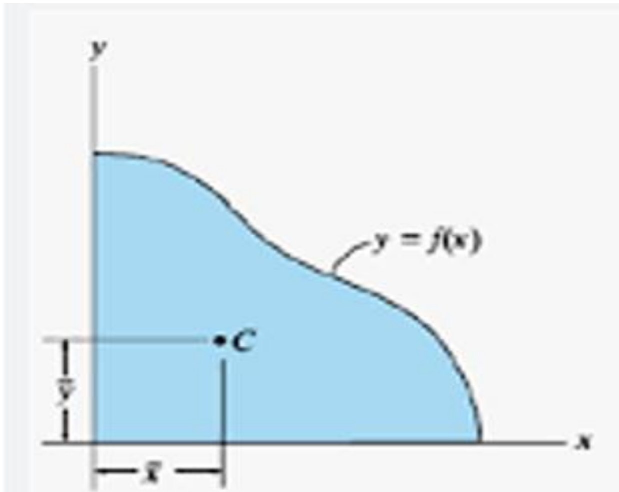
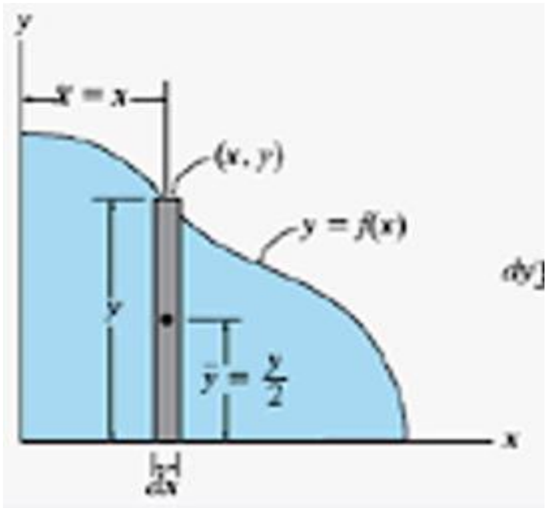
CENTRO DE MASA O CENTROIDE DE UNA REGIÓN PLANA

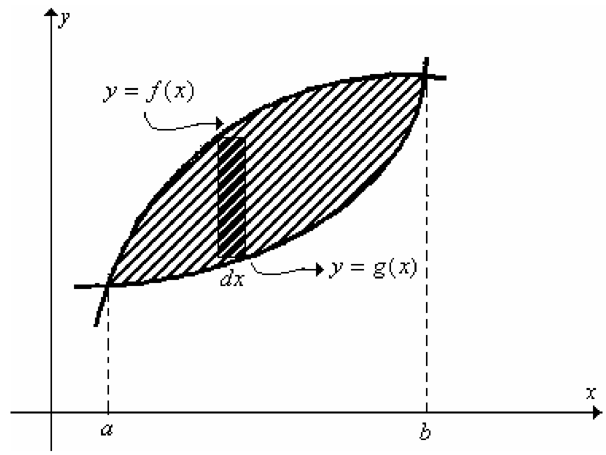
El centro de masa o centroide está dada por : $C = \left(\overline{X}, \overline{Y}\right)$



Caso 1

Si la región plana D está limitada por: $y = f(x), \quad y = g(x) \quad \wedge \quad x = a, \quad x = b.$





El centro de masa o centroide está dada por:

$$(\overline{X}, \overline{Y}) = \left(\frac{M_x}{A} \; , \; \frac{M_y}{A} \right)$$

donde

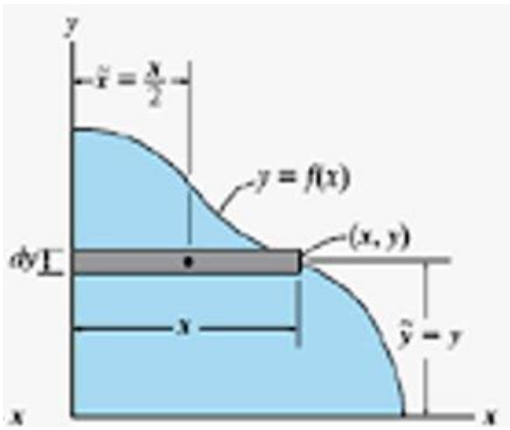
$$M_x = \int_a^b x\,[f(x) - g(x)]\,dx$$

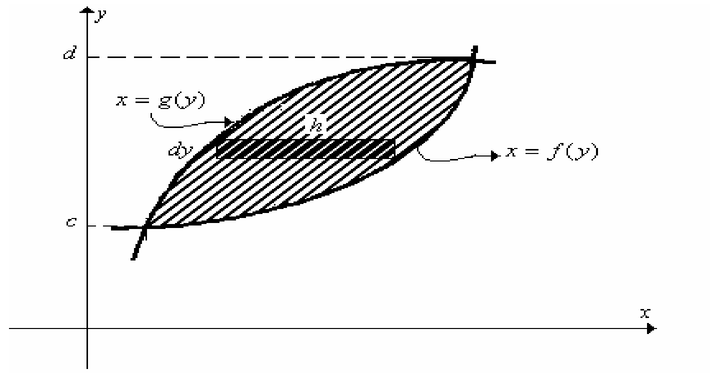
$$M_y = \frac{1}{2}\int_a^b \left[(f(x))^2 - (g(x))^2\right]dx$$

$A = A(D) = \int_a^b [f(x) - g(x)]\,dx$ (Área de la región plana)

Caso 2

Si la región plana D está limitada por: $x = f(y), \quad x = g(y) \quad \wedge \quad y = c, \quad y = d.$





El centro de masa o centroide está dada por:

$$(\overline{X}, \overline{Y}) = \left(\frac{M_x}{A}, \frac{M_y}{A} \right)$$

donde

$$M_x = \frac{1}{2} \cdot \int_c^d \left[(f(y))^2 - (g(y))^2 \right] dy$$

$$M_y = \int_c^d y \cdot [f(y) - g(y)] dy$$

$$A = A(D) = \int_c^d [f(y) - g(y)] dy$$ (Área de la región plana)

EJEMPLO

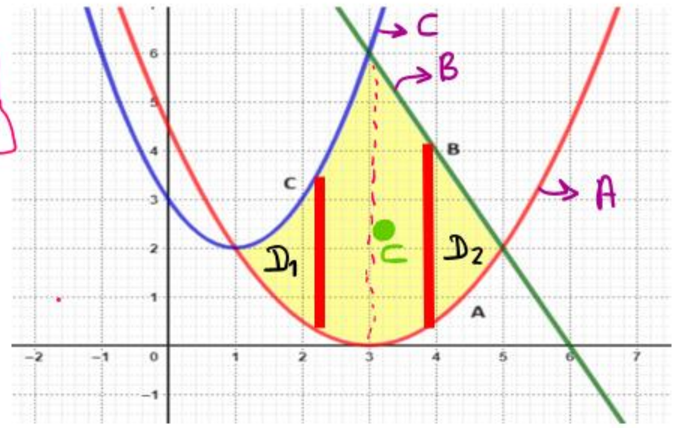
2) Hallar el centroide o centro de masa de la región R limitada por las curvas:

$$y=\frac{(x-3)^2}{2} \quad , \quad y+2x-12=0 \quad , \quad y-2=(x-1)^2$$

*) A: $\boxed{y=\dfrac{(x-3)^2}{2}}$

*) B: $y+2x-12=0 \Rightarrow \boxed{y=12-2x}$

*) C: $y-2=(x-1)^2 \Rightarrow \boxed{y=(x-1)^2+2}$



$$A=A(D) = \int_a^b [f(x)-g(x)]\,dx$$

i) $A(D)=A(D_1)+A(D_2)$

$$A(D)=\int_1^3 \Big[\overset{C\ -\ A}{(x-1)^2+2-\Big(\frac{(x-3)^2}{2}\Big)}\Big]dx+\int_3^5\Big[\overset{B-A}{12-2x-\frac{(x-3)^2}{2}}\Big]dx$$

$$A(D)=\Big[\frac{(x-1)^3}{3}+2x-\frac{1}{2}\cdot\frac{(x-3)^3}{3}\Big|_1^3+\Big[12x-x^2-\frac{1}{2}\cdot\frac{(x-3)^3}{3}\Big|_3^5$$

$$A(D)=\frac{8}{3}+6-\Big(2-\frac{1}{6}(-8)\Big)+60-25-\frac{1}{6}(8)-(36-9)$$

$$A(D)=\frac{16}{3}+\frac{20}{3}=\frac{36}{3}$$

$$\boxed{A=A(D)=12\,u^2}$$

ii) $$M_x = \int_a^b x\,[f(x)-g(x)]\,dx$$

$$M_x(D)=M_x(D_1)+M_x(D_2)$$

$$=\int_1^3 x\Big[\overset{C\ -\ A}{(x-1)^2+2-\frac{(x-3)^2}{2}}\Big]dx+\int_3^5 x\Big[\overset{B\ -\ A}{12-2x-\frac{(x-3)^2}{2}}\Big]dx$$

$$=\int_1^3 x\Big[x^2-2x+1+2-\frac{(x^2-6x+9)}{2}\Big]dx+\int_3^5 x\Big[12-2x-\frac{(x^2-6x+9)}{2}\Big]dx$$

$$=\frac{1}{2}\int_1^3 x\,[x^2+2x-3]\,dx+\frac{1}{2}\int_3^5 x\,[-x^2+2x+15]\,dx$$

Continental

CÁLCULO INTEGRAL

$$= \frac{1}{2}\int_1^3 \left[x^3+2x^2-3x\right]dx + \frac{1}{2}\int_3^5 \left[-x^3+2x^2+15x\right]dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4}+\frac{2x^3}{3}-\frac{3x^2}{2}\right]\Big|_1^3 + \frac{1}{2}\left[-\frac{x^4}{4}+\frac{2x^3}{3}+\frac{15x^2}{2}\right]\Big|_3^5$$

$$= \frac{1}{2}\left[\frac{81}{4}+18-\frac{27}{2}-\left(\frac{1}{4}+\frac{2}{3}-\frac{3}{2}\right)\right] + \frac{1}{2}\left[-\frac{625}{4}+\frac{250}{3}+\frac{375}{2}-\left(-\frac{81}{4}+18+\frac{135}{2}\right.\right.$$

$$= \frac{38}{3} + \frac{74}{3}$$

$$\boxed{M_x(D) = \frac{112}{3}}$$

iii) $M_y = \frac{1}{2}\int_a^b \left[(f(x))^2-(g(x))^2\right]dx$

$$M_y(D) = M_y(D_1) + M_y(D_2)$$

$$M_y(D) = \frac{1}{2}\int_1^3 \Big[\overbrace{((x-1)^2+2)^2}^{C^2} - \overbrace{\left(\left(\frac{(x-3)^2}{2}\right)^2\right)}^{A^2}\Big]dx + \frac{1}{2}\int_3^5 \Big[\overbrace{(12-2x)^2}^{B^2} - \overbrace{\left(\left(\frac{(x-3)^2}{2}\right)^2\right)}^{A^2}\Big]dx$$

$$M_y(D) = \frac{1}{2}\int_1^3 \left[(x-1)^4+4(x-1)^2+4-\frac{(x-3)^4}{4}\right]dx + \frac{1}{2}\int_3^5 \left[(12-2x)^2-\frac{(x-3)^4}{4}\right]dx$$

$$M_y(D) = \frac{1}{2}\left[\frac{(x-1)^5}{5}+4\frac{(x-1)^3}{3}+4x-\frac{(x-3)^5}{20}\right]\Big|_1^3 + \frac{1}{2}\left[\frac{(12-2x)^3}{3}\cdot\left(-\frac{1}{2}\right)-\frac{(x-3)^5}{20}\right]\Big|_3^5$$

$$M_y(D) = \frac{1}{2}\left[\frac{32}{5}+\frac{32}{3}+12-\left(4+\frac{8}{5}\right)\right] + \frac{1}{2}\left[-\frac{4}{3}-\frac{8}{5}-\left(-\frac{108}{3}\right)\right]$$

$$M_y(D) = \frac{176}{15} + \frac{248}{15} \Rightarrow \boxed{M_y(D) = \frac{424}{15}}$$

Luego, $\bar{C} = (\bar{x}, \bar{y}) = \left(\frac{M_x}{A}, \frac{M_y}{A}\right) = \left(\frac{\frac{112}{3}}{12}, \frac{\frac{424}{15}}{12}\right) = \left(\frac{28}{9}, \frac{106}{45}\right) \approx (3{,}1;\ 2{,}4)$

2) Hallar el centroide o centro de masa de la región R limitada por las curvas:

$$y=\sqrt{x+3},\ \ x=-2y,\ \ x=y+3.$$

A: $\boxed{y=\sqrt{x+3}}$ ✓

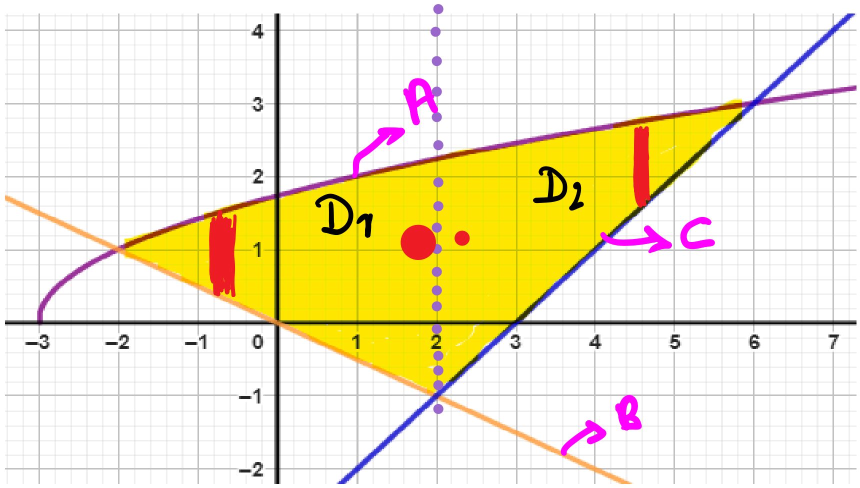
B: $x=-2y$

$\Rightarrow \boxed{y=-\frac{x}{2}}$

C: $x=y+3$

$\boxed{y=x-3}$

$D=D_1\cup D_2$



i) $A=A(D)=A(D_1)+A(D_2)$

$$A=\int_{-2}^{2}\left[\sqrt{x+3}-\left(\frac{-x}{2}\right)\right]dx+\int_{2}^{6}\left[\sqrt{x+3}-(x-3)\right]dx$$

$$A=\left[\frac{2}{3}(x+3)^{3/2}+\frac{x^2}{4}\right|_{-2}^{2}+\left[\frac{2}{3}(x+3)^{3/2}-\frac{x^2}{2}+3x\right|_{2}^{6}$$

$$A=\cancel{\frac{2}{3}(\sqrt{5})^3}+1-\left(\frac{2}{3}+1\right)+\cancel{18}-\cancel{18}+18-\left(\cancel{\frac{2}{3}(\sqrt{5})^3}-2+6\right)$$

$$\boxed{A=\frac{40}{3}\,u^2}$$

ii) $M_x = \int_a^b x[f(x)-g(x)]dx$

$$M_x = M_x(D) = M_x(D_1) + M_x(D_2)$$

$$M_x = \int_{-2}^{2} x\left[\sqrt{x+3} - \left(-\frac{x}{2}\right)\right]dx + \int_{2}^{6} x\left[\sqrt{x+3} - (x-3)\right]dx$$

$$M_x = \int_{-2}^{2} \left[x\sqrt{x+3} + \frac{x^2}{2}\right]dx + \int_{2}^{6} \left[x\sqrt{x+3} - x^2 + 3x\right]dx$$

$$\int x\sqrt{x+3}\,dx$$

Cambio de variable

$z = \sqrt{x+3}$

$z^2 = x+3$

$x = z^2 - 3$

$dx = 2z\,dz$

$$= \int (z^2-3)\,z \cdot 2z\,dz$$

$$= 2\int (z^2-3)\,z^2\,dz$$

$$= 2\int \left[z^4 - 3z^2\right]dz$$

$$= 2\left[\frac{z^5}{5} - \frac{3z^3}{3}\right]$$

$$= 2\left[\frac{(\sqrt{x+3})^5}{5} - (\sqrt{x+3})^3\right]$$

$$M_x = \left[2\left(\frac{(\sqrt{x+3})^5}{5} - (\sqrt{x+3})^3\right) + \frac{x^3}{6}\right|_{-2}^{2} + \left[2\left(\frac{(\sqrt{x+3})^5}{5} - (\sqrt{x+3})^3\right) - \frac{x^3}{3} + \frac{3x^2}{2}\right|_{2}^{6}$$

$$M_x = \cancel{2\left(\frac{(\sqrt{5})^5}{5} - (\sqrt{5})^3\right)} + \frac{4}{3} - \left(2\left(\frac{1}{5} - 1\right)\right) + 2\left(\frac{243}{5} - 27\right) - 72 + 54 - \left(\cancel{2\left(\frac{(\sqrt{5})^5}{5} - (\sqrt{5})^3\right)} - \frac{8}{3} + 6\right)$$

$$\boxed{M_x = \frac{124}{5}}$$

ii) $$M_y = \frac{1}{2}\int_a^b \left[(f(x))^2 - (g(x))^2\right]dx$$

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{n+1}\cdot\frac{1}{a}$$

$$M_y = M_y(D) = M_y(D_1) + M_y(D_2)$$

$$M_y = \frac{1}{2}\int_{-2}^{2}\left[\cancel{(\sqrt{x+3}\,)^2} - \left(-\frac{x}{2}\right)^2\right]dx + \frac{1}{2}\int_2^6\left[(\sqrt{x+3}\,)^2 - (x-3)^2\right]dx$$

$$M_y = \frac{1}{2}\int_{-2}^{2}\left[x+3-\frac{x^2}{4}\right]dx + \frac{1}{2}\int_2^6\left[x+3-(x-3)^2\right]dx$$

$$M_y = \frac{1}{2}\left[\frac{x^2}{2}+3x-\frac{x^3}{12}\right]_{-2}^{2} + \frac{1}{2}\left[\frac{x^2}{2}+3x-\frac{(x-3)^3}{3}\right]_2^6$$

$$M_y = \frac{1}{2}\left[2+6-\frac{2}{3}-\left(2-6+\frac{2}{3}\right)\right] + \frac{1}{2}\left[18+18-9-\left(2+6+\frac{1}{3}\right)\right]$$

$$\boxed{M_y = \frac{44}{3}}$$

Luego, $(\bar{x}, \bar{y}) = \left(\frac{M_x}{A}, \frac{M_y}{A}\right)$

$$= \left(\frac{\frac{124}{5}}{\frac{40}{3}}, \frac{\frac{44}{3}}{\frac{40}{3}}\right)$$

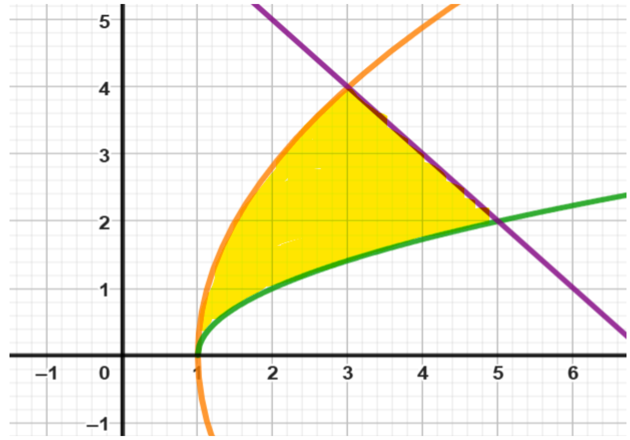
$$= (93/50,\ 11/10)$$

$$\approx (1{,}86\ ;\ 1{,}1)$$

3) Dada la región D limitada por las gráficas:

$$\mathrm{x+y=7}\ ,\quad \mathrm{y=\sqrt{x-1}},\quad \mathrm{x=\frac{y^2}{8}+1}\ ,\quad \mathrm{IC.}$$

Hallar el centroide o centro de masa de la región D



4) Dada la región D limitada por las gráficas:

$$\mathrm{x}=2+\sqrt{y}\quad ,\quad \mathrm{y}=8-(x-2)^2\quad ,\quad \mathrm{2x+y=4}\quad ,\quad \mathrm{IC.}$$

Hallar el centroide o centro de masa de la región D