Low-lying zeros in a family of holomorphic cusp forms

Lucile Devin
joint work with Daniel Fiorilli and Anders Södergren

Chalmers och Göteborgs Universitet

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Zeros of L-functions

Let L be an L-function.
Where are the (non-trivial) zeros of L?

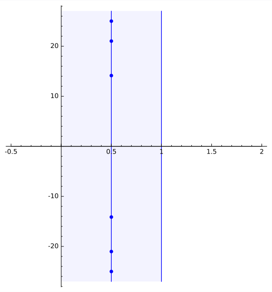

FIGURE – Zeros of ζ

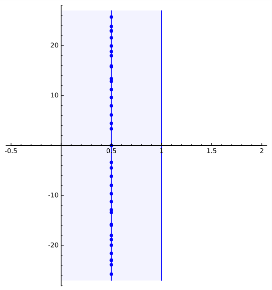

FIGURE – Zeros of some L function from LMFDB

Zeros of L-functions

Let L be an L-function.
How many zeros of L are close to $\frac{1}{2}$?

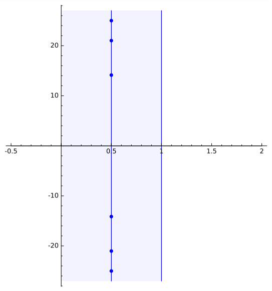

FIGURE – Zeros of ζ

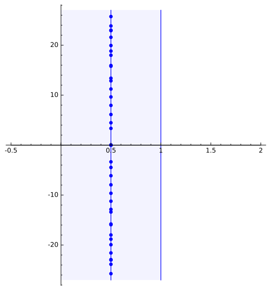

FIGURE – Zeros of some L function from LMFDB

Zeros of L-functions

Let L be an L-function.
How many zeros of L are close to $\frac{1}{2}$?
$\approx \frac{20}{2\pi} \log \frac{20C}{2\pi}$ zeros of L with imaginary part ≤ 20, where C is the conductor.

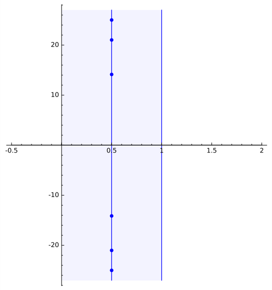

FIGURE – Zeros of ζ

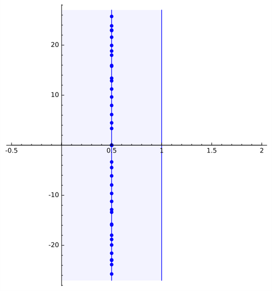

FIGURE – Zeros of some L function from LMFDB

A normalization is necessary : we multiply the imaginary part by $\frac{\log C}{2\pi}$

Counting zeros close to $\frac{1}{2}$

Let $L(f,s)$ be an L-function of conductor c_f, then $L(f,s)$ has $\approx 20\frac{\log c_f}{2\pi}$ zeros of imaginary part smaller than 20.

Definition (One-level density for one L-function)

Let $L(f,s)$ be an L-function of conductor c_f and let ϕ be an even Schwartz function we define

$$D(f,\phi) = \sum_{\substack{\gamma \\ L(f,\frac{1}{2}+i\gamma)=0}} \phi(\frac{\gamma}{2\pi}\log c_f)$$

Counting zeros close to $\frac{1}{2}$ in a family

Let $\mathcal{F}$ be a *family* of L-functions, and

$$\mathcal{F}(Q) = \begin{cases} \{f \in \mathcal{F} : c_f \leq Q\} \text{ or} \\ \{f \in \mathcal{F} : c_f = Q\} \text{ or} \\ \{f \in \mathcal{F} : Q \leq c_f \leq 2Q\} \text{ or } \ldots \end{cases}$$

with $|\mathcal{F}(Q)| \xrightarrow[Q \to \infty]{} \infty$.
Average over the family :

Definition (One-level density for a family of L-function)

Let $\mathcal{F}$ be a family of L-functions, let ϕ be an even Schwartz function we define

$$\mathcal{D}(\mathcal{F}(Q), \phi) = \frac{1}{|\mathcal{F}(Q)|} \sum_{f \in \mathcal{F}(Q)} \sum_{\substack{\gamma \\ L(f, \frac{1}{2} + i\gamma) = 0}} \phi(\frac{\gamma}{2\pi} \log Q)$$

The Katz–Sarnak heuristic

Inspired by ideas of Dyson, Montgomery and Odlyzko, using function fields analogues and random matrix models,

Conjecture (Katz–Sarnak)

For any family $\mathcal{F}$, there exist a symmetry type $G(\mathcal{F})$, such that for any even Schwartz function ϕ, one has

$$\lim_{Q\to\infty} \mathcal{D}(\mathcal{F}(Q), \phi) = \int \phi(x) W(G(\mathcal{F}))(x)\, \mathrm{d}x$$

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For any family $\mathcal{F}$, there exist a symmetry type $G(\mathcal{F})$, such that for any even Schwartz function ϕ, one has

$$\begin{aligned}\lim_{Q\to\infty} \mathcal{D}(\mathcal{F}(Q),\phi) &= \int \phi(x) W(G(\mathcal{F}))(x)\,\mathrm{d}x \\ &= \int \hat{\phi}(x) \widehat{W(G(\mathcal{F}))}(x),\end{aligned}$$

where $\widehat{\phi}(\xi) := \int_{\mathbf{R}} \phi(x) e^{-2\pi i \xi x} dx$.

Katz–Sarnak heuristic – symmetry types

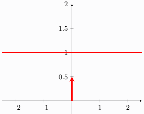
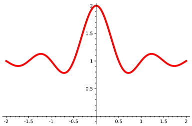
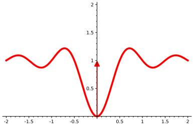
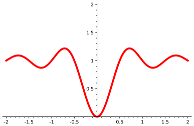
Symmetry type	$\int \phi W$	graph of W
O	$\int \phi + \frac{1}{2}\phi(0)$	
$SO(\text{even})$	$\int \phi + \int \phi(x)\frac{\sin(2\pi x)}{2\pi x}\,\mathrm{d}x$	
$SO(\text{odd})$	$\int \phi + \phi(0) - \int \phi(x)\frac{\sin(2\pi x)}{2\pi x}\,\mathrm{d}x$	
Sp	$\int \phi - \int \phi(x)\frac{\sin(2\pi x)}{2\pi x}\,\mathrm{d}x$	

TABLE – Possible symmetry types in real families

Katz–Sarnak heuristic – symmetry types

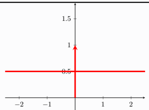
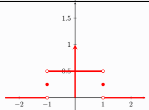
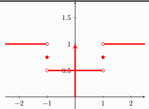
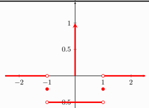
Symmetry type	$\int \hat{\phi}\hat{W}$	graph of $\hat{W}$
O	$\hat{\phi}(0) + \frac{1}{2}\phi(0)$	
$SO(\text{even})$	$\hat{\phi}(0) + \frac{1}{2}\int_{-1}^{1}\hat{\phi}$	
$SO(\text{odd})$	$\hat{\phi}(0) + \phi(0) - \frac{1}{2}\int_{-1}^{1}\hat{\phi}$	
Sp	$\hat{\phi}(0) - \frac{1}{2}\int_{-1}^{1}\hat{\phi}$	

Table – Possible symmetry types in real families

Determining the symmetry type

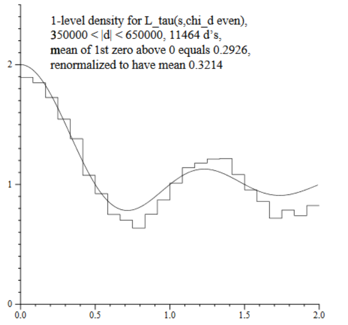
- Function fields analogue, Katz–Sarnak (1999)
- Direct calculation of $\lim_{Q\to\infty} \mathcal{D}(\mathcal{F}(Q), \phi)$:
    - Asymptotics with restriction on the support of $\hat{\phi}$ (maybe also under GRH). If one can extend the support of $\hat{\phi}$ up to $[-1-\epsilon, 1+\epsilon]$ then the conjectural symmetry type is determined.
      Özlük–Snyder (1999), Iwaniec–Luo–Sarnak (2000),...
    - Ratios conjecture of Conrey–Farmer–Zirnbauer, Conrey–Snaith (2007), Miller (2008–),...
- Study the n-level density for $n \geq 2$:
  $$\frac{1}{|\mathcal{F}(Q)|} \sum_{f\in\mathcal{F}(Q)} \sum_{\substack{\gamma_1,\ldots,\gamma_n \text{ distinct} \\ L(f,\frac{1}{2}+i\gamma_j)=0}} \phi\left(\frac{\gamma_1}{2\pi}\log Q, \ldots, \frac{\gamma_1}{2\pi}\log Q\right) \text{ where } \phi : \mathbf{R}^n \to \mathbf{R}$$
  with some restriction on the support of $\hat{\phi}$.
  Rubinstein (1998), Cho–Kim (2015),...
- Determine the symmetry type directly from invariants of the family, Duéñez–Miller (2006), Sarnak–Shin–Templier (2016).

Lower order terms

Study

$$\mathcal{D}(\mathcal{F}(Q),\phi) - \int \phi(x) W(G(\mathcal{F}))(x)\,\mathrm{d}x.$$

How does this depend on the family $\mathcal{F}$?



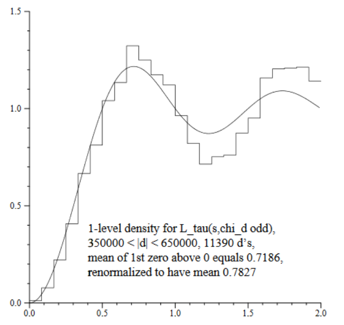


FIGURE – Rubinstein (1998) – 1-level densities for two families with symmetry type SO(even) and SO(odd)

Lower order terms

Study

$$\mathcal{D}(\mathcal{F}(Q),\phi) - \int \phi(x) W(G(\mathcal{F}))(x)\,\mathrm{d}x.$$

How does this depend on the family $\mathcal{F}$? Is there also a transition when $\mathrm{supp}(\hat{\phi})$ reaches 1 ?

- Symplectic families : Fouvry–Iwaniec (2003), Rudnick (2010), Fiorilli–Parks–Södergren (2017), Waxman ... lower order terms involving $\hat{\phi}(1)$
- Orthogonal families : Miller (2009), Ricotta–Royer (2010)... the lower order terms do not have a transition at 1.
- Special Orthogonal families : Miller–Montague (2011), D.–Fiorilli–Södergren : lower order terms involving $\hat{\phi}(1)$ again.

Holomorphic cusp forms of level 1

We fix a basis B_k of Hecke new eigenforms in the space H_k of holomorphic modular forms of level 1 and even weight k. For $\mathrm{Re}(s) > 1$ the L-function of $f \in B_k$ takes the form

$$\begin{aligned} L(s,f) &= \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \\ &= \prod_p \Big(1 - \frac{\alpha_f(p)}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_f(p)}{p^s}\Big)^{-1} \qquad (\mathrm{Re}(s) > 1), \end{aligned}$$

where $\lambda_f(n)$ are the Hecke eigenvalues of f, and $|\alpha_f(p)| = |\beta_f(p)| = 1$.
One has $c_f = k^2$, $L(s,f)$ extends to an entire function and satisfies a functional equation relating the values at s to those at $1-s$ with sign $(-1)^{\frac{k}{2}}$.

One-level density over the family

For an even Schwartz function ϕ,
for h a non-negative, not identically zero smooth weight function with compact support in $\mathbf{R}_{>0}$,

One-level densities over families with constant sign of the functional equation :

$$\mathscr{D}^{\pm}_{K,h}(\phi) := \frac{1}{H^{\pm}(K)} \sum_{k\equiv 3\pm 1 \bmod 4} h\Big(\frac{k-1}{K}\Big)\frac{1}{\Omega_k}\sum_{f\in B_k}\omega_f \sum_{\gamma_f}\phi\Big(\gamma_f \frac{\log K^2}{2\pi}\Big),$$

where $H^{\pm}(K) = \sum_{k\equiv 3\pm 1 \bmod 4} h\Big(\frac{k-1}{K}\Big)$,

$$\omega_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}}\left(\int_{SL_2(\mathbf{Z})\backslash\mathbf{H}} y^{k-2}|f(z)|^2 dxdy\right)^{-1} \approx k^{-1},$$

(harmonic weights) and $\Omega_k := \sum_{f\in B_k}\omega_f = 1 + O(2^{-k})$.

Symmetry type

Theorem (Iwaniec–Luo–Sarnak, 2000)

Assuming the Generalized Riemann Hypothesis. The symmetry type is special orthogonal even or odd depending on the sign of the functional equation. More precisely, if $\mathrm{supp}(\hat{\phi}) \subset (-2, 2)$, *one has*

$$\lim_{K\to\infty} \mathscr{D}^{\pm}_{K,h}(\phi) = \int_{\mathbf{R}} \widehat{\phi} \cdot \widehat{W}^{\pm},$$

with

$$\widehat{W}^{+}(t) = \widehat{W}(SO(even))(t); \quad \widehat{W}^{-}(t) = \widehat{W}(SO(odd))(t).$$

Lower order terms

Theorem (D.–Fiorilli–Södergren)

Let ϕ be an even Schwartz test function for which $\mathrm{supp}(\hat{\phi}) \subset (-2,2)$, and let h be a non-negative, not identically zero smooth weight function with compact support in $\mathbf{R}_{>0}$. Assuming the Riemann Hypothesis for Dirichlet L-functions, we have the estimate

$$\mathscr{D}^{\pm}_{K,h}(\phi) = \int_{\mathbf{R}} \widehat{\phi} \cdot \widehat{W}^{\pm} + \sum_{1 \le j \le J} \frac{R_{j,h}\widehat{\phi}^{(j-1)}(0) \pm S_{j,h}\widehat{\phi}^{(j-1)}(1)}{(\log K)^j} + O_{\phi,h,J}\Big(\frac{1}{(\log K)^{J+1}}\Big),$$

where the constants $R_{j,h}$ and $S_{j,h}$ appearing in the lower-order terms can be made explicit and only depend on the weight function h.

Lower order terms

Theorem (D.–Fiorilli–Södergren)

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$$\mathscr{D}^{\pm}_{K,h}(\phi) = \int_{\mathbf{R}} \widehat{\phi} \cdot \widehat{W}^{\pm} + \frac{\frac{\int_0^\infty h \cdot \log}{\int_0^\infty h} - \log(4\pi) - \gamma - \sum_p \frac{\log p}{p(p-1)}}{\log K} \big(\widehat{\phi}(0) \pm \widehat{\phi}(1)\big) + O_{\phi,h}\Big(\frac{1}{(\log K)^2}\Big),$$

Explicit formula

Lemma

Let ϕ be an even Schwartz test function. For $f \in B_k$, we have the formula

$$\sum_{\gamma_f} \phi\Big(\gamma_f \frac{\log K^2}{2\pi}\Big) = -2\widehat{\phi}(0)\frac{\log \pi}{\log K^2} + \frac{1}{\log K^2}\int_{\mathbf{R}} \Big(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{k+1}{4} + \frac{\pi i t}{\log K^2}\Big) + \frac{\Gamma'}{\Gamma}\Big(\frac{1}{4} + \frac{k-1}{4} + \frac{\pi i t}{\log K^2}\Big)\Big)\phi(t)dt - 2\sum_{p,\nu} \frac{\alpha_f^\nu(p) + \beta_f^\nu(p)}{p^{\frac{\nu}{2}}}\widehat{\phi}\Big(\frac{\nu \log p}{\log K^2}\Big)\frac{\log p}{\log K^2}.$$

Here, $\alpha_f(p), \beta_f(p)$ are the local coefficients of the L-function $L(s,f)$, in particular we have $|\alpha_f(p)| = |\beta_f(p)| = 1$.

Petersson trace formula

We study $\sum_{f \in B_k} \omega_f(\alpha_f^\nu(p) + \beta_f^\nu(p))$,
by the Hecke relations, one has

$$\alpha_f^\nu(p) + \beta_f^\nu(p) = \begin{cases} \lambda_f(p), \text{ if } \nu = 1 \\ \lambda_f(p^\nu) - \lambda_f(p^{\nu-2}), \text{ if } \nu \geq 2. \end{cases}$$

Lemma (Petersson Trace formula)

Let $m, k \in \mathbf{N}$, with $2 \mid k$. We have the exact formula

$$\sum_{f \in B_k} \omega_f \lambda_f(m) = \delta(m,1) + 2\pi i^k \sum_{c \geq 1} c^{-1} S(m,1;c) J_{k-1}\Big(\frac{4\pi\sqrt{m}}{c}\Big),$$

where $S(m,1;c) = \sum_{\substack{x \mod c \\ (x,c)=1}} e(\frac{mx+\bar{x}}{c})$.

Bounds on Bessel functions

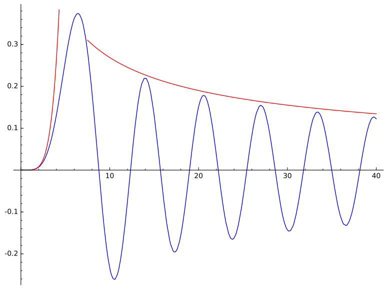

Figure – Comparing J_5 to $\frac{1}{5!}\left(\frac{x}{2}\right)^5$ and $x^{-1/2}$

Lemma

Let $k \in \mathbf{N}$. We have the bound

$$J_{k-1}(x) \ll \min\left(\frac{1}{(k-1)!}\left(\frac{x}{2}\right)^{k-1}, x^{-\frac{1}{4}}(|x-k+1| + k^{\frac{1}{3}})^{-\frac{1}{4}}\right).$$

Petersson trace formula

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$$= \delta(m,1) + \begin{cases} O_\epsilon\left(\frac{m^{\frac{1}{4}+\epsilon}}{k} + \frac{k^{\frac{1}{6}}}{m^{\frac{1}{4}-\epsilon}}\right) \\ O_\epsilon\left(2^{-k} m^{\frac{1}{4}+\epsilon}\right) \text{ if } m \leq \frac{k^2}{(4\pi e)^2}. \end{cases}$$

Petersson trace formula

Lemma (Petersson Trace formula)

Let $m, k \in \mathbf{N}$, with $2 \mid k$. We have the exact formula

$$\sum_{f \in B_k} \omega_f \lambda_f(m) = \delta(m,1) + 2\pi i^k \sum_{c \geq 1} c^{-1} S(m,1;c) J_{k-1}\Big(\frac{4\pi\sqrt{m}}{c}\Big),$$
$$= \delta(m,1) + \begin{cases} O_\epsilon\Big(\frac{m^{\frac{1}{4}+\epsilon}}{k} + \frac{k^{\frac{1}{6}}}{m^{\frac{1}{4}-\epsilon}}\Big) \\ O_\epsilon\Big(2^{-k} m^{\frac{1}{4}+\epsilon}\Big) \ \textit{if } m \leq \frac{k^2}{(4\pi e)^2}. \end{cases}$$

Use this in the term $-2\sum_{p,\nu} \frac{\lambda_f(p^\nu) - \lambda_f(p^{\nu-2})}{p^{\frac{\nu}{2}}} \widehat{\phi}\Big(\frac{\nu \log p}{\log K^2}\Big) \frac{\log p}{\log K^2}$

Small support

Theorem (D.–Fiorilli–Södergren)

Let ϕ be an even Schwartz test function for which supp$(\hat{\phi}) \subset (-1,1)$, we have

$$\frac{1}{\Omega_k}\sum_{f\in B_k}\omega_f\sum_{\gamma_f}\phi\Big(\gamma_f\frac{\log k^2}{2\pi}\Big) = -\widehat{\phi}(0)\frac{\log \pi}{\log k}$$

$$+\frac{1}{\log(k^2)}\int_{\mathbb{R}}\bigg(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi i t}{\log(k^2)}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi i t}{\log(k^2)}\Big)\bigg)\phi(t)dt$$

$$+2\sum_{p}\frac{1}{p}\widehat{\phi}\Big(\frac{2\log p}{\log(k^2)}\Big)\frac{\log p}{\log(k^2)}+O\big(k^{\frac{3}{2}}2^{-k}\big).$$

Gamma factors

Lemma

Let $\epsilon > 0$ and let ϕ be an even Schwartz test function. In the range $k \leq K^5$, we have the estimate

$$\frac{1}{\log K^2}\int_{\mathbf{R}}\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k+1}{4}+\frac{\pi i t}{\log X}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+\frac{k-1}{4}+\frac{\pi i t}{\log K^2}\Big)\Big)\phi(t)dt$$
$$= \widehat{\phi}(0)\Big(\frac{\log(k^2)-\log 16}{\log K^2}\Big)+O_\epsilon(k^{-1+\epsilon}).$$

Estimating the sum with 1

Lemma

Let ϕ be an even Schwartz test function. For any fixed $J \geq 1$, we have the estimate

$$2\sum_p \frac{1}{p}\widehat{\phi}\Big(\frac{\log p}{\log K}\Big)\frac{\log p}{\log K^2} = \frac{\phi(0)}{2} + \sum_{1\leq j\leq J}\frac{c_j\widehat{\phi}^{(j-1)}(0)}{(\log K)^j} + O_J\Big(\frac{1}{(\log K)^{J+1}}\Big),$$

where

$$c_1 := \int_1^\infty \frac{\theta(t)-t}{t^2}dt + 1 = -\gamma - \sum_p \frac{\log p}{p(p-1)}$$

and for $j \geq 2$,

$$c_j := \frac{1}{(j-2)!}\int_1^\infty (\log t)^{j-2}\Big(\frac{\log t}{j-1} - 1\Big)\frac{\theta(t)-t}{t^2}dt.$$

Extended support

For $\sigma > 1$, we obtain, under the condition that $\mathrm{supp}(\hat{\phi}) \subset [-\sigma, \sigma]$,

$$\begin{aligned}
\mathscr{D}^{\pm}_{K,h}(\phi) = \widehat{\phi}(0) &+ \frac{-\log 4\pi + \frac{\int_{\mathbf{R}^+} h\cdot\log}{\int_{\mathbf{R}^+} h}}{\log K}\widehat{\phi}(0) \\
&+ \frac{\phi(0)}{2} + \sum_{1\leq j\leq J} \frac{c_j \widehat{\phi}^{(j-1)}(0)}{(\log K)^j} + O_J\Big(\frac{1}{(\log K)^{J+1}}\Big) \\
- \frac{1}{H^{\pm}(K)} &\sum_{k\equiv 3\pm 1 \bmod 4} h\Big(\frac{k-1}{K}\Big)\frac{2}{\Omega_k}\sum_p \sum_{f\in B_k} \omega_f \frac{\lambda_f(p)}{p^{\frac{1}{2}}}\widehat{\phi}\Big(\frac{\log p}{\log K^2}\Big)\frac{\log p}{\log K^2} \\
&+ O_\epsilon\Big(K^{\frac{\sigma}{2}-1+\epsilon} + K^{-\frac{1}{3}+\epsilon}\Big)
\end{aligned}$$

Estimating the sum with $\lambda_f(p)$

Lemma

Let ϕ be an even Schwartz test function. Assume the Riemann Hypothesis for Dirichlet L-functions. For any fixed $J \geq 1$, we have the estimate

$$\frac{1}{H^{\pm}(K)} \sum_{k \equiv 3 \pm 1 \bmod 4} h\Big(\frac{k-1}{K}\Big) \frac{2}{\Omega_k} \sum_{p} \sum_{f \in B_k} \omega_f \frac{\lambda_f(p)}{p^{\frac{1}{2}}} \widehat{\phi}\Big(\frac{\log p}{\log K^2}\Big) \frac{\log p}{\log K^2}$$

$$= \pm \int_1^{\infty} \hat{\phi} \pm \sum_{1 \leq j \leq J} \frac{S_j \hat{\phi}^{(j-1)}(1)}{(\log K)^j} + O((\log K)^{-J-1})$$

Estimating the sum with $\lambda_f(p)$

Now we use the exact form in Petersson formula :

Lemma (Petersson Trace formula)

Let p be a prime and $k \in \mathbf{N}$, with $2 \mid k$. We have the exact formula

$$\sum_{f \in B_k} \omega_f \lambda_f(p) = 2\pi i^k \sum_{c \geq 1} c^{-1} S(p,1;c) J_{k-1}\Big(\frac{4\pi\sqrt{p}}{c}\Big).$$

we obtain

$$- \frac{4\pi}{H^{\pm}(K)} \sum_p p^{-\frac{1}{2}} \widehat{\phi}\Big(\frac{\log p}{\log K^2}\Big) \frac{\log p}{\log K^2}$$

$$\sum_{c \geq 1} \frac{S(p,1;c)}{c} \sum_{k \equiv 3 \pm 1 \bmod 4} h\Big(\frac{k-1}{K}\Big) \frac{i^k}{\Omega_k} J_{k-1}\Big(\frac{4\pi\sqrt{p}}{c}\Big)$$

Averaging over k

Lemma (Iwaniec)

For h a non-negative, smooth function with compact support in $\mathbf{R}_{>0}$ and for any $K \geq 2$, we have the estimates

$$2 \sum_{k \equiv 0 \bmod 2} h\Big(\frac{k-1}{K}\Big) J_{k-1}(x) = h\Big(\frac{x}{K}\Big) + O\Big(\frac{x}{K^3}\Big);$$

$$2 \sum_{k \equiv 0 \bmod 2} i^k h\Big(\frac{k-1}{K}\Big) J_{k-1}(x) = O_A\Big(\frac{x^A}{K^{2A}} + \frac{x^{\frac{1}{2}}}{K^5}\Big).$$

This gives

$$\sum_{k \equiv 3 \pm 1 \bmod 4} h\Big(\frac{k-1}{K}\Big) \frac{i^k}{\Omega_k} J_{k-1}\Big(\frac{4\pi\sqrt{p}}{c}\Big) \quad = \quad \pm h\Big(\frac{4\pi\sqrt{p}}{Kc}\Big) \; + \; O\Big(\; \star \; \Big)$$

where $\star$ sums well over c and p if $\sigma < 2$.

Estimating the sum over primes

Lemma (Iwaniec–Luo–Sarnak)

Assume the Riemann Hypothesis for Dirichlet L-functions. Then

$$\sum_{p\le t} S(p,1;c)\log p = t\frac{\mu^2(c)}{\varphi(c)} + O(\varphi(c)t^{\frac{1}{2}}(\log(ct))^2),$$

where φ is Euler's totient function.

By integration by parts

$$\sum_p p^{-\frac{1}{2}}\widehat{\phi}\Big(\frac{\log p}{\log K^2}\Big)\frac{\log p}{\log K^2}S(p,1;c)h\Big(\frac{4\pi\sqrt{p}}{Kc}\Big)$$
$$= \int_0^\sigma K^u\widehat{\phi}(u)\frac{\mu^2(c)}{\varphi(c)}h\Big(\frac{4\pi K^{u-1}}{c}\Big)du + O(K^{\sigma-1}(\log K)^3).$$

Estimating the integral

$$I_{0,\sigma} := \frac{\pi}{H^{\pm}(K)} \int_0^{\sigma} K^u \widehat{\phi}(u) \sum_{c\geq 1} \frac{\mu^2(c)}{c\varphi(c)} h\Big(\frac{4\pi K^{u-1}}{c}\Big) du.$$

$I_{0,1-\delta_K} = 0$ thanks to the hypothesis on the support of h.
By integration by parts

$$I_{1+\delta_K,\sigma} = \int_{1+\delta_K}^{\sigma} \hat{\phi} + O(K^{-\frac{\delta_K}{2}})$$

By developing in Taylor series

$$I_{1-\delta_K,1+\delta_K} = \int_1^{1+\delta_K} \hat{\phi} + \sum_{1\leq j\leq J} \frac{S_j \hat{\phi}^{(j-1)}(1)}{(\log K)^j} + O((\log K)^{-J-1})$$

Conclusion

We obtain, under the conditions that $\operatorname{supp}(\hat{\phi}) \subset (-2, 2)$, that h has compact support in $\mathbf{R}_{>0}$, and the Riemann Hypothesis for Dirichlet L-functions

$$\mathscr{D}^{\pm}_{K,h}(\phi) = \widehat{\phi}(0) + \frac{\phi(0)}{2} \mp \int_1^\infty \hat{\phi} + \frac{-\log 4\pi + \frac{\int_{\mathbf{R}^+} h \cdot \log}{\int_{\mathbf{R}^+} h}}{\log K}\widehat{\phi}(0) + \sum_{1 \le j \le J} \frac{c_j \widehat{\phi}^{(j-1)}(0)}{(\log K)^j} \mp \sum_{1 \le j \le J} \frac{S_j \hat{\phi}^{(j-1)}(1)}{(\log K)^j} + O_J\Big(\frac{1}{(\log X)^{J+1}}\Big)$$

Thank you!