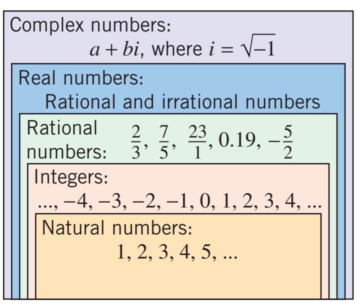
E

REAL NUMBERS, INTERVALS, AND INEQUALITIES

REAL NUMBERS



▲ **Figure E.1**

Figure E.1 describes the various categories of numbers that we will encounter in this text. The simplest numbers are the ***natural numbers***

$$1, \ 2, \ 3, \ 4, \ 5, \ldots$$

These are a subset of the ***integers***

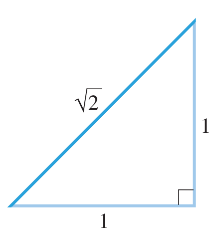
$$\ldots, \ -4, \ -3, \ -2, \ -1, \ 0, \ 1, \ 2, \ 3, \ 4, \ldots$$

and these in turn are a subset of the ***rational numbers***, which are the numbers formed by taking ratios of integers (avoiding division by 0). Some examples are

$$\frac{2}{3}, \quad \frac{7}{5}, \quad 23 = \frac{23}{1}, \quad 0.19 = \frac{19}{100}, \quad -\frac{5}{2} = \frac{-5}{2} = \frac{5}{-2}$$

The early Greeks believed that every measurable quantity had to be a rational number. However, this idea was overturned in the fifth century B.C. by Hippasus of Metapontum who demonstrated the existence of ***irrational numbers***, that is, numbers that cannot be expressed as the ratio of two integers. Using geometric methods, he showed that the length of the hypotenuse of the triangle in Figure E.2 could not be expressed as a ratio of integers, thereby proving that $\sqrt{2}$ is an irrational number. Some other examples of irrational numbers are

$$\sqrt{3}, \quad \sqrt{5}, \quad 1 + \sqrt{2}, \quad \sqrt[3]{7}, \quad \pi, \quad \cos 19^\circ$$



▲ **Figure E.2**

The rational and irrational numbers together comprise what is called the ***real number system***, and both the rational and irrational numbers are called ***real numbers***.

COMPLEX NUMBERS

Because the square of a real number cannot be negative, the equation

$$x^2 = -1$$

has no solutions in the real number system. In the eighteenth century mathematicians remedied this problem by inventing a new number, which they denoted by

$$i = \sqrt{-1}$$

and which they defined to have the property $i^2 = -1$. This, in turn, led to the development of the ***complex numbers***, which are numbers of the form

$$a + bi$$

Hippasus of Metapontum (circa 500 B.C.) Greek Pythagorean philosopher. According to a legend, Hippasus made his discovery at sea and was thrown overboard by fanatic Pythagoreans because his result contradicted their doctrine. The discovery of Hippasus is one of the most fundamental in the entire history of science.

where a and b are real numbers. Some examples are

$2+3i$	$3-4i$	$6i$	$\frac{2}{3}$
$a=2, b=3$	$a=3, b=-4$	$a=0, b=6$	$a=\frac{2}{3}, b=0$

Observe that every real number a is also a complex number because it can be written as

$$a = a + 0i$$

Thus, the real numbers are a subset of the complex numbers. Although we will be concerned primarily with real numbers in this text, complex numbers will arise in the course of solving equations. For example, the solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

which are given by the ***quadratic formula***

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

are not real if the quantity $b^2 - 4ac$ is negative.

DIVISION BY ZERO

Division by zero is not allowed in numerical computations because it leads to mathematical inconsistencies. For example, if $1/0$ were assigned some numerical value, say p, then it would follow that $0 \cdot p = 1$, which is incorrect.

DECIMAL REPRESENTATION OF REAL NUMBERS

Rational and irrational numbers can be distinguished by their decimal representations. Rational numbers have decimals that are ***repeating***, by which we mean that at some point in the decimal some fixed block of numbers begins to repeat indefinitely. For example,

$$\underbrace{\tfrac{4}{3} = 1.333\ldots}_{3 \text{ repeats}}, \quad \underbrace{\tfrac{3}{11} = .272727\ldots}_{27 \text{ repeats}}, \quad \underbrace{\tfrac{1}{2} = .50000\ldots}_{0 \text{ repeats}}, \quad \underbrace{\tfrac{5}{7} = .714285714285714285\ldots}_{714285 \text{ repeats}}$$

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3.141592653589793238462643383279502884197169
39937510582097494459230781640628620899862803
48253421170679821480865132823066470938446095
50582231725359408128481117450284102701938521
10555964462294895493038196442881097566593344
61284756482337867831652712019091456485669234
60348610454326648213393607260249141273724587
00660631558817488152092096282925409171536436
78925903600113305305488204665213841469519415
11609433057270365759591953092186117381932611
79310511854807446237996274956735188575272489
12279381830119491298336733624406566430860213
94946395224737190702179860943702770539217176
29317675238467481846766940513200056812714526
35608277857713427577896091736371787214684409
01224953430146549585371050792279689258923542
01995611212902196086403441815981362977477130
99605187072113499999983729780499510597317328
16096318595024459455346908302642522308253344
68503526193118817101000313783875288658753320
83814206171776691473035982534904287554687311
59562863882353787593751957781857780532171226
80661300192787661119590921642019 89
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▲ Figure E.3

Decimals in which zero repeats from some point on are called ***terminating decimals***. For brevity, it is usual to omit the repetitive zeros in terminating decimals and for other repeating decimals to write the repeating digits only once but with a bar over them to indicate the repetition. For example,

$$\tfrac{1}{2} = .5, \quad \tfrac{12}{4} = 3, \quad \tfrac{8}{25} = .32, \quad \tfrac{4}{3} = 1.\overline{3}, \quad \tfrac{3}{11} = .\overline{27}, \quad \tfrac{5}{7} = .\overline{714285}$$

Irrational numbers have nonrepeating decimals, so we can be certain that the decimals

$$\sqrt{2} = 1.414213562373095\ldots \quad \text{and} \quad \pi = 3.141592653589793\ldots$$

do not repeat from some point on. Moreover, if we stop the decimal expansion of an irrational number at some point, we get only an approximation to the number, never an exact value. For example, even if we compute π to 1000 decimal places, as in Figure E.3, we still have only an approximation.

REMARK Beginning mathematics students are sometimes taught to approximate π by $\frac{22}{7}$. Keep in mind, however, that this is only an approximation, since

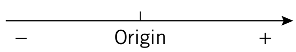
$$\tfrac{22}{7} = 3.\overline{142857}$$

is a rational number whose decimal representation begins to differ from π in the third decimal place.

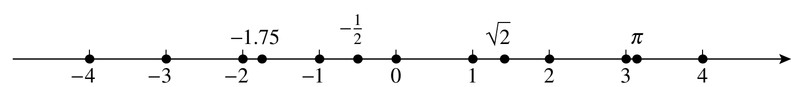
COORDINATE LINES

In 1637 René Descartes published a philosophical work called *Discourse on the Method of Rightly Conducting the Reason*. In the back of that book was an appendix that the British philosopher John Stuart Mill described as "the greatest single step ever made in the progress of the exact sciences." In that appendix René Descartes linked together algebra and geometry, thereby creating a new subject called ***analytic geometry***; it gave a way of describing algebraic formulas by geometric curves and, conversely, geometric curves by algebraic formulas.

The key step in analytic geometry is to establish a correspondence between real numbers and points on a line. To do this, choose any point on the line as a reference point and call it the ***origin***; and then arbitrarily choose one of the two directions along the line to be the ***positive direction*** and let the other be the ***negative direction***. It is usual to mark the positive direction with an arrowhead, as in Figure E.4, and to take the positive direction to the right when the line is horizontal. Next, choose a convenient unit of measure and represent each positive number r by the point that is r units from the origin in the positive direction, each negative number $-r$ by the point that is r units from the origin in the negative direction from the origin, and 0 by the origin itself (Figure E.5). The number associated with a point P is called the ***coordinate*** of P, and the line is called a ***coordinate line***, a ***real number line***, or a ***real line***.



▲ **Figure E.4**



▶ **Figure E.5**

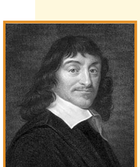
INEQUALITY NOTATION

The real numbers can be ordered by size as follows: If $b - a$ is positive, then we write either $a < b$ (read "a is less than b") or $b > a$ (read "b is greater than a"). We write $a \leq b$ to mean $a < b$ or $a = b$, and we write $a < b < c$ to mean that $a < b$ and $b < c$. As one traverses a coordinate line in the positive direction, the real numbers increase in size, so on a horizontal coordinate line the inequality $a < b$ implies that a is to the left of b, and the inequalities $a < b < c$ imply that a is to the left of c, and b lies between a and c. The meanings of such symbols as

$$a \leq b < c, \quad a \leq b \leq c, \quad \text{and} \quad a < b < c < d$$

should be clear. For example, you should be able to confirm that all of the following are true statements:

$$3 < 8, \quad -7 < 1.5, \quad -12 \leq -\pi, \quad 5 \leq 5, \quad 0 \leq 2 \leq 4,$$
$$8 \geq 3, \quad 1.5 > -7, \quad -\pi > -12, \quad 5 \geq 5, \quad 3 > 0 > -1 > -3$$



René Descartes (1596–1650) Descartes, a French aristocrat, was the son of a government official. He graduated from the University of Poitiers with a law degree at age 20. After a brief probe into the pleasures of Paris he became a military engineer, first for the Dutch Prince of Nassau and then for the German Duke of Bavaria. It was during his service as a soldier that Descartes began to pursue mathematics seriously and develop his analytic geometry. After the wars, he returned to Paris where he stalked the city as an eccentric, wearing a sword in his belt and a plumed hat. He lived in leisure, seldom arose before 11 A.M., and dabbled in the study of human physiology, philosophy, glaciers, meteors, and rainbows. He eventually moved to Holland, where he published his *Discourse on the Method*, and finally to Sweden where he died while serving as tutor to Queen Christina. Descartes is regarded as a genius of the first magnitude. In addition to major contributions in mathematics and philosophy, he is considered, along with William Harvey, to be a founder of modern physiology.

REVIEW OF SETS

In the following discussion we will be concerned with certain sets of real numbers, so it will be helpful to review the basic ideas about sets. Recall that a ***set*** is a collection of objects, called ***elements*** or ***members*** of the set. In this text we will be concerned primarily with sets whose members are numbers or points that lie on a line, a plane, or in three-dimensional space. We will denote sets by capital letters and elements by lowercase letters. To indicate that a is a member of the set A we will write $a \in A$ (read "a belongs to A"), and to indicate that a is not a member of the set A we will write $a \notin A$ (read "a does not belong to A"). For example, if A is the set of positive integers, then $5 \in A$, but $-5 \notin A$. Sometimes sets arise that have no members (e.g., the set of odd integers that are divisible by 2). A set with no members is called an ***empty set*** or a ***null set*** and is denoted by the symbol $\varnothing$.

Some sets can be described by listing their members between braces. The order in which the members are listed does not matter; so, for example, the set A of positive integers that are less than 6 can be expressed as

$$A = \{1, 2, 3, 4, 5\} \quad \text{or} \quad A = \{2, 3, 1, 5, 4\}$$

We can also write A in ***set-builder notation*** as

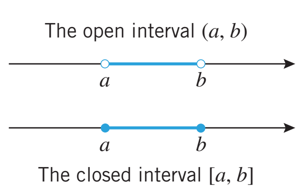
$$A = \{x : x \text{ is an integer and } 0 < x < 6\}$$

which is read "A is the set of all x such that x is an integer and $0 < x < 6$." In general, to express a set S in set-builder notation we write $S = \{x : \underline{\qquad\quad}\}$ in which the line is replaced by a property that identifies exactly those elements in the set S.

If every member of a set A is also a member of a set B, then we say that A is a ***subset*** of B and write $A \subseteq B$. For example, if A is the set of positive integers and B is the set of all integers, then $A \subseteq B$. If two sets A and B have the same members (i.e., $A \subseteq B$ and $B \subseteq A$), then we say that A and B are ***equal*** and write $A = B$.

INTERVALS

In calculus we will be concerned with sets of real numbers, called ***intervals***, that correspond to line segments on a coordinate line. For example, if $a < b$, then the ***open interval*** from a to b, denoted by (a, b), is the line segment extending from a to b, *excluding* the endpoints; and the ***closed interval*** from a to b, denoted by $[a, b]$, is the line segment extending from a to b, *including* the endpoints (Figure E.6). These sets can be expressed in set-builder notation as



▲ **Figure E.6**

$$(a, b) = \{x : a < x < b\} \qquad \boxed{\text{The open interval from } a \text{ to } b}$$

$$[a, b] = \{x : a \le x \le b\} \qquad \boxed{\text{The closed interval from } a \text{ to } b}$$

REMARK Observe that in this notation and in the corresponding Figure E.6, parentheses and open dots mark endpoints that are excluded from the interval, whereas brackets and closed dots mark endpoints that are included in the interval. Observe also that in set-builder notation for the intervals, it is understood that x is a real number, even though it is not stated explicitly.

As shown in Table E.1, an interval can include one endpoint and not the other; such intervals are called ***half-open*** (or sometimes ***half-closed***). Moreover, the table also shows that it is possible for an interval to extend indefinitely in one or both directions. To indicate that an interval extends indefinitely in the positive direction we write $+\infty$ (read "positive infinity") in place of a right endpoint, and to indicate that an interval extends indefinitely in the negative direction we write $-\infty$ (read "negative infinity") in place of a left endpoint. Intervals that extend between two real numbers are called ***finite intervals***, whereas intervals that extend indefinitely in one or both directions are called ***infinite intervals***.

Table E.1

INTERVAL NOTATION	SET NOTATION	GEOMETRIC PICTURE	CLASSIFICATION
(a, b)	$\{x : a < x < b\}$		Finite; open
$[a, b]$	$\{x : a \le x \le b\}$		Finite; closed
$[a, b)$	$\{x : a \le x < b\}$		Finite; half-open
$(a, b]$	$\{x : a < x \le b\}$		Finite; half-open
$(-\infty, b]$	$\{x : x \le b\}$		Infinite; closed
$(-\infty, b)$	$\{x : x < b\}$		Infinite; open
$[a, +\infty)$	$\{x : x \ge a\}$		Infinite; closed
$(a, +\infty)$	$\{x : x > a\}$		Infinite; open
$(-\infty, +\infty)$	$\mathbb{R}$		Infinite; open and closed

REMARK By convention, infinite intervals of the form $[a, +\infty)$ or $(-\infty, b]$ are considered to be closed because they contain their endpoint, and intervals of the form $(a, +\infty)$ and $(-\infty, b)$ are considered to be open because they do not include their endpoint. The interval $(-\infty, +\infty)$, which is the set of all real numbers, has no endpoints and can be regarded as both open and closed. This set is often denoted by the special symbol $\mathbb{R}$. To distinguish verbally between the open interval $(0, +\infty) = \{x : x > 0\}$ and the closed interval $[0, +\infty) = \{x : x \ge 0\}$, we will call x ***positive*** if $x > 0$ and ***nonnegative*** if $x \ge 0$. Thus, a positive number must be nonnegative, but a nonnegative number need not be positive, since it might possibly be 0.

UNIONS AND INTERSECTIONS OF INTERVALS

If A and B are sets, then the ***union*** of A and B (denoted by $A \cup B$) is the set whose members belong to A or B (or both), and the ***intersection*** of A and B (denoted by $A \cap B$) is the set whose members belong to both A and B. For example,

$$\{x : 0 < x < 5\} \cup \{x : 1 < x < 7\} = \{x : 0 < x < 7\}$$
$$\{x : x < 1\} \cap \{x : x \ge 0\} = \{x : 0 \le x < 1\}$$
$$\{x : x < 0\} \cap \{x : x > 0\} = \varnothing$$

or in interval notation,

$$(0, 5) \cup (1, 7) = (0, 7)$$
$$(-\infty, 1) \cap [0, +\infty) = [0, 1)$$
$$(-\infty, 0) \cap (0, +\infty) = \varnothing$$

ALGEBRAIC PROPERTIES OF INEQUALITIES

The following algebraic properties of inequalities will be used frequently in this text. We omit the proofs.

E.1 THEOREM ***(Properties of Inequalities)*** *Let a, b, c, and d be real numbers.*

(*a*) *If $a < b$ and $b < c$, then $a < c$.*

(*b*) *If $a < b$, then $a + c < b + c$ and $a - c < b - c$.*

(*c*) *If $a < b$, then $ac < bc$ when c is positive and $ac > bc$ when c is negative.*

(*d*) *If $a < b$ and $c < d$, then $a + c < b + d$.*

(*e*) *If a and b are both positive or both negative and $a < b$, then $1/a > 1/b$.*

If we call the direction of an inequality its *sense*, then these properties can be paraphrased as follows:

(*b*) *The sense of an inequality is unchanged if the same number is added to or subtracted from both sides.*

(*c*) *The sense of an inequality is unchanged if both sides are multiplied by the same positive number, but the sense is reversed if both sides are multiplied by the same negative number.*

(*d*) *Inequalities with the same sense can be added.*

(*e*) *If both sides of an inequality have the same sign, then the sense of the inequality is reversed by taking the reciprocal of each side.*

The properties in Theorem E.1 remain true if the symbols $<$ and $>$ are replaced by $\leq$ and $\geq$ respectively.

Example 1

STARTING INEQUALITY	OPERATION	RESULTING INEQUALITY
$-2 < 6$	Add 7 to both sides.	$5 < 13$
$-2 < 6$	Subtract 8 from both sides.	$-10 < -2$
$-2 < 6$	Multiply both sides by 3.	$-6 < 18$
$-2 < 6$	Multiply both sides by -3.	$6 > -18$
$3 < 7$	Multiply both sides by 4.	$12 < 28$
$3 < 7$	Multiply both sides by -4.	$-12 > -28$
$3 < 7$	Take reciprocals of both sides.	$\frac{1}{3} > \frac{1}{7}$
$-8 < -6$	Take reciprocals of both sides.	$-\frac{1}{8} > -\frac{1}{6}$
$4 < 5, -7 < 8$	Add corresponding sides.	$-3 < 13$

SOLVING INEQUALITIES

A ***solution*** of an inequality in an unknown x is a value for x that makes the inequality a true statement. For example, $x = 1$ is a solution of the inequality $x < 5$, but $x = 7$ is not. The set of all solutions of an inequality is called its ***solution set***. It can be shown that if one does not multiply both sides of an inequality by zero or an expression involving an unknown, then the operations in Theorem E.1 will not change the solution set of the inequality. The process of finding the solution set of an inequality is called ***solving*** the inequality.

Example 2 Solve $3 + 7x \leq 2x - 9$.

Solution. We will use the operations of Theorem E.1 to isolate x on one side of the inequality.

$$3 + 7x \leq 2x - 9 \qquad \boxed{\text{Given.}}$$

$$7x \leq 2x - 12 \qquad \boxed{\text{We subtracted 3 from both sides.}}$$

$$5x \leq -12 \qquad \boxed{\text{We subtracted } 2x \text{ from both sides.}}$$

$$x \leq -\tfrac{12}{5} \qquad \boxed{\text{We multiplied both sides by } \tfrac{1}{5}.}$$

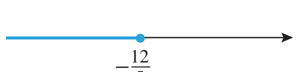


Figure E.7

Because we have not multiplied by any expressions involving the unknown x, the last inequality has the same solution set as the first. Thus, the solution set is the interval $\left(-\infty, -\frac{12}{5}\right]$ shown in Figure E.7.

Example 3 Solve $7 \le 2 - 5x < 9$.

Solution. The given inequality is actually a combination of the two inequalities

$$7 \le 2 - 5x \quad \text{and} \quad 2 - 5x < 9$$

We could solve the two inequalities separately, then determine the values of x that satisfy both by taking the intersection of the two solution sets. However, it is possible to work with the combined inequalities in this problem:

$7 \le 2 - 5x < 9$	Given.
$5 \le -5x < 7$	We subtracted 2 from each member.
$-1 \ge x > -\frac{7}{5}$	We multiplied by $-\frac{1}{5}$ and reversed the sense of the inequalities.
$-\frac{7}{5} < x \le -1$	For clarity, we rewrote the inequalities with the smaller number on the left.

Figure E.8

Thus, the solution set is the interval $\left(-\frac{7}{5}, -1\right]$ shown in Figure E.8. ◀

Example 4 Solve $x^2 - 3x > 10$.

Solution. By subtracting 10 from both sides, the inequality can be rewritten as

$$x^2 - 3x - 10 > 0$$

Factoring the left side yields

$$(x+2)(x-5) > 0$$

The values of x for which $x + 2 = 0$ or $x - 5 = 0$ are $x = -2$ and $x = 5$. These points divide the coordinate line into three open intervals,

$$(-\infty, -2), \quad (-2, 5), \quad (5, +\infty)$$

on each of which the product $(x + 2)(x - 5)$ has constant sign. To determine those signs we will choose an *arbitrary* point in each interval at which we will determine the sign; these are called ***test points***. As shown in Figure E.9, we will use -3, 0, and 6 as our test points. The results can be organized as follows:

INTERVAL	TEST POINT	SIGN OF $(x+2)(x-5)$ AT THE TEST POINT
$(-\infty, -2)$	-3	$(-)(-) = +$
$(-2, 5)$	0	$(+)(-) = -$
$(5, +\infty)$	6	$(+)(+) = +$

The pattern of signs in the intervals is shown on the number line in the middle of Figure E.9. We deduce that the solution set is $(-\infty, -2) \cup (5, +\infty)$, which is shown at the bottom of Figure E.9. ◀

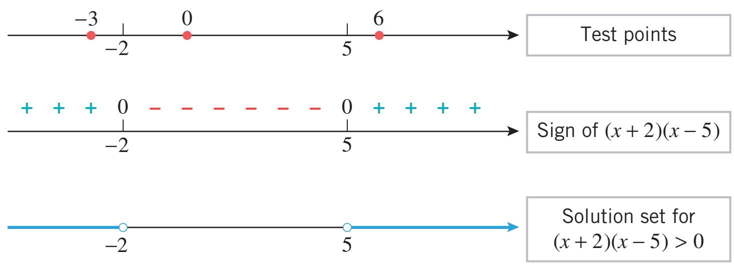


Figure E.9

Example 5 Solve $\dfrac{2x-5}{x-2} < 1$.

Solution. We could start by multiplying both sides by $x-2$ to eliminate the fraction. However, this would require us to consider the cases $x-2>0$ and $x-2<0$ separately because the sense of the inequality would be reversed in the second case, but not the first. The following approach is simpler:

$$\frac{2x-5}{x-2} < 1 \qquad \boxed{\text{Given.}}$$

$$\frac{2x-5}{x-2} - 1 < 0 \qquad \boxed{\text{We subtracted 1 from both sides to obtain a 0 on the right.}}$$

$$\frac{(2x-5)-(x-2)}{x-2} < 0 \qquad \boxed{\text{We combined terms.}}$$

$$\frac{x-3}{x-2} < 0 \qquad \boxed{\text{We simplified.}}$$

The quantity $x-3$ is zero if $x=3$, and the quantity $x-2$ is zero if $x=2$. These points divide the coordinate line into three open intervals,

$$(-\infty, 2), \quad (2, 3), \quad (3, +\infty)$$

on each of which the quotient $(x-3)/(x-2)$ has constant sign. Using 0, 2.5, and 4 as test points (Figure E.10), we obtain the following results:

INTERVAL	TEST POINT	SIGN OF $(x-3)/(x-2)$ AT THE TEST POINT
$(-\infty, 2)$	0	$(-)/(-) = +$
$(2, 3)$	2.5	$(-)/(+) = -$
$(3, +\infty)$	4	$(+)/(+) = +$

The signs of the quotient are shown in the middle of Figure E.10. From the figure we see that the solution set consists of all real values of x such that $2 < x < 3$. This is the interval $(2, 3)$ shown at the bottom of Figure E.10. ◀

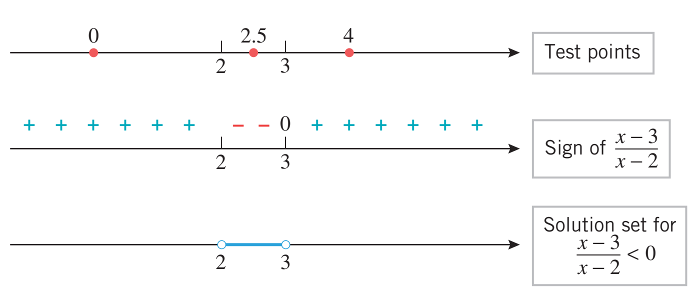


Figure E.10

EXERCISE SET E

1. Among the terms *integer*, *rational*, and *irrational*, which ones apply to the given number?

(a) $-\frac{3}{4}$ (b) 0 (c) $\frac{24}{8}$ (d) 0.25
(e) $-\sqrt{16}$ (f) $2^{1/2}$ (g) $0.020202\ldots$ (h) $7.000\ldots$

2. Which of the terms *integer*, *rational*, and *irrational* apply to the given number?

(a) $0.31311311131111\ldots$ (b) $0.729999\ldots$
(c) $0.376237623762\ldots$ (d) $17\frac{4}{5}$

3. The repeating decimal $0.137137137\ldots$ can be expressed as a ratio of integers by writing

$$x = 0.137137137\ldots$$
$$1000x = 137.137137137\ldots$$

and subtracting to obtain $999x = 137$ or $x = \frac{137}{999}$. Use this idea, where needed, to express the following decimals as ratios of integers.

(a) $0.123123123\ldots$ (b) $12.7777\ldots$
(c) $38.07818181\ldots$ (d) $0.4296000\ldots$

4. Show that the repeating decimal $0.99999\ldots$ represents the number 1. Since $1.000\ldots$ is also a decimal representation of 1, this problem shows that a real number can have two different decimal representations. [*Hint:* Use the technique of Exercise 3.]

5. The Rhind Papyrus, which is a fragment of Egyptian mathematical writing from about 1650 B.C., is one of the oldest known examples of written mathematics. It is stated in the papyrus that the area A of a circle is related to its diameter D by

$$A = \left(\tfrac{8}{9}D\right)^2$$

(a) What approximation to π were the Egyptians using?
(b) Use a calculating utility to determine if this approximation is better or worse than the approximation $\frac{22}{7}$.

6. The following are all famous approximations to π:

$\dfrac{333}{106}$ Adrian Athoniszoon, c. 1583

$\dfrac{355}{113}$ Tsu Chung-Chi and others

$\dfrac{63}{25}\left(\dfrac{17+15\sqrt{5}}{7+15\sqrt{5}}\right)$ Ramanujan

$\dfrac{22}{7}$ Archimedes

$\dfrac{223}{71}$ Archimedes

(a) Use a calculating utility to order these approximations according to size.
(b) Which of these approximations is closest to but larger than π?
(c) Which of these approximations is closest to but smaller than π?
(d) Which of these approximations is most accurate?

7. In each line of the accompanying table, check the blocks, if any, that describe a valid relationship between the real numbers a and b. The first line is already completed as an illustration.

a	b	$a<b$	$a \le b$	$a>b$	$a \ge b$	$a=b$
1	6	✓	✓			
6	1					
−3	5					
5	−3					
−4	−4					
0.25	$\frac{1}{3}$					
$-\frac{1}{4}$	$-\frac{3}{4}$					

▲ **Table Ex-7**

8. In each line of the accompanying table, check the blocks, if any, that describe a valid relationship between the real numbers a, b, and c.

a	b	c	$a<b<c$	$a \le b \le c$	$a<b \le c$	$a \le b<c$
−1	0	2				
2	4	−3				
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{4}$				
−5	−5	−5				
0.75	1.25	1.25				

▲ **Table Ex-8**

9. Which of the following are always correct if $a \le b$?

(a) $a-3 \le b-3$ (b) $-a \le -b$ (c) $3-a \le 3-b$
(d) $6a \le 6b$ (e) $a^2 \le ab$ (f) $a^3 \le a^2 b$

10. Which of the following are always correct if $a \le b$ and $c \le d$?

(a) $a+2c \le b+2d$ (b) $a-2c \le b-2d$
(c) $a-2c \ge b-2d$

11. For what values of a are the following inequalities valid?

(a) $a \le a$ (b) $a < a$

12. If $a \le b$ and $b \le a$, what can you say about a and b?

13. (a) If $a < b$ is true, does it follow that $a \le b$ must also be true?
(b) If $a \le b$ is true, does it follow that $a < b$ must also be true?

14. In each part, list the elements in the set.

(a) $\{x : x^2 - 5x = 0\}$
(b) $\{x : x \text{ is an integer satisfying } -2 < x < 3\}$

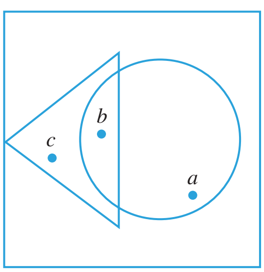
15. In each part, express the set in the notation $\{x : \_\_\_\_\_\_\}$.
(a) $\{1, 3, 5, 7, 9, \ldots\}$ (b) the set of even integers
(c) the set of irrational numbers
(d) $\{7, 8, 9, 10\}$

16. Let $A = \{1, 2, 3\}$. Which of the following sets are equal to A?
(a) $\{0, 1, 2, 3\}$ (b) $\{3, 2, 1\}$
(c) $\{x : (x-3)(x^2-3x+2) = 0\}$

17. In the accompanying figure, let

S = the set of points inside the square
T = the set of points inside the triangle
C = the set of points inside the circle

and let a, b, and c be the points shown. Answer the following as true or false.
(a) $T \subseteq C$ (b) $T \subseteq S$ (c) $a \notin T$
(d) $a \notin S$ (e) $b \in T$ and $b \in C$
(f) $a \in C$ or $a \in T$ (g) $c \in T$ and $c \notin C$



◀ **Figure Ex-17**

18. List all subsets of
(a) $\{a_1, a_2, a_3\}$ (b) $\varnothing$.

19. In each part, sketch on a coordinate line all values of x that satisfy the stated condition.
(a) $x \le 4$ (b) $x \ge -3$ (c) $-1 \le x \le 7$
(d) $x^2 = 9$ (e) $x^2 \le 9$ (f) $x^2 \ge 9$

20. In parts (a)–(d), sketch on a coordinate line all values of x, if any, that satisfy the stated conditions.
(a) $x > 4$ and $x \le 8$
(b) $x \le 2$ or $x \ge 5$
(c) $x > -2$ and $x \ge 3$
(d) $x \le 5$ and $x > 7$

21. Express in interval notation.
(a) $\{x : x^2 \le 4\}$ (b) $\{x : x^2 > 4\}$

22. In each part, sketch the set on a coordinate line.
(a) $[-3, 2] \cup [1, 4]$ (b) $[4, 6] \cup [8, 11]$
(c) $(-4, 0) \cup (-5, 1)$ (d) $[2, 4) \cup (4, 7)$
(e) $(-2, 4) \cap (0, 5]$ (f) $[1, 2.3) \cup (1.4, \sqrt{2})$
(g) $(-\infty, -1) \cup (-3, +\infty)$ (h) $(-\infty, 5) \cap [0, +\infty)$

23–44 Solve the inequality and sketch the solution on a coordinate line. ■

23. $3x - 2 < 8$ **24.** $\frac{1}{5}x + 6 \ge 14$
25. $4 + 5x \le 3x - 7$ **26.** $2x - 1 > 11x + 9$
27. $3 \le 4 - 2x < 7$ **28.** $-2 \ge 3 - 8x \ge -11$

29. $\dfrac{x}{x-3} < 4$ **30.** $\dfrac{x}{8-x} \ge -2$ **31.** $\dfrac{3x+1}{x-2} < 1$
32. $\dfrac{\frac{1}{2}x - 3}{4+x} > 1$ **33.** $\dfrac{4}{2-x} \le 1$ **34.** $\dfrac{3}{x-5} \le 2$
35. $x^2 > 9$ **36.** $x^2 \le 5$
37. $(x-4)(x+2) > 0$ **38.** $(x-3)(x+4) < 0$
39. $x^2 - 9x + 20 \le 0$ **40.** $2 - 3x + x^2 \ge 0$
41. $\dfrac{2}{x} < \dfrac{3}{x-4}$ **42.** $\dfrac{1}{x+1} \ge \dfrac{3}{x-2}$
43. $x^3 - x^2 - x - 2 > 0$ **44.** $x^3 - 3x + 2 \le 0$

45–46 Find all values of x for which the given expression yields a real number. ■

45. $\sqrt{x^2 + x - 6}$ **46.** $\sqrt{\dfrac{x+2}{x-1}}$

47. Fahrenheit and Celsius temperatures are related by the formula $C = \frac{5}{9}(F - 32)$. If the temperature in degrees Celsius ranges over the interval $25 \le C \le 40$ on a certain day, what is the temperature range in degrees Fahrenheit that day?

48. Every integer is either even or odd. The even integers are those that are divisible by 2, so n is even if and only if $n = 2k$ for some integer k. Each odd integer is one unit larger than an even integer, so n is odd if and only if $n = 2k + 1$ for some integer k. Show:
(a) If n is even, then so is n^2
(b) If n is odd, then so is n^2.

49. Prove the following results about sums of rational and irrational numbers:
(a) rational + rational = rational
(b) rational + irrational = irrational.

50. Prove the following results about products of rational and irrational numbers:
(a) rational · rational = rational
(b) rational · irrational = irrational (provided the rational factor is nonzero).

51. Show that the sum or product of two irrational numbers can be rational or irrational.

52. Classify the following as rational or irrational and justify your conclusion.
(a) $3 + \pi$ (b) $\frac{3}{4}\sqrt{2}$ (c) $\sqrt{8}\sqrt{2}$ (d) $\sqrt{\pi}$
(See Exercises 49 and 50.)

53. Prove: The average of two rational numbers is a rational number, but the average of two irrational numbers can be rational or irrational.

54. Can a rational number satisfy $10^x = 3$?

55. Solve: $8x^3 - 4x^2 - 2x + 1 < 0$.

56. Solve: $12x^3 - 20x^2 \ge -11x + 2$.

57. Prove: If a, b, c, and d are positive numbers such that $a < b$ and $c < d$, then $ac < bd$. (This result gives conditions under which inequalities can be "multiplied together.")

58. Is the number represented by the decimal

$$0.101001000100001000001\ldots$$

rational or irrational? Explain your reasoning.