p-adic equidistribution of CM points

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Motivation

Let $\mathbb{K}$ be a complete, algebraically closed field of characteristic 0.
Let

$$Y(\mathbb{K}) = \{\text{elliptic curves defined over } \mathbb{K}, \text{ up to } \mathbb{K}-\text{isomorphism}\}.$$

A *CM point* is an elliptic curve $E \in Y(\mathbb{K})$ with complex multiplication. That is, $End(E)$ is isomorphic to an order in an imaginary quadratic number field.
Such orders are determined by the discriminant $D = disc(End(E))$, which is a negative integer with $D \equiv 0$ or $1 \mod 4$.
Set

$$\mathcal{H}_D = \{E \in Y(\mathbb{K}) : disc(End(E)) = D\}.$$

Main question

How are the sets $\mathcal{H}_D$ distributed inside $Y(\mathbb{K})$ as $|D| \to \infty$?

The complex case

In the case $\mathbb{K} = \mathbb{C}$, the space $Y(\mathbb{C})$ is uniformized by the upper half plane

$$\mathbb{H} = \{\tau \in \mathbb{C} : Im(\tau) > 0\}.$$

For any $\tau \in \mathbb{H}$, the complex torus $\mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ has a projective model E_τ defined over $\mathbb{C}$. The map

$$\mathbb{H} \to Y(\mathbb{C}), \quad \tau \mapsto E_\tau$$

induces a bijection $SL_2(\mathbb{Z}) \backslash \mathbb{H} \simeq Y(\mathbb{C})$.
An elliptic curve E_τ is a CM point if and only if τ is a quadratic algebraic number. In particular, CM points are dense in $Y(\mathbb{C})$.
Seeking to refine this information, we introduce for any discriminant D, the probability measure

$$\overline{\delta}_D = \frac{1}{|\mathcal{H}_D|} \sum_{E \in \mathcal{H}_D} \delta_E.$$

Here, δ_x is the Dirac measure supported at x.

The complex case

Besides the measures

$$\overline{\delta}_D = \frac{1}{|\mathcal{H}_D|} \sum_{E \in \mathcal{H}_D} \delta_E,$$

consider also the hyperbolic measure $\mu_{hyp} = \frac{3}{\pi} \cdot \frac{dxdy}{y^2}$ on $\mathbb{H}$. It is invariant under the action of $SL_2(\mathbb{Z})$ and induces a probability measure on $Y(\mathbb{C})$.

Theorem (Duke 1988, Clozel-Ullmo 2004)

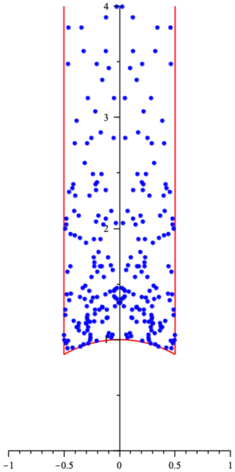
The sets $\{\mathcal{H}_D\}_{D \text{ discriminant}}$ are equidistributed with respect to μ_{hyp} as $|D| \to \infty$. In other words, the sequence of measures $\overline{\delta}_D$ weakly converges to μ_{hyp} as $|D| \to \infty$.

Weak convergence means that for any continuous, compactly supported function $f : Y(\mathbb{C}) \to \mathbb{R}$, we have that

$$\lim_{\substack{D \to -\infty \\ D \text{ discriminant}}} \frac{1}{|\mathcal{H}_D|} \sum_{E \in \mathcal{H}_D} f(E) = \int_{SL_2(\mathbb{Z}) \backslash \mathbb{H}} f(E_\tau) d\mu_{hyp}(\tau).$$

The complex case

$d = -418916$



(Picture stolen from a talk by Philippe Michel)

The p-adic case

Fix a prime number p. The p-adic norm $|\cdot|_p$ on $\mathbb{Q}_p$ extends in an unique way to $\overline{\mathbb{Q}}_p$. This is not a complete field, so we take its completion

$$\mathbb{C}_p := \text{ completion of } \overline{\mathbb{Q}}_p.$$

Then, $\mathbb{C}_p$ is both algebraically closed and complete.
For any elliptic curve $E_{/\mathbb{C}_p}$ we denote by $\tilde{E}$ its "reduction $\mod p$".
The space $Y(\mathbb{C}_p)$ is divided in three subspaces

$$\begin{aligned} Y_{bad}(\mathbb{C}_p) &= \{E \in Y(\mathbb{C}_p) : \tilde{E} \text{ is a singular curve}\} \\ Y_{ord}(\mathbb{C}_p) &= \{E \in Y(\mathbb{C}_p) : \tilde{E} \text{ is an ordinary elliptic curve}\} \\ Y_{ss}(\mathbb{C}_p) &= \{E \in Y(\mathbb{C}_p) : \tilde{E} \text{ is a supersingular elliptic curve}\}. \end{aligned}$$

For any discriminant D, we have that

$$\mathcal{H}_D \subseteq \begin{cases} Y_{ord}(\mathbb{C}_p) & \text{if } p \text{ splits in } \mathbb{Q}(\sqrt{D}) \\ Y_{ss}(\mathbb{C}_p) & \text{if } p \text{ is inert or ramified in } \mathbb{Q}(\sqrt{D}) \end{cases}$$

Transient behaviour

Set $|D|_{p-ss} = \begin{cases} 0 & \text{if } \mathcal{H}_D \subseteq Y_{ord}(\mathbb{C}_p) \\ |D|_p & \text{if } \mathcal{H}_D \subseteq Y_{ss}(\mathbb{C}_p) \end{cases}$

Transient theorem (H,M,R-L)

Let (D_n) be a sequence of discriminants such that $D_n \to -\infty$ as $n \to \infty$. Then, we have that

$$\lim_{n\to\infty} |D_n|_{p-ss} = 0$$

if and only if the following holds: for all $x \in Y(\mathbb{C}_p)$, there exists a neighborhood U of x such that

$$\lim_{n\to\infty} \frac{|\mathcal{H}_{D_n} \cap U|}{|\mathcal{H}_{D_n}|} = 0.$$

In this case, we say that the distribution of the sets $\mathcal{H}_{D_n}$ is **transient**.

Transient behaviour

The transient behaviour can be expressed as an equidistribution statement using the language of Berkovich spaces: the j-(absolute) invariant induces an analytic bijection $j : Y(\mathbb{C}_p) \to \mathbb{C}_p$.

- Consider $\mathbb{C}_p$ as embedded in the Berkovich affine line $\mathbb{A}^{1,an}_{\mathbb{C}_p}$.
- Let $\xi \in \mathbb{A}^{1,an}_{\mathbb{C}_p}$ be the Gauss point attached to the unit disc.
- Let x_{can} the unique point in $Y(\mathbb{C}_p)^{an}$ which is sent to ξ by the map induced by j.

Transient theorem (H,M,R-L), alternative formulation

Let (D_n) be a sequence of discriminants such that $D_n \to -\infty$ as $n \to \infty$. Then, the sequence of measures $\overline{\delta}_{D_n}$ weakly converges to $\delta_{x_{can}}$ if and only if

$$\lim_{n\to\infty} |D_n|_{p-ss} = 0.$$

The supersingular locus

Given $E \in Y(\mathbb{C}_p)$, we can consider its formal group $\mathcal{F}_E$. Then, $End(\mathcal{F}_E)$ is either isomorphic to $\mathbb{Z}_p$ or to an order in a quadratic extension of $\mathbb{Q}_p$. In the latter case we say that E has *formal CM*. Examples and non-example:

1. If $E \in Y_{ord}(\mathbb{C}_p)$, then $End(\mathcal{F}_E) \simeq \mathbb{Z}_p$
2. If $E \in Y_{ss}(\mathbb{C}_p)$ has CM, then $End(\mathcal{F}_E) \simeq End(E) \otimes \mathbb{Z}_p$ is an order in a quadratic extension of $\mathbb{Q}_p$
3. Non CM with formal CM: let e be a supersingular elliptic curve over $\mathbb{F}_{p^2}$ with $Aut(e) = \{\pm 1\}$. Let $E_{/\mathbb{Q}_{p^2}}$ be such that $\tilde{E} \simeq e$. Then, $Frob_{p^2} = [\pm p]_e$ lifts to $End(E)$. A theorem of Lubin-Tate then ensures that $End(\mathcal{F}_E) \neq \mathbb{Z}_p$. Since the set of CM points is countable, and $\{E_{/\mathbb{Q}_{p^2}} : \tilde{E} = e\}$ is not, there are many non CM elliptic curves with formal CM.

p-adic discriminants

Let $\mathcal{O}$ be an order in a quadratic extension of $\mathbb{Q}_p$. For any $\mathbb{Z}_p$-basis $B = \{u, v\}$ of $\mathcal{O}$, there is a discriminant $\mathrm{disc}(B) \in \mathbb{Z}_p$. Two such discriminants differ by an element in $(\mathbb{Z}_p^\times)^2$. Hence, the set

$$\Delta(\mathcal{O}) = disc(\mathcal{O}) = \{disc(B) : B \text{ is a } \mathbb{Z}_p - \text{basis of } \mathcal{O}\}$$

is a class in $\mathbb{Q}_p^\times/(\mathbb{Z}_p^\times)^2$, which happens to be contained in $\mathbb{Z}_p$. We call any such class a **p-adic discriminant**. Moreover, $\Delta(\mathcal{O})$ is called a **fundamental p-adic discriminant** if $\mathcal{O}$ is the maximal order.

For $p \geq 3$, the p-adic fundamental discriminants are

$$\Delta^{unr} = \mathbb{Z}_p^\times - (\mathbb{Z}_p^\times)^2, \quad \Delta_1^r = p(\mathbb{Z}_p^\times)^2, \quad \Delta_2^r = p(\mathbb{Z}_p^\times - (\mathbb{Z}_p^\times)^2).$$

Any p-adic discriminant can be written as

$$\Delta = \Delta_0 \cdot p^{2m}, \quad \Delta_0 \text{ fundamental and } m \geq 0.$$

For any p-adic discriminant Δ, set

$$\Lambda_\Delta = \{E \in Y_{ss}(\mathbb{C}_p) : disc(End(\mathcal{F}_E)) = \Delta\}.$$

For any discriminant D,

$$\mathcal{H}_D \cap \Lambda_\Delta \neq \emptyset \Leftrightarrow \mathcal{H}_D \subset \Lambda_\Delta \Leftrightarrow D \in \Delta$$

Consider Katz's valuation $v : Y_{ss}(\mathbb{C}_p) \to \mathbb{Q}$. Then (Coleman, H,M,R-L)

$$v(\Lambda_{\Delta_0 \cdot p^{2m}}) \begin{cases} \geq & \frac{p}{p+1} & \text{if } \Delta_0 = \Delta^{unr} \text{ and } m = 0 \\ = & \frac{1}{p^{m-1}(p+1)} & \text{if } \Delta_0 = \Delta^{unr} \text{ and } m \geq 1 \\ = & \frac{1}{2p^m} & \text{if } \Delta_0 \neq \Delta^{unr} \end{cases}$$

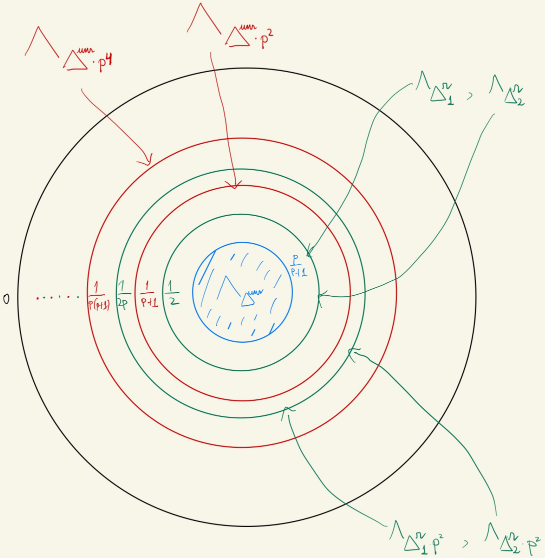
Fix a supersingular elliptic curve e and set

$$\mathbf{D}_e = \{E \in Y_{ss}(\mathbb{C}_p) : \tilde{E} \simeq e\}, \quad Y_{ss}(\mathbb{C}_p) = \bigsqcup_{\substack{e \text{ supersingular} \\ \text{elliptic curve over } \overline{\mathbb{F}}_p}} \mathbf{D}_e$$

is a finite union of discs.

$D_e = \{ E/\mathbb{C}_p : \tilde{E} \simeq e \}$

///: "too supersingular" locus



Prime discriminants

A discriminant $d < 0$ is called a **prime discriminant** d is fundamental and divisible by only one prime number.

- When $p \geq 3$, the only prime discriminant divisible by p is $d = -p$ with $p \equiv 3 \mod 4$.
- The prime discriminants divisible by 2 are -4 and -8.

Lemma

Let Δ_0 be a ramified fundamental p-adic discriminant. Let $d \in \Delta_0$ be a fundamental discriminant divisible by p. Then, d is a prime discriminant if and only if for all prime numbers $q \nmid d$ we have that

$$q \in \mathrm{nr}\Delta_0 \Leftrightarrow \left(\frac{d}{q}\right) = 1.$$

The supersingular locus

Theorem A (H,M,R-L)

For any p-adic discriminant Δ, there exists a non atomic measure ν_Δ with support equal to Λ_Δ with the following property. For any sequence of discriminants D_n contained in Δ, with $D_n \to -\infty$ and such that the fundamental discriminant of D_n is either not divisible by p or not a prime discriminant, we have that $\overline{\delta}_{D_n}$ weakly converges to ν_Δ. Moreover,

- Λ_Δ is a compact subset of $Y(\mathbb{C}_p)$
- The set $\{\mathcal{H}_D : D \in \Delta\}$ is dense in Λ_Δ.

The supersingular locus

Theorem B (H,M,R-L)

Let d be a prime discriminant divisible by p. Let $m \geq 1, D = d \cdot p^{2m}$ and Δ the p-adic discriminant containing D. Then, there exists a non atomic measure ν_Δ with support equal to Λ_Δ and a partition of Λ_Δ into compact sets

$$\Lambda_\Delta = \Lambda_\Delta^+ \sqcup \Lambda_\Delta^-$$

such that

$$\nu_\Delta^+ := 2 \cdot \nu_\Delta|_{\Lambda_\Delta^+}, \quad \nu_\Delta^- := 2 \cdot \nu_\Delta|_{\Lambda_\Delta^-}$$

are both probability measures. Moreover, if $(f_n)_{n\geq 1}$ is a sequence of prime-to-p integers tending to infinity, we have that

- if $\left(\frac{d}{f_n}\right) = 1$ for all $n \geq 1$, then $\overline{\delta}_{Df_n^2}$ weakly converges to ν_Δ^+
- if $\left(\frac{d}{f_n}\right) = -1$ for all $n \geq 1$, then $\overline{\delta}_{Df_n^2}$ weakly converges to ν_Δ^-

The supersingular locus, ideas of the proof

Recall

$$\mathbf{D}_e = \{E \in Y_{ss}(\mathbb{C}_p) : \tilde{E} \simeq e\}, \quad Y_{ss}(\mathbb{C}_p) = \bigsqcup_{\substack{e \text{ supersingular} \\ \text{elliptic curve over } \overline{\mathbb{F}}_p}} \mathbf{D}_e.$$

We are thus led to understand the distribution of $\Lambda_D \cap \mathbf{D}_e$, as D varies, for all e.

The main ideas in the proof of Theorems A and B are the same. In this talk we will focus on Theorem A. In order to avoid technicalities we will assume:

- $Aut(e) = \{\pm 1\}$
- D runs through fundamental discriminants
- $p \nmid D$ (unramified case)

We have that $End(e) \otimes_{\mathbb{Z}} \mathbb{Z}_p$ is isomorphic to the maximal order in the quaternion division algebra over $\mathbb{Q}_p$.

Deformation theory affords an action of $G_e := (End(e) \otimes \mathbb{Z}_p)^*$ on $\mathbf{D}_e$.

Traceless spheres

For any $\ell \in \mathbb{Z}_p^*$, consider the *traceless sphere of radius* ℓ

$$S_e(\ell) = \{g \in End(e) \otimes_{\mathbb{Z}} \mathbb{Z}_p : \mathrm{tr}(g) = 0 \text{ and } \mathrm{nr}(g) = -\ell\}.$$

(mind the switch form ℓ to $-\ell$)

- Since $p \nmid \ell$, any $g \in S_e(\ell)$ is an invertible element in $End(e) \otimes \mathbb{Z}_p$, therefore it acts on $\mathbf{D}_e$. Moreover, it has a unique fixed point in $\mathbf{D}_e$.
- There is a distinguished measure μ_ℓ on $S_e(\ell)$ constructed as follows: the sphere

$$S_e^1 := \{g \in End(e) \otimes_{\mathbb{Z}} \mathbb{Z}_p : \mathrm{nr}(g) = 1\}$$

is a compact subgroup of G_e and acts on $S_e(\ell)$ by conjugation. This action is isometric. Hence, there is an unique probability measure on $S_e(\ell)$ left invariant by this action, and this is μ_ℓ.

The supersingular locus

Fixed points formula. (H,M,R-L)

Let d be a fundamental discriminant such that $p \nmid d$. Set

$$V_e(d) = \{\varphi \in End(e) : \varphi^2 = d\}.$$

The, we have that

$$\Lambda_d \cap \mathbf{D}_e = \bigcup_{\varphi \in V_e(d)} Fix(\varphi).$$

Remarks:

- We remark that the elements in $V_e(d)$ have trace 0, and they form the set of integral points inside the traceless sphere $S_e(d)$
- A similar, but slightly more complicated formula holds when $p|d$ (but still d is assumed to be fundamental)
- The proof uses Deuring's lifting theorem.

Strategy

Theorem A, soft proof strategy: let (d_n) be a sequence of fundamental discriminants, not divisible by p, contained in the p-adic discriminant Δ and tending to $-\infty$.
Auxiliary hipotesis (to be removed later): assume that (d_n) converges p-adically to some p-adic integer $\ell \in \mathbb{Z}_p^\times$.
Since

$$\Lambda_{d_n} \cap \mathbf{D}_e = \bigcup_{\varphi \in V_e(d_n)} Fix(\varphi),$$

we are led to understand how the sets $V_e(d_n)$ distribute as $d_n \to \ell$ p-adically.

The set $V_e(d_n)$ can be seen as the set of integral points in the traceless sphere $S_e(d_n)$.

Key fact (H,M,R-L)

The sets $V_e(d_n)$ equidistribute with respect to the measure μ_ℓ on the traceless sphere

$$S_e(\ell) = \{g \in End(e) \otimes \mathbb{Z}_p : \mathrm{tr}(g) = 0 \text{ and } \mathrm{nr}(g) = -\ell\}$$

as $d_n \to \ell$ p-adically.

On the other hand, the map $Fix : S_e(\ell) \to \mathbf{D}_e$ sending $g \in S_e(\ell) \subset G_e$ to its unique fixed point, is continuous. Then, the sets

$$\Lambda_{d_n} \cap \mathbf{D}_e$$

equidistribute with respect to the measure

$$\nu_\Delta := Fix_* \mu_\ell$$

as $d_n \to \ell$.

The measure ν_Δ does not depend on ℓ, only on Δ. This is why the auxiliary hypothesis $d_n \to \ell$ can be removed.

p-adic Linnik equidistribution

Key fact (H,M,R-L)

The sets $V_e(d_n)$ equidistribute with respect to the measure μ_ℓ on the traceless sphere

$$S_e(\ell) = \{g \in End(e) \otimes \mathbb{Z}_p : \mathrm{tr}(g) = 0 \text{ and } \mathrm{nr}(g) = -\ell\}$$

as $d_n \to \ell$ p-adically.

This fact is an instance of a "p-adic equidistribution result of Linnik type", analogous to the classical Linnik problem on equidistribution of integral points on (real) spheres. Our proof hinges on Duke's bounds on fourier coefficients of cuspidal modular forms of weight $\frac{3}{2}$, just like the classical proof in the complex case!

Fix an integer $n \geq 3$ and a positive definite quadratic form Q in $\mathbb{Z}[X_1, X_2, \ldots, X_n]$. For m in $\mathbb{N}$ put

$$V_m(Q) = \{x \in \mathbb{Z}^n : Q(x) = m\}$$

and for ℓ in $\mathbb{Z}_p$ define the sphere

$$S_\ell(Q) = \{x \in \mathbb{Z}_p^n : Q(x) = \ell\}.$$

Note that the orthogonal group of Q

$$\mathrm{O}_Q(\mathbb{Z}_p) = \{T \in \mathrm{GL}_n(\mathbb{Z}_p) : Q(T \cdot X) = Q(X)\}$$

is compact, acts on $\mathbb{Z}_p^n$ and for every ℓ in $\mathbb{Z}_p$ it preserves the sphere $S_\ell(Q)$.
For every u in $\mathbb{Z}_p^\times$ denote by M_u the element of $\mathrm{GL}_n(\mathbb{Z}_p)$ defined by

$$M_u(X_1, \ldots, X_n) = (uX_1, \ldots, uX_n)$$

Note that for every ℓ in $\mathbb{Z}_p$ we have $M_u(S_\ell(Q)) = S_{\ell u^2}(Q)$.

Theorem (H,M,R-L)

Let κ_n be equal to $\frac{1}{2}$ if n is even and to $\frac{2}{7}$ if n is odd and fix $c > \frac{n}{4} - \kappa_n$. Let ℓ in $\mathbb{Z}_p - \{0\}$ be such that $S_\ell(Q)$ is nonempty and $\mathrm{O}_Q(\mathbb{Z}_p)$ acts transitively on $S_\ell(Q)$ and denote by μ_ℓ be the unique Borel probability measure on $S_\ell(Q)$ that is invariant under the action of $\mathrm{O}_Q(\mathbb{Z}_p)$. Moreover, let $(m_j)_{j=1}^{\infty}$ be a sequence in $\mathbb{N}$ tending to ∞ that is contained in the coset $\ell(\mathbb{Z}_p^\times)^2$ of $\mathbb{Q}_p^\times/(\mathbb{Z}_p^\times)^2$ and such that for every sufficiently large j we have $\# V_{m_j}(Q) \geq m_j^c$. For each $j \geq 1$, let u_j in $\mathbb{Z}_p^\times$ be such that $m_j = \ell u_j^2$. If $n = 3$, then assume in addition that the there is $S \geq 1$ such that for each j the largest square diving m_j is less than or equal to S. Then we have the weak convergence of measures

$$\frac{1}{\# V_{m_j}(Q)} \sum_{x \in V_{m_j}(Q)} \delta_{M_{u_j}^{-1}(x)} \to \mu_\ell \text{ as } j \to \infty.$$

Key fact: the sets $V_e(d_n)$ equidistribute with respect to the measure μ_ℓ on the traceless sphere

$$S_e(\ell) = \{g \in R_e : \mathrm{tr}(g) = 0 \text{ and } \mathrm{nr}(g) = -\ell\}$$

as $d_n \to \ell$ p-adically.
In this application, the action of S_e^1 on $S_e(\ell)$ is transitive due to the Skolem-Noether theorem. Also, there are sufficiently many points in $V_e(d_n)$ because of Siegel's lower bound on the class numbers of $\mathbb{Q}(\sqrt{d_n})$ and to the previously known equidistribution mod p of CM points.

A diophantine application

Theorem (H,M,R-L)

Let S be a finite set of primes. Set

$$J_S := \{j(E) : E \text{ is CM and } j(E) \text{ is a } S\text{-unit}\}.$$

Then, J_S is a finite set.

- Habegger proved in 2015 that $J_\emptyset$ (i.e. singular moduli which are algebraic units) is finite. Latter, Bilu-Habegger-Kühne showed that actually $J_\emptyset$ is empty!
- Our proof follows the strategy of Habegger's 2015 result, and is non effective.

S-units, proof idea

Using Colmez's formula and work by Nakkajima-Taguchi, Habegger shows that for any discriminant D and $E_{/\overline{\mathbb{Q}}} \in \mathcal{H}_D$, we have that

$$h\big(j(E)\big) \geq A \log |D| + B,$$

for some constants A, B with $A > 0$.
Assume for contradiction that J_S is infinite. Let d_n be a sequence of discriminants tending to $-\infty$ such that for all n, there is $E_n \in \mathcal{H}_{d_n}$ with $j(E_n)$ a S-unit. Then

$$h\big(j(E_n)\big) \geq A \log |d_n| + B.$$

The idea is to show that $h\big(j(E_n)\big)$ grows at a slower rate. This will be a reflection of a certain "repulsion" phenomenon between CM points.

We have that

$$h\big(j(E_n)\big) = \frac{1}{|\mathcal{H}_{d_n}|}\sum_{v|\infty}\log^+|j(E_n)|_v = \frac{1}{|\mathcal{H}_{d_n}|}\sum_{\substack{v|\infty\\ |j(E_n)|_v>1}}\log|j(E_n)|_v.$$

Using that $j(E_n)$ is a S-unit, we see that

$$1 = \prod_v |j(E_n)|_v = \prod_{v|S\cup\{\infty\}}|j(E_n)|_v.$$

Hence

$$\begin{aligned} h\big(j(E_n)\big) &= \frac{1}{|\mathcal{H}_{d_n}|}\sum_{\substack{v|\infty\\ |j(E_n)|_v>1}}\log|j(E_n)|_v \\ &= -\frac{1}{|\mathcal{H}_{d_n}|}\left(\sum_{v|S}\log|j(E_n)|_v + \sum_{\substack{v|\infty\\ |j(E_n)|_v<1}}\log|j(E_n)|_v\right) \end{aligned}$$

$$
\begin{aligned}
h\big(j(E_n)\big) &= -\frac{1}{|\mathcal{H}_{d_n}|}\left(\sum_{v|S}\log|j(E_n)|_v + \sum_{\substack{v|\infty\\ |j(E_n)|_v<1}}\log|j(E_n)|_v\right)\\
&= -\frac{1}{|\mathcal{H}_{d_n}|}\left(\sum_{\substack{v|S\cup\{\infty\}\\ |j(E_n)|_v<r}}\log|j(E_n)|_v + \sum_{\substack{v|S\cup\{\infty\}\\ r\le|j(E_n)|_v<1}}\log|j(E_n)|_v\right)
\end{aligned}
$$

On the other hand, we have the estimate

$$
-\log|j(E_n)|_v \le A_v\log|d_n| + B_v,\quad A_v>0,\qquad \text{for all } v.
$$

(This is due to Habegger when $v|\infty$).
Hence

$$
h\big(j(E_n)\big) \le \sum_{v|S\cup\infty}\frac{|\mathcal{H}_{d_n}\cap D_v(0,r)|}{|\mathcal{H}_{d_n}|}(A_v\log|d_n|+B_v)+(|S|+1)\log\frac{1}{r}.
$$

$$h\big(j(E_n)\big) \leq \sum_{v|S\cup\infty} \frac{|\mathcal{H}_{d_n} \cap D_v(0,r)|}{|\mathcal{H}_{d_n}|}(A_v \log|d_n|+B_v)+(|S|+1)\log\frac{1}{r}.$$

Since the possible limit measures in the equidistribution of CM points (complex or p-adic) are non atomic, we can choose r such that the fraction

$$\frac{|\mathcal{H}_{d_n} \cap D_v(0,r)|}{|\mathcal{H}_{d_n}|}$$

is small enough so as to contradict the lower bound

$$h\big(j(E_n)\big) \geq A\log|d_n| + B.$$

for n big enough. ■

Thanks!