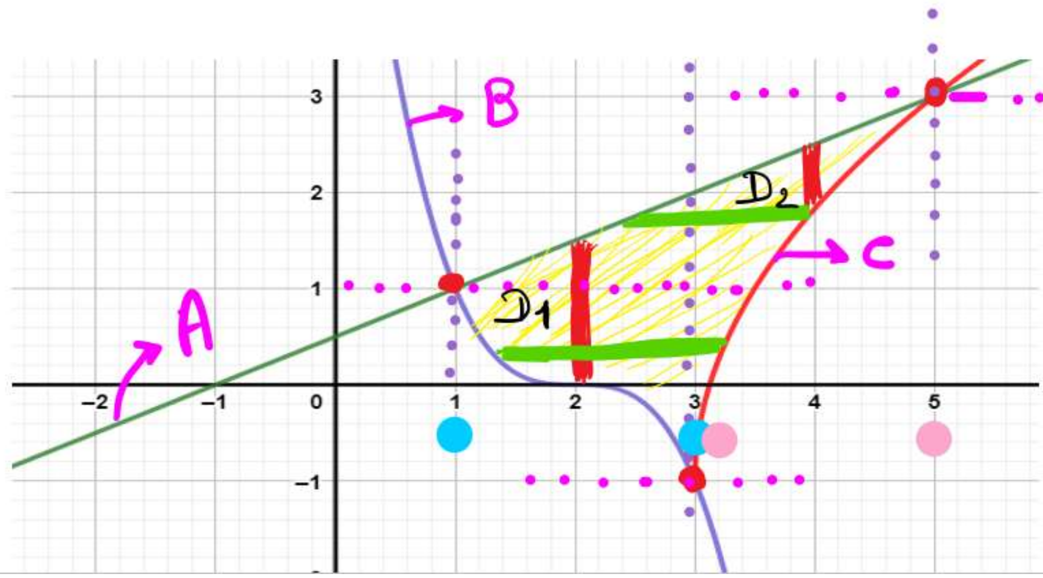
$$= \frac{7}{3} + \frac{7}{3}$$

$$\boxed{A(D) = \frac{14}{3}\ u^2}$$

EJEMPLO

Calcular el área de la región D limitada por las curvas:

1) $x-2y=-1$, $y=-(x-2)^3$, $y=-1+\sqrt{8x-24}$



Tipo I

A: $x-2y=-1 \Rightarrow \boxed{y=\frac{x+1}{2}}$

B: $y=-(x-2)^3$

C: $y=-1+\sqrt{8x-24}$

i) Región D1

$$A(D_1)=\int_1^3\left[\underbrace{\frac{x+1}{2}}_{A}-\underbrace{\left(-(x-2)^3\right)}_{B}\right]dx$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1}\cdot\frac{1}{a}$$

$$=\int_1^3\left[\frac{1}{2}(x+1)^1+(x-2)^3\right]dx$$

$$=\left[\frac{1}{2}\cdot\frac{(x+1)^2}{2}+\frac{(x-2)^4}{4}\right|_1^3$$

$$= 4+\frac{1}{4}-\left(1+\frac{1}{4}\right)$$

$$\boxed{A(D_1)=3}$$

ii) Region D_2 $\quad$ A − C

$$A(D_2) = \int_3^5 \left[\frac{x+1}{2} - \left(-1+\sqrt{8x-24}\right)\right] dx$$

$$= \int_3^5 \left[\frac{1}{2}(x+1)^1 + 1 - (8x-24)^{1/2}\right] dx$$

$$= \left[\frac{1}{2}\cdot\frac{(x+1)^2}{2} + x - \frac{\cancel{2}^{\,1}}{3}(8x-24)^{3/2}\cdot\frac{1}{\cancel{8}_{\,4}}\right]_3^5$$

$$= 9+5-\frac{1}{12}(64) - \left(4+3-\frac{1}{12}(0)\right)$$

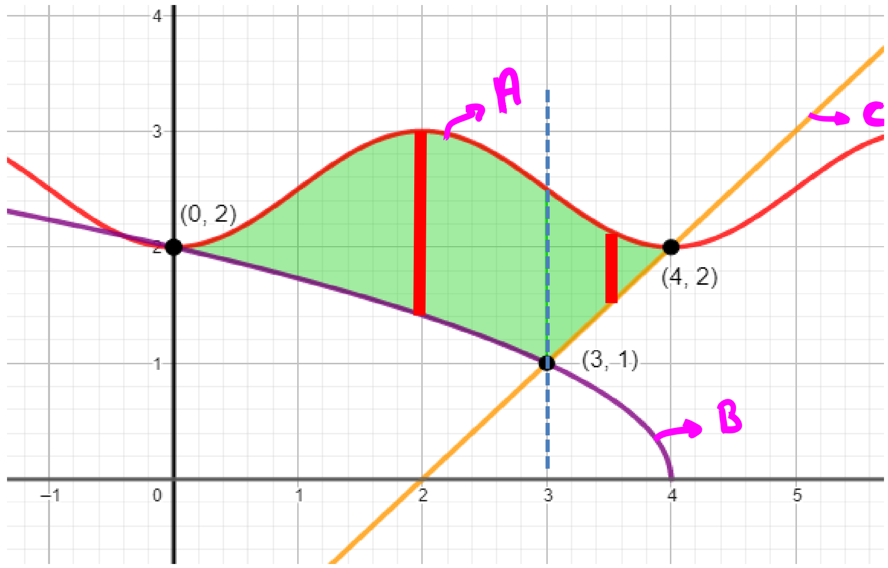
$$\boxed{A(D_2) = 5/3}$$

Luego, $A(D) = A(D_1) + A(D_2)$

$$= 3+\frac{5}{3}$$

$$A(D) = \frac{14}{3}\,u^2$$

2) Halle el área de la región limitada por las curvas: $y = sen^2\left(\frac{\pi}{4}x\right) + 2$; $y = \sqrt{4-x}$; $y = x - 2$.



Sol

Tipo I

$A: \ y = sen^2\left(\frac{\pi}{4}x\right) + 2$

$B: \ y = \sqrt{4-x}$

$C: \ y = x - 2$

i) Región D1

$$A(D_1) = \int_0^3 \overbrace{\left[\underbrace{sen^2\left(\frac{\pi}{4}x\right)}_{\text{Identidad}} + 2 - \sqrt{4-x}\right]}^{A - B} dx$$

Identidad: $Sen^2 x = \dfrac{1 - \cos(2x)}{2}$

$$= \int_0^3 \left[\frac{1}{2}\left(1 - \cos\left(\frac{\pi x}{2}\right)\right) - (4-x)^{1/2} \right] dx$$

$$= \left[\frac{1}{2}\left(x - sen\left(\frac{\pi x}{2}\right)\cdot\frac{2}{\pi}\right) - \frac{2}{3}(4-x)^{3/2}(-1) + 2x \right|_0^3$$

$$= \left[\frac{1}{2}x - \frac{1}{\pi} sen\left(\frac{\pi x}{2}\right) + \frac{2}{3}(4-x)^{3/2} + 2x \right|_0^3$$

$$= \frac{3}{2} - \frac{1}{\pi}\underbrace{sen\left(\frac{3\pi}{2}\right)}_{-1} + \frac{2}{3} + 6 - \left(-\frac{1}{\pi}\underbrace{sen(0)}_{0} + \frac{16}{3}\right)$$

$$= \frac{3}{2} - \frac{1}{\pi}(-1) + \frac{2}{3} + 6 - \frac{16}{3} \Rightarrow \boxed{A(D_1) = \frac{1}{\pi} + \frac{17}{6}}$$

ii) Región D_2

A − C

$$A(D_2) = \int_3^4 \left[\operatorname{sen}^2\left(\frac{\pi}{4}x\right) + 2 - (x-2)\right] dx$$

$$= \int_3^4 \left[\operatorname{sen}^2\left(\frac{\pi x}{4}\right) - x + 4\right] dx$$

$$= \left[\frac{1}{2}\left(x - \operatorname{sen}\left(\frac{\pi x}{2}\right)\cdot\frac{2}{\pi}\right) - \frac{x^2}{2} + 4x \right|_3^4$$

$$= \left[\frac{x}{2} - \frac{1}{\pi}\operatorname{sen}\left(\frac{\pi x}{2}\right) - \frac{x^2}{2} + 4x \right|_3^4$$

$$= 2 - \frac{1}{\pi}\underbrace{\operatorname{sen}(2\pi)}_{0} - 8 + 16 - \left(\frac{3}{2} - \frac{1}{\pi}\underbrace{\operatorname{sen}\left(\frac{3\pi}{2}\right)}_{-1} - \frac{9}{2} + 12\right)$$

$$= 10 - \left(9 - \frac{1}{\pi}(-1)\right)$$

$$\boxed{A(D_2) = 1 - \frac{1}{\pi}}$$

Luego, $A(D) = A(D_1) + A(D_2)$

$$= \cancel{\frac{1}{\pi}} + \frac{17}{6} + 1 - \cancel{\frac{1}{\pi}}$$

$$\boxed{A(D) = \frac{23}{6}\ u^2}$$

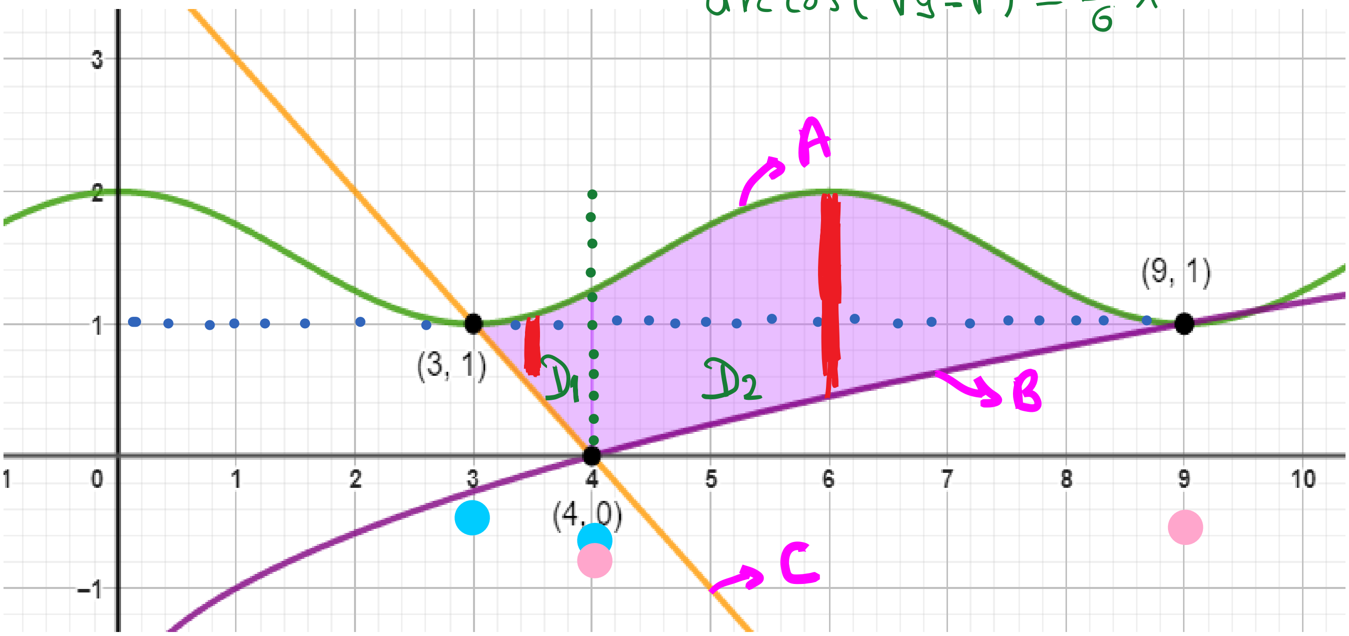
3) Halle el área de la región limitada por las curvas:

$$y = \cos^2\left(\frac{\pi}{6}x\right) + 1 \quad , \quad y = \sqrt{x} - 2, \quad y = -x + 4$$

$y = \left(\cos\left(\frac{\pi}{6}x\right)\right)^2 + 1$

$\sqrt{y-1} = \cos\left(\frac{\pi}{6}x\right)$

$\arccos\left(\sqrt{y-1}\right) = \frac{\pi}{6}x$



Tipo I

A: $y = \cos^2\left(\frac{\pi}{6}x\right) + 1$

B: $y = \sqrt{x} - 2$

C: $y = -x + 4$

i) **Región D_1**

A - C

$$A(D_1) = \int_3^4 \left[\cos^2\left(\frac{\pi}{6}x\right) + 1 - (-x+4)\right] dx$$

$$= \int_3^4 \left[\underbrace{\cos^2\left(\frac{\pi}{6}x\right)}_{\text{identidad}} + x - 3\right] dx \qquad \cos^2 x = \frac{1+\cos(2x)}{2}$$

$$\int \cos(kx)\,dx = \operatorname{sen}(kx)\frac{1}{k}$$

$$= \int_3^4 \left[\frac{1}{2}\left(1 + \cos\left(\frac{\pi}{3}x\right)\right) + x - 3\right] dx$$

$$= \left[\frac{1}{2}\left(x + \operatorname{sen}\left(\frac{\pi}{3}x\right)\cdot\frac{3}{\pi}\right) + \frac{x^2}{2} - 3x\right|_3^4$$

$$= \left[\frac{x}{2} + \frac{3}{2\pi} \operatorname{Sen}\left(\frac{\pi}{3}x\right) + \frac{x^2}{2} - 3x \right|_3^4$$

$$= 2 + \frac{3}{2\pi} \operatorname{Sen}\left(\frac{4\pi}{3}\right) + 8 - 12 - \left(\frac{3}{2} + \frac{3}{2\pi} \cancel{\operatorname{Sen}(\pi)} + \frac{9}{2} - 9 \right)$$

$-\frac{\sqrt{3}}{2}$, 0

$$= \frac{3}{2\pi}\left(-\frac{\sqrt{3}}{2}\right) + 1$$

$$\boxed{A(D_1) = \frac{-3\sqrt{3}}{4\pi} + 1}$$

ii) Región D_2

$$A \quad - \quad B$$

$$A(D_2) = \int_4^9 \left[\cos^2\left(\frac{\pi}{6}x\right) + 1 - \left(\sqrt{x} - 2\right) \right] dx$$

$$= \int_4^9 \left[\cos^2\left(\frac{\pi}{6}x\right) - x^{1/2} + 3 \right] dx$$

$$= \left[\frac{1}{2}\left(x + \operatorname{Sen}\left(\frac{\pi}{3}x\right) \cdot \frac{3}{\pi}\right) - \frac{2}{3}x^{3/2} + 3x \right|_4^9$$

$$= \left[\frac{x}{2} + \frac{3}{2\pi} \operatorname{Sen}\left(\frac{\pi}{3}x\right) - \frac{2}{3}x^{3/2} + 3x \right|_4^9$$

$$= \frac{9}{2} + \frac{3}{2\pi} \cancel{\operatorname{Sen}(3\pi)} - 18 + 27 - \left(2 + \frac{3}{2\pi} \operatorname{Sen}\left(\frac{4\pi}{3}\right) - \frac{16}{3} + 12 \right)$$

0 , $-\frac{\sqrt{3}}{2}$

$$= -\frac{3}{2\pi}\left(-\frac{\sqrt{3}}{2}\right) + \frac{29}{6}$$

$$\boxed{A(D_2) = \frac{3\sqrt{3}}{4\pi} + \frac{29}{6}}$$

Luego, $A(D) = A(D_1) + A(D_2)$

$$= \cancel{\frac{-3\sqrt{3}}{4\pi}} + 1 + \cancel{\frac{3\sqrt{3}}{4\pi}} + \frac{29}{6}$$

$$\boxed{A(D) = \frac{35}{6}\,u^2} \approx 5{,}8333$$

4) Dada la región D limitada por las gráficas:

$(x-4)^2 = \frac{5}{y} - 1$; $(x-4)^2 = 4y$; $2x + 4y = 8.$

$(x-4)^2 = \frac{5}{y} - 1$

$x - 4 = \pm\sqrt{\frac{5}{y} - 1} \Rightarrow x = \pm\sqrt{\frac{5}{y} - 1} + 4$

Calcular el valor de: $Q = A(D) - 10\arctan(2) + 10$, donde $A(D)$ es el área de la región D.

SOLUCIÓN

a) Hallando $A(D)$

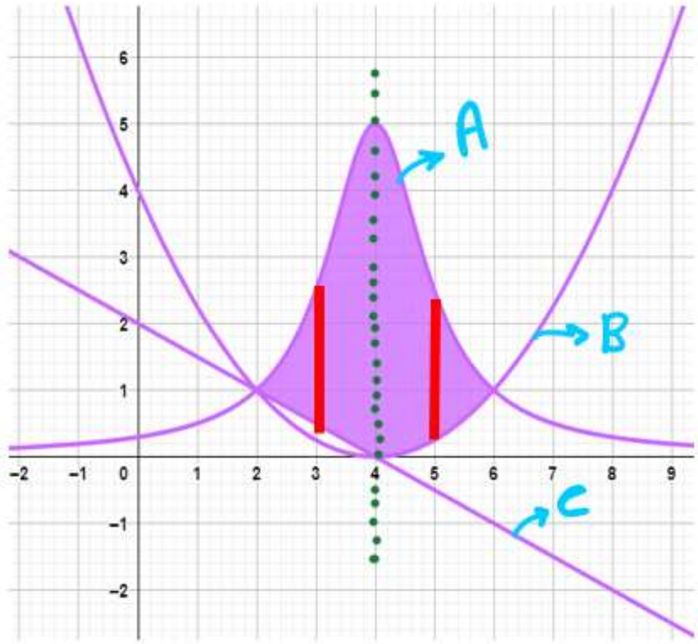
Tipo I

•) A : $(x-4)^2 = \frac{5}{y} - 1$

$\Rightarrow \frac{5}{y} = (x-4)^2 + 1$

$\Rightarrow \frac{y}{5} = \frac{1}{(x-4)^2 + 1}$

$\Rightarrow \boxed{y = \frac{5}{(x-4)^2 + 1}}$



•) B : $(x-4)^2 = 4y$

$\Rightarrow \boxed{y = \frac{1}{4}(x-4)^2}$

•) C : $2x + 4y = 8$

$\Rightarrow x + 2y = 4$

$\Rightarrow \boxed{y = \frac{4-x}{2}}$

i) Región D1 :

$$A(D_1) = \int [A - C]\, dx$$

$$A(D_1) = \int_2^4 \left[\frac{5}{(x-4)^2+1} - \frac{4-x}{2}\right] dx$$

$$A(D_1) = \int_2^4 \left[5\cdot \frac{1}{\underbrace{(x-4)}_{u}{}^2+1} - 2 + \frac{x}{2}\right] dx$$

fórmula: $\int \frac{1}{u^2+a^2}\,du = \frac{1}{a}\operatorname{arctg}\left(\frac{u}{a}\right)$

$$= \left[5\operatorname{arctg}(x-4) - 2x + \frac{x^2}{4}\right|_2^4$$

$$= 5\cancel{\operatorname{arctg}(0)}^{\,0} - 8 + \frac{16}{4} - \left(5\operatorname{arctg}(-2) - 4 + \frac{4}{4}\right)$$

$$= -8 + 4 - 5\operatorname{arctg}(-2) + 3$$

$$= -1 - 5\operatorname{arctg}(-2)$$

$$\boxed{A(D_1) = -1 + 5\operatorname{arctg}(2)}$$

$\boxed{\operatorname{arctg}(-2) = -\operatorname{arctg}(2)}$ función impar

ii) Región D_2:

$$A(D_2) = \int [A - B]\,dx$$

$$A(D_2) = \int_4^6 \left[\underbrace{\frac{5}{(x-4)^2+1}}_{\text{fórmula}} - \frac{1}{4}\underbrace{(x-4)^2}_{\text{fórmula}}\right] dx$$

$$A(D_2) = \left[5\operatorname{arctg}(x-4) - \frac{1}{4}\cdot\frac{(x-4)^3}{3}\right|_4^6$$

$$= 5\operatorname{arctg}(2) - \frac{1}{4}\cdot\frac{(8)}{3} - \left(5\cancel{\operatorname{arctg}(0)}^{\,0} - \cancel{\frac{0}{4}}^{\,0}\right)$$

$$\boxed{A(D_2) = 5\operatorname{arctg}(2) - \frac{2}{3}}$$

Luego, $A(D) = A(D_1) + A(D_2)$

$$A(D) = -1 + 5\operatorname{arctg}(2) + 5\operatorname{arctg}(2) - \frac{2}{3}$$

$$A(D) = -\frac{5}{3} + 10\operatorname{arctg}(2)$$

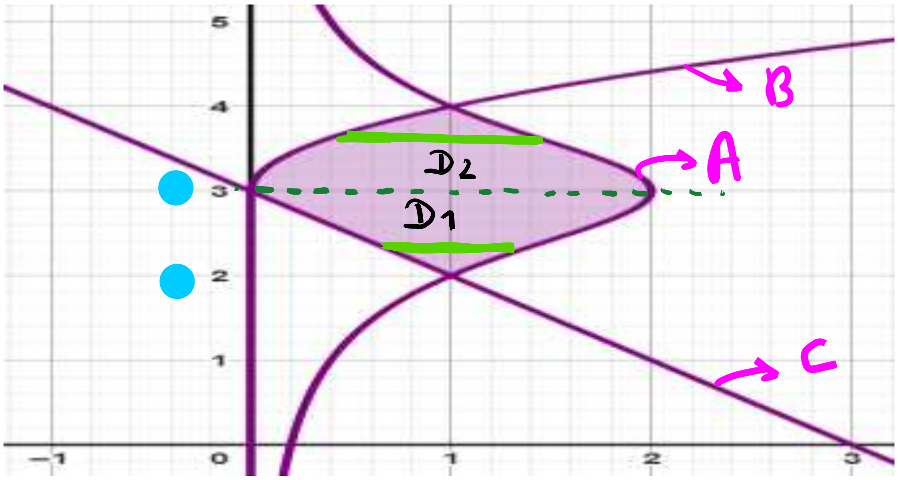
b) Hallando $Q = A(D) - 10\operatorname{arctg}(2) + 10$

$$Q = -\frac{5}{3} + \cancel{10\operatorname{arctg}(2)} - \cancel{10\operatorname{arctg}(2)} + 10$$

$$\boxed{Q = \frac{25}{3}}$$

5) Dada la región D limitada por las gráficas: $(y-3)^2 = \frac{2}{x} - 1$; $y = 3 + \sqrt{x}$; $x + y = 3$.

Calcular el valor de: $Q = A(D) - \pi - \frac{7}{6}$, donde $A(D)$ es el área de la región D.



Tipo II

•) A : $(y-3)^2 = \frac{2}{x} - 1$

$\Rightarrow (y-3)^2 + 1 = \frac{2}{x}$

$\Rightarrow \frac{1}{(y-3)^2+1} = \frac{x}{2}$

$\Rightarrow \boxed{x = \frac{2}{(y-3)^2+1}}$

•) B : $y = 3 + \sqrt{x}$

$\Rightarrow y - 3 = \sqrt{x}$

$\Rightarrow \boxed{x = (y-3)^2}$

•) C : $x + y = 3$

$\Rightarrow \boxed{x = 3 - y}$

i) Región D1

$$A(D_1) = \int_2^3 \Big[\underbrace{\frac{2}{(y-3)^2+1}}_{A} - \underbrace{(3-y)}_{C} \Big] dy$$

$$= \int_2^3 \Big[\frac{2}{(\underbrace{y-3}_{u})^2+1} - 3 + y \Big] dy$$

fórmula

$$\int \frac{1}{u^2+a^2} du = \frac{1}{a} \operatorname{arctg}\left(\frac{u}{a}\right)$$

$$= \left[2 \cdot \operatorname{arctg}(y-3) - 3y + \frac{y^2}{2} \right|_2^3$$

$$= 2\cancel{\operatorname{arctg}(0)}^{0} - 9 + \frac{9}{2} - \left(2\cancel{\operatorname{arctg}(-1)}_{-\frac{\pi}{4}} - 6 + 2 \right)$$

$$= -2\left(-\frac{\pi}{4}\right) - \frac{1}{2}$$

$$\boxed{A(D_1) = \frac{\pi}{2} - \frac{1}{2}}$$

ii) Región D_2 :

$$A(D_2) = \int [A - B]\, dy$$

$$A(D_2) = \int_3^4 \left[\frac{2}{(y-3)^2+1} - (y-3)^2 \right] dy$$

$$A(D_2) = \left[2\operatorname{arctg}(y-3) - \frac{(y-3)^3}{3} \right|_3^4$$

$$A(D_2) = 2\cancel{\operatorname{arctg}(1)}_{\pi/4} - \frac{1}{3} - \left(2\cancel{\operatorname{arctg}(0)}_{0} - \frac{0}{3} \right)$$

$$A(D_2) = 2\left(\frac{\pi}{4}\right) - \frac{1}{3}$$

$$\boxed{A(D_2) = \frac{\pi}{2} - \frac{1}{3}}$$

Luego,
$$A(D) = A(D_1) + A(D_2)$$
$$A(D) = \frac{\pi}{2} - \frac{1}{2} + \frac{\pi}{2} - \frac{1}{3}$$
$$\boxed{A(D) = \pi - \frac{5}{6}}$$

b) Hallando $Q = A(D) - \pi - \frac{7}{6}$

$$Q = \pi - \frac{5}{6} - \pi - \frac{7}{6}$$

$$Q = -\frac{12}{6}$$

$$\boxed{Q = -2}$$

6) Dada la región D limitada por las gráficas:

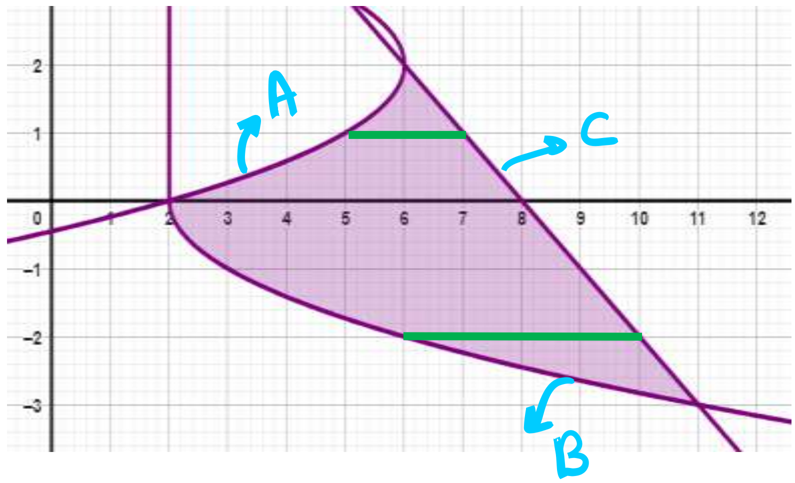
$$x=6-(y-2)^2 \quad , \quad y=-\sqrt{x-2} \quad , \quad y+x-8=0$$

Determinar el valor de la constante "T" si $6\cdot\left|A(D)+T^3\right|=271$, donde $A(D)$ es el área de la región D.

SOLUCIÓN

a) Hallando $A(D)$

Tipo II



•) A : $\boxed{X=6-(y-2)^2}$

•) B: $y=-\sqrt{x-2}$

$\Rightarrow \quad y^2 = x-2$

$\Rightarrow \quad \boxed{x = y^2+2}$

•) C: $y+x-8=0$

$\Rightarrow \quad \boxed{x = 8-y}$

i) Región D_1

$$A(D_1) = \int_{-3}^{0} [\overset{C}{8-y} - \overset{B}{(y^2+2)}]\,dy$$

$$A(D_1) = \int_{-3}^{0} [-y^2-y+6]\,dy$$

$$A(D_1) = \left[-\frac{y^3}{3}-\frac{y^2}{2}+6y\right|_{-3}^{0}$$

$$A(D_1) = 0-\left(-\frac{(-3)^3}{3}-\frac{(-3)^2}{2}+6(-3)\right)$$

$$\boxed{A(D_1) = \frac{27}{2}}$$

ii) Región D_2

$$A(D_2) = \int_{0}^{2} [\overset{C}{8-y} - \overset{A}{(6-(y-2)^2)}]\,dy$$

$$A(D_2) = \int_0^2 \left[(y-2)^2 - y + 2 \right] dy$$

$$A(D_2) = \left[\frac{(y-2)^3}{3} - \frac{y^2}{2} + 2y \right|_0^2$$

$$A(D_2) = 0 - \frac{4}{2} + 4 - \left(\frac{(-2)^3}{3} \right)$$

$$\boxed{A(D_2) = \frac{14}{3}}$$

Luego, $A(D) = A(D_1) + A(D_2)$

$$A(D) = \frac{27}{2} + \frac{14}{3}$$

$$\boxed{A(D) = \frac{109}{6} u^2}$$

b) Hallando $T = ?$

Si $6\left[A(D) + T^3\right] = 271$

$\Rightarrow \quad A(D) + T^3 = \frac{271}{6}$

$\Rightarrow \quad \frac{109}{6} + T^3 = \frac{271}{6}$

$\Rightarrow$

$\Rightarrow \quad T^3 = \frac{271}{6} - \frac{109}{6}$

$\Rightarrow \quad T^3 = 27$

$\Rightarrow \quad T = \sqrt[3]{27}$

Luego, $\boxed{T = 3}$