Fabiha → Mat216 (LC-1,2,3) (Full)

Linear Algebra and Fourier Analysis

Equation, Variable, Equation, Unknown, Parameters, Constant

Equation: $2x + 5 = 7x - 3y$

Variable: $2x + 3 = 7$ → defined solution

Unknown/Leader: $2x + 3 = 7y + 5$

Parameter: $2x - b = c$ → Parameter

$\Rightarrow x = \dfrac{b + c}{2}$

x → Unknown

Constant: 1, 2, 4 → all detected number.

Linear Equation:-

→ Power of unknown 1

→ no product of unknowns

$\sin(2) \cdot x + \ln(3) \cdot y = \tan 5$

$2x + 3y = 5$ ✓

$6x - 7y = 9$ ✓

$6x^2 - 5y = 7$ ✗ → $x \cdot x =$ ✗

$6x - 7xy = 5$ ✗ → $x \cdot y =$ ✗

$2\sqrt{x} + 3y = 9$ ✗ → $x^{1/2} =$ ✗

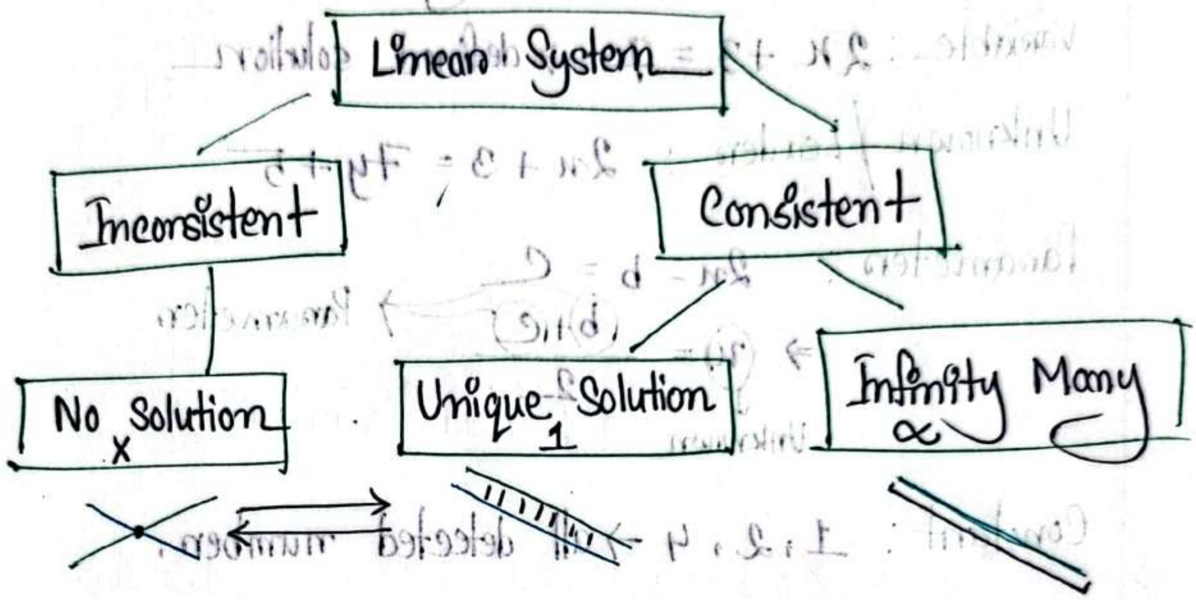
↓ power 1 হতে হবে

$$\boxed{ax + by = c}$$

$\sin(x) + 3y = 5$ → sin এর সাথে x আছে। x কে একা থাকতে হবে।

m linear equations in n unknown

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots\dots + a_{mn}x_n = b_m$$



$$\left.\begin{aligned} 2x + 3y &= 1 \\ x - y &= 3 \end{aligned}\right] \rightarrow \text{Unique solution}$$

$$x - y = 3 \,] \rightarrow \text{Infinite many solution}$$

$$\left.\begin{aligned} 2x + 3y &= 1 \\ 4x + 6y &= 1 \end{aligned}\right] \rightarrow \parallel \rightarrow \text{Parallal line that never meet} - \text{No solution.}$$

$$\left.\begin{aligned} 2x + 3y &= 1 \\ 4x + 6y &= 2 \end{aligned}\right] \Rightarrow \begin{aligned} 2 &= 2 \\ 0 &= 0 \end{aligned}$$ দুইটা similar বা 0 আসলে অনেক many solution ।

Row Echelon form of a Matrix :-

$$
\begin{aligned}
10y - 4z + w &= 1 \\
x + 4y - z + w &= 2 \\
3x + 2y + z + 2w &= 5 \\
-2x - 8y + 2z - 2w &= -4 \\
x - 6y + 3z &= 1
\end{aligned}
\qquad
\left(\begin{array}{cccc|c}
0 & 10 & -4 & 1 & 1 \\
1 & 4 & -1 & 1 & 2 \\
3 & 2 & 1 & 2 & 5 \\
-2 & -8 & 2 & -2 & -4 \\
-1 & -6 & 3 & 0 & 1
\end{array}\right)
$$

Idea of Gaussian Elimination Method :-

→ Unknown ~~ele~~ এর coefficient কে eliminate করে অনেক equation solve করানো easy করে দেওয়া।

- Swapping two equation.
- Multiplying an equation by non-zero constant.
- Adding or substracting two equations (rows).

Row Echelon REF

→ Matrix এর কোনো row 0 হলে তা একদম নিচে থাকবে।

→ For each row: The first non-zero entry (pivot) is strictly to the right of the first non-zero entry of the row above.

⇒ যেকোনো row এর pivot আগের row এর pivot এর ডানে থাকবে।

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \times \qquad \begin{pmatrix} 2 & 5 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \checkmark$$

REF+RREF ... REF

$$\begin{pmatrix} 1 & 3 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \checkmark \qquad \begin{pmatrix} 1 & 2 & 3 \\ \boxed{0 \quad 0 \quad 0} \\ 0 & 0 & 1 \end{pmatrix} \times \qquad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

REF ... RREF

In Row Echelon → Pivot এর নিচের সব element গুলো 0 হবে।

Reduced Row Echelon Form [RREF]

→ আগের দুইটা শর্ত থাকবে।

→ Pivot element 1 হবে।

→ Pivot এর উপরে সব zero হবে।

$$\begin{pmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \qquad \begin{pmatrix} 1 & -7 & 5 & 5 \\ 0 & 1 & 3 & 2 \end{pmatrix}$$

RREF ... RREF ... not RREF but REF

No solution:

Row

zero = non-zero

$$\left(\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \boxed{0 \quad 0 \quad 0} & & & 1 \end{array}\right)$$

zero non zero

Unique Solution:

number of unknowns = numbers of pivots

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 \end{array}\right)$$

3 টা unknown = 3 টা pivot.

Infinity Many:-

number of unknowns > number of pivots

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{array}\right)$$

pivot 3 টা < variable 4 টা

Augmented Matrix:-

$$\left.\begin{array}{l} 2x + 3y = 5 \\ 7x - y = 9 \end{array}\right\} \Leftrightarrow \left(\begin{array}{cc|c} 2 & 3 & 5 \\ 7 & -1 & 9 \end{array}\right)$$

main matrix — Extended.

Practise problem

Solve the system from the reduced row-echelon form metrix

Q1.

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & 3 & 2 \\ 0 & 0 & 1 & 1 & -5 \end{array}\right)$$

[ω is a free variable]

⇒ যে column এ pivot থাকবে না

$x - 7w = 8$

$y + 3w = 2$

$z + w = -5$

ধরি,

$w = t$

$(x, y, z, w) = (8+7t, 2-3t, -5-t, t) \quad t \in \mathbb{R}$

Q2. $\left(\begin{array}{ccccc|c} 1 & -6 & 0 & 0 & 3 & -2 \\ 0 & 0 & 1 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$

$x_2 = t_1$
$x_5 = t_2$
t_1 and $t_2 \in \mathbb{R}$

দুটাই ~~first~~ ti count করবো as free variable

$$x_1 - 6x_2 + 3x_5 = -2$$
$$x_3 + 4x_5 = 7$$
$$x_4 + 5x_5 = 8$$

$x_1 = -2 + 6t_1 - 3t_2$
$x_2 = t_1$
$x_3 = 7 - 4t_2$
$x_4 = 8 - 5t_2$
$x_5 = t_2$

☐ শুধুমাত্র Row Echelon এ থাকলে, Back-substitution করতে হয়

$\left(\begin{array}{ccc|c} 1 & -3 & 4 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 5 \end{array}\right)$

$$x - 3y + 4z = 7$$
$$y + 2z = 2$$
$$z = 5$$

$z = 5$
$y = 2 - 2z = -8$
$x = 7 + 3y - 4z$
$= -37$

CS CamScanner

Mat216

Combined operation of operation 2(row change) and 3(+ , - করা) then solve REF and RREF.

Practise :-

(1)

$$2x - y + z = -3$$
$$4x - y + z = -5$$
$$x + 3y - 2z = 4$$

$\longrightarrow$ convert to Augment

$$\left(\begin{array}{ccc:c} 2 & -1 & 1 & -3 \\ 4 & -1 & 1 & -5 \\ 1 & 3 & -2 & 4 \end{array}\right) \rightarrow \text{REF করবো}$$

find pivot from here

[every row তে pivot লাগবে]

$$\Rightarrow \left(\begin{array}{ccc:c} \overset{\text{Pivot}}{2} & -1 & 1 & -3 \\ 0 & 1 & -1 & -2 \\ 1 & 3 & -2 & 4 \end{array}\right) \quad [R_2' = R_2 - 2R_1]$$

$\Rightarrow$ Pivot এর সাথে (+ , -) করবো।

$\rightarrow$ pivot এর নিচে সব element REF এ 0 হবে

$$\Rightarrow \left(\begin{array}{ccc:c} 2 & -1 & 1 & -3 \\ 0 & 1^{\text{Pivot}} & -1 & 1 \\ 0 & 7 & -5 & 11 \end{array}\right) \quad [R_3' = 2R_3 - R_1]$$

$$\Rightarrow \left(\begin{array}{ccc:c} 2 & -1 & 1 & -3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right) \quad [R_3' = R_3' - 7R_1] \rightarrow \text{converted to REF}$$

$$\Rightarrow 2x - y + z = -3$$
$$y - z = 1$$
$$2z = 4$$

Back substitution ব্যবহার,

$z = 2$

$y = 1 + z = 3$

$2x = -3 + y - z = -2$

$\Rightarrow x = -1$

$(x, y, z) = (-1, 3, 2)$

(2) Use of gaussian Elimination method to solve the Linear System.

$$y + 2z = 3$$
$$2x + 4y - 3z = -4$$
$$3x + 5y - 5z = -6$$

[always swap rows not column]

$$= \left(\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ 2 & 4 & -3 & -4 \\ 3 & 5 & -5 & -6 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} \overset{\rightarrow \text{Pivot}}{2} & 4 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 3 & 5 & -5 & -6 \end{array}\right) \quad [R_1 \leftrightarrow R_2]$$

$$= \left(\begin{array}{ccc|c} 2 & 4 & -3 & -4 \\ 0 & 1^{\rightarrow \text{Pivot}} & 2 & 3 \\ 0 & -2 & -1 & 0 \end{array}\right) \quad [R_3' = 2R_3 - 3R_1]$$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & 4 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 \end{array}\right) \quad [R_3' = R_3 + 2R_2]$$

$$\Rightarrow \quad \left.\begin{array}{l} 2x + 4y - 3z = -4 \\ y + 2z = 3 \\ 3z = 6 \end{array}\right\}$$

Back substitution

$\Rightarrow$ $z = 2$

$y = -1$

$x =$

(3) $$\begin{cases} x - 4y + 5z = 8 \\ 3x + 7y - z = 3 \\ x + 15y - 11z = -14 \end{cases}$$

$$\Rightarrow \left(\begin{array}{ccc|c} \underset{\text{Pivot}}{1} & -4 & 5 & 8 \\ 3 & 7 & -1 & 3 \\ 1 & 15 & -11 & -14 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -4 & 5 & 8 \\ 0 & 19 & -16 & -21 \\ 1 & 15 & -11 & -14 \end{array}\right) \quad [R_2' = R_2 - 3R_1]$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -4 & 5 & 8 \\ 0 & 19 & -16 & -21 \\ 0 & 19 & -16 & -22 \end{array}\right) \quad [R_3' = R_3 - R_1]$$

$$\Rightarrow \left(\begin{array}{ccc|c} 1 & -4 & 5 & 8 \\ 0 & 19 & -16 & -21 \\ 0 & 0 & 0 & -1 \end{array}\right) \quad [R_3' = R_3 - R_2]$$

$\rightarrow$ pivot

no solution

(4) $$\left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{array}\right)$$

$$= \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 3 & -6 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - R_1 \\ R_3' = R_3 + R_1 \\ R_4' = R_4 - 3R_1 \end{array}$$

$$= \left(\begin{array}{cccc|c} 1 & -1 & 2 & -1 & -1 \\ 0 & 3 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{array}{l} R_3' = 3R_3 - R_2 \\ R_4' = R_4 - R_2 \end{array}$$

$z \quad w$

free variable.

Here z and ω are free variable,

$z = t_1$ and $\omega = t_2$

$x - y + 2z - \omega = -1$

$3y - 6z = 0$

$\Rightarrow y = 2t_1$

$\Rightarrow x = -1 + t_2$

Find the general solution of the homogeneous system, $Ax = 0$ where,

5

$$A = \begin{pmatrix} 1 & 2 & 2 & 1 & 4 \\ 3 & 7 & 7 & 3 & 13 \\ 2 & 5 & 5 & 2 & 9 \end{pmatrix}$$

$Ax = 0$ means

$$\left(\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 4 & 0 \\ 3 & 7 & 7 & 3 & 13 & 0 \\ 2 & 5 & 5 & 2 & 9 & 0 \end{array}\right)$$ → homogeneous eqt where All right values are 0

$$= \left(\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 4 & 0 \\ 3 & 7 & 7 & 3 & 13 & 0 \\ 2 & 5 & 5 & 2 & 9 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 4 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 3R_1 \\ R_3' = R_3 - 2R_1 \end{array}$$

$$= \left(\begin{array}{ccccc|c} 1 & 2 & 2 & 1 & 4 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_3' = R_3 - R_2$$

$\uparrow \quad \uparrow \quad \uparrow$

$x_1 + 2x_2 + 2x_3 + x_4 + 4x_5 = 0$

$x_2 + x_3 + x_5 = 0$

Hence x_3, x_4, x_5 are free variable.

$x_2 = -t_1 - t_3$

$x_1 = -4t_1 - t_2 - 2t_3$ $[t_1, t_2, t_3 \in \mathbb{R}]$

* Gaussian Elemination method REF এ convert করে।
* Gauss - Jorden method RREF এ convert করে।

Gauss- Jorden method by RREF :-

REF
* All zero rows are at the bottom.
* Pivot will be in right then the first non-zero entry.

RREF
* All entries above pivot must be zero.
* pivot element must be 1.

Step 1. Row-echelon form.

Step 2. All entries above pivot must be zero.

step 3. All pivots must be 1

Practise

(6) $2x - y + z = -3$

$4x - y + z = -5$

$x + 3y - 2z = 4$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & -1 & 1 & -3 \\ 4 & -1 & 1 & -5 \\ 1 & 3 & -2 & 4 \end{array}\right) \quad R_2' = R_2 - 2R_1, \quad R_3' = 2R_3 - R_1$$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & -1 & 1 & -3 \\ 0 & 1 & -1 & 1 \\ 0 & 7 & -5 & 11 \end{array}\right) \quad \cancel{R_3' = R_3 - 7R_2}$$

$$\Rightarrow \left(\begin{array}{ccc|c} 2 & -1 & 1 & -3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right) \quad R_3' = R_3 - 7R_2$$

$$\Rightarrow \left(\begin{array}{ccc|c} 4 & -2 & 0 & -10 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 4 \end{array}\right) \quad R_2' = 2R_2 + R_3, \quad R_1' = 2R_1 - R_3$$

$$\Rightarrow \left(\begin{array}{ccc|c} 4 & 0 & 0 & -4 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 4 \end{array}\right) \quad R_1' = R_1 + R_2$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right) \quad \begin{array}{l} R_1' = R_1/4 \\ R_2' = R_2/2 \\ R_3' = R_3/2 \end{array}$$

$x = -1$
$y = 3$
$z = 2$

7) $x - 4y + 5z = 8$
$3x + 7y - z = 3$
$x + 15y - 11z = -14$

$$= \left(\begin{array}{ccc|c} 1 & -4 & 5 & 8 \\ 3 & 7 & -1 & 3 \\ 1 & 15 & -11 & -14 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & -4 & 5 & 8 \\ 0 & 19 & -16 & -21 \\ 0 & 19 & -16 & -22 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 3R_1 \\ R_3' = R_3 - R_2 \end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & -4 & 5 & 8 \\ 0 & 19 & -16 & -21 \\ 0 & 0 & 0 & -1 \end{array}\right) \quad R_3' = R_3 - R_2$$

No solution

(8) $-3x_2 + 2x_3 + x_4 = -1$

$x_1 + 2x_3 - 10x_4 = 10$

$-x_1 + x_2 - 2x_3 + 4x_4 = -7$

$$= \left(\begin{array}{cccc|c} 0 & -3 & 2 & 1 & -1 \\ 1 & 0 & 2 & -10 & 10 \\ -1 & 2 & -2 & 7 & -7 \end{array}\right)$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & -10 & 10 \\ 0 & -3 & 2 & 1 & -1 \\ -1 & 1 & -2 & 7 & -7 \end{array}\right) \quad [R_1 \leftrightarrow R_2]$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & -10 & 10 \\ 0 & -3 & 2 & 1 & -1 \\ 0 & 1 & 0 & -3 & 3 \end{array}\right) \quad [R_3' = R_3 + R_2]$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 2 & -10 & 10 \\ 0 & -3 & 2 & 1 & -1 \\ 0 & 0 & 2 & -8 & 8 \end{array}\right) \quad R_3' = 3R_3 + R_2$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 2 \\ 0 & -3 & 0 & 9 & -9 \\ 0 & 0 & 2 & -8 & 8 \end{array}\right) \quad \begin{array}{l} R_1' = R_1 - R_3 \\ R_2 - R_3 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & -3 & 3 \\ 0 & 0 & 1 & -4 & 4 \end{array}\right) \quad \begin{array}{l} R_2' = R_2/-3 \\ R_3' = R_3/2 \end{array}$$

$$\Rightarrow \begin{aligned} x_1 - 2x_4 &= 2 \\ x_2 - 3x_4 &= 3 \\ x_3 - 4x_4 &= 4 \end{aligned}$$

x_4 is a free variable. $x_4 = t$

$$\begin{aligned} x_1 &= 2+2t \\ x_2 &= 3+3t \\ x_3 &= 4+4t \\ x_4 &= t \end{aligned} \qquad t \in \mathbb{R}$$

(9) $x_1 - x_2 - x_3 + 2x_4 = 1$

$2x_1 - 2x_2 - x_3 + 3x_4 = 3$

$-x_1 + x_2 - x_3 = -3$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 2 & -2 & -1 & 3 & 3 \\ -1 & 1 & -1 & 0 & -3 \end{array}\right)$$

$C_1 + 2C_2$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 2 & -2 \end{array}\right) \quad \begin{array}{l} R_2' = R_2 - 2R_1 \\ R_3' = R_3 + R_1 \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_3' = R_3 + 2R_2$$

$$\sim \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \quad R_1' = R_1 + R_2$$

$$x_1 - x_2 + x_4 = 2$$

$$x_3 - x_4 = 1$$

Here, x_2 and x_4 are free variable, let $x_2 = t_1$ and $x_4 = t_2$

$$x_1 = 2 + t_1 - t_2$$

$$x_2 = t_1$$

$$x_3 = 1 + t_2$$

$$x_4 = t_2$$

t_1 and $t_2 \in \mathbb{R}$.

Mat216 → Mid.

1. Solving system.
2. Vector spaces
3. Linear subspaces.
4. Linear combination and span.
5. Linear Independence and Dependence.
6. Basis, Dimension, Fundamental Subspaces.
7. Linear Transformation.

Linear combination :-

$$\alpha_1 \bar{v}_1 + \alpha_2 \bar{v}_2 + \alpha_3 \bar{v}_3 + \cdots \alpha_n \vec{v}_n = \vec{b} / \vec{v}$$

Problem 1.

$b = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ a linear combination of $\bar{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\bar{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$?

Solution:-

$$\alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$$

So $\vec{b}$ is a linear combination of $\bar{v}_1$ and $\bar{v}_2$.

i) If consistent → Yes → Unique / Many

ii) If inconsistent → No

Span of set of vectors :-

→ Span means set of all possible linear combination of v's.

→ For 2 vector $\bar{v}_1, \bar{v}_2$

$$\text{Span}\{\bar{v}_1, \bar{v}_2\} = \{\alpha_1 v_1 + \alpha_2 v_2 ; \alpha_1, \alpha_2 \in \mathbb{R}\}$$

Problem :-

$$\text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = ?$$

$$\alpha_1\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\end{pmatrix}$$

$$= \begin{pmatrix}\alpha_1\\0\end{pmatrix} + \begin{pmatrix}0\\\alpha_2\end{pmatrix}$$

$$= \begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix}$$

$$\text{span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\} = \left\{\begin{pmatrix}\alpha_1\\\alpha_2\end{pmatrix} ; \alpha_1, \alpha_2 \in \mathbb{R}\right\}$$
$$= \mathbb{R}^2$$

problem :- Is $\bar{b} = \begin{pmatrix}1\\1\\2\end{pmatrix}$ is the span of $\bar{v}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\bar{v}_2 = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $v_3 = \begin{pmatrix}1\\0\\-1\end{pmatrix}$ = ?

→ Is $\bar{b}$ a linear combination of v_1, v_2, v_3

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

Vector Space :-

$A_1 \Rightarrow \bar{u} + \bar{v} \in V$

$A_2 \Rightarrow \bar{u} + \bar{v} = \bar{v} + \bar{u}$

$A_3 \Rightarrow (\bar{w} + \bar{v}) + \bar{u} = (\bar{w} + \bar{u}) + \bar{v}$

$A_4 \Rightarrow \bar{u} + 0 = \bar{u}$

$A_5 \Rightarrow \bar{u} + \bar{u}' = 0$

$M_1 \Rightarrow \alpha\bar{u} \in V$

$M_2 \Rightarrow \alpha(\bar{u} + \bar{v}) = \alpha\bar{u} + \alpha\bar{v}$

$M_3 \Rightarrow (\alpha+\beta)\bar{u} = \alpha\bar{u} + \beta\bar{u}$

$M_4 \Rightarrow \alpha\beta\,\bar{u} = \alpha(\beta\bar{u})$

$M_5 \Rightarrow 1\,\bar{u} = \bar{u}$

$F = \mathbb{R}$ → standared.

$F = \mathbb{C}$

Linear Subspace :-

1. $\bar{a}$ A_4 : $\vec{0} \in W$
2. A_1 : $\bar{u}+\bar{v} \in W$
3. M_1 : $\bar{a}\, \alpha \vec{u} \in W$

Problem :-

$M_{2\times2}(\mathbb{R})$.

Determine whether the set S is subspace of $M_{2\times2}$ or not.

$$S = \left\{ \begin{pmatrix} a & d \\ c & b \end{pmatrix} : a+b+c+d=0 \right\}$$

Clearly $S \subseteq M_{2\times2}$

(i) $\bar{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in S$ as $0+0+0+0=0$

ii) $\bar{u} \in S$ and $\bar{v} \in S$

$\bar{u} = \begin{pmatrix} a_1 & d_1 \\ c_1 & b_1 \end{pmatrix}$ $\quad \bar{v} = \begin{pmatrix} a_2 & d_2 \\ c_2 & b_2 \end{pmatrix}$

$$\bar{u}+\bar{v} = \begin{pmatrix} a_1+a_2 & d_1+d_2 \\ c_1+c_2 & b_1+b_2 \end{pmatrix}$$

$= a_1+a_2+b_1+b_2+c_1+c_2+d_1+d_2$

$= 0+0 = 0$

$\bar{u} + \bar{v} \in S$

iii) $\alpha \bar{u} = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} = \alpha a + \alpha b + \alpha c + \alpha d$

$= \alpha (a + b + c + d)$

$= 0$

$\therefore$ So, S is a subspace of vector space $M_{2\times 2}$.

Problem

Is the set of vectors $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : y = -4x - z, z = 8x \right\}$ a subspace of vector space $V = \mathbb{R}^3$?

i) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad y = -4x - z = 0$

$z = 8x = 0$

ii) $\bar{u} + \bar{v} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \\ \boxed{z_1 + z_2} \end{pmatrix} \begin{matrix} \to x \\ \to y \\ \to z \end{matrix}$

$y = -4x - z$

$= -4x_1 - 4x_2 - z_1 - z_2$ মিলল।

$= y_1 + y_2$

$z = 8(x_1 + x_2) = 8x_1 + 8x_2 = z_1 + z_2$

iii) $\alpha \bar{u} = \begin{pmatrix} \alpha x \\ \alpha y \\ \alpha z \end{pmatrix} \begin{matrix} \to x \\ \to y \\ \to z \end{matrix}$

Linear Dependence and Independence :-

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0 - \left[\begin{array}{l} \overbrace{\text{Unique}}^{\text{vowels}} = \text{Independent} \\ \text{Many} = \text{Dependent} \end{array} \right.$

Problem

Show that the set of vectors $\{(2,1,2),(0,1,-1),(4,3,4)\}$ in $\mathbb{R}^3$ Linearly Independent.

$$\alpha_1(2,1,2) + \alpha_2(0,1,-1) + \alpha_3(4,3,4) = (0,0,0)$$

$$\Rightarrow (2\alpha_1, \alpha_1, 2\alpha_1) + (0, \alpha_2, -\alpha_2) + (4\alpha_3 + 3\alpha_3 + 4\alpha_3) = (0,0,0)$$

$$\Rightarrow \left.\begin{array}{l} 2\alpha_1 + 4\alpha_3 = 0 \\ \alpha_1 + \alpha_2 + 3\alpha_3 = 0 \\ 2\alpha_1 - \alpha_2 + 4\alpha_3 = 0 \end{array}\right\} - (1)$$

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 1 & 1 & 3 & 0 \\ 2 & -1 & 4 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{array}\right)$$

Same

$$\sim \left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$$

pivot = 3 , variable = 3

Unique solution = Independent.

Basis :-

i) S is linearly independent

ii) S spans vector space V

Problem :-

Is $S = \{(1,2),(3,1)\}$ a basis of the space $\mathbb{R}^2$

i) S linearly independent.

ii) $\text{Span}(S) = \mathbb{R}^2$

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = (a,b)$$

$$\Rightarrow \alpha_1 + 3\alpha_2 = a$$

$$2\alpha_1 + 2\alpha_2 = b$$

$$\sim \left(\begin{array}{cc|c} 1 & 3 & a \\ 0 & -5 & b-2a \end{array}\right)$$

= Unique solution

Basis in space V এর কথা বললে , pivot আলাগুলো হবে Basis.

MAT216

After Midterm

Eigenvalue

Linear Transformation.

Matrix as Linear Transformation:-

~~$T(x$~~ $T(x, y) = (3x-2y, 2x-2y)$

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x-2y \\ 2x-2y \end{pmatrix} \qquad T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Idea of Eigenvalue and Eigenvector:-

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

→ Eigenvalue

যে পরিমাণ scale হয়।

$$T\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

যে যে vector নিজেই নিজের multiple. Scale হচ্ছে But rotate (Eigenvetor) না থাকে Eigenvector বলে।

Eigenvalue and Eigenvector of a Linear Transformation:-

Let T be linear transformation from V to V. A scaler number λ is called eigen~~vector~~value of T if there exists a vector v such that $\vec{v} \neq 0$ whith $T(\vec{v}) = \lambda\vec{v}$

Here the vector $\vec{v}$ is called an eigen vector of the λ.

Square Matrix হতে হবে।

Eigenvalue বের করার condition :-

$$A.\vec{V} = \lambda.\vec{V}$$

$$\Rightarrow A\vec{V} - \lambda\vec{V} = \vec{0}$$

$$\Rightarrow \vec{V}(A-\lambda) = \vec{0}$$

$$\Rightarrow (A - I\lambda)\vec{V} = \vec{0}$$

$\hookrightarrow \det(A - I\lambda) = \vec{0} \Rightarrow$ Charechteristic equation.

Maths :

$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \rightarrow$ Find the chareeteristic equation and eigenvalues of the matrix A.

$\Rightarrow$

Let λ be an eigenvalue of A.

$$\det(A - \lambda I) = 0$$

$$\Rightarrow \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 3-\lambda & 2 \\ 2 & 2-\lambda \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -6 - 3\lambda + 2\lambda + \lambda^2 + 4 = 0$$

$$\Rightarrow \lambda^2 - \lambda + 2 = 0$$

$$\Rightarrow \lambda = -1, 2$$

2. $A \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 \\ -1 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & 2-\lambda \\ -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(4 - 5\lambda + \lambda^2) - 1(2) + 2(2-\lambda) = 0$$

$$\Rightarrow 4 - 5\lambda + \lambda^2 - 4\lambda + 5\lambda^2 - \lambda^3 - 2 + 4 - 2\lambda = 0$$

$$\Rightarrow -\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\Rightarrow \underbrace{\lambda^3 - 6\lambda^2 + 11\lambda - 6}_{\text{Charecteristic polinomial}} = 0 \quad \text{C.equation}$$

$\lambda = 1, 3, 2$

□ Trace $A = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$

Determinant $= \lambda_1 \cdot \lambda_2 \cdot \lambda_3 \cdot \cdots \cdot \lambda_n$

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ Trace

Eigenvalues of a Triangular Matrix :-

Let A be an Upper or Lower Triangular matrix then the diagonal elements are the eigenvalues.

$$A = \begin{pmatrix} \lambda_1 & * & * \\ 0 & \lambda_2 & * \\ 0 & 0 & \lambda_3 \end{pmatrix} \text{ or } A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ * & \lambda_2 & 0 \\ * & * & \lambda_3 \end{pmatrix}$$

□ Find eigenvalues of $A = \begin{pmatrix} 6 & 0 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$ - (Upper T)

$\lambda_1 = 6, 2, 4.$

Eigenvector :

$A\vec{v} = \lambda\vec{v}$

$\Rightarrow A\vec{v} - \lambda\vec{v} = 0$

$\Rightarrow \boxed{(A - \lambda I)\vec{v} = 0}$

↓

Solve

null space of $A - \lambda I$ means non-zero element.

- There exists infinity many eigenvectors for each e.values
- If $\vec{v}$ is an eigenvector then multiple of $\vec{v}$ is also an eigenvector.

Math :-

1. Find the eigenvalues and eigenvectors of A.

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$$

Let λ be an eigenvalue of A.

$$A - \lambda I = 0$$

$$\begin{vmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\lambda = -1, 2$$

For $\lambda = -1$, let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigenvector,

$$(A - \lambda I)\vec{v} = 0$$

$$\Rightarrow \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad [\lambda = -1]$$

Augmented,

$$\left(\begin{array}{cc|c} 4 & -2 & 0 \\ 2 & -1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{cc|c} 4 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad R_2' = 2R_2 - R_1$$

$$= \left(\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$= x_1 - \frac{1}{2} x_2 = 0 \quad | \; x_1 = 1/2\, x_2$

$x_2 = t$ (free variable)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2\, t \\ t \end{pmatrix} = t \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

$$= 1/2\, t \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

For $\lambda = -1$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector.

For $\lambda = 2$,

$(A - \lambda I)\vec{v} = \vec{0}$

$$\Rightarrow \begin{pmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Augmented,

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad R_2' = R_2 - 2R_1$$

$x_1 - 2x_2 = 0 \quad | \; x_1 = 2x_2$

$x_2 = \cancel{free}\ t$ (free variable)

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2t \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda = 2$, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector.

2. 3 टा parameter হলে,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t_1 + t_2 \\ t_1 \\ t_2 \end{pmatrix} = t_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

2 টা

Similar matrix :-

$$A = PBP^{-1}$$

A and B similar হলে,

1. Eigenvalue same.
2. Det same.
3. Trace same!

Diagonalization :-

$A = P\underset{\downarrow}{D}P^{-1}$

Diagonal Matrix

if,

1. A $n \times n$ is diagonalizable if and only if it has n linearly independent eigenvectors.

Where D is diagonal matrix with eigenvalue in the main diagonal and P is the matrix with corresponding eigenvectors in the same columns.

Math:-

Check whether that A is diagonalizable or not. If then diagonalize A -

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

*Eigenvector আসতে হবে ৩ টা।

Let λ be an eigenvalue of A

$A - \lambda I$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda) \cdot \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 2 \\ -1 & 3-\lambda \end{vmatrix} + 2 \begin{vmatrix} 0 & 2-\lambda \\ -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4 - 5\lambda + \lambda^2 - 4\lambda + 5\lambda^2 - \lambda^3 - 2 + 4 - 2\lambda = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 3, 2, 1$$

For,

$\lambda = 3$, let $v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$(A - \lambda I)\vec{v} = 0$$

$$\Rightarrow \begin{pmatrix} 1-\lambda & 1 & 2 \\ 0 & 2-\lambda & 2 \\ -1 & 1 & 3-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -2 & 1 & 2 \\ 0 & -1 & 2 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The augmented,

$$\left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 1 & 0 & 0 \end{array}\right)$$

RREF

$$\Rightarrow \left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - 2x_3 = 0$ $\quad x_3 = t$

$x_2 - 2x_3 = 0$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2t \\ 2t \\ t \end{pmatrix} = t \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$$

For, $\lambda = 3$, $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$ is an eigenvector.

⋮

Same way to $\lambda = 2$ an $\lambda = 1$,

For, $\lambda = 2$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

For, $\lambda = 1$, $\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ is " "

$\therefore$ A is diagonalizable as $n = 3$ ~~and~~ = total E.V.

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$$

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P = \begin{pmatrix} \vdots & \vdots & \vdots \\ V_1 & V_2 & V_3 \\ \vdots & \vdots & \vdots \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & -2 \\ 1 & 0 & 1 \end{pmatrix}$$

By calculator,

$$P^{-1} = \begin{pmatrix} -1/2 & 1/2 & +1 \\ 2 & -1 & -2 \\ 1/2 & -1/2 & 0 \end{pmatrix}$$

Calculating power on a Diagonal Matrix:-

$$D = D^n$$

$$\begin{pmatrix} a & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{pmatrix} = \begin{pmatrix} a^n & 0 & 0 \\ 0 & 5^n & 0 \\ 0 & 0 & 8^n \end{pmatrix}$$

$A = PDP^{-1}$

$A^2 = PD^2P^{-1}$

$A^n = P \cdot D^n P^{-1}$

A^{1000}, $\sqrt{A}$ এমন করে বাড়াতে দিলে PD^nP^{-1} বারে বারে বাড়াবো,

Inner Product:-

Generalization of dot Product ; Takes two vector give one output.

$\vec{U} \cdot \vec{V}$ = Real number / scaler number

Positivity :- $\langle \vec{u}, \vec{u} \rangle \geq 0$ for all $\vec{u} \in \vec{V}$

Definiteness :- $\langle \vec{u}, \vec{u} \rangle = 0$ if $\vec{u} = 0$;

Additivity in first slot :→ $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$, $\vec{u}, \vec{v}, \vec{w} \in$

Homogeneity in first slot :- $\langle \lambda \vec{u}, \vec{v} \rangle = \lambda \langle \vec{u}, \vec{v} \rangle$ for all $\lambda \in \mathbb{R}$

Symmetry :- $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$ for all $\vec{u}, \vec{v} \in V$

Standard Inner Product :-

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = x_1y_1 + x_2y_2 \Big\} \text{ dot product}$$

Math :-

1. Show that $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = 2x_1y_1 + 3x_2y_2$ is an Inner product in $\mathbb{R}^2$.

Positivity :-

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle = 2x_1^2 + 3x_2^2 \geqslant 0$$

Symmetry :-

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle \qquad \left\langle \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle$$

$$= 2x_1y_1 + 3x_2y_2 \qquad = 2x_1y_1 + 3x_2y_2$$

$\therefore$ Inner product.

2. $\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = 2x_1y_1 - 3x_2y_2$

Positivity :-

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle = 2x_1^2 - 3x_2^2 \not\geqslant 0$$

$\therefore$ Not an Inner Product

Standard Inner Product in $\mathbb{C}^2$:-

$$\left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle = z_1 \overline{w_1} + z_2 \overline{w_2}$$

$$\left\langle \begin{pmatrix} 2+i \\ 7 \end{pmatrix}, \begin{pmatrix} 3 \\ 1+3i \end{pmatrix} \right\rangle = (2+i)\overline{3} + 7 \cdot (\overline{1+3i})$$

$$= (2+i) \cdot 3 + 7 \cdot (1-3i)$$

$$= 6 + 3i + 7 - 21i$$

$$= 13 - 18i$$

Inner product in Matrix Spaces:-

standard inner product of 2x3 matrix:-

$$\left\langle \begin{pmatrix} 4 & 5 & 2 \\ 1 & 0 & 3 \end{pmatrix}, \begin{pmatrix} 2 & 1 & 5 \\ 4 & 2 & 3 \end{pmatrix} \right\rangle$$

$$\boxed{\begin{gathered} \langle A, B \rangle \\ \text{Trace}(A^T B) \end{gathered}}$$

~~Tra~~, or

$\text{Trace}(AB^T)$

$$= \text{Trace} \begin{pmatrix} 4 & 1 \\ 5 & 0 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 5 \\ 4 & 2 & 3 \end{pmatrix}$$

$$= \text{Trace} \begin{pmatrix} 12 & 6 & 23 \\ 10 & 5 & 25 \\ 16 & 8 & 19 \end{pmatrix}$$

$$12 + 5 + 19 = 36$$

Inner product in function Spaces:-

$$\langle f(x), g(x) \rangle = \int_a^b f(x) \cdot g(x) \, dx$$

Complex: $\langle f(z), g(z) \rangle = \int_a^b f(z) \cdot g(z) \, dz$

NorM :-

Abstraction of Length.

$\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$

Length $\|\vec{u}\| = \sqrt{a^2+b^2} \rightarrow \mathbb{R}^n$ Euclidean Norm.

$= \sqrt{\langle \bar{u}, \bar{u} \rangle}$

Positivity : $\|\vec{u}\| \geq 0$ for all $\vec{u} \in V$;

Definiteness : $\|\vec{u}\| = 0$ if and only if $\vec{u} = 0$;

Triangle Inequality : $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ for all $\vec{u}, \vec{v} \in V$;

Homogeneity : $\|\lambda \vec{u}\| = |\lambda| \|\vec{v}\|$ for all $\lambda \in F$ and for all $\vec{u} \in V$;

$\boxplus$ Length from inner product,

$\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

Matric :

Abstruction of Distance.

$d((a,b),(c,d)) = \sqrt{(a-c)^2+(b-d)^2}$

$d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$

Positivity :- $d(\bar{u}, \bar{v}) \geq 0$ for all $\bar{u}, \bar{v} \in V$;

Definiteness : $d(\bar{u}, \bar{v}) = 0$ if and only if $\bar{u} = \bar{v}$.

Triangle Inequality :- $d(\bar{u}, \bar{w}) \leq d(\bar{u}, \bar{v}) + d(\bar{v}, \bar{w})$

Symmetry :- $d(\bar{u}, \bar{v}) = d(\bar{v}, \bar{u})$ for all $\bar{u}, \bar{v} \in V$.

All together

$\langle \bar{u}, \bar{v} \rangle$ inner product

$\|\bar{u}\| = \sqrt{\langle \bar{u}, \bar{u} \rangle}$ norm

$d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$ metric.

Inner product function :-

* $\langle f(x), g(x) \rangle = \int_a^b f(x) \cdot g(x)\, dx$

complex :

$$\langle f(z), g(z) \rangle = \int_a^b f(z) \cdot g(z)\, dz$$

Orthogonality :-

Abstraction of 90°

$\langle \bar{u}, \bar{v} \rangle = 0$ [inner/dot product 0]

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = x_1 y_1 + x_2 y_2 + \cdots x_n y_n = 0 \rightarrow \text{orthogonal}$$

$(x_1, x_2 \ldots x_n)$ and $(y_1, y_2, \ldots y_n)$

function এর ক্ষেত্রে,

$$\langle f(x), g(x) \rangle = \int_a^b f(x) \cdot g(x)\, dx = 0$$

Math:

1. Consider the inner product

$$\langle f(x), g(x) \rangle = \int_{-1}^{1} f(x)\, g(x)\, dx$$

Show that the functions $3x^2-1$ and $5x^3-3x$ are orthogonal.

$$\langle 3x^2-1, 5x^3-3x \rangle = \int_{-1}^{1} (3x^2-1)(5x^3-3x)\, dx$$

$$= \int_{-1}^{1} (15x^5 - 9x^3 - 5x^3 + 3x)\, dx$$

$$= \left[15\frac{x^6}{6} - 9\frac{x^4}{4} + 5\frac{x^4}{4} + 3\frac{x^2}{2}\right]_{-1}^{1}$$

$$= \left[15\frac{1}{6} - 9\frac{1}{4} + 5\frac{1}{4} + \frac{3}{2}\right] - \left(\frac{15}{6} - \frac{9}{4} + \frac{5}{4} + \frac{3}{2}\right)$$

$$= 0$$

$\therefore$ function are orthogonal.

2. $$\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$$

$\sin(x)$ and $\cos(2x)$ are orthogonal.

$$\langle \sin x, \cos 2x \rangle = \int_{-\pi}^{\pi} \sin x \cdot \cos 2x \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\sin(x)\cos(2x) \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left[\sin(x+2x) + \sin(x-2x)\right] dx$$

$$= \frac{1}{2}\left[\frac{-\cos x}{3}\right]_{-\pi}^{\pi} - \frac{1}{2}\left[-\cos x\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[-\frac{\cos 3\pi}{3} - \frac{-\cos(-3\pi)}{3}\right] - \frac{1}{2}\left[-\cos\pi - (-\cos(-\pi))\right]$$

$$= 0$$

Necessary Trigonometry Formulas :-

$2\sin A\cos B = \sin(A+B) + \sin(A-B)$

$2\sin A \sin B = \cos(A-B) - \cos(A+B)$

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$

$2\sin^2 A = 1 - \cos 2A$

$2\cos^2 A = 1 + \cos 2A$

$\cancel{\sin(n\pi)} = 0$

ec

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\sin(n\pi) = 0 \quad n \in \mathbb{Z}$

$\cos(n\pi) = (-1)^n \quad n \in \mathbb{Z}$

Orthogonal and Orthonormal set :-

Orthogonal vector :-

$\langle \bar{v}_m , \bar{v}_n \rangle = 0$ for all $m \neq n$; 90°.

Orthonormal vector :-

(Orthogonal) + (norm (length) = 1)

$\langle \bar{v}_m , \bar{v}_n \rangle = 0$ for all $m \neq n$.

$\langle \bar{v}_m , \bar{v}_n \rangle = 1$ for all $m = n$.

Math :-

1. Show that

$B = \{(0,1,0), (-4/5, 0, 3/5), (3/5, 0, 4/5)\}$

is an orthonormal set.

$\therefore \langle (0,1,0), (-\frac{4}{5}, 0, \frac{3}{5}) \rangle = 0+0+0$

$\therefore \langle (0,1,0), (\frac{3}{5}, 0, \frac{4}{5}) \rangle = 0+0+0$

$\therefore \langle (-\frac{4}{5}, 0, \frac{3}{5}), (\frac{3}{5}, 0, \frac{4}{5}) \rangle = -\frac{12}{25} + 0 + \frac{12}{25} = 0$

$\therefore$ B is orthogonal.

$\langle (0,1,0),(0,1,0)\rangle = 0+1+0 = 1$

$\langle (4/5, 0, 3/5),(-4/5, 0, 3/5)\rangle = \frac{16}{25} + \frac{9}{25} + 0 = 1$

$\langle (3/5, 0, 4/5),(3/5, 0, 4/5)\rangle = \frac{9}{25} + 0 + \frac{16}{25} = 1$

$\therefore$ B is orthonormal.

2. $B = \{(1,1,1),(1,-1,0),(1,1,-2)\}$

is orthogonal set but not orthonormal. Convert this to an orthonormal set.

$\langle (1,1,1),(1,-1,0)\rangle = 1-1+0 = 0$

$\therefore \langle (1,1,1),(1,1,-2)\rangle = 1+1-2 = 0$

$\therefore \langle (1,-1,0),(1,1,-2)\rangle = 1-1+0 = 0$

$\therefore$ B is orthogonal.

$\langle (1,1,1),(1,1,1)\rangle = 3 \neq 1$

$\therefore$ B is not orthonormal.

Convert :-

$\|\bar{v}\| = \sqrt{\langle (1,1,1),(1,1,1)\rangle}$

$= \sqrt{3}$

$\bar{e}_1 = \frac{\bar{v}_1}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\|\bar{v}_2\| = \sqrt{\langle \bar{v}_2, \bar{v}_2 \rangle} = \sqrt{\langle (1,-1,0),(1,-1,0) \rangle} = \sqrt{2}$

$\bar{e}_2 = \frac{v_2}{\sqrt{2}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right)$

$\|\bar{v}_3\| = \sqrt{\langle \bar{v}_3, \bar{v}_3 \rangle} = \sqrt{\langle (1,1,-2),(1,1,-2) \rangle} = \sqrt{6}$

$\bar{e}_3 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

Orthonormal set $\left\{ \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0\right), \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right) \right\}$

Important Maths:-

3. Prove that for $m, n \in \mathbb{Z}$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2.\sin(nx)\cdot\cos(mx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left[\sin(nx+mx) + \sin(nx-mx)\right] dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \sin\left((n+m)x\right)dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin\left((n-m)x\right)dx$$

$$= \frac{1}{2}\left[\frac{-\cos((n+m)x)}{n+m}\right]_{-\pi}^{\pi} + \frac{1}{2}\left[\frac{-\cos((n-m)x)}{n-m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\cos((n+m)x)}{n+m}\right]_{\pi}^{-\pi} + \frac{1}{2}\cdot\left[\frac{\cos((n-m)x)}{n-m}\right]_{\pi}^{-\pi}$$

$$= \frac{1}{2}\left[\frac{\cos(-(n+m)\pi)}{n+m} - \frac{\cos((n+m)\pi)}{n+m}\right] + \frac{1}{2}\left[\frac{\cos(-(n-m)\pi)}{n-m} - \frac{\cos((n-m)\pi)}{n-m}\right]$$

$$= \frac{1}{2}\left[\frac{\cos(n+m)\pi}{n+m} - \frac{\cos(n+m)\pi}{n+m}\right] + \frac{1}{2}\left[\frac{\cos(n-m)\pi}{n-m} - \frac{\cos(n-m)\pi}{n-m}\right]$$

$$= 0$$

$\therefore$ $\sin(nx)$ and $\cos(mx)$ are orthogonal.

4. Prove that for $m, n \in \mathbb{Z}$

$$\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\Rightarrow \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\sin(nx)\cdot\sin(mx)\,dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(nx - mx) - \cos(nx + mx)\right] dx$$

Case 1.

$n \neq m$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)x)\,dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)x)\,dx$$

$$= \frac{1}{2}\left[\frac{\sin((n-m)x)}{n-m}\right]_{-\pi}^{\pi} - \frac{1}{2}\left[\frac{\sin((n+m)x)}{n+m}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\sin((n-m)\pi)}{n-m} - \frac{\sin((n-m)(-\pi))}{n-m}\right] - \frac{1}{2}\left[\frac{\sin((n+m)\pi)}{n+m} - \frac{\sin((n+m)(-\pi))}{n+m}\right]$$

$$= \frac{\sin((n-m)\pi)}{n-m} - \frac{\sin((n+m)\pi)}{n+m}$$

$$= \frac{0}{n-m} - \frac{0}{n+m} = 0 \qquad [\sin(n\pi) = 0]$$

Case 2 $n = m$;

$$\int_{-\pi}^{\pi} \sin(nx) \cdot \sin(nx)\, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} 2 \cdot \sin^2(nx)\, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos(2nx))\, dx$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} 1\, dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos(2nx)\, dx$$

$$= \frac{1}{2} [x]_{-\pi}^{\pi} - \frac{1}{2} \left[\frac{\sin(2nx)}{2n} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2} [\pi - (-\pi)] - \frac{1}{2} \left[\frac{\sin((2n)\pi)}{2n} - \frac{\sin(2n)(-\pi)}{2n} \right]$$

$$\Rightarrow \frac{1}{2} \cdot 2\pi - \frac{1}{2} [0 - 0]$$

$$= \pi$$

(Proved)

5. Show that the set of the function $(\cos kx)$, $k = 1, 2, 3 \ldots$ is orthogonal over the interval $(-\pi, \pi)$. Also check weather they are orthonormal or not. If not, construct an orthonormal set of function from the above set of function.

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0 \quad \text{for } m \neq n.$$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi} 2\cos(mx)\cdot\cos(nx)\,dx$$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(mx+nx)+\cos(mx-nx)\right]dx$$

$$\Rightarrow \frac{1}{2}\int_{-\pi}^{\pi}\cos((m+n)x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi}\cos((m-n)x)\,dx$$

$$= \frac{1}{2}\left[\frac{\sin((m+n)x)}{m+n}\right]_{-\pi}^{\pi} + \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\sin((m+n)\pi)}{m+n} - \frac{s((m+n)(-\pi))}{m+n}\right] + \frac{1}{2}\left[\frac{\sin((m-n)\pi)}{m-n} - \frac{\sin((m-n)(-\pi))}{m-n}\right]$$

$$= 0$$

$\therefore \{\cos(kx) : k \in \mathbb{N}\}$ is orthogonal.

Now,

$$\langle \cos kx, \cos kx \rangle = \int_{-\pi}^{\pi} \cos kx . \cos kx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} 2\cos^2 kx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left[1 + \cos(2kx)\right] dx$$

$$= \frac{1}{2}\left[x\right]_{-\pi}^{\pi} + \frac{1}{2}\left[\frac{\sin(2kx)}{2k}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left(\pi - (-\pi)\right) + \frac{1}{2}\left[\frac{\sin 2k\pi}{2k} - \frac{\sin(2k(-\pi))}{2k}\right]$$

$$= \pi$$

Norm $= \| \cos(kx) \| = \sqrt{\pi}$

$\left\{ \frac{\cos kx}{\sqrt{\pi}} : k \in \mathbb{N} \right\}$ is orthonormal.

Orthogonal Complement :-

Let S be a subset of a vector space V and $\langle \rightarrow, \rightarrow \rangle$ be an inner product. The orthogonal complement of S is denoted by $S^{\perp}$ and defined by

$$\langle \bar{u}, \bar{v} \rangle = 0 \text{ for all } \vec{u} \in S \text{ and } \vec{v} \in S^{\perp}$$

$S = \{ \cdot \cdot \quad \cdot \cdot \}$ $\qquad$ $S^{\perp} = \{ \cdot \quad \cdot \cdot \quad \cdot \}$

perpendicular

90°

Math :-

1. Calculate the orthogonal complement, $W^{\perp}$, of the subspace.

$$W = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\}$$

Let $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in W^{\perp}$

$\therefore \left\langle \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\rangle = 0 \Rightarrow x_1 + 2x_2 - x_3 = 0$

$\therefore \left\langle \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\rangle = 0 \Rightarrow 2x_1 + 4x_2 + x_3 = 0$

Augmented,

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right) \quad R_2' = R_2 - 2R_1$$

$$= \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 3 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \quad R_1' = 3R_1 + R_2$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$x_1 + 2x_2 = 0$

$x_3 = 0$

$x_2 = t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2t \\ t \\ 0 \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\therefore W^{\perp} = \text{span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \right\}$$

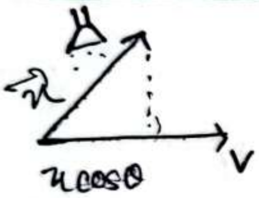
(Ans)

Diff type :-

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} t_1 + 2t_2 \\ -5t_2 \\ t_1 \\ t_2 \end{pmatrix} \qquad W^{\perp} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$= t_1 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ -5 \\ 0 \\ 1 \end{pmatrix}$$

Orthogonal projection :-



V be a subset of a vector space over F and $\langle \rightarrow , \rightarrow \rangle$ be an inner product. The orthogonal projection of the vector $\vec{u}$ onto $\bar{v}$ is define

$$\boxed{\text{proj}_{\bar{v}}(\vec{u}) = \frac{\langle \bar{u}, \bar{v} \rangle}{\langle \bar{v}, \bar{v} \rangle} \bar{v}}$$

$$\bar{u} \cdot \bar{v} = |\bar{u}| \cdot |\bar{v}| \cos\theta$$

$$= |\bar{v}| \cdot \underbrace{|\bar{u}| \cos\theta}_{\text{projection}}$$

$$\boxed{\text{projection} = \frac{\bar{u} \cdot \bar{v}}{|\bar{v}|}}$$

with direction, $\dfrac{\bar{u} \cdot \bar{v}}{\|\bar{v}\|} \cdot \dfrac{\bar{v}}{\|v\|}$

Math :-

1. Let $u = (8, \sqrt{3}, \sqrt{7}, -1, 1)$ and $v = (1, -1, 0, 2, \sqrt{3})$. If the orthogonal projection of u onto v is $\frac{a}{b} v$, then determine a and b.

$$\text{Proj}_{\bar{v}}(\bar{u}) = \frac{\langle \bar{u}, \bar{v} \rangle}{\langle \bar{v}, \bar{v} \rangle} \bar{v}$$

$$= \frac{8 - \sqrt{3} + 0 - 2 + \sqrt{3}}{1^2 + (-1)^2 + 0^2 + 2^2 + (\sqrt{3})^2} \bar{v}$$

$$= \frac{6}{9} \bar{v}$$

$(a, b) = (6, 9)$

Gram Schimidt Procedure :

Given $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3 \dots \bar{v}_n\}$ a linearly independent set. The gram-schmidt procedure constructs an Orthogonal set $S' = \{\bar{w}_1, \bar{w}_2, \bar{w}_3, \dots \bar{w}_n\}$ such that $\text{span}\{\bar{v}_1, \bar{v} \dots \bar{v}_n\}$ $= \text{span}\{\bar{w}_1, \bar{w}_2, \bar{w}_3 \dots \bar{w}_n\}$.

Furthermore, we can obtain orthonormal set $s'' = \{\vec{e}_1, \vec{e}_2, \vec{e}_3 \dots \vec{e}_n\}$ by dividing each vector by its own norm.

Formulas : Orthogonal vectors :

$\boxed{1}$ $\bar{w}_1 = \bar{v}_1$

$\boxed{2}$ $\bar{w}_2 = \bar{v}_2 - \dfrac{\langle \bar{v}_2, \bar{w}_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle} \vec{w}_1$

$\boxed{3}$ $\bar{w}_3 = \bar{v}_3 - \dfrac{\langle \bar{v}_3, \bar{w}_1 \rangle}{\langle \bar{w}_1, \bar{w}_1 \rangle} \bar{w}_1 - \dfrac{\langle \bar{v}_3, \bar{w}_2 \rangle}{\langle \bar{w}_2, \bar{w}_2 \rangle} \bar{w}_2$

Math :

Use gram - Schmidt process to transform the basis $\{v_1, v_2, v_3\}$ into an orthonormal basis.

$$v_1 = (1,1,1) \quad , \quad v_2 = (0,1,1) \quad , \quad v_3 = (0,0,1)$$

find orthogonal,

$$w_1 = (1,1,1)$$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= (0,1,1) - \frac{0+1+1}{1+1+1}(1,1,1)$$

$$= (0,1,1) - (2/3, 2/3, 2/3)$$

$$= (-2/3, 1/3, 1/3)$$

$$w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1/3}{4/9 + 1/9 + 1/9}(-2/3, 1/3, 1/3)$$

$$= (0,0,1) - (1/3, 1/3, 1/3) - \frac{1/3}{6/9}\left(\frac{-2}{3}, 1/3, 1/3\right)$$

$$= (0, -1/2, \cancel{+}1/2$$

Orthonormal,

$$\bar{e}_1 = \frac{w_1}{\|w_1\|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$\bar{e}_2 = \frac{w_2}{\|w_2\|} = \frac{(-2/3, 1/3, 1/3)}{\sqrt{6}/3} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$\bar{e}_3 = \frac{w_3}{\|w_3\|} = \frac{(0, -1/2, 1/2)}{\frac{1}{\sqrt{2}}} = \left(0, \frac{-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

Finding Linear combination from Orthogonal Basis.

Math: Consider the orthogonal basis $B = \{\underset{v_1}{(1,1,1)}, \underset{v_2}{(1,-1,0)}, \underset{v_3}{(1,1,-2)}\}$. Express $(3,9,0)$ as a linear combination of the basis vector.

$$\bar{v} = \alpha_1 \bar{v}_1 + \alpha_2 \bar{v} + \alpha_3 \bar{v}_3$$

$$\langle \bar{v}, \bar{v}_1 \rangle = \alpha_1 \langle \bar{v}_1, \bar{v}_1 \rangle + \alpha_2 \langle \bar{v}_2, \bar{v}_1 \rangle + \alpha_3 \langle \bar{v}_3, \bar{v}_1 \rangle$$

$$\alpha_1 = \frac{\langle \bar{v}, \bar{v}_1 \rangle}{\langle \bar{v}_1, \bar{v}_1 \rangle} \qquad \alpha_2 = \frac{\langle \bar{v}, \bar{v}_2 \rangle}{\langle \bar{v}_2, \bar{v}_2 \rangle}$$

$$\alpha_3 = \frac{\langle \bar{v}, \bar{v}_3 \rangle}{\langle \bar{v}_3, \bar{v}_3 \rangle}$$

$$\alpha_1 = \frac{\langle (3,9,0), (1,1,1) \rangle}{\langle (1,1,1), (1,1,1) \rangle} = \frac{12}{3} = 4$$

$$\alpha_2 = \frac{-6}{2} = -3$$

$$\alpha_3 = \frac{12}{6} = 2$$

$$(3,9,0) = 4 \cdot (1,1,1) - 3 \cdot (1,-1,0) + 2(1,1,-2)$$

Periodic Function :-

A function $f(x)$ is said to be periodic if there is a $p > 0$ such that $f(x+p) = f(x)$ for all x.

The value ~~of~~ (least) of $p > 0$ is called the period.

- $\sin(x)$ এর period 2π
- $\cos(x)$ " " 2π
- $\tan(x)$ " " π
- $\cot(x)$ " " π
- $\operatorname{cosec}(x)$ " " 2π
- $\sec(x)$ " " 2π

$a \cdot \sin^b(cx+d) + f$

a = amplitude

d = Phase shift.

f =

c = New period $= \frac{\text{original}}{c}$

b = even $\rightarrow$ sin or cos period π.

b = odd $\rightarrow$ sin or cos period 2π.

Math:-

1. Determine the period of the following function,

$$f(t) = 1952\cos^{\underline{\underline{1971}}}(\underline{\underline{2025}}\,x + 2026).$$

will create difference.

$\sin(x)$ period 2π

$\sin^{1971}(x)$ " 2π

$\sin^{1971}(2025x)$ period $= \frac{2\pi}{2025}$

2. $f(x) = \sin(6x) - \sin(8x)$

$\sin(8x) = \frac{2\pi}{8}$

$\sin(6x) = \frac{2\pi}{6}$

$L.cm\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \frac{\pi}{1} = \pi$

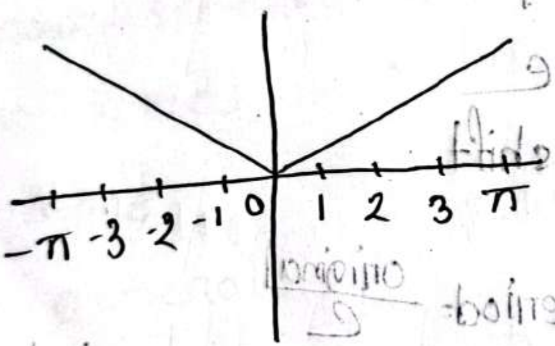
Sketch of function:

1. Sketch the function,

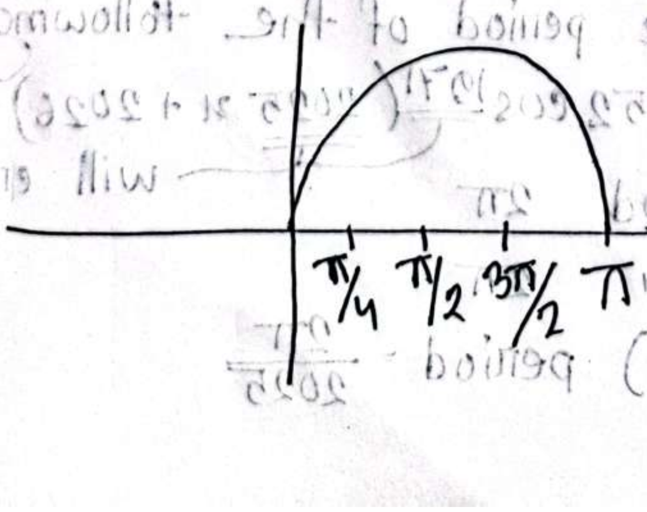
$$f(x) = \begin{cases} -x & ; -\pi < x < 0 \\ x & ; 0 \leq x < \pi \end{cases} \quad \text{or } f(x) = |x| , -\pi < x < \pi$$

$x = 1, 2, 3, 4$

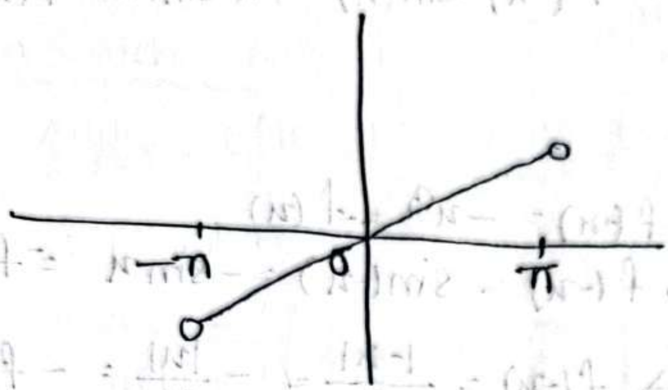
$-1, 2, -3, 4$



2. $f(x) = \begin{cases} 2\sin x & 0 < x < \pi \\ 0 & \text{otherwise} \end{cases}$



3. $f(x) = \frac{x}{\pi} \quad -\pi < x < \pi$



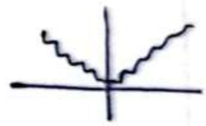
$x = \pi$
$x = -\pi$

Even and Odd Function:-

A function must be defined on $(-L, L)$ and

$$f(-x) = f(x)$$

* Symmetric about y-axis.

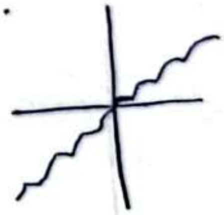


$\Downarrow$

For Even

$$f(-x) = -f(x)$$

* $f(x)$ is symmetric about origin.



$\Downarrow$

Odd

Even function:-

1. $f(x) = x^4 \longrightarrow f(-x) = (-x)^4 = x^4 = f(x)$
2. $f(x) = \cos x \longrightarrow f(-x) = \cos(-x) = \cos x = f(x)$
3. $f(x) = |x| \longrightarrow f(-x) = |-x| = x = f(x)$
4. $f(x) = 3 \longrightarrow f(-x) = 3 = f(x)$

5. $f(x) = 3|\sin(x)| \longrightarrow f(-x) = 3 \cdot |\sin(-x)| = 3 \cdot |-\sin(x)| = 3|\sin(x)|$
$= f(x)$

6. $f(x) = x^3 \sin(x) \longrightarrow (-x)^3 \cdot \sin(-x) = x^3 \sin(x) = f(x)$

<u>Odd function :-</u>

1. $f(x) = x^3 \longrightarrow f(-x) = -x^3 = -f(x)$
2. $f(x) = \sin(x) \longrightarrow f(-x) = \sin(-x) = -\sin x = -f(x)$
3. $f(x) = \frac{|x|}{x} \longrightarrow f(-x) = \frac{|-x|}{-x} = -\frac{|x|}{x} = -f(x)$
4. $f(x) = \frac{x}{\pi} \longrightarrow f(-x) = \frac{-x}{\pi} = -\frac{x}{\pi} = -f(x)$
5. $f(x) = x^3 \cos(x) \longrightarrow f(-x) = (-x)^3 \cos(-x) = -f(x)$

Fourier Series :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

General

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{2}\right)dx \quad [n = 0, 1, 2 \ldots] \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx \quad [n = 1, 2, 3 \ldots]$$

$$\boxed{f(x) = \cancel{\text{costa}}\ \text{constant} + (\text{cosine terms}) + (\text{sine term})}$$

- Signal or Function কে sine/cosine a convert করাকে fourier series বলে।

- Period = Interval length (if not given) = $2L$
- period = $2L$ (if given)

Math:-

1. Find the Fourier series expension of the periodic function $f(x) = \pi - x$ on interval $(-\pi, \pi)$.

$2L = 2\pi$

$L = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx,$$

$$\boxed{b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx}$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} (\pi - x) dx$$

$$= \frac{1}{\pi} \left[\pi x - \frac{x^2}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\left(\pi^2 - \frac{\pi^2}{2} \right) - \left(-\pi^2 - \frac{\pi^2}{2} \right) \right]$$

$$= \frac{1}{\pi} \cdot 2\pi^2 = 2\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)\cos(nx)\,dx$$

	u derivative	v intigrate
+	$\pi - x$	$\cos(nx)$
−	-1	$\frac{\sin(nx)}{n}$
	0	$\frac{-\cos(nx)}{n^2}$

$$= \frac{1}{\pi}\left[(\pi - x)\cdot\frac{\sin(nx)}{n} - (-1)\frac{-\cos(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[0\cdot\underbrace{\cancel{\frac{\sin n\pi}{n}}}_{0} - (-1)\frac{-\cos(n\pi)}{n^2}\right] - \left[(2\pi)\cdot\underbrace{\cancel{\frac{\sin(-n\pi)}{n}}}_{0} - (-1)\frac{-\cos(-n\pi)}{n^2}\right]$$

$\sin(k\cdot\pi) = 0$ for all integer k.
$\cos(k\cdot\pi) = (-1)^k$ for all integer k

$$= \frac{1}{\pi}\left[\cancel{1\cdot\frac{-(-1)^n}{n^2} + \frac{(-1)^{-n}}{n^2}}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n\pi)}{n^2} + \frac{\cos(n\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} + \frac{(-1)^n}{n^2}\right] = \cancel{\frac{2(-1)^n}{\pi n^2}} \quad 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx$$

	derivative u	Integration v
+	$\pi - x$	$\sin(nx)$
−	-1	$\frac{-\cos nx}{n}$
	0	$-\frac{\sin nx}{n^2}$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[(\pi - x)\frac{-\cos nx}{n} - (-1)\frac{-\sin(nx)}{n^2}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(0.\frac{\cos\pi}{n} - \frac{\sin n\pi}{n^2}\right) - \left((-2\pi)\cdot\frac{\cos(-n\pi)}{n} - \frac{\sin(-n\pi)}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[2\pi.\frac{(-1)^n}{n}\right] = \frac{2.(-1)^n}{n}$$

$a_0 = 2\pi$, $a_n = 0$, $b_n = \frac{2.(-1)^n}{n}$ → work done here.

$$f(x) = \pi + (-2)\cdot\sin x + (1)\cdot\sin 2x + \frac{-2}{3}\sin 3x + \frac{1}{2}\sin(4x) \ldots\ldots$$

2. Find the Fourier series expansion of the periodic function :- $f(x) = x^2$ $0 \le x \le 2\pi$ where the period of f is 2π.

$2L = 2\pi$

$\Rightarrow L = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} \right) + \left(b_n \sin \frac{n\pi x}{L} \right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \cos(nx)\, dx, \quad b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cdot \sin(nx)\, dx.$$

$$a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{0}^{2\pi} x^2\, dx$$

$$= \frac{1}{\pi} \left[\frac{x^3}{3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left(\frac{8\pi^3}{3} - 0 \right)$$

$$= 8\pi^2/3$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cdot \cos(nx)\, dx$$

x^2	$\cos nx$
$+ \; 2x$	$\frac{\sin nx}{n}$
$- \; 2$	$-\cos nx / n^2$
$+ \; 0$	$-\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi} \left[x^2 \cdot \frac{\sin(nx)}{n} + 2x \frac{\cos nx}{n^2} - 2 \frac{\sin nx}{n^3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\left(4\pi^2 \cdot \frac{\sin(2\pi n)}{n} + 4\pi \cdot \frac{\cos(2n\pi)}{n^2} - 2 \frac{\sin(2n\pi)}{n^3} \right) - (0 + 0 - 0 \right]$$

$$= \frac{1}{\pi} \left[4\pi \cdot \frac{(-1)^{2n}}{n^2} \right]$$

$$= \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x^2 \cdot \sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[x^2 \cdot \frac{-\cos(nx)}{n} - (2x)\frac{-\sin nx}{n^2} + 2\cdot\frac{\cos(nx)}{n^3}\right]_0^{2\pi}$$

x^2	$\sin(nx)$
$2x$	$-\frac{\cos nx}{n}$
2	$-\frac{\sin(nx)}{n^2}$
0	$\frac{\cos(nx)}{n^3}$

$$= \frac{1}{\pi}\left[-4\pi^2\frac{\cos(2n\pi)}{n} + 4\pi\cancel{\frac{\sin(2n\pi)}{n}} + 2\cdot\frac{\cos(2n\pi)}{n^3}\right) - \left(0+0+\frac{2}{n^3}\right)\Big]$$

$$= \frac{1}{\pi}\left[-4\pi^2\cdot\frac{1}{n} + \cancel{\frac{2}{n^3}} - \cancel{\frac{2}{n^3}}\right] = \frac{-4\pi}{n}$$

$$a_0 = \frac{8\pi^2}{3}, \quad a_n = \frac{4}{n^2}, \quad b_n = \frac{-4\pi}{n}$$

Half Range Fourier Series :-

Even $\longrightarrow$ cosine. $f(-x) = f(x)$ $[-L, L]$

Odd $\longrightarrow$ sine. $f(-x) = -f(x)$

Cosine Series :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L}\right) \quad \text{(Even)}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n = 0, 1, 2, 3 \ldots$$

Sine Series :-

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi x}{L} \right) \qquad \text{(odd)}$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \qquad n = 1, 2, 3 \ldots$$

Calculate L when half/cosine/sine series :-

L = given half range.

Math:-

1. Expand $f(x) = \cos x$, $0 < x < \pi$ in Fourier sine series.

$\therefore L = \pi$

$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin (nx) \right)$$

$$b_n = \frac{2}{L} \int_0^{\pi} f(x) \sin (nx)$$

$$= \frac{2}{\pi} \int_0^{\pi} \cos x \cdot \sin (nx) \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} 2 \sin (nx) \cos (x) \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \left[\sin (nx + x) + \sin (nx - x) \right] dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos(nx+x)}{n+1} + \frac{-\cos(nx-x)}{n-1} \right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos(nx+x)}{n+1} + \frac{\cos(nx+x)}{n-1}\right]_{\pi}^{0}$$

$$= \frac{1}{\pi}\left[\left(\left(\frac{1}{n+1}\right)+\left(\frac{1}{n-1}\right)\right) - \left(\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n+1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1} - \frac{(-1)^{n+1}}{n+1} - \frac{(-1)^{n-1}}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2n}{n^2-1} - \frac{(-1)(-1)^n}{n+1} - \frac{(-1)^n\cdot(-1)^{-1}}{n-1}\right] \text{ (Ans}$$

$$= \frac{2n(1+(-1)^n)}{\pi(n^2-1)}$$

2. Expand $f(x)=x$, $0<x<2$ in a half range series of cosine.

$L=2$

$\therefore$ Fourier cosine series. $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(\frac{n\pi x}{2}\right)\right)$

$$a_n = \frac{2}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{2}\int_{0}^{2} f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

x	$\cos\left(\frac{n\pi x}{2}\right)$
1	$\frac{\sin(n\pi x/2)}{n\pi/2}$
0	$-\frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{2}}$

$$= \left[x \cdot \frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} - 1 \cdot \frac{-\cos\left(\frac{n\pi x}{2}\right)}{\frac{n^2\pi^2}{4}} \right]_0^2$$

$$= \left[\frac{2x}{n\pi} \sin\left(\frac{n\pi x}{2}\right) + \frac{4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) \right]_0^2$$

$$= \left[\left(\frac{4}{n\pi} \sin(n\pi) + \frac{4}{n^2\pi^2} \cos(\pi n) \right) - \left(0 + \frac{4}{n^2\pi^2} 1 \right) \right]$$

$$= \frac{4}{n^2\pi^2} (-1)^n - \frac{4}{n^2\pi^2} = \frac{4(-1)^n - 4}{n^2\pi^2}$$

$$a_0 = \int_0^2 f(x)\, dx$$

$$= \left[\frac{x^2}{2} \right]_0^2 = 2$$

Some General Fourier Series Problem:-

3. Find fourier series expansion of the periodic function,

$$f(x) = \begin{cases} 2\sin x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$$

$2L = 2\pi$

$\Rightarrow L = \pi$

$$f(x) = \frac{a_0}{2} \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right) \right]$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx \qquad b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin(nx)\, d$$

$$\therefore a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} f(x)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin x\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\,dx$$

$$= \frac{1}{\pi}\Big[-2\cos x\Big]_0^{\pi}$$

$$= \frac{1}{\pi}\Big[2\cos x\Big]_{\pi}^{0} = \frac{1}{\pi}\big(2-2\cos(\pi)\big)$$

$$= \frac{1}{\pi}(2+2) = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 2\sin x\cdot\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \big(\sin(x+nx)+\sin(x-nx)\big)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi}\Big[\sin(1+n)x + \sin(1-n)x\Big]\,dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(1+n)x}{1+n} + \frac{-\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{1}{1+n} + \frac{1}{1-n}\right) - \left(\frac{\cos(1+n)\pi}{1+n} + \cos\frac{(1-n)\pi}{1-n}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{1-n^2} - \frac{(-1)^{1+n}}{1+n} - \frac{(-1)^{1-n}}{1-n}\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{1-n^2} - \frac{(-1)^1 \cdot (-1)^n}{1+n} - \frac{(-1)^1 \cdot (-1)^{-n}}{1-n}\right]$$

$$= \frac{2 \cdot (1+(-1)^n)}{\pi(1-n^2)} \text{ (Ans)}$$

$a_1 =$

Even - Odd Fourier Series :-

Even Case :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} \right)$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 0, 1, 2 \ldots$$

Odd Case :-

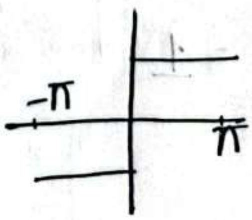
$$f(x) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi x}{L} \right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, 3,$$

Math:

1. Find the Fourier series expansion of the periodic function

$$f(x) \begin{cases} -2 & -\pi < x < 0 \\ 2 & 0 < x < \pi \end{cases}$$



Odd function.

$$2L = 2\pi \Rightarrow L = \pi$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 2\sin(nx)\,dx$$

$$= \frac{4}{\pi}\int_0^{\pi} \sin(nx)\,dx$$

$$= \frac{4}{\pi}\left[\frac{-\cos(nx)}{n}\right]_0^{\pi} = \frac{4}{\pi}\left[\frac{\cos(nx)}{n}\right]_{\pi}^{0}$$

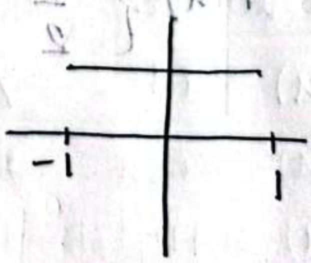
$$= \frac{4}{\pi}\left[\frac{1}{n} - \frac{\cos(n\pi)}{n}\right] = \frac{4}{\pi} \times \frac{1-(-1)^n}{n}$$

$$= \frac{4[1-(-1)^n]}{n\pi}$$

2. $f(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{Otherwise} \end{cases}$

$2L = 2$

$\Rightarrow L = \underline{1}$



Even function

$$f(x) = \frac{a_0}{2} + f(x)\cos\left(\frac{n\pi x}{2}\right)$$

$$= \frac{a_0}{2} + f(x)\cos(n\pi x)$$

$$a_0 = 2\int_0^1 f(x)\,dx$$

$$= 2\,[x]_0^1$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x)\,dx$$

$$= \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$= 2\int_0^1 \underline{x}\cos(n\pi x)\,dx$$

x	$\cos(n\pi x)$
1	$\frac{\sin(n\pi x)}{n\pi}$
0	$-\cos$

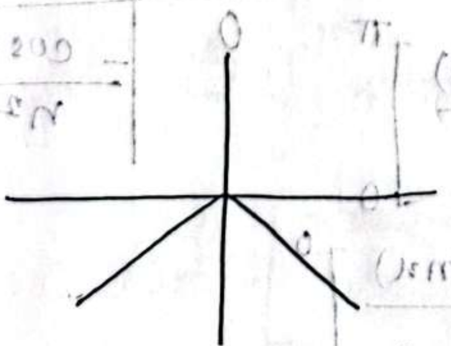
$$= 2\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1$$

$$= 2\left[\cancel{\frac{\sin(n\pi)}{n\pi}}^{\,0} - 0\right] = 0$$

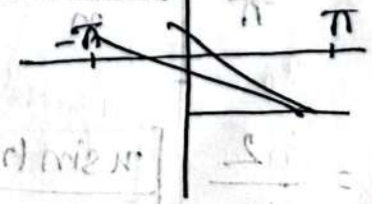
3. $f(x) = \begin{cases} x ; & -\pi < x < 0 \\ -x ; & 0 < x < \pi \end{cases}$

$f(x) = x \quad [-1, -2,$

$f(-x) = -x$



Even function.



Odd

$2L = 2\pi$

$L = \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (-x)\, dx = \frac{-2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = -\pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} -x\cos(nx)\, dx$$

$-x$	$\cos(nx)$
-1	$\frac{\sin nx}{n}$
0	$-\frac{\cos(nx)}{n^2}$

$$= \frac{2}{\pi}\left[\frac{-x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_{\pi}^{0}$$

$$= \frac{2}{\pi}\left[\left(0+\frac{1}{n^2}\right)-\left(0+\frac{\cos(n\pi)}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{1}{n^2}-\frac{(-1)^n}{n^2}\right)$$

4. $f(x) = |x| \qquad -\pi < x < \pi$

mod means even function.

$$2L = 2\pi$$
$$L = \pi$$

$* \ |x| = x, \ x \geq 0$

$* \ |x| = -x, \ x < 0$

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\,dx$$

$$= \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi x\cos(nx)\,dx$$

x	$\cos(nx)$
1	$\frac{\sin(nx)}{n}$
0	$\frac{-\cos(nx)}{n^2}$

$$\Rightarrow \frac{2}{\pi}\left[\frac{x\sin(nx)}{n} + \cancel{\cos}\frac{\cos(nx)}{n^2}\right]_0^\pi$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi \cancel{\sin(n\pi)}^{\to 0}}{n} + \frac{\cos(n\pi)}{n^2}\right) - \left(0 + \frac{1}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left(\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right) = \frac{2}{\pi}\left(\frac{(-1)^n - 1}{n^2}\right)$$

5. $f(x) = x^2$. over the interval $(-3, 3)$

$f(-x) = (-x)^2 = x$ Even.

$2L = 6$

$\Rightarrow L = 3$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{3}\right)$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{3}\int_{0}^{3} f(x)\cos\left(\frac{n\pi x}{3}\right)dx$$

$$a_0 = \frac{2}{3} f(x)\, dx$$

$$= \frac{2}{3} \cancel{f(x} \int_0^3 x^2\, dx$$

$$= \frac{2}{3}\left[\frac{x^3}{3}\right]_0^3 = \frac{2}{3} \times \frac{27}{3} = 6$$

$$a_n = \frac{2}{3}\int_0^3 x^2 \cos\frac{n\pi x}{3}\, dx$$

x^2	$\cos\left(\frac{n\pi x}{3}\right)$
$2x$	$\frac{\sin\left(\frac{n\pi x}{3}\right)}{n\pi/3}$
2	$\frac{-\cos\left(\frac{n\pi x}{3}\right)}{n^2\pi^2/9}$
0	$\frac{-\sin\left(\frac{n\pi x}{3}\right)}{\frac{n^3\pi^3}{27}}$

$$= \frac{2}{3}\left[\frac{x^2\sin\left(\frac{n\pi x}{3}\right)}{n\pi/3} + \frac{2x\cdot\cos\left(\frac{n\pi x}{3}\right)}{\frac{n^2\pi^2}{3}} - \frac{\sin\left(\frac{n\pi x}{3}\right)}{\frac{n^3\pi^3}{27}}\right]_0^3$$

$$= \frac{2}{3}\left[\left(\cancelto{0}{\frac{9\sin(n\pi)}{n\pi/3}} + \frac{6\cos(n\pi)}{\frac{n^2\pi^2}{3}} - \cancelto{0}{\frac{\sin(n\pi)}{\frac{n^3\pi^3}{27}}}\right) - (0+0-0)\right]$$

$$= \frac{2}{3} \times \frac{6(-1)^n}{\frac{n^2\pi^2}{3}} = \frac{36(-1)^n}{n^2\pi^2} \text{ (Ans)}$$

6. $f(x) = \begin{cases} x & -4 < x < 0 \\ -x & 0 < x < 4 \end{cases}$

$f(x) = x$

$f(-x) = -x$

Even

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$2L = 8$$

$$\Rightarrow L = 4$$

$$a_n = \frac{2}{4}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)dx$$

$$a_0 = \frac{2}{4}\int_0^4 f(x)\,dx$$

$$= \frac{2}{4}\left[\frac{-x^2}{2}\right]_0^4 = \frac{2}{4} \times \frac{-16}{2} = -4$$

$$a_n = \frac{2}{4}\int_0^4 f(x)\cos\left(\frac{n\pi x}{4}\right)dx$$

$-x$	$\cos\left(\frac{n\pi x}{4}\right)$	
-1	$\dfrac{\sin\left(\frac{n\pi x}{4}\right)}{n\pi/4}$	
0	$\dfrac{-\cos\left(\frac{n\pi x}{4}\right)}{n^2\pi^2/4^2}$	

$$= \frac{2}{4}\int_0^4 -x\cos\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{2}{4}\left[\frac{-x\sin\left(\frac{n\pi x}{4}\right)}{n\pi/4} - \frac{\cos\left(\frac{n\pi x}{4}\right)}{n^2\pi^2/4^2}\right]_0^4$$

$$= \frac{2}{4}\left[\left(\cancel{\frac{-4\sin(n\pi)}{n\pi/4}}^{\,0} - \frac{\cos(n\pi)}{n^2\pi^2/4}\right) - \left(0 - \frac{1}{n^2\pi^2/4}\right)\right]$$

$$= \frac{2}{4}\left(\frac{-(-1)^n}{n^2\pi^2/4} + \frac{1}{n^2\pi^2/4}\right) \text{(Ans)}$$

General Fourier Transform :-

Complex fourier series on $(-L,L)$:

Let $f(x)$ be defined in an interval with period $2L$. The complex fourier series expansion of $f(x)$ is defined to be,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$$

where,

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-i\frac{n\pi x}{L}}\, dx \qquad n = \dots -3,-2,-1,1,2\dots$$

Idea of a Fourier Transform :-

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, dx \rightarrow \text{Inverse fourier Transform}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx \rightarrow \text{fourier transform.}$$

Math :-

01. Find the fourier transform of the function,

$$f(x) = \begin{cases} \pi, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

Evaluate $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\, dx$

Fourier transform,

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-1}^{1} \pi\, e^{-i\omega x}\, dx + 0$$

$$= \pi \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-1}^{1}$$

$$= \pi \left[\left(\frac{e^{-i\omega}}{-i\omega} \right) - \left(\frac{e^{i\omega}}{-i\omega} \right) \right]$$

$$= \pi \cdot \frac{e^{i\omega} - e^{-i\omega}}{i\omega} = \pi \cdot \frac{2i \sin\omega}{i\omega}$$

$$= \frac{2\pi \sin\omega}{\omega}$$

$$\boxed{\begin{aligned} e^{i\theta} + e^{-i\theta} &= 2\cos\theta \\ e^{i\theta} - e^{-i\theta} &= 2i\sin\theta \end{aligned}}$$

Now applying Inverse fourier transform :-

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\omega)\, e^{i\omega x}\, d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2\pi \sin\omega}{\omega}\, e^{i\omega x}\, d\omega$$

substituting $x=0$, $f(0) = \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \frac{2\pi \sin \omega}{\omega} e^0 d\omega$

$$\pi = \frac{1}{\cancel{2\pi}} \int_{-\alpha}^{\alpha} \frac{\cancel{2\pi} \sin \omega}{\omega} d\omega$$

$$\Rightarrow \int_{-\alpha}^{\alpha} \frac{\sin \omega}{\omega} d\omega = \pi$$

$$\Rightarrow \int_{-\alpha}^{\alpha} \frac{\sin x}{x} dx = \pi$$

$$\Rightarrow 2 \cdot \int_{0}^{\alpha} \frac{\sin x}{x} dx = \pi$$

$$\Rightarrow \int_{0}^{\alpha} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

STEPS :-

* Formula লিখবো।
* Non zero element টা integrate করবো।
* e^0 এর ~~rate~~ value বসায় fourier transfer পাবো।
* Inverse portion টা বের করবো।
* Omega (ω) replace করে দিলাম।

Q2. $f(x) = \begin{cases} \frac{1}{m}, & |x| < m \\ 0, & |x| > m \end{cases}$ $-m < x < m$

evaluate, $\int_0^{\infty} \frac{\sin x}{x} dx$

Fourier Transform,

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x} dx$$

$$= \int_{-m}^{m} \frac{1}{m} e^{-i\omega x} dx$$

$$= \frac{1}{m} \int_{-m}^{m} \frac{e^{-i\omega x}}{-i\omega} dx$$

$$= \frac{1}{m} \left[\frac{e^{-i\omega x}}{-i\omega} \right]_{-m}^{m}$$

$$= \frac{1}{m} \left[\frac{e^{-i\omega m}}{-i\omega} - \frac{e^{i\omega m}}{-i\omega} \right]$$

$$= \frac{1}{m} \left(\frac{e^{i\omega m}}{i\omega} - \frac{e^{-i\omega m}}{i\omega} \right)$$

$$= \frac{1}{m} \frac{e^{i\omega m} - e^{-i\omega m}}{i\omega}$$

$$= \frac{1}{m} \frac{2i\sin(\omega m)}{i\omega}$$

$$= \frac{2\sin(m\omega)}{m\omega}$$

Applying I.F.T,

$$f(x) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} F(\omega)\, e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2\sin(m\omega)}{m\omega}\, e^{i\omega x}\, dx$$

At $x = 0$,

$$f(0) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2\sin(\omega m)}{(\omega m)}\, 1\, dx$$

$$\Rightarrow \frac{1}{\cancel{m}} = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2\sin(m\omega)}{\cancel{m}.\omega}\, d\omega$$

$$\Rightarrow 1 = \frac{1}{\pi}\int_{-\alpha}^{\alpha} \frac{\sin(m\omega)}{\omega}\, d\omega$$

$$\Rightarrow \int_{-\alpha}^{\alpha} \frac{\sin(m\omega)}{m\omega}\, d\omega = \pi$$

$$\Rightarrow 2\int_{-\alpha}^{\alpha} \frac{\sin(m\omega)}{\omega}\, d\omega = \pi$$

$m\omega = x$
$m\, d\omega = dx$

$$\Rightarrow \int_{0}^{\alpha} \frac{\sin x}{\frac{x}{m}} \cdot dx/m = \pi/2$$

$$\therefore \int_0^{\infty} \frac{\sin x}{x}\, dx = \pi/2$$

Q3. $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

$$\int_0^{\infty} \frac{\sin(ax)\cos(ax)}{x}\, dx$$

Fourier transform,

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-a}^{a} 1\, e^{-i\omega x}\, dx$$

$$= \left[\frac{e^{-i\omega x}}{-i\omega}\right]_{-a}^{a} = \frac{e^{-i\omega a}}{-i\omega} - \frac{e^{i\omega a}}{-i\omega}$$

$$= \frac{e^{i\omega a} - e^{-ia\omega}}{i\omega}$$

$$= \frac{2i\sin(a\omega)}{i\omega}$$

$$= \frac{2\sin(a\omega)}{\omega}$$

I.F.T,

$$f(x) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} f(\omega)\, e^{i\omega x}\, dx$$

Oiler form,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2\sin(a\omega)}{\omega}\, e^{i\omega x}\, dx$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2\sin(a\omega)}{\omega}\,(\cos \omega x + i\cdot\sin \omega x)\, dx$$

$$\underbrace{f(x) + i.0}_{0} = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2\sin(a\omega)}{\omega}\cos(\omega x)\, dx + \underbrace{i\frac{1}{2\pi}\int_{-\alpha}^{\alpha}\frac{2\sin(a\omega)}{\omega}\cdot\sin(\omega x)\, dx}_{0}$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2\sin(a\omega)}{\omega}\cos(\omega x)\, d\omega = f(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2\sin(a\omega)}{\omega}\cdot\cos(a\omega)\, d\omega = f(a)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{2\sin(a\omega)}{\omega}\cos(a\omega)\, d\omega = \frac{1+0}{2}$$

$$\Rightarrow 2\int_{0}^{\alpha} \frac{\sin(a\omega)}{\omega}\cos(a\omega) = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\alpha} \frac{\sin(a\omega)}{\omega}\cos(a\omega)\, d\omega = \pi/4$$

04. $f(x) = \begin{cases} 1-x^2, & |x|<1 \\ 0, & |x|>1 \end{cases}$ — $-1<x<1$

$$\int_0^{\alpha} \left(\frac{x\cos x - \sin x}{x^3}\right) \cos\left(\frac{x}{2}\right) dx.$$

Fourier Transform,

$$f(\omega) = \int_{-\alpha}^{\alpha} f(x)\, e^{-i\omega x}\, dx$$

$$= \int_{-1}^{1} (1-x^2)\, e^{-i\omega x}\, dx$$

$1-x^2$	$e^{-i\omega x}$
$-2x$	$\frac{e^{-i\omega x}}{i\omega}$
-2	$\frac{e^{-i\omega x}}{i^2\omega^2}$
0	$\frac{e^{-i\omega x}}{i^3\omega^3}$

$$= \left[(1-x^2)\frac{e^{-i\omega x}}{-i\omega} - (-2x)\frac{e^{-i\omega x}}{i^2\omega^2} + (-2)\frac{e^{-i\omega x}}{-i^3\omega^3}\right]_{-1}^{1}$$

$$= \left[(1-x^2)\frac{e^{-i\omega x}}{-i\omega} - 2x\cdot\frac{e^{-i\omega x}}{\omega^2} - 2\cdot\frac{e^{-i\omega x}}{i\omega^3}\right]_{-1}^{1}$$

$$= \left[(1-x^2)\frac{e^{-i\omega x}}{i\omega} + 2x\,\frac{e^{-i\omega x}}{\omega^2} + 2\cdot\frac{e^{-i\omega x}}{i\omega^3}\right]_{1}^{-1}$$

$$= \left(0\cdot\frac{e^{i\omega}}{i\omega} - 2\cdot\frac{e^{i\omega}}{\omega^2} + 2\cdot\frac{e^{i\omega}}{i\omega^3}\right) - \left(0\cdot\frac{e^{-i\omega}}{i\omega} + 2\cdot\frac{e^{i\omega}}{\omega^2} + 2\,\frac{e^{-i\omega}}{i\omega^3}\right)$$

$$= -2 \cdot \frac{e^{i\omega}}{\omega^2} - 2 \cdot \frac{e^{-i\omega}}{\omega^2} + 2 \cdot \frac{e^{i\omega}}{i\omega^3} - 2 \cdot \frac{e^{i\omega}}{i\omega^3}$$

$$= -2 \cdot \frac{e^{i\omega} + e^{-i\omega}}{\omega^2} + 2 \cdot \frac{e^{i\omega} - e^{i\omega}}{i\omega^3}$$

$$= -2 \cdot \frac{2\cos\omega}{\omega^2} + 2 \cdot \frac{2i\sin\omega}{i\omega^3}$$

$$= \frac{4\sin\omega - 4\omega\cos\omega}{\omega^3}$$

Applying I.F.T,

$$f(x) = \frac{1}{2\pi}\int_{-\alpha}^{\alpha} f(\omega) \cdot e^{i\omega x} \, dx$$

$$= \frac{1}{2\pi}\int_{-\alpha}^{\alpha} \frac{4\sin\omega - 4\omega\cos\omega}{\omega^3} e^{i\omega x} \, dx$$

$$= \frac{4}{2\pi}\int_{-\alpha}^{\alpha} \frac{\sin\omega - \omega\cos\omega}{\omega^3} (\cos\omega + i\sin\omega) \, dx$$

$$\Rightarrow f(x) + i \cdot 0 = \frac{2}{\pi}\int_{-\alpha}^{\alpha} \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\omega x \, d\omega + 0$$

$$\Rightarrow \frac{2}{\pi}\int_{-\alpha}^{\alpha} \frac{\sin\omega - \omega\cos\omega}{\omega^3} \cos\omega x \, d\omega = f(x)$$

substitute $x = \frac{1}{2}$,

$$\Rightarrow \frac{2}{\pi}\int_{-\alpha}^{\alpha} \frac{\sin w - w\cos w}{w^3} \cos\left(w \cdot \frac{1}{2}\right) dw = f\left(\frac{1}{2}\right)$$

$$\Rightarrow \int_{-\alpha}^{\alpha} \frac{\sin w - w\cos w}{w^3} \cos\left(\frac{w}{2}\right) dw = \frac{\pi}{2} f\left(\frac{1}{2}\right)$$

$$\Rightarrow \int_{-\alpha}^{\alpha} \frac{\sin x - x\cos x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{\pi}{2}\left(1 - \left(\frac{1}{2}\right)^2\right)$$

$$\Rightarrow \int_{0}^{\alpha} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16} \text{ (Ans)}$$

Cosine & Sin fourier Transform :-

Fourier Cosine :- Even

$$F_c(w) = \int_0^{\alpha} f(x) \cos(wx)\, dx$$

$$f(x) = \frac{2}{\pi}\int_0^{\alpha} F_c(w) \cos(wx)\, dw$$

Fourier Sine :- Odd

$$F_s(\omega) = \int_0^{\infty} f(x) \sin(\omega x)\, dx$$

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\omega) \sin(\omega x)\, d\omega$$

UV Integration short cut :- (Recurring Type) :-

$$I = \int e^{2x} \cos 3x\, dx$$

$+$	e^{2x}	$\cos 3x$
$-$	$2e^{2x}$	$\frac{\sin 3x}{3}$
$+$	$4e^{2x}$	$\frac{-\cos 3x}{9}$

$$= e^{2x} \cdot \frac{\sin 3x}{3} - 2e^{2x} \frac{-\cos 3x}{9} + \int 4e^{2x} \cdot \frac{-\cos 3x}{9}\, dx$$

$$= \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} \underbrace{\int e^{2x} \cos 3x\, dx}_{I}$$

$$I = \frac{1}{3} e^{2x} \sin 3x + \frac{2}{9} e^{2x} \cos 3x - \frac{4}{9} I$$

$\hookrightarrow$ need to calculate I now by solving the equation.

Math :-

1. Find the Fourier Cosine Transformation of e^{-x}, $x \geq 0$,

$$\int_0^\infty \frac{\cos(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m},\ m>0$$

Fourier Cosine Transform

$$F_c(\omega) = \int_0^\infty f(x)\cos\omega x\,dx$$

$$I = \int_0^\infty e^{-x}\cos\omega x\,dx \quad \text{[uv applying without shortcut]}$$

$$= \left[e^{-x}\frac{\sin\omega x}{\omega}\right]_0^\infty - \int_0^\infty (-e^{-x})\frac{\sin\omega x}{\omega}\,dx$$

$$= 0 - 0 + \int_0^\infty e^{-x}\frac{\sin\omega x}{\omega}\,dx$$

$$= \frac{1}{\omega}\int_0^\infty e^{-x}\sin\omega x\,dx$$

$$= \frac{1}{\omega}\left[\left[e^{-x}\frac{-\cos\omega x}{\omega}\right]_0^\infty - \int_0^\infty (-e^{-x})\frac{-\cos\omega x}{\omega}\,dx\right]$$

$$= \frac{1}{\omega}\left[0 - 1\cdot\frac{-1}{\omega} - \frac{1}{\omega}\int_0^\infty e^{-x}\cos\omega x\,dx\right.$$

$$I = \frac{1}{\omega^2} - \frac{1}{\omega^2}I$$

$\omega^2 I = 1 - I$

$\Rightarrow I = \frac{1}{1+\omega^2}$

$F_c(\omega) = \frac{1}{1+\omega^2}$

I.F.T

$f(x) = \frac{2}{\pi}\int_0^\infty F_c(\omega)\cos(\omega x)\,d\omega$

$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\omega^2}\cdot\cos\omega x\,d\omega$

$\Rightarrow \int_0^\infty \frac{\cos(\omega x)}{1+\omega^2}\,d\omega = \frac{\pi}{2}e^{-x}$

substitute x by m.

$\int_0^\infty \frac{\cos(m\omega)}{1+\omega^2}\,d\omega = \frac{\pi}{2}e^{-m}$

$\Rightarrow \int_0^\infty \frac{\cos(mx)}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$

02. $\int_0^\infty \frac{x\sin(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m},\ m>0$ $\left[\begin{array}{l} e^{-x},\ x \geq 0 \\ \text{sine transform.}\end{array}\right.$

Fourier sine Transform,

$$F_s(\omega) = \int_0^{\infty} f(x) \sin \omega x \, dx$$

$$I = \int_0^{\infty} e^{-x} \sin \omega x \, dx$$

e^{-x}	$\sin \omega x$
$-e^{-x}$	$\frac{-\cos \omega x}{\omega}$
e^{-x}	$\frac{-\sin \omega x}{\omega^2}$

$$= \left[e^{-x} \frac{-\cos x\omega}{\omega} - (-e^{-x}) \frac{-\sin x\omega}{\omega^2} \right]_0^{\infty} + \int_0^{\infty} e^{-x} \cdot \frac{-\sin \omega x}{\omega^2} \, dx$$

[$\underset{\text{limit}}{\infty}$ হলে 0 হয়ে যায়]

$$I = \left[(-0-0) - \left(\frac{-1}{\omega} - 0\right) \right] - \frac{1}{\omega^2} \int_0^{\infty} e^{-x} \sin \omega x \, dx$$

$$I = \frac{1}{\omega} - \frac{1}{\omega^2} \cdot I$$

$$\Rightarrow \omega^2 I = \omega - I$$

$$\Rightarrow I = \frac{\omega}{1+\omega^2}$$

Applying I.F.S.T

$$f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(\omega) \sin(\omega x) \, dx$$

$$e^{-x} = \frac{2}{\pi}\int_0^{\infty} \frac{\omega}{\omega^2+1} \sin(\omega x)\, d\omega$$

$$\Rightarrow \int_0^{\infty} \frac{\omega \sin \omega x}{\omega^2+1}\, d\omega = \frac{\pi}{2} e^{-x}$$

Substituting $x = m$,

$$\int_0^{\infty} \frac{\omega . \sin(m\omega)}{\omega^2+1}\, d\omega = \frac{\pi}{2} e^{-m}$$

$$\Rightarrow \int_0^{\infty} \frac{x \sin(mx)}{x^2+1}\, dx = \frac{\pi}{2} e^{-m}$$

———————— x ————————

End of MAT216

Linear Algebra and Fourier Aanlysis

Fabiha Tarannum