Heights and periodic points for one-parameter families of Hénon maps

(Joint work with Liang-Chung Hsia)

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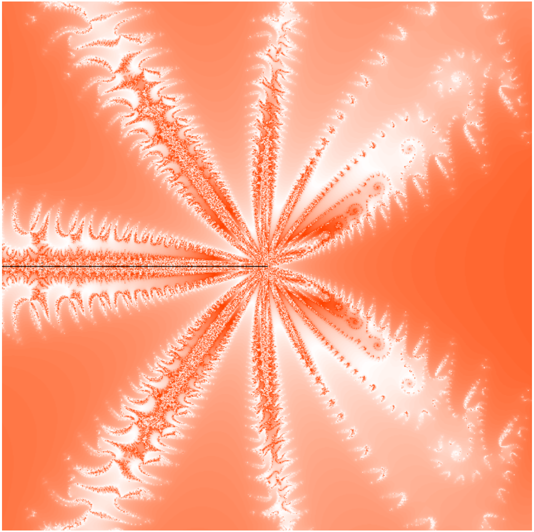
September 4, 2018

Plan of the talk

1. Background and motivation (surveyal):
   Variation of Néron–Tate heights for families of elliptic curves after Silverman, Tate, Masser–Zannier, DeMarco–Wang–Ye …
2. From elliptic curves to dynamical systems
3. Arithmetic properties of families of Hénon maps:
   Definition of a Hénon map, a height on the parameter space, the set of periodic parameter values, unlikely intersection …



Part 1 Variation of Néron–Tate heights on elliptic curves

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(For simplicity, we consider elliptic curves. For the general case of abelian varieties, see e.g. Call '89, Green '89, Holms–de Jong '15, '17)

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$\pi : \mathcal{E} \to B$ is an elliptic surface defined over a number field K,

B is a smooth projective curve

$E_t := \pi^{-1}(t)$ is a smooth elliptic curve

except for finitely many $t \in B(\overline{K})$.

Define $B^\circ \subseteq B$ to be the maximal Zariski open subset over which π is smooth.

Assume that π has a section $O : B \to \mathcal{E}$, which we regard as zero section.

Variation of Néron–Tate heights (continued)

$\mathcal{E} \to B$: an elliptic surface over a number field K
with zero section $O : B \to \mathcal{E}$

K is equipped with absolute values satisfying the product formula.

So is the function field $K(B)$ of B

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$\rightsquigarrow$ We have Néron–Tate heights (η: generic point of B)

- $\widehat{h}_{E_t} : E_t(\overline{K}) \to \mathbb{R}_{\geq 0}$ on each fiber E_t for $t \in B^\circ(\overline{K})$
- $\widehat{h}_{\mathcal{E}_\eta} : \mathcal{E}_\eta(\overline{K(B)}) \to \mathbb{R}_{\geq 0}$ on the generic fiber $\mathcal{E}_\eta$

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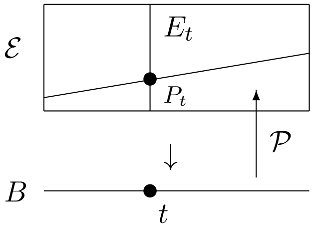
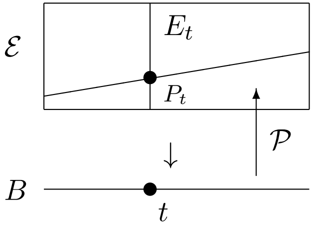
Let $\mathcal{P} : B \to \mathcal{E}$ be a section.
Set $P_t := \mathcal{P}(t)$ for $t \in B^\circ(\overline{K})$
How $\widehat{h}_{E_t}(P_t)$ and $\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta)$ are related?



Variation of Néron–Tate heights (continued)

$\mathcal{E} \to B$: an elliptic surface over a number field K



Theorem (Silverman '83, Tate '83)

Let $\mathcal{P} : B \to \mathcal{E}$ be a section with $\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta) \neq 0$.

Let h_B be a height on $B(\overline{K})$ associated to a degree 1 divisor. Then

$$\widehat{h}_{E_t}(P_t) = \widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta)\, h_B(t) + O(\sqrt{h_B(t)}) \qquad \textit{for any } t \in B^\circ(\overline{K}).$$

The error term $O(\sqrt{h_B(t)})$ is replaced by $O(1)$ if $B = \mathbb{P}^1$.

Variation of Néron–Tate heights (continued)

Assume $B = \mathbb{P}^1$, for simplicity of explanation.

$\mathcal{E} \to \mathbb{P}^1$: an elliptic surface over a number field K with zero section

$h_{\mathrm{std}} : \mathbb{P}^1(\overline{K}) \to \mathbb{R}_{\geq 0}$ standard logarithmic Weil height

$\mathcal{P} : \mathbb{P}^1 \to \mathcal{E}$ a section with $\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta) \neq 0$

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Results of Silverman and Tate assert that

$$\widehat{h}_{E_t}(P_t) = \widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta)\, h_{\mathrm{std}}(t) + O(1) \quad \text{for any } t \in (\mathbb{P}^1)^\circ(\overline{K}).$$

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We put $h_{\mathcal{P}} := \widehat{h}_{E_t}(P_t)/\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta)$ for $t \in (\mathbb{P}^1)^\circ(\overline{K})$.

Then

$$h_{\mathcal{P}} = h_{\mathrm{std}} + O(1) \quad \text{on } (\mathbb{P}^1)^\circ(\overline{K}).$$

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Question What if $t \notin (\mathbb{P}^1)^\circ(\overline{K})$? (That is, what if E_t is singular?)

Variation of Néron–Tate heights (continued)

Theorem (DeMarco–Mavraki '17+ based on Silverman '92, '94)

Let $\mathcal{E} \to \mathbb{P}^1$ *be an elliptic surface over a number field* K *with zero section. Let* $\mathcal{P} : \mathbb{P}^1 \to \mathcal{E}$ *be a section with* $\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta) \neq 0$*. Set*

$$h_{\mathcal{P}}(t) := \widehat{h}_{E_t}(P_t)/\widehat{h}_{\mathcal{E}_\eta}(\mathcal{P}_\eta) \qquad \text{for } t \in (\mathbb{P}^1)^\circ(\overline{K}).$$

Then $h_{\mathcal{P}}$ *is the restriction of a semipositive adelically metrized line bundle* $\overline{\mathcal{L}_{\mathcal{P}}} = (\mathcal{O}_{\mathbb{P}^1}(1), \{\| \cdot \|_v\})$ *on* $\mathbb{P}^1$ *(in the sense of Zhang).*

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- Theorem says that $h_{\mathcal{P}}$ extends "nicely" to $t \notin (\mathbb{P}^1)^\circ(\overline{K})$ (That is, for t with singular E_t).

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- Theorem says that $h_{\mathcal{P}}$ extends "nicely" to $t \notin (\mathbb{P}^1)^\circ(\overline{K})$ (That is, for t with singular E_t).
- The base curve need not be $\mathbb{P}^1$. DeMarco–Mavraki showed that for any elliptic surface $\mathcal{E} \to B$ (B a smooth projective curve), $h_{\mathcal{P}}$ is the restriction of a semipositive adelically metrized line bundle on B.

Application to unlikely intersection of Masser–Zannier

Let $\mathcal{E} = \{y^2 z = x(x-z)(x-tz)\}$

Legendre family of elliptic curves over $t \in \mathbb{P}^1$.

(At $t = 0, 1, \infty$, E_t is singular)

For $a \in \overline{\mathbb{Q}} \setminus \{0,1\}$,

consider a section $\mathcal{P}_a : \mathbb{P}^1 \to \mathcal{E}$,

$$t \mapsto (a : \sqrt{a(a-1)(a-t)} : 1).$$

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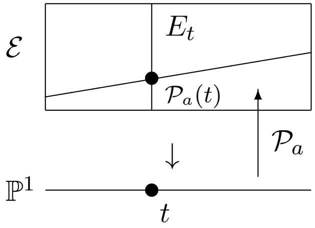

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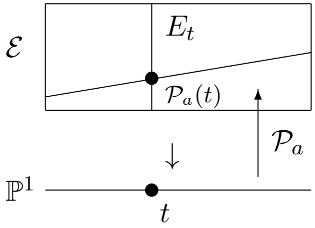

Note: $\Sigma(\mathcal{P}_a)$ is an infinite set (Masser–Zannier).

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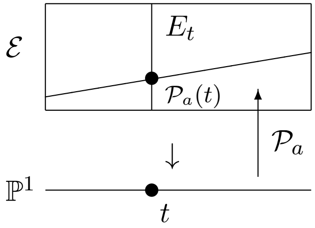

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Theorem (Masser–Zannier '08, '10, '12)

Let $a, b \in \overline{\mathbb{Q}} \setminus \{0,1\}$. If there are infinitely many parameter values $t \in \overline{\mathbb{Q}}$ such that both $\mathcal{P}_a(t)$ and $\mathcal{P}_b(t)$ are torsion points on E_t (i.e., if $\Sigma(\mathcal{P}_a) \cap \Sigma(\mathcal{P}_b)$ is an infinite set), then $a = b$.

Application to unlikely intersection of Masser–Zannier (continued)

DeMarco–Wang–Ye '14 give an alternate proof of Masser–Zannier's theorem using the fact that $h_{\mathcal{P}}$ is semipositive adelically metrized.

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- $\Sigma(\mathcal{P}_a) := \{t \mid \mathcal{P}_a(t) \text{ torsion on } E_t\} = \{t \mid h_{\mathcal{P}_a}(t) = 0\}$.
- Since $h_{\mathcal{P}_a} = h_{\overline{\mathcal{L}_{\mathcal{P}_a}}}$ with semipositive adelically metrized line bundle $\overline{\mathcal{L}_{\mathcal{P}_a}}$, one can use the equidistribution theorem (such as by Yuan).

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- If $|\Sigma(\mathcal{P}_a) \cap \Sigma(\mathcal{P}_b)| = \infty$, then equidistribution theorem implies that $c_1(\overline{\mathcal{L}_{\mathcal{P}_a}})_v = c_1(\overline{\mathcal{L}_{\mathcal{P}_b}})_v$ for any $v \in M_K$.

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- Then $h_{\overline{\mathcal{L}_{\mathcal{P}_a}}} = h_{\overline{\mathcal{L}_{\mathcal{P}_b}}}$, $\Sigma(\mathcal{P}_a) = \Sigma(\mathcal{P}_b)$, and (with more arguments) $a = b$.

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Part 2 From elliptic curves to dynamical systems

Canonical heights

E is an elliptic curve over a number field K

$L = \mathcal{O}_E([0])$, h_L is any height function associated to L

$[2]\colon E \to E$ twice multiplication map. Note that $[2]^*(L) \cong L^{\otimes 4}$.

Néron–Tate height $\widehat{h}_E : E(\overline{K}) \to \mathbb{R}_{\geq 0}$ is defined by

$$\widehat{h}_E(P) = \lim_{n\to\infty} \frac{1}{4^n} h_L([2]^n P)$$

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In place of $(E, [2], \mathcal{O}_E([0]))$, this construction of a height is generalized to the case (X, f, L)

(continued ...)

Canonical heights (continued)

X is a projective variety over a number field K

L is an ample line bundle over X

h_L is any height function associated to L

$f\colon X \to X$ a morphism

Assume that $f^*(L) \cong L^{\otimes d}$ for some $d > 1$

Such a triple (X, f, L) is called a polarized dynamical system

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Canonical height $\widehat{h}_f : X(\overline{K}) \to \mathbb{R}_{\geq 0}$ is defined by

$$\widehat{h}_f(P) = \lim_{n\to\infty} \frac{1}{d^n} h_L(f^n P)$$

(Call–Silverman '93, Zhang '95)

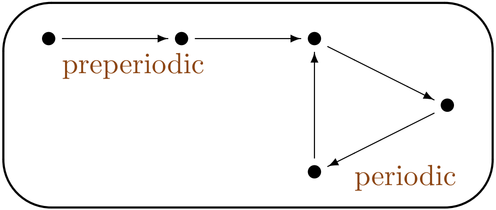
Néron–Tate height is when $(X, f, L) = (E, [2], \mathcal{O}_E([0]))$.

Torsion points, preperiodic points

$f : X \to X$ a morphism over a field K

A point $P \in X(\overline{K})$ is periodic if $f^n(P) = P$ for some $n \geq 1$

$P \in X(\overline{K})$ is preperiodic if $f^m(P)$ is periodic for some $m \geq 1$



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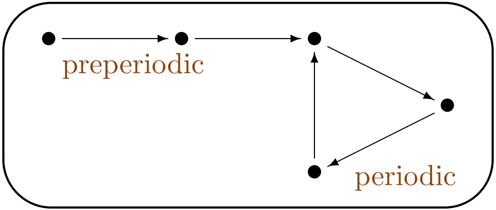

For an elliptic curve E, it is easy to see that a point $P \in E(\overline{K})$ is torsion if and only if P is preperiodic under [2].

Torsion points, preperiodic points (continued)

For an elliptic curve E over a number field K

$$\begin{aligned}\{\text{torsion point}\} &= \{\text{preperiodic point under } [2]\} \\ &= \{P \in E(\overline{K}) \mid \widehat{h}_E(P) = 0\}\end{aligned}$$

$$\Downarrow$$

Torsion points, preperiodic points (continued)

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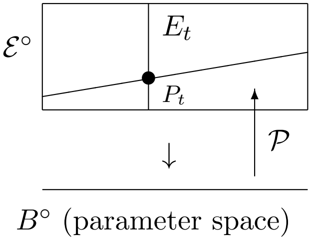
$$\Downarrow$$

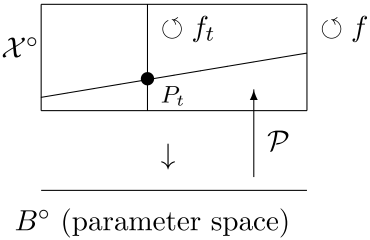
In a polarized dynamical system (X, f, L), in place of torsion points, we consider

$$\{\text{preperiodic point under } f\} = \{P \in X(\overline{K}) \mid \widehat{h}_f(P) = 0\}$$

(The equality follows from Northcott's finiteness theorem.)

Families (continued)





elliptic surface	polarized dynamical system
torsion point	preperiodic point
$\Sigma(\mathcal{P}) = \{t \mid P_t \text{ is torsion on } E_t\}$	$\Sigma(\mathcal{P}) = \{t \mid P_t \text{ is preperiodic under } f_t\}$
Néron–Tate height	canonical height

Families (continued)

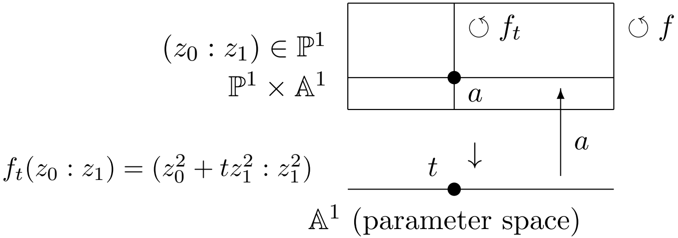
Baker–DeMarco obtained the the first result in a dynamical setting.

$$\mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{A}^1, \quad ((z_0 : z_1), t) \mapsto t$$

$$f : \mathbb{P}^1 \times \mathbb{A}^1 \to \mathbb{P}^1 \times \mathbb{A}^1, \quad ((z_0 : z_1), t) \mapsto ((z_0^2 + tz_1^2 : z_1^2), t)$$

a constant section $a : \mathbb{A}^1 \to \mathbb{P}^1 \times \mathbb{A}^1, \quad t \mapsto ((a : 1), t)$

$$\Sigma(a) := \{t \in \mathbb{A}^1 \mid (a : 1) \text{ is preperiodic under } f_t\}$$



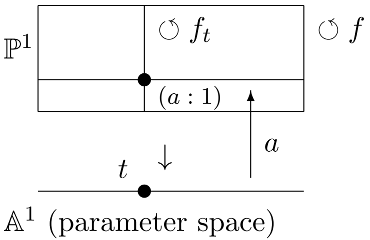
Families (continued)

$f_t(z_0 : z_1) = (z_0^2 + tz_1^2 : z_1^2)$

$\Sigma(a) = \{t \in \mathbb{A}^1(\mathbb{C})$

$\mid (a : 1)$ is preperiodic under $f_t\}$

Note: $\Sigma(a)$ is an infinite set.



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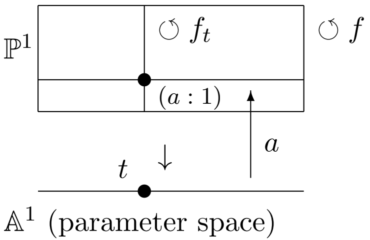

Theorem (Baker–DeMarco '11)

Let $f_t(z) = z^2 + t$. Let $a, b \in \mathbb{C}$. Suppose that there exist infinitely many $t \in \mathbb{C}$ such that a and b are both preperiodic under f_t. Then $a^2 = b^2$.

Families (continued)

$f_t(z_0 : z_1) = (z_0^2 + t z_1^2 : z_1^2)$

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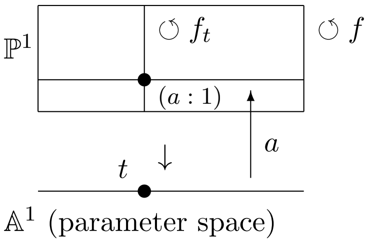

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- This answered a question of Zannier. The assumption says that $\#(\Sigma(a) \cap \Sigma(b)) = \infty$. Using equidistribution theorem, they show $\Sigma(a) = \Sigma(b)$. Properties of the Bötthcher coordinate then imply $a^2 = b^2$.

More comments on Baker–DeMarco's theorem

- Further generalizations (in relation to the dynamical Pink–Zilber conjecture) have been obtained by Baker–DeMarco, Ghioca–Hsia–Tucker, Favre–Gauthier, DeMarco–Wang–Ye …
- Families of rational maps of $\mathbb{P}^1$ have been mostly studied. Our talk is about families of higher-dimensional maps.

Plan of the talk

1. Background and motivation (surveyal):
   Variation of Néron–Tate heights for families of elliptic curves after Silverman, Tate, Masser–Zannier, DeMarco–Wang–Ye …
2. From elliptic curves to dynamical systems
3. Arithmetic properties of families of Hénon maps:
   Definition of a Hénon map, a height on the parameter space, the set of periodic parameter values, unlikely intersection …

Part 3 Arithmetic properties of families of Hénon maps

Hénon maps

$\mathbb{A}^2$: affine plane

A Hénon map over a field K is an automorphism of the form

$$H : \mathbb{A}^2 \to \mathbb{A}^2, \quad H\,(x, y) = (\delta y + f(x), x)$$

for some $\delta \in K \setminus \{0\}$ and $f(x) \in K[x]$ with $d := \deg(f) \geq 2$.

The inverse is given by

$$H^{-1} : \mathbb{A}^2 \to \mathbb{A}^2, \quad H^{-1}\,(x, y) = \left(y, \frac{1}{\delta}(x - f(y))\right)$$

Note: H extends to a birational map $H : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, but not an isomorphism.

Hénon maps (continued)

Some remarks on Hénon maps

- Hénon '76 showed that a Hénon map has a strange attractor.

Hénon maps (continued)

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- If δ in $H(x, y) = (\delta y + f(x), x)$ is very small (nearly 0), then the map looks like

$$(x, y) \mapsto (f(x), x) \mapsto (f^2(x), f(x)) \mapsto \dots$$

So, Hénon maps are more complicated than one-variable polynomial maps.

Hénon maps (continued)

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  So, Hénon maps are more complicated than one-variable polynomial maps.
- Friedland–Milnor '89 showed that any automorphism $F : \mathbb{A}^2 \to \mathbb{A}^2$ over $\mathbb{C}$ is, up to conjugacy of $\mathrm{Aut}(\mathbb{C}^2)$, either triangularizable or the composition of Hénon maps. (Dynamically, triangularizable maps are not interesting.) So, the class of Hénon maps consists of a fundamental class of plane automorphisms.

Hénon maps (continued)

- Hénon maps over $\mathbb{C}$ are deeply studied in Bedford–Smilie '91, Fornæss–Sibony '92, Hubbard –Oberste-Vorth '94 among from the vast literature.

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- Arithmetic properties of Hénon maps are also studied. I think that they are first studied by Silverman '94.
- For a Hénon map $H(x, y) = (y, \delta x + f(x))$, up to conjugacy of $\mathrm{Aut}(\overline{K})$, we may assume that $f(x)$ is monic.

Canonical heights for Hénon maps

Hénon maps are not polarized dynamical systems,
but one can define canonical heights for Hénon maps

- Ω: an algebraically closed field complete with respect to an absolute value $|\cdot|$
- $\|(a_1, \ldots, a_n)\| := \max_i\{|a_i|\}$
- $\log^+(r) := \log\max\{r, 1\}$ for $r \in \mathbb{R}$
- $H\colon \mathbb{A}^2 \to \mathbb{A}^2$ a Hénon map over Ω, $P \in \mathbb{A}^2(\Omega)$

Definition (Green functions on $\mathbb{A}^2(\Omega)$)

$$G_H^+(P) := \lim_{n\to+\infty} \frac{1}{d^n}\log^+\|H^n(P)\|, \; G_H^-(P) := \lim_{n\to+\infty} \frac{1}{d^n}\log^+\|H^{-n}(P)\|$$

$$G_H(P) := \max\{G_H^+(P), G_H^-(P)\}$$

Canonical heights for Hénon maps (continued)

$H\colon \mathbb{A}^2 \to \mathbb{A}^2$ a Hénon map over a number field K

For each place $v \in M_K$ with absolute value $|\cdot|_v$,

K_v: completion of K with respect to $|\cdot|_v$

$\mathbb{K}_v$: completion of an algebraic closure of K_v

We have the v-adic Green function

$$G_{H,v} := \max\{G^+_{H,v}, G^-_{H,v}\} : \mathbb{A}^2(\mathbb{K}_v) \to \mathbb{R}_{\geq 0}$$

Definition (canonical height)

$$\widetilde{h}_H\colon \mathbb{A}^2(\overline{K}) \to \mathbb{R}_{\geq 0}, \quad \widetilde{h}_H(P) := \sum_{v \in M_K} n_v G_{H,v}(P)$$

Here n_v is a usual normalizing constant.

For example, if $p \mid v$, then $n_v = [K_v : \mathbb{Q}_p]/[K : \mathbb{Q}]$.

Canonical heights for Hénon maps (continued)

$H : \mathbb{A}^2 \to \mathbb{A}^2$: a Hénon map of degree $d \geq 2$ over a number field K.

Canonical heights for Hénon maps (continued)

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- $G_{H,v} \colon \mathbb{A}^2(\mathbb{K}_v) \to \mathbb{R}_{\geq 0}$ (v-adic Green function)
  $G_{H,v}(P) := \max\left\{\lim \frac{1}{d^n} \log^+ \|H^n(P)\|, \lim \frac{1}{d^n} \log^+ \|H^{-n}(P)\|\right\}$
- When $\mathbb{K}_v = \mathbb{C}$, the Green function is extremely useful in Bedford–Smilie '91, Farnæss–Sibony '92, Hubbard–Oberste-Vorth '94.
- $\widetilde{h}_H \colon \mathbb{A}^2(\overline{K}) \to \mathbb{R}_{\geq 0}$ (canonical height), $\quad \widetilde{h}_H(P) := \sum_{v \in M_K} n_v G_{H,v}$

Canonical heights for Hénon maps (continued)

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Theorem (K– '06, '13)

1. *The limits defining $G_{H,v}$ exist for all $v \in M_K$.*
2. $\widetilde{h}_H = h_{\mathrm{std}} + O(1)$ *on* $\mathbb{A}^2(\overline{K})$
3. $\{$*periodic point under* $H\} = \{P \in \mathbb{A}^2(\overline{K}) \mid \widetilde{h}_H(P) = 0\}$

Note: Hénon map is an autom, so preperiodic = periodic.

Families (our setting)

K a number field, $\delta \in K \setminus \{0\}$,

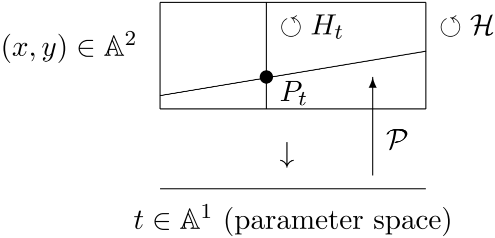
$f_t(x) \in K[t,x]$ monic in x, degree $d \geq 2$ in x

$$\mathbb{A}^2 \times \mathbb{A}^1 \to \mathbb{A}^1, \quad ((x,y),t) \mapsto t$$

$$\mathcal{H} : \mathbb{A}^2 \times \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1, \quad ((x,y),t) \mapsto ((\delta y + f_t(x), x), t)$$

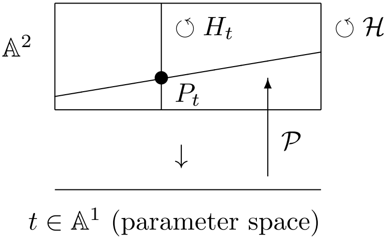
a section $\mathcal{P} \colon \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1, \quad t \mapsto ((a(t), b(t)), t)$

$$\Sigma(\mathcal{P}) := \{t \in \mathbb{A}^1(\overline{K}) \mid P_t = (a(t), b(t)) \text{ is periodic under } H_t\}$$



Families (continued)

K: a number field



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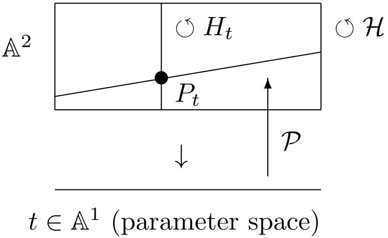


$\rightsquigarrow$ We have canonical heights (η: generic point of $\mathbb{A}^1$)

- $\widetilde{h}_{H_t} : \mathbb{A}^2(\overline{K}) \to \mathbb{R}_{\geq 0}$ for each $t \in \mathbb{A}^1(\overline{K})$
- $\widetilde{h}_{\mathcal{H}_\eta} : \mathbb{A}^2(\overline{K})(\overline{K(t)}) \to \mathbb{R}_{\geq 0}$ on the generic fiber $\mathbb{A}^2_\eta$

Let $\mathcal{P} : \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1$ be a section with $\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \neq 0$. Set

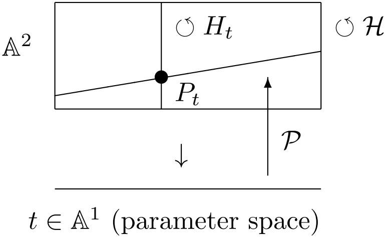
$$h_{\mathcal{P}}(t) := \widetilde{h}_{H_t}(P_t)/\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta).$$

Question 1 Is $h_{\mathcal{P}} : \mathbb{A}^1(\overline{K}) \to \mathbb{R}$ a “nice” height function?

Families (continued)

K: a number field

$\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \neq 0$



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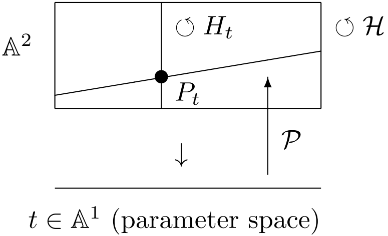


Theorem (Hsia–K)

$h_{\mathcal{P}}$ is the restriction of the height function associated to a semipositive adelically metrized line bundle $(\mathcal{O}_{\mathbb{P}^1}(1), \{\|\cdot\|_v\})$ on $\mathbb{P}^1$ (in the sense of Zhang).

Remark

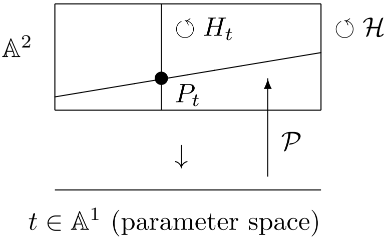
A weaker estimate $\widetilde{h}_{H_t} = \widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) h_{\mathrm{std}} + O(1)$ on $\mathbb{A}^1(\overline{K})$ (Silverman and Tate type) was previously obtained by Ingram '14.

Families (continued)

K: a number field

$\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \neq 0$

$v \in M_K$



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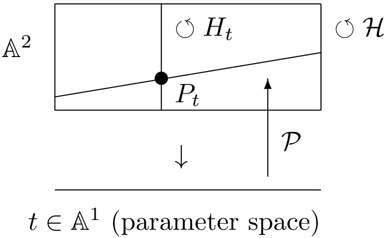

$$G_{\mathcal{P},v} \colon \mathbb{A}^1(\mathbb{K}_v) \to \mathbb{R}_{\geq 0}, \qquad G_{\mathcal{P},v}(t) := G_{H_t,v}(P_t)$$

$$\mathcal{K}_{\mathcal{P},v} := \{t \in \mathbb{A}^1(\mathbb{K}_v) \mid \{H^n(P_t)\}_{n\in\mathbb{Z}} \text{ is bounded}\}$$

$$\mathcal{W}_{\mathcal{P},v} := \{t \in \mathbb{A}^1(\mathbb{K}_v) \mid \lim_{n\to+\infty} \|(H^n(P_t), H^{-n}(P_t))\| = +\infty\}$$

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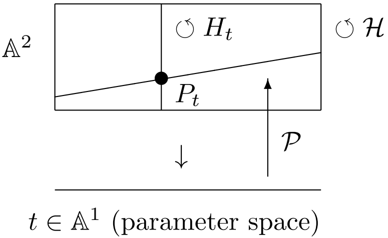

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Proposition (Hsia–K)

1. $h_{\mathcal{P}} = c \sum_{v \in M_K} n_v G_{\mathcal{P},v}$ with $c := 1/\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \in \mathbb{Q}_{>0}$.
2. $\mathcal{K}_{\mathcal{P},v} = \{t \in \mathbb{A}^1(\mathbb{K}_v) \mid G_{\mathcal{P}}(t) = 0\}$
3. $\mathbb{A}^1(\mathbb{K}_v) = \mathcal{K}_{\mathcal{P},v} \amalg \mathcal{W}_{\mathcal{P},v}$

Example

$\mathcal{H} = (H_t)_t \colon \mathbb{A}^2 \to \mathbb{A}^2, \quad (x, y) \mapsto (y + x^2 + t, x)$

(family of quadratic Hénon maps parametrized by t)

$\mathcal{P}_{(0,0)} : \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1$, $t \mapsto ((0, 0), t)$ a constant family of initial points

$\mathcal{K}_{(0,0),\mathbb{C}} := \{t \in \mathbb{C} \mid \{H_t^n((0, 0))\}_{n\in\mathbb{Z}}$ is bounded$\}$

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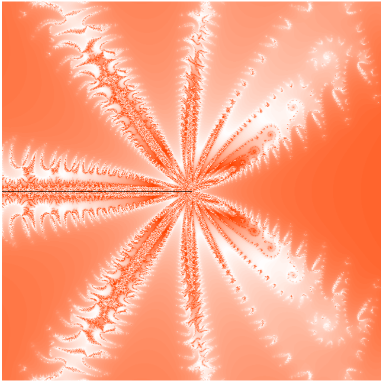
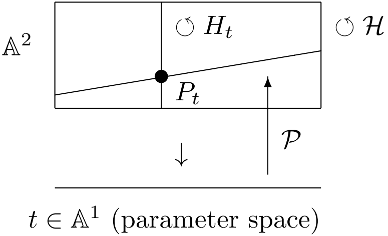


Figure: $|\mathrm{Re}(t)| \leq 0.1$, $|\mathrm{Im}(t)| \leq 0.1$

The set $\Sigma(\mathcal{P})$ of periodic parameter values

K: a number field

$\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) \neq 0$



$$\Sigma(\mathcal{P}) := \{t \in \mathbb{A}^1(\overline{K}) \mid P_t \text{ is periodic under } H_t\}$$

Here, we have a phenomena that was not observed in families of one-dimensional dynamics.

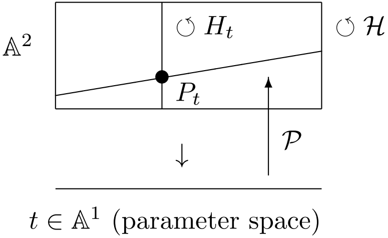
$\Sigma(\mathcal{P})$ may not be an infinite set.

Result on infiniteness of $\Sigma(\mathcal{P})$

K a field (of any characteristic)

$\mathcal{H}(x, y) = (\delta y + f_t(x), x)$

$\Sigma(\mathcal{P}) = \{t \in \mathbb{A}^1(\overline{K})$
$\mid P_t$ is periodic under $H_t\}$



Inspired by Dujardin–Favre's result on dynamical Mordell–Lang conjecture for plane automorphisms, we consider a **reversible** Hénon map. Thus we assume that $\delta = \pm 1$ and that, if $\delta = 1$, then $f_t(x)$ is an even polynomial in x. Then, via the involution $\iota\colon (x, y) \mapsto (-\delta y, -\delta x)$, we have

$$\iota \circ \mathcal{H} \circ \iota = \mathcal{H}^{-1}.$$

Further ι has the fixed curve $C = \{x + \delta y = 0\}$ in $\mathbb{A}^2$.

Result on infiniteness of $\Sigma(\mathcal{P})$ (continued)

K a field (of any characteristic)

$\mathcal{H}(x, y) = (\delta y + f_t(x), x)$

$\Sigma(\mathcal{P}) = \{t \in \mathbb{A}^1(\overline{K})$
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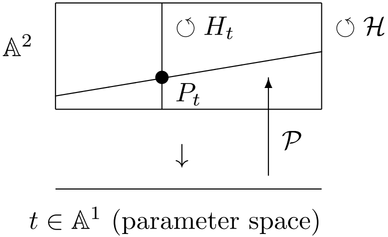


Theorem (Hsia–K)

We assume that $\delta = \pm 1$ and that, if $\delta = 1$, then $f_t(x)$ is an even polynomial in x. If the family of initial points $\mathcal{P} = (a(t), b(t))$ lie on the fixed curve of the involution $\iota\colon (x, y) \mapsto (-\delta y, -\delta x)$, i.e., $a(t) + \delta b(t) = 0$, then $\Sigma(\mathcal{P})$ is an infinite set.

Example

Let $\mathcal{H}(x, y) = (y + x^2 + t, x)$. Then, for any $a \in K$, $|\Sigma((a, -a))| = \infty$.

Result on finiteness/emptyness of $\Sigma(\mathcal{P})$

As a complimentary result, we point out that $\Sigma(\mathcal{P})$ can be the empty set.

Proposition (Hsia–K)

We consider the family of quadratic Hénon maps over $\mathbb{C}$

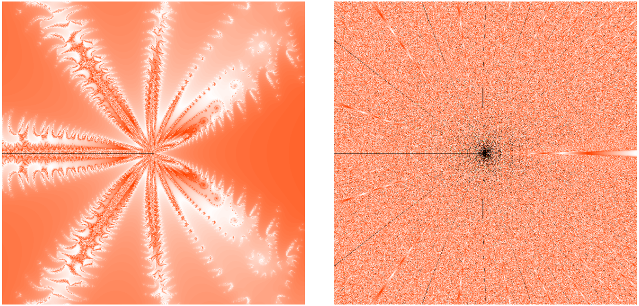
$$\mathcal{H}(x, y) = (y + x^2 + t, x).$$

Let $b \in \mathbb{C}$, and assume that $b \notin \overline{\mathbb{Z}}$. Then

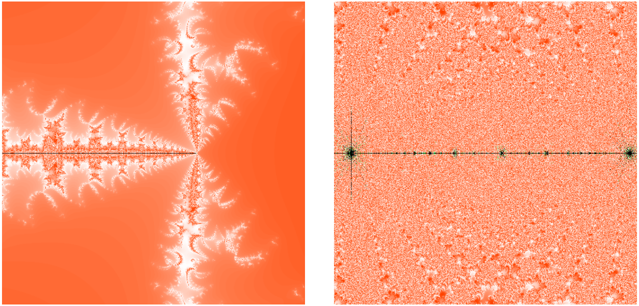
$$\Sigma((0, b)) := \{t \in \mathbb{A}^1(\overline{K}) \mid (0, b) \text{ is periodic under } H_t\} = \emptyset.$$

Example

We have $\Sigma((0, 1/2)) = \emptyset$, while $|\Sigma((a, -a))| = \infty$ for any $a \in \mathbb{C}$.



$\mathcal{P} = (0,0)$ near $t = 0$. (Right) enlarged 10^3 times. $|\Sigma(\mathcal{P})| = \infty$



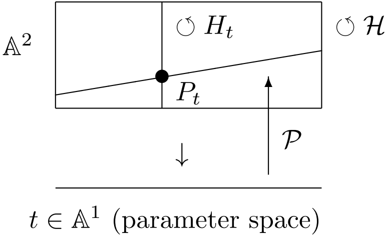
$\mathcal{P} = (0,1/2)$ near $t = -0.1$. (Right) enlarged 10^3 times. $\Sigma(\mathcal{P}) = \emptyset$

Unlikely intersection

K: a number field

$\Sigma(\mathcal{P}) = \{t \in \mathbb{A}^1(\overline{K})$
$\quad | \; P_t$ is periodic under $H_t\}$

We have instances
that $\Sigma(\mathcal{P})$ is infinite.



Unlikely intersection

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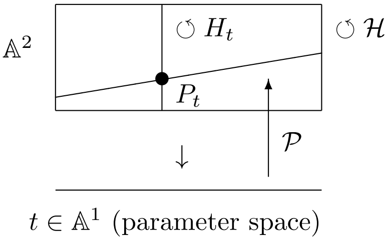

Let $\mathcal{Q} : \mathbb{A}^1 \to \mathbb{A}^2 \times \mathbb{A}^1$ be another section with $\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{Q}_\eta) \neq 0$.

Unlikely intersection

K: a number field

$\Sigma(\mathcal{P}) = \{t \in \mathbb{A}^1(\overline{K})$
$\mid P_t$ is periodic under $H_t\}$

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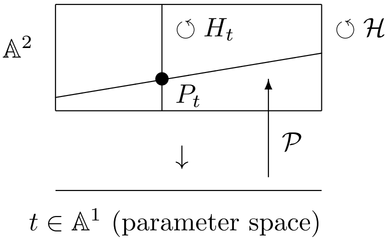

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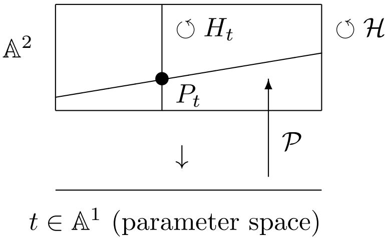

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We would like to consider when there are infinitely many parameter values t such that both P_t and Q_t are periodic under H_t.

Recall that we have shown

- $h_{\mathcal{P}}(t) := \widetilde{h}_{H_t}(P_t)/\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta)$ is the restriction of the semipositive adelically metrized line bundle $\overline{\mathcal{L}_{\mathcal{P}}} := (\mathcal{O}_{\mathbb{P}^1}(1), \{\| \cdot \|\}_v)$.
- $\Sigma(P) = \{t \in \mathbb{A}^1(\overline{K}) \mid h_{\mathcal{P}}(t) = 0\}$.

Unlikely intersection (continued)

$h_{\mathcal{P}} \colon \mathbb{A}^1(\overline{K}) \to \mathbb{R}_{\geq 0}$ defined by $h_{\mathcal{P}}(t) := \widetilde{h}_{H_t}(P_t)/\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta)$

We have $h_{\mathcal{P}} = c \sum_{v \in M_K} n_v G_{\mathcal{P},v}$ with $c = 1/\widetilde{h}_{\mathcal{H}_\eta}(\mathcal{P}_\eta) > 0$

$$\begin{aligned}\Sigma(\mathcal{P}) &:= \{t \in \mathbb{A}^1(\overline{K}) \mid P_t \text{ is periodic under } H_t\} \\ &= \{t \in \mathbb{A}^1(\overline{K}) \mid h_{\mathcal{P}}(t) = 0\}\end{aligned}$$

Theorem (Hsia–K)

Assume that $\Sigma(\mathcal{P})$ and $\Sigma(\mathcal{Q})$ are infinite.

Then the following are equivalent.

1. $\Sigma(\mathcal{P}) \cap \Sigma(\mathcal{Q})$ *is infinite. Namely, there are infinitely many periodic values t such that P_t and Q_t are both periodic under H_t.*
2. $\Sigma(\mathcal{P}) = \Sigma(\mathcal{Q})$
3. $G_{\mathcal{P},v} = G_{\mathcal{Q},v}$ *for all $v \in M_K$.*
4. $h_{\mathcal{P}} = h_{\mathcal{Q}}$.

Unlikely intersection (continued)

Proof uses Yuan's equidistribution theorem. (Or, as the parameter space is 1-dimensional, so we can also use equidistribution theorem due to Autisser, Thuillier, Chambert-Loir, Baker–Rumely, Favre–Rivera-Letlier . . .).

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Indeed, suppose that $\Sigma(\mathcal{P}) \cap \Sigma(\mathcal{Q})$ is infinite.
Let $\{x_n\}_{n\geq 1}$ be a sequence of distinct points with $\{x_n\} \subseteq \Sigma(\mathcal{P}) \cap \Sigma(\mathcal{Q})$.

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Let $\{x_n\}_{n\geq 1}$ be a sequence of distinct points with $\{x_n\} \subseteq \Sigma(\mathcal{P}) \cap \Sigma(\mathcal{Q})$.
Since $\{x_n\}_{n\geq 1}$ has height 0 with respect to $h_{\mathcal{P}} = h_{\overline{\mathcal{L}_{\mathcal{P}}}}$, the equidistribution theorem implies that, as $n \to \infty$, the Galois orbit of x_n will be equidistributed on $\mathbb{P}^1(\mathbb{K}_v)$ with respect to the measure $c_1(\overline{\mathcal{L}_{\mathcal{P}}})_v$ for any $v \in M_K$.

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It follows that $G_{\mathcal{P},v} = G_{\mathcal{Q},v}$ for all $v \in M_K$, and $h_{\mathcal{P}} = h_{\mathcal{Q}}$. Then $\Sigma(\mathcal{P}) = \Sigma(\mathcal{Q})$.

Unlikely intersection (continued)

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Question

Suppose that $h_{\mathcal{P}} = h_{\mathcal{Q}}$. Then does there exist an automorphism $\sigma : \mathbb{A}^2 \to \mathbb{A}^2$ over $\mathbb{A}^1$ and a positive integer $m \geq 1$ with $\sigma^{-1} \circ \mathcal{H}^m \circ \sigma = \mathcal{H}^m$ or $\sigma^{-1} \circ \mathcal{H}^m \circ \sigma = \mathcal{H}^{-m}$ such that $\mathcal{Q} = \mathcal{H}^n(\sigma(\mathcal{P}))$ for some $n \geq 1$?