INTEGRALES IMPROPIAS

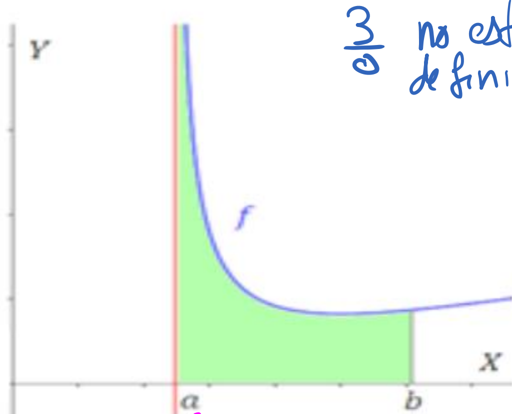
1. Si $f:(a,b]\to\Re$ es una función continua en $(a,b]$ y con asíntota vertical $x=a$, entonces la integral impropia $\int_a^b f(x)dx$ se define como: $\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx.$

Si existe el limite diremos que la integral impropia es convergente, caso contrario es divergente.

$\int_0^3 (x^2+2x)\,dx$

$\frac{3}{0}$ no está definida

$\int_0^1 \frac{3x+1}{x^2+2x}\,dx$



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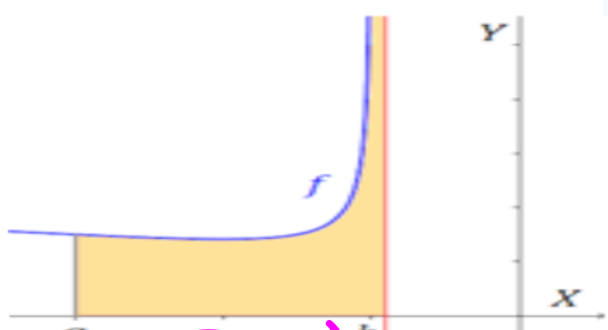
$\int_{-2}^5 \frac{3}{\sqrt{4-x}}\,dx$

$\int_0^3 \frac{3}{\sqrt[3]{2+x}}\,dx$

2. Si $f:[a,b)\to\Re$ es una función continua en $[a,b)$ y con asíntota vertical $x=b$, entonces la integral impropia $\int_a^b f(x)dx$ se define como: $\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx.$

Si existe el limite diremos que la integral impropia es convergente, caso contrario es divergente.



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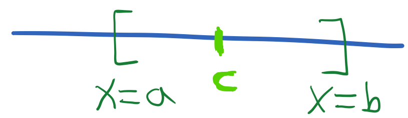
3. Si $f:[a,b]\to\Re$ es una función continua en $[a,b]$ excepto en $x=c$, donde $a<c<b$, entonces la integral impropia $\int_a^b f(x)dx$ se define como:

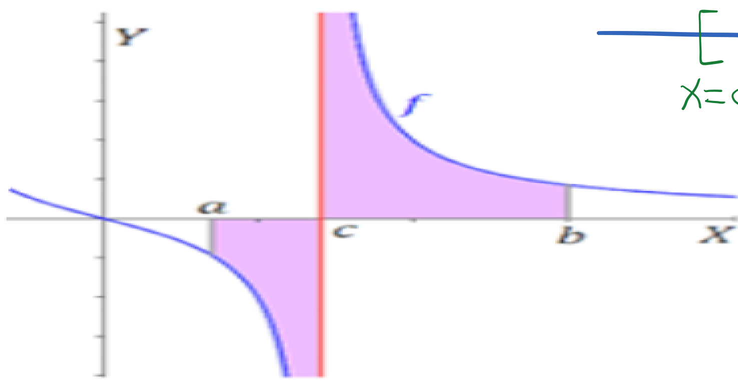
$$\int_a^b f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx$$

$$\int_a^b f(x)dx=\lim_{t\to c^-}\int_a^t f(x)dx+\lim_{w\to c^+}\int_w^b f(x)dx.$$

Si ambas integrales impropias de la derecha son convergentes, entonces $\int_a^b f(x)dx$ es convergente. Si una o ambas divergen entonces la $\int_a^b f(x)dx$ es divergente.

$[a, b]$





EJEMPLOS

Determine la convergencia o divergencia de las siguientes integrales impropias:

1) $\displaystyle\int_0^1 \frac{1}{(2-x)\sqrt{1-x}}\,dx$

Sol.

i) $f(x) = \dfrac{1}{(2-x)\sqrt{1-x}}$

f no está definida en $\begin{cases} x = 2 \notin [0,1] \\ x = 1 \in [0,1] \end{cases}$

$\Rightarrow$ f no es continua en $\boxed{x=1}$

$x=0$ $x=1$

ii) $$\int_0^1 \frac{1}{(2-x)\sqrt{1-x}}\,dx = \lim_{t\to 1^-} \int_0^t \underbrace{\frac{1}{(2-x)\sqrt{1-x}}}_{\text{cambio de variable}}\,dx$$

$z = \sqrt{1-x}$
$z^2 = 1-x$
$x = 1-z^2$
$dx = -2z\,dz$

$$= \lim_{t\to 1^-} \int_0^t \frac{1}{[2-(1-z^2)]\cancel{z}} \cdot (-2\cancel{z}\,dz)$$

$$= \lim_{t\to 1^-} -2\int_0^t \underbrace{\frac{1}{z^2+1}}_{\text{fórmula}}\,dz$$

$$= \lim_{t\to 1^-} -2\left[\operatorname{arctg}(z)\right]\Big|_0^t$$

$$= \lim_{t\to 1^-} -2\left[\operatorname{arctg}(\sqrt{1-x})\right]\Big|_0^t$$

$$= \lim_{t\to 1^-} -2\left[\operatorname{arctg}(\sqrt{1-t}) - \operatorname{arctg}(\sqrt{1-0})\right] \quad \operatorname{arctg}(\sqrt{1-0}) \to \pi/4$$

$$= \lim_{t\to 1^-} -2\left[\operatorname{arctg}(\sqrt{1-t}) - \frac{\pi}{4}\right] = -2\left[0 - \frac{\pi}{4}\right] = \frac{\pi}{2} \text{ (converge)}$$

2) $\displaystyle\int_0^2 \frac{x^3}{\sqrt{4-x^2}}\,dx$

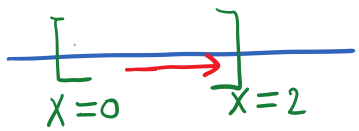
Sol.

i) $f(x) = \dfrac{x^3}{\sqrt{4-x^2}}$

f no está definida si $\begin{cases} x = 2 \in [0,2] \\ x = -2 \notin [0,2] \end{cases}$

f no es continua en $x=2$

ii) Luego,



$$\int_0^2 \frac{x^3}{\sqrt{4-x^2}}\,dx = \lim_{t\to 2^-} \underbrace{\int_0^t \frac{x^3}{\sqrt{4-x^2}}\,dx}_{\text{Cambio de variable}}$$

Cambio de variable

$z = \sqrt{4-x^2}$

$z^2 = 4-x^2$

$\Rightarrow x^2 = 4-z^2$

$\Rightarrow 2x\,dx = -2z\,dz$

$\Rightarrow x\,dx = -z\,dz$

$$= \lim_{t\to 2^-} \int_0^t \frac{x^2 \cdot x}{\sqrt{4-x^2}}\,dx$$

$$= \lim_{t\to 2^-} \int_0^t \frac{(4-z^2)}{\cancel{z}} (-\cancel{z}\,dz)$$

$$= -\lim_{t\to 2^-} \int_0^t (4-z^2)\,dz$$

$$= -\lim_{t\to 2^-} \left[4z - \frac{z^3}{3}\right|_0^t$$

$$= -\lim_{t\to 2^-} \left[4\sqrt{4-x^2} - \frac{(\sqrt{4-x^2})^3}{3}\right|_0^t$$

$$= -\lim_{t\to 2^-} \left[4\sqrt{4-t^2} - \frac{(\sqrt{4-t^2})^3}{3} - \left(4\sqrt{4} - \frac{(\sqrt{4})^3}{3}\right)\right]$$

$$= -\lim_{t\to 2^-}\left[4\cancelto{0}{\sqrt{4-t^2}} - \frac{\cancelto{0}{(\sqrt{4-t^2})}}{3} - \left(8-\frac{8}{3}\right)\right]$$

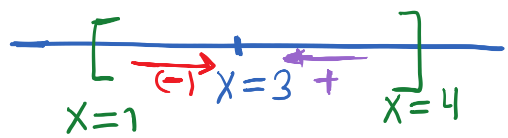
$$= -\left[0-0-\frac{16}{3}\right] = \frac{16}{3} \rightarrow \text{converge}$$

3) $\displaystyle\int_1^4 \frac{x^2}{\sqrt[3]{3-x}}\,dx$

Sol.

i) $f(x) = \dfrac{x^2}{\sqrt[3]{3-x}}$ no está definida en $x=3 \in [1,4]$

$\Rightarrow f$ no es continua en $x=3$



ii) Luego,

$$\int_1^4 \frac{x^2}{\sqrt[3]{3-x}}\,dx = \int_1^3 \frac{x^2}{\sqrt[3]{3-x}}\,dx + \int_3^4 \frac{x^2}{\sqrt[3]{3-x}}\,dx$$

$$= \lim_{t\to 3^-}\int_1^t \frac{x^2}{\sqrt[3]{3-x}}\,dx + \lim_{t\to 3^+}\int_t^4 \frac{x^2}{\sqrt[3]{3-x}}\,dx$$

Cambio de variable

$z = \sqrt[3]{3-x}$

$\Rightarrow z^3 = 3-x$

$\Rightarrow x = 3-z^3$

$dx = -3z^2\,dz$

$$= \lim_{t\to 3^-}\int_1^t \frac{(3-z^3)^2}{\cancel{z}}(-3z^{\cancel{2}})\,dz + \lim_{t\to 3^+}\int_t^4 \frac{(3-z^3)^2}{\cancel{z}}(-3z^{\cancel{2}})\,dz$$

$$= \lim_{t\to 3^-} -3\int_1^t \left(9-6z^3+z^6\right) z\,dz + \lim_{t\to 3^+} -3\int_t^4 \left(9-6z^3+z^6\right) z\,dz$$

$$= \lim_{t\to 3^-} -3\int_1^t \left(9z-6z^4+z^7\right) dz + \lim_{t\to 3^+} -3\int_t^4 \left(9z-6z^4+z^7\right) dz$$

$$= \lim_{t\to 3^-} -3\left[\frac{9z^2}{2}-\frac{6z^5}{5}+\frac{z^8}{8}\right|_1^t + \lim_{t\to 3^+} -3\left[\frac{9z^2}{2}-\frac{6z^5}{5}+\frac{z^8}{8}\right|_t^4$$

$$= \lim_{t\to 3^-} -3\left[\frac{9}{2}\left(\sqrt[3]{3-x}\right)^2-\frac{6}{5}\left(\sqrt[3]{3-x}\right)^5+\frac{1}{8}\left(\sqrt[3]{3-x}\right)^8\right|_1^t$$

$$+ \lim_{t\to 3^+} -3\left[\frac{9}{2}\left(\sqrt[3]{3-x}\right)^2-\frac{6}{5}\left(\sqrt[3]{3-x}\right)^5+\frac{1}{8}\left(\sqrt[3]{3-x}\right)^8\right|_t^4$$

$$= \lim_{t\to 3^-} -3\left[\frac{9}{2}\underbrace{\left(\sqrt[3]{3-t}\right)^2}_{0}-\frac{6}{5}\underbrace{\left(\sqrt[3]{3-t}\right)^5}_{0}+\frac{1}{8}\underbrace{\left(\sqrt[3]{3-t}\right)^8}_{0}-\left(\frac{9}{2}\left(\sqrt[3]{2}\right)^2-\frac{6}{5}\left(\sqrt[3]{2}\right)^5+\frac{1}{8}\left(\sqrt[3]{2}\right)^8\right)\right]$$

$$+ \lim_{t\to 3^+} -3\left[\frac{9}{2}(-1)^2-\frac{6}{5}(-1)^5+\frac{1}{8}(-1)^8-\left(\frac{9}{2}\underbrace{\left(\sqrt[3]{3-t}\right)^2}_{0}-\frac{6}{5}\underbrace{\left(\sqrt[3]{3-t}\right)^5}_{0}+\frac{1}{8}\underbrace{\left(\sqrt[3]{3-t}\right)^8}_{0}\right)\right]$$

$$= -3\left[0-\left(\frac{9}{2}\left(\sqrt[3]{2}\right)^2-\frac{6}{5}\left(\sqrt[3]{2}\right)^5+\frac{1}{8}\left(\sqrt[3]{2}\right)^8\right)\right]-3\left[\frac{9}{2}+\frac{6}{5}+\frac{1}{8}\right]$$

$\frac{3}{0}=\infty$ ↳ no existe

Luego, $\int_1^4 \frac{x^2}{\sqrt[3]{3-x}}\,dx$ es convergente

4) $\displaystyle\int_{\frac{1}{2}}^{1} \frac{1}{x\sqrt[7]{\ln^2 x}}\,dx,$

$\ln(1) = 0$

Sol. $\quad f(x) = \dfrac{1}{x\sqrt[7]{\ln^2 x}}$ $\quad$ f no está definida en: $\begin{cases} x = 0 \notin [1/2, 1] \\ x = 1 \in [1/2, 1] \end{cases}$

f no es continua en $x = 1$

$$\int_{1/2}^{1} \frac{1}{x(\ln x)^{2/7}}\,dx$$

$$= \lim_{t\to 1^-} \int_{1/2}^{t} \frac{1}{x(\ln x)^{2/7}}\,dx$$

sust. simple

$m = \ln x$

$dm = \dfrac{1}{x}\,dx$

$$\int_{1/2}^{t} \frac{1}{(\ln x)^{2/7}} \cdot \frac{1}{x}\,dx$$

$$= \lim_{t\to 1^-} \int_{1/2}^{t} \frac{1}{m^{2/7}}\,dm$$

$$= \lim_{t\to 1^-} \int_{1/2}^{t} m^{-2/7}\,dm$$

$$= \lim_{t\to 1^-} \left[\frac{7}{5} m^{5/7}\right|_{1/2}^{t}$$

$$= \lim_{t\to 1^-} \frac{7}{5}\left[(\ln x)^{5/7}\right|_{1/2}^{t}$$

$$= \lim_{t\to 1^-} \frac{7}{5}\left[(\ln(t))^{5/7} - (\ln(1/2))^{5/7}\right]$$

$$= \frac{7}{5}\left[0 - (\ln(1/2))^{5/7}\right] = -\frac{7}{5}(\ln(1/2))^{5/7}$$ existe

Luego, $\displaystyle\int_{1/2}^{1} \frac{1}{x\sqrt[7]{\ln^2 x}}\,dx$ converge

5) $\displaystyle\int_0^{\sqrt{\frac{2}{\pi}}} \frac{1}{x^3}\cos\left(\frac{1}{x^2}\right)dx,$ $\quad$ *Rpta. diverge*

6) $\displaystyle\int_{\sqrt{2}}^{\sqrt{10}} \frac{2x\ln\left(x^2-2\right)}{\sqrt[3]{x^2-2}}dx$

7) $\displaystyle\int_1^{e} \frac{1}{x\ln^3 x}dx,$ $\quad$ *Rpta. diverge*

8) $\displaystyle\int_0^{\frac{1}{2}} \frac{1}{x\sqrt[3]{\ln x}}dx,$

9) $\displaystyle\int_2^{4} \frac{1}{\sqrt{6x-x^2-8}}dx,$ $\quad$ *Rpta.* π

10) $\displaystyle\int_0^{\pi/4} \left(\frac{1}{x}-\frac{1}{senx.\cos x}\right)dx$

11) $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx,$

12) $\displaystyle\int_0^{2} \frac{x}{\left(x^2-1\right)^{\frac{4}{3}}}dx$

13) $\displaystyle\int_0^{6} \frac{2x}{\left(x^4-4\right)^{\frac{2}{3}}}dx,$

14) $\displaystyle\int_0^{1} \frac{1}{x^3-5x^2}dx,$ $\quad$ *Rpta. diverge*