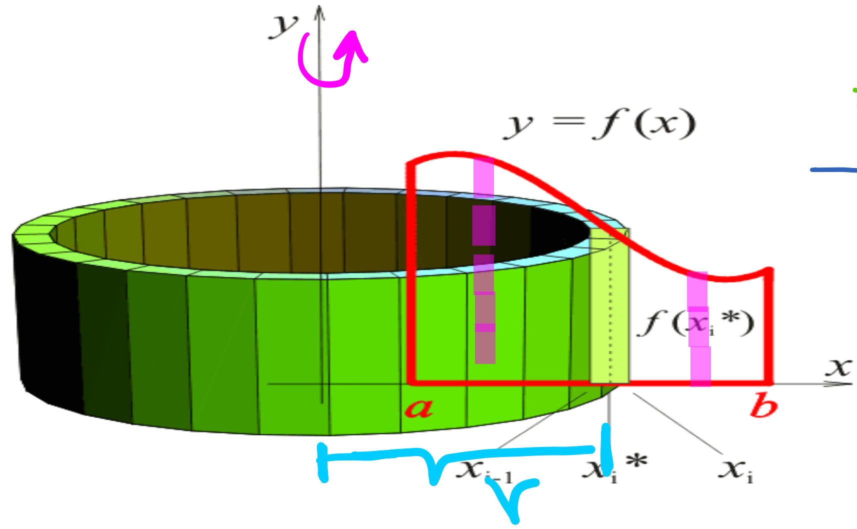
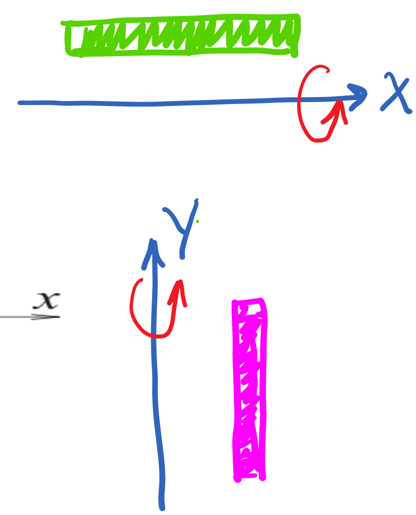
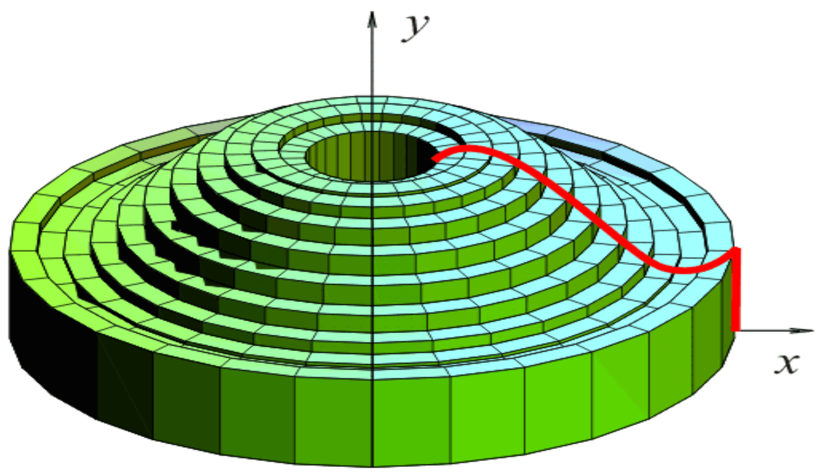
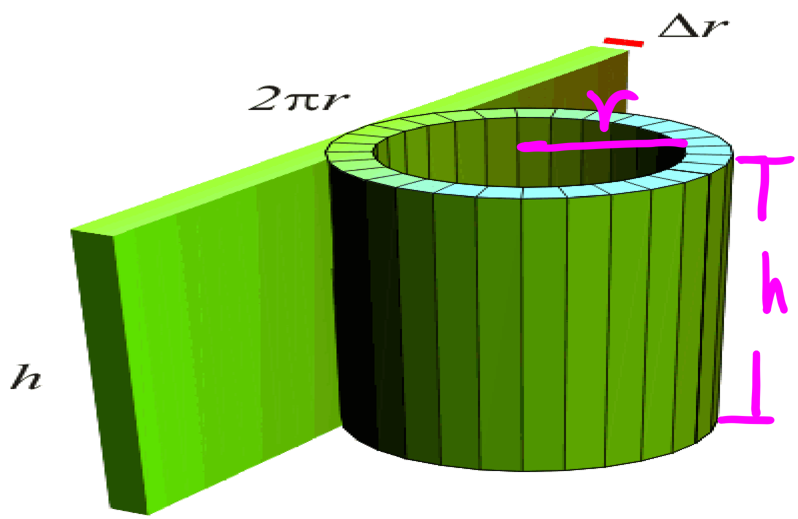
MÉTODO DE LAS CAPAS CILÍNDRICAS









$2\pi r h$

El volumen del sólido de revolución es: $V(S) = 2\pi \int_a^b r \cdot h \, dx.$

Donde r : *radio del cilindro,* h : *altura del cilindro.*

EJEMPLOS

Calcular el volumen del sólido generado por la rotación de la región D limitada por las curvas dadas alrededor de la recta dada

1) $y=\sqrt{x},\quad x=4,\ y=0,\quad L: eje\, Y,$

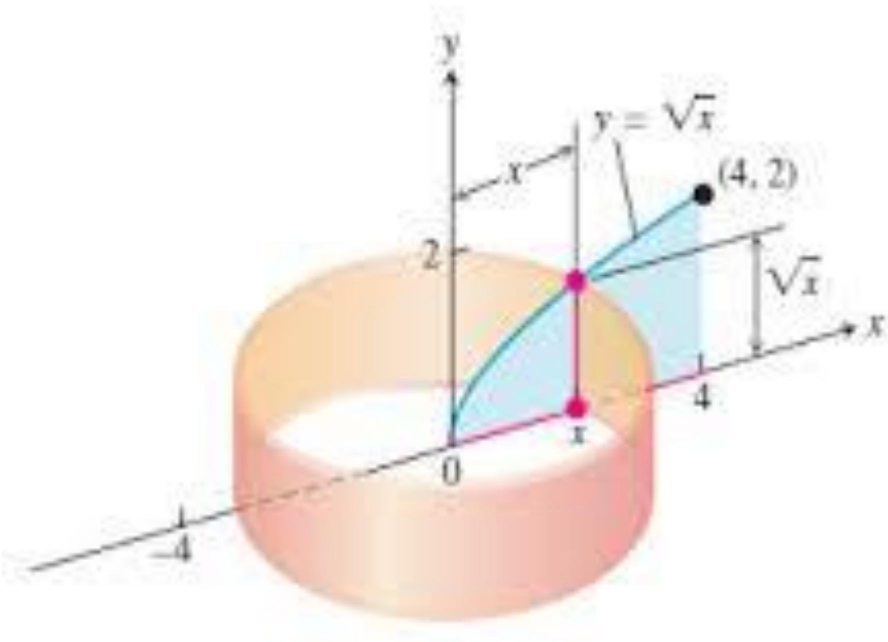
Solución

Radio: $r=x$

Altura: $h=\sqrt{x}$

Entonces

$$V(S)=2\pi\int_0^4 r\cdot h dx=2\pi\int_0^4 (x)(\sqrt{x})dx=2\pi\int_0^4 x^{3/2}dx=2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4=\frac{4\pi}{5}\left[\left(\sqrt{4}\right)^5\right]=\frac{128\pi}{5}$$



2) $y=\sqrt{8x}, \quad y=x^2, \ L: eje\, Y,$

Solución

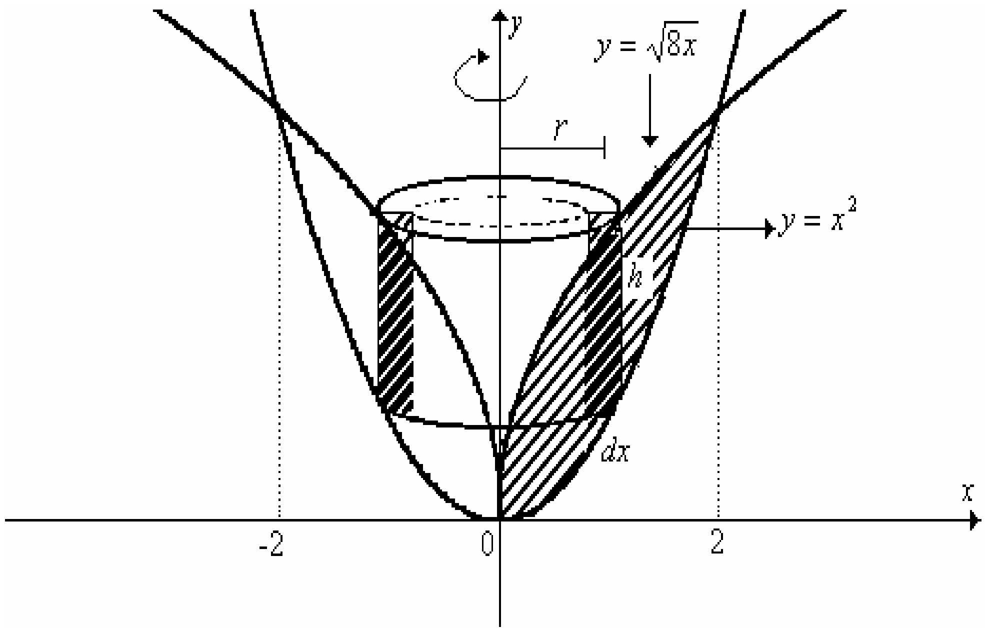
Radio: $r=x$

Altura: $h=\sqrt{8x}-x^2$

Entonces

$$V(S)=2\pi\int_0^2 r\cdot h dx=2\pi\int_0^2 (x)\left(\sqrt{8x}-x^2\right)dx=2\pi\int_0^2 (x)\left(\sqrt{8}\sqrt{x}-x^2\right)dx=2\pi\int_0^2 \left[\sqrt{8}\,x^{3/2}-x^3\right]dx$$

$$=2\pi\left[\sqrt{8}\,\frac{2}{5}x^{5/2}-\frac{x^4}{4}\right]_0^2=2\pi\left[\frac{2\sqrt{8}}{5}\left(\sqrt{2}\right)^5-\frac{(2)^4}{4}\right]=2\pi\left[\frac{32}{5}-\frac{16}{4}\right]=\frac{24\pi}{5}u^3$$



3) $y=\sqrt{x}, \quad x=4,\ y=0, \qquad L: eje\ X$

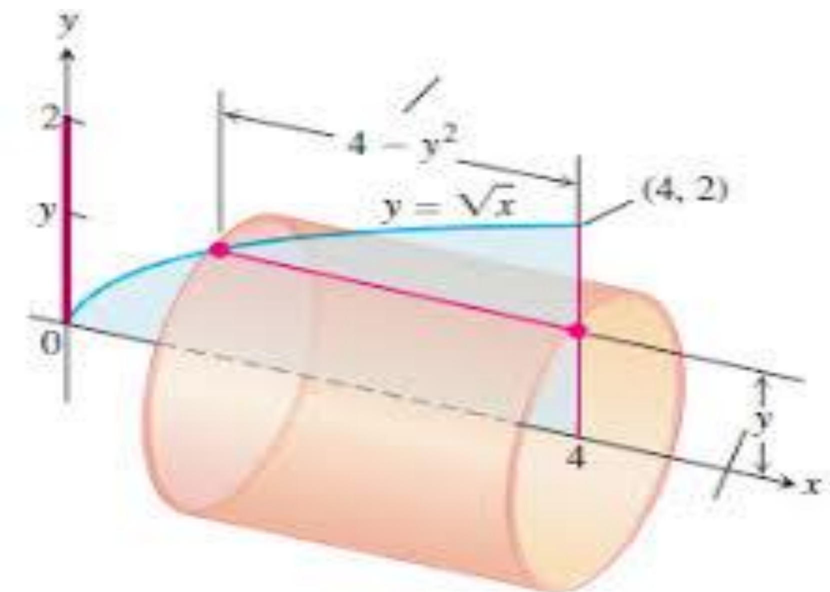
Solución

Radio: $r=y$

Altura: $h=4-y^2$

Entonces $V(S)=2\pi\int_0^2 r\cdot h dy=2\pi\int_0^2 (y)(4-y^2)dx=2\pi\int_0^2 (4y-y^3)dx=2\pi\left[2y^2-\frac{y^3}{3}\right]_0^2$

$=2\pi\left[2(2)^2-\frac{(2)^3}{3}\right]=\frac{32\pi}{3}u^3$



EJEMPLO

1) Dada las siguientes curvas: $y = 2^x$; $y = \frac{4}{1+x^2}$; $(x-1)^2 + \frac{(y-4)^2}{4} = 1$.

a) Grafique la región ***R*** comprendida por dichas curvas (**Primer cuadrante)**

b) Plantee el volumen del sólido de revolución, cuando la región ***R*** gira alrededor del eje "Y" usando el método de capas cilíndricas, luego calcule el volumen del sólido.

A: $\boxed{y=2^x}$

B: $\boxed{y=\frac{4}{1+x^2}}$

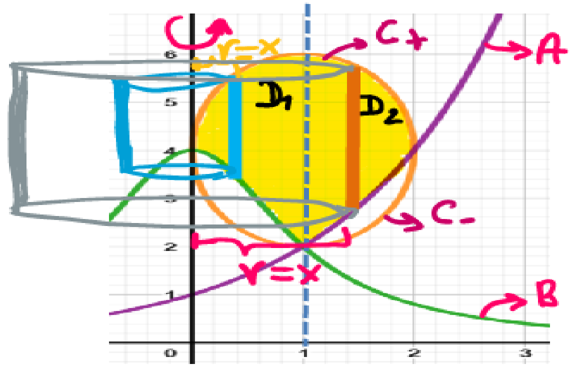
C: $(x-1)^2+\frac{(y-4)^2}{4}=1$

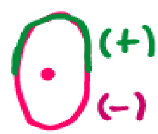
$$\Rightarrow \frac{(y-4)^2}{4}=1-(x-1)^2$$

$$\Rightarrow (y-4)^2=4\left[1-(x-1)^2\right]$$

$$\Rightarrow y-4=\pm\sqrt{4\left[1-(x-1)^2\right]}$$

$$\Rightarrow \boxed{y=\pm 2\sqrt{1-(x-1)^2}+4}$$





Región D1

Radio: $r = x$

Altura: $h = C_+ - B$

$$h = 2\sqrt{1-(x-1)^2}+4-\frac{4}{1+x^2}$$

Región D2

Radio: $r = x$

Altura: $h = C_+ - A$

$$h = 2\sqrt{1-(x-1)^2}+4-2^x$$

Luego,

$$V(S) = V(S_1) + V(S_2)$$

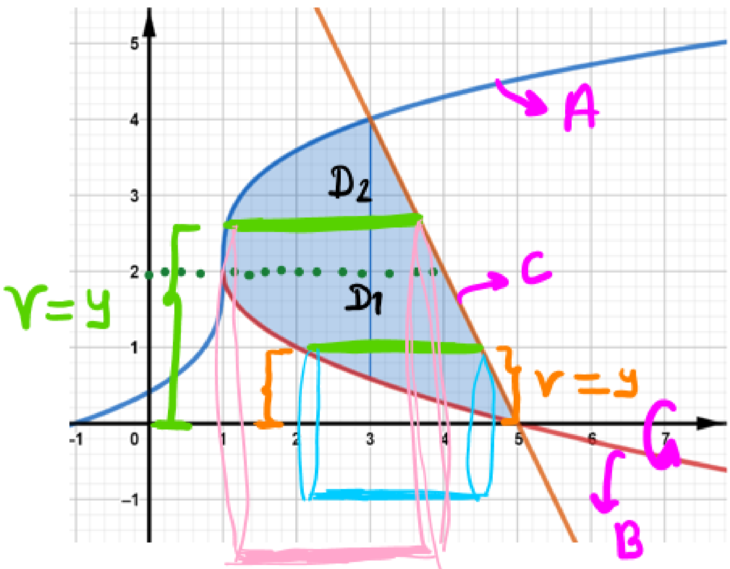
$$V(S) = 2\pi\int_{D_1} r\,h\,dx + 2\pi\int_{D_2} r\cdot h\,dx$$

$$V(S) = 2\pi\int_0^1 x\left[2\sqrt{1-(x-1)^2}+4-\frac{4}{1+x^2}\right]dx + 2\pi\int_1^2 x\left[2\sqrt{1-(x-1)^2}+4-2^x\right]dx$$

EJEMPLO

1) Calcular el volumen del sólido de revolución que se genera al girar la región D alrededor del eje X, limitado por las ecuaciones:

$$4x-4=(y-2)^3\ ,\quad y=-\sqrt{x-1}+2,\quad 2x+y=10$$



*) A: $4x-4=(y-2)^3$

$\Rightarrow \boxed{x=\dfrac{(y-2)^3+4}{4}}$

*) B: $y=-\sqrt{x-1}+2$

$\Rightarrow y-2=-\sqrt{x-1}$

$\Rightarrow (y-2)^2=x-1$

$\Rightarrow \boxed{x=(y-2)^2+1}$

*) C: $2x+y=10$

$\Rightarrow \boxed{x=\dfrac{10-y}{2}}$

i) Región D1

Radio: $r=y$

Altura: $h=\overset{C-B}{\dfrac{10-y}{2}-((y-2)^2+1)}$

$h=\frac{1}{2}\left[10-y-2\left(y^2-4y+4+1\right)\right]$

$h=\frac{1}{2}\left[-2y^2+7y+8\right]$

$V(S_1)=2\pi\int r h\,dy$

$V(S_1)=2\pi\int_0^2 y\cdot\frac{1}{2}\left[-2y^2+7y+8\right]dy$

$V(S_1)=\frac{\cancel{2}\pi}{\cancel{2}}\int_0^2\left[-2y^3+7y^2+8y\right]dy$

$V(S_1)=\pi\left[-\frac{\cancel{2}^{1}y^4}{\cancel{4}_{2}}+\frac{7y^3}{3}+\frac{\cancel{8}^{4}y^2}{\cancel{2}_{1}}\right]_0^2$

$V(S_1)=\frac{80\pi}{3}$

ii) Región D2

Radio: $r=y$

Altura: $h=\overset{C-A}{\dfrac{10-y}{2}-\dfrac{(y-2)^3+4}{4}}$

$\Rightarrow h=\frac{1}{4}\left[2(10-y)-\left((y-2)^3+4\right)\right]$

$\Rightarrow h=\frac{1}{4}\left[20-2y-\left(y^3-3y^2(2)+3y(2)^2-8+4\right)\right]$

$\Rightarrow h=\frac{1}{4}\left[-y^3+6y^2-14y+24\right]$

Luego,

$V(S_2)=2\pi\int r.h\,dy$

$V(S_2)=2\pi\int_2^4 y\cdot\frac{1}{4}\left[-y^3+6y^2-14y+24\right]dy$

$V(S_2)=\frac{\cancel{2}^{1}\pi}{\cancel{4}_{2}}\int_2^4\left[-y^4+6y^3-14y^2+24y\right]dy$

$$V(S_2)=\frac{\pi}{2}\left[-\frac{y^5}{5}+\frac{\cancelto{3}{6}y^4}{\cancelto{2}{4}}-\frac{14y^3}{3}+\frac{\cancelto{12}{24}y^2}{\cancelto{1}{2}}\right]\Bigg|_2^4$$

$$V(S_2)=\frac{\pi}{2}\left[\frac{1088}{15}-\left(\frac{424}{15}\right)\right]$$

$$V(S_2)=\frac{\pi}{\cancelto{1}{2}}\left[\frac{\cancel{664}}{15}\right]$$

$$V(S_2)=\frac{332}{15}\pi$$

Luego, $V(S)=V(S_1)+V(S_2)$

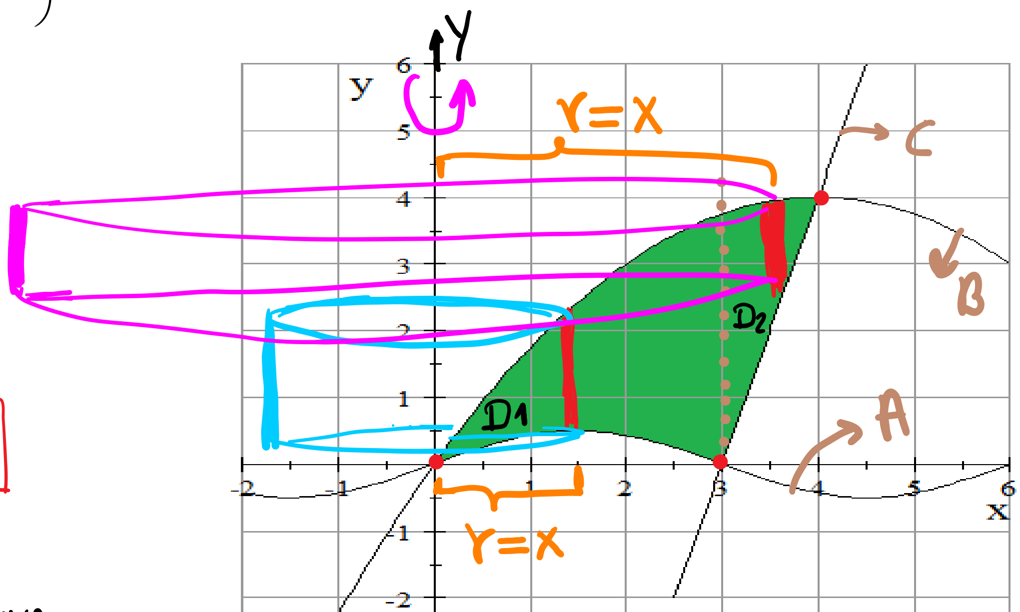
$$=\frac{80\pi}{3}+\frac{332}{15}\pi$$

$$=\frac{244}{5}\pi\ u^3$$

EJEMPLOS

1) Calcular el volumen del sólido de revolución que se genera al girar la región D alrededor del eje Y, limitado por las ecuaciones:

$$y=\frac{1}{2}sen\left(\frac{\pi}{3}x\right);\; 16-4y=(x-4)^2;\; 4x-y-12=0$$



$A: \boxed{y=\frac{1}{2}sen\left(\frac{\pi}{3}x\right)}$

$B: 16-4y=(x-4)^2$

$\Rightarrow 16-(x-4)^2=4y$

$\Rightarrow \boxed{y=4-\frac{1}{4}(x-4)^2}$

$C: 4x-y-12=0 \Rightarrow \boxed{y=4x-12}$

$D=D_1\cup D_2$

i) Región D_1

Radio: $r=x$

Altura: $h=\underbrace{4-\frac{1}{4}(x-4)^2-\frac{1}{2}sen\left(\frac{\pi}{3}x\right)}_{B-A}$

$h=\frac{1}{4}\left[16-(x^2-8x+16)-2sen\left(\frac{\pi}{3}x\right)\right]$

$h=\frac{1}{4}\left[-x^2+8x-2sen\left(\frac{\pi}{3}x\right)\right]$

$V(S_1)=2\pi\int r\,h\,dx$

$V(S_1)=\cancel{2}\pi\int_0^3 x\cdot\frac{1}{\cancel{4}_{2}}\left[-x^2+8x-2sen\left(\frac{\pi}{3}x\right)\right]dx$

ii) Región D_2

Radio: $r=x$

Altura: $h=\underbrace{4-\frac{1}{4}(x-4)^2-(4x-12)}_{B-C}$

$\Rightarrow h=\frac{1}{4}\left[16-(x-4)^2-4(4x-12)\right]$

$\Rightarrow h=\frac{1}{4}\left[16-(x^2-8x+16)-16x+48\right.$

$\Rightarrow h=\frac{1}{4}\left[-x^2+8x-16x+48\right]$

$\Rightarrow h=\frac{1}{4}\left[-x^2-8x+48\right]$

$V(S_2)=2\pi\int r\,h\,dx$

$V(S_2)=2\pi\int_3^4 x\cdot\frac{1}{4}\left[-x^2+8x+48\right]dx$

$$V(S_1)=\frac{\pi}{2}\int_0^3\left[-x^3+8x^2-2x\,\text{sen}\left(\frac{\pi}{3}x\right)\right]dx$$

por partes

Derivar		Integrar
$2x$	$+$	$\text{sen}\left(\frac{\pi}{3}x\right)$
2	$-$	$-\cos\left(\frac{\pi}{3}x\right)\frac{3}{\pi}$
0		$-\text{sen}\left(\frac{\pi}{3}x\right)\frac{9}{\pi^2}$

$$V(S_1)=\frac{\pi}{2}\left[-\frac{x^4}{4}+\frac{8x^3}{3}-2\left(2x\left(-\cos\left(\frac{\pi}{3}x\right)\cdot\frac{3}{\pi}\right)-2\left(-\text{sen}\left(\frac{\pi}{3}x\right)\right)\cdot\frac{9}{\pi^2}\right)\right]_0^3$$

$$V(S_1)=\frac{\pi}{2}\left[-\frac{x^4}{4}+\frac{8x^3}{3}+\frac{12x}{\pi}\cos\left(\frac{\pi}{3}x\right)+\frac{18}{\pi^2}\text{sen}\left(\frac{\pi}{3}x\right)\right|_0^3$$

$$V(S_1)=\frac{\pi}{2}\left[-\frac{81}{4}+72+\frac{36}{\pi}\underbrace{\cos(\pi)}_{-1}+\frac{18}{\pi^2}\underbrace{\text{sen}(\pi)}_{0}\right]$$

$$V(S_1)=\frac{\pi}{2}\left[-\frac{81}{4}+72-\frac{36}{\pi}\right]$$

$$V(S_2)=\frac{\pi}{2}\left[\frac{207}{4}-\frac{36}{\pi}\right]$$

$$V(S_2)=\frac{207}{8}\pi-18$$

Luego,

$$V(S)=V(S_1)+V(S_2)$$

$$V(S)=\frac{207}{8}\pi-18+\frac{2675}{24}\pi$$

$$V(S)=\left(\frac{412}{3}\pi-18\right)u^3$$

$$V(S_2)=\frac{2\pi}{4}\int_3^4\left[-x^3+8x^2+48x\right]dx$$

$$V(S_2)=\frac{\pi}{2}\left[-\frac{x^4}{4}+\frac{8x^3}{3}+24x^2\right|_3^4$$

$$V(S_2)=\frac{\pi}{2}\left[-64+\frac{512}{3}+384-\left(-\frac{81}{4}+72+216\right)\right]$$

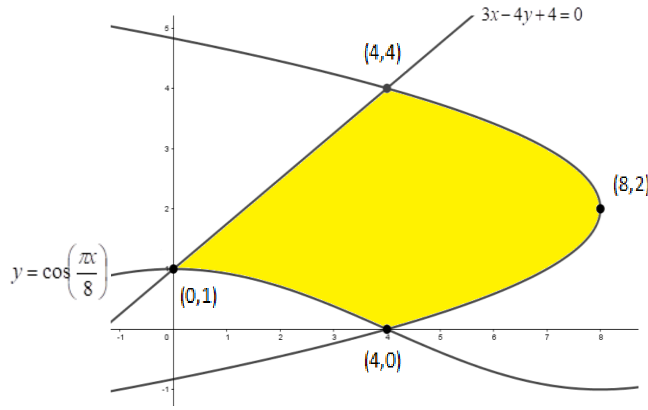
$$V(S_2)=\frac{\pi}{2}\left[\frac{1472}{3}-\frac{1071}{4}\right]$$

$$V(S_2)=\frac{\pi}{2}\left[\frac{2675}{12}\right]$$

$$\boxed{V(S_2)=\frac{2675\pi}{24}}$$

2) Calcular el volumen del sólido de revolución que se genera al girar la región D alrededor del eje X, limitado por las ecuaciones: $x-4=-y^2+4y;\ y=\cos\left(\frac{\pi x}{8}\right);\ 3x-4y+4=0$

SOLUCIÓN



•) A: $x-4=-y^2+4y$

$\Rightarrow \boxed{x=-y^2+4y+4}$

•) B: $y=\cos\left(\frac{\pi x}{8}\right)$

$\Rightarrow \arccos(y)=\frac{\pi x}{8}$

$\Rightarrow \boxed{x=\frac{8}{\pi}\arccos(y)}$

•) C: $3x-4y+4=0$

$\Rightarrow 3x=4y-4$

$\Rightarrow \boxed{x=\frac{4}{3}y-\frac{4}{3}}$

i) Región D_1

Radio: $r=y$

Altura: $h=-y^2+4y+4-\frac{8}{\pi}\arccos(y)$

$$V(S_1)=2\pi\int r\,h\,dy$$

$$V(S_1)=2\pi\int_0^1 y\left[-y^2+4y+4-\frac{8}{\pi}\arccos(y)\right]dy$$

$$V(S_1)=2\pi\int_0^1\left[-y^3+4y^2+4y-\frac{8}{\pi}y\arccos(y)\right]dy$$

$$V(S_1)=2\pi\left\{\int_0^1\left[-y^3+4y^2+4y\right]dy-\frac{8}{\pi}\underbrace{\int_0^1 y\arccos(y)\,dy}_{I}\right\}$$

$$V(S_1)=2\pi\left[-\frac{y^4}{4}+\frac{4y^3}{3}+2y^2-\frac{8}{\pi}\left(\frac{y^2}{2}\arccos(y)+\frac{1}{4}\operatorname{arcsen}(y)-\frac{y}{4}\sqrt{1-y^2}\right)\right|_0^1$$

$$=2\pi\left[-\frac{y^4}{4}+\frac{4y^3}{3}+2y^2-\frac{4y^2}{\pi}\arccos(y)-\frac{2}{\pi}\operatorname{arcsen}(y)+\frac{2y}{\pi}\sqrt{1-y^2}\right|_0^1$$

$$V(S_1) = 2\pi\left[\frac{37}{12} - \frac{4}{\pi}\underbrace{\arccos(1)}_{\to 0} - \frac{2}{\pi}\underbrace{\text{arcsen}(1)}_{\to \pi/2} - (0)\right]$$

$$V(S_1) = 2\pi\left[\frac{37}{12} - \frac{\cancel{2}}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{\cancel{2}}\right]$$

$$V(S_1) = 2\pi\left[\frac{37}{12} - 1\right] \Rightarrow V(S_1) = 2\pi\left[\frac{25}{12}\right] \Rightarrow \boxed{V(S_1) = \frac{25\pi}{6}}$$

ii) Región D_2

Radio : $r = y$

Altura : $h = -y^2 + 4y + 4 - \left(\frac{4y}{3} - \frac{4}{3}\right)$

$$h = -y^2 + \frac{8}{3}y + \frac{16}{3}$$

$$V(S_2) = 2\pi\int r h\, dy$$

$$V(S_2) = 2\pi\int_1^4 y\left[-y^2 + \frac{8}{3}y + \frac{16}{3}\right]dy$$

$$V(S_2) = 2\pi\int_1^4\left[-y^3 + \frac{8}{3}y^2 + \frac{16}{3}y\right]dy = 2\pi\left[-\frac{y^4}{4} + \frac{8}{9}y^3 + \frac{8}{3}y^2\right]\Big|_1^4 = 2\pi\left[\frac{320}{9} - \frac{119}{36}\right]$$

$$V(S_2) = 2\pi\left[\frac{129}{4}\right]$$

$$\boxed{V(S_2) = \frac{129\pi}{2}}$$

Por lo tanto, $V(S) = V(S_1) + V(S_2)$

$$V(S) = \frac{25\pi}{6} + \frac{129\pi}{2} = \frac{206\pi}{3}\ u^3$$

$$I = \int y\arccos(y)\,dy$$

por partes

$u = \arccos(y)$ $\quad dv = y\,dy$

$du = -\frac{1}{\sqrt{1-y^2}}dy$ $\quad v = \frac{y^2}{2}$

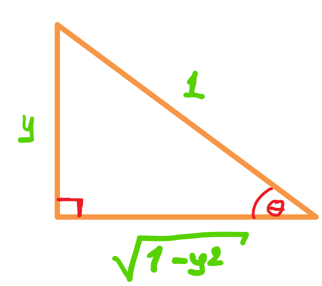
$$I = \frac{y^2}{2}\arccos(y) - \int\frac{y^2}{2}\left(-\frac{1}{\sqrt{1-y^2}}\right)dy$$

$$I = \frac{y^2}{2}\arccos(y) + \frac{1}{2}\int\frac{y^2}{\sqrt{1-y^2}}dy$$

Sust. trigonométrica

$y = \text{sen}\,\theta$

$dy = \cos\theta\, d\theta$



$$I = \frac{y^2}{2}\arccos(y) + \frac{1}{2}\int\frac{\text{sen}^2\theta}{\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\,d\theta$$

$$I = \frac{y^2}{2}\arccos(y) + \frac{1}{2}\int\text{sen}^2\theta\, d\theta$$

$$I = \frac{y^2}{2}\arccos(y) + \frac{1}{2}\int\frac{1-\cos(2\theta)}{2}d\theta$$

$$I = \frac{y^2}{2}\arccos(y) + \frac{1}{4}\left[\theta - \frac{\text{sen}(2\theta)}{2}\right]$$

$$I = \frac{y^2}{2}\arccos(y) + \frac{1}{4}\left[\theta - \frac{\cancel{2}\,\text{sen}\theta\cos\theta}{\cancel{2}}\right]$$

$$I = \frac{y^2}{2}\arccos(y) + \frac{1}{4}\left[\text{arcsen}(y) - y\sqrt{1-y^2}\right]$$

$$\boxed{I = \frac{y^2}{2}\arccos(y) + \frac{1}{4}\text{arcsen}(y) - \frac{y}{4}\sqrt{1-y^2}}$$