The Foundations: Logic and Proofs

Chapter 1, Part 2: Predicate

Predicates and Quantifiers

Section 1.4

Introduction

Propositional logic does not express **meaning** completely of statements and natural language.

"Ahmed is a Fast Nuces student"

We can't conclude the truth of the statement with propositional logic. Therefore more powerful logic is used instead, called **Predicate logic.**

Limitations of Propositional Logic(1)

Statements that must be repeated for many objects

Example: – If Ahmed is a CS graduate then Ahmed has passed cs220

Translation:

Ahmed is a CS graduate → Ahmed has passed cs220

Similar statements can be written for other cs graduates:

Ali is a CS graduate → Ali has passed cs220

Sara is a CS graduate → Sara has passed cs220

- **Solution**: make statements with **variables**

If x is a CS graduate then x has passed cs220

x is a CS graduate → x has passed cs220

First-Order Logic

FOL is a logical system for reasoning about properties of Objects.

Augments the logical connectives from propositional logic with

- ***Predicates:* describe** Properties of Objects
- ***Functions*** that connects objects to another object
- ***Quantifiers*** that allow to reason about multiple objects

FOL use Predicates to reason about objects.

Predicate Logic

- Express meaning of statements in mathematics and CS
- Allows to reason and explore properties or relationships between objects.
- **“X > 3”, “x = y + 3”, “x + y = z”**
- **“Computer x is under attack by an intruder”**
- **“Computer x is functioning properly”**

Example

To identify the meaning of statement, break it into two parts, **Subject and predicate**

Statement= “X is greater than 3”

Unknown part is the variable “x” is **subject.**

The other part “ is greater than 3” is the **predicate**.

<u>Predicates</u> refers to the property that the subject of the statement can have.

Denote the statement with ***P(x), P*** denotes Predicate while ***x*** itself is the unknown variable in statement.

Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Statement with more Variables

$x=y+3$
Statement= $Q(x,y)$

Q is predicate.
x,y terms are called constant symbols (referred as objects)

Predicates take objects as an arguments and evaluate them to be true or false
What are the truth values of the propositions
$Q(1, 2)$ and $Q(3, 0)$?

What are the truth values of the propositions R(1, 2, 3) and R(0, 0, 1)?

Given statement: ***R(x,y,z)*** denote the statement **"x + y = z"**

Predicate with compound Propositions

- If $P(x)$ denotes "$x > 0$," find these truth values:

  $P(3) \vee P(-1)$ **Solution**: T

  $P(3) \wedge P(-1)$ **Solution**: F

  $P(3) \rightarrow P(-1)$ **Solution**: F

Predicates and Objects

Working in FOL: keep objects (actual things) and propositions (True, False) separate.

Do not apply connectives to objects

$X \rightarrow y$ or $2 \rightarrow 4$

The Type Checking Table

	… operate on …	… and produce
Connectives ($\leftrightarrow$, $\wedge$, etc.) …	propositions	a proposition
Predicates ($=$, etc.) …	objects	a proposition
Functions …	objects	an object

Limitations of Propositional Logic(2)

Statements that define the property of the group of objects

Example:

- **All** new cars must be registered.
- **Some** of the CS graduates graduate with honors.

Quantifiers

Quantifiers are phrases that refer to given quantities, such as **"for some" or "for all" or "for every"**, indicating how many objects have a certain property.

Types of Quantifiers:

Universal Quantifier: represented by $\forall$,“for all”, “for every”, “for each”, or “for any”.

Existential Quantifier: represented by $\exists$,“for some”, “there exists”, “there is a”, or “for at least one”.

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

Universal Quantifier

The universal quantification of P(x) is the proposition:

$\forall$ xP(x) denotes that universal quantification of P(x).

*An element for which P(x) is false is called a **counterexample to $\forall$xP(x).***

For every x, P(x):

- **Let P(x) denote “x > x -1”**
- **What is truth value of $\forall$x P(x)? //x is all real numbers.**
- **Let P(x) be the statement“x + 1 > x.”**
- **What is the truth value of the quantification $\forall$xP(x)?**

Remember that the truth value of $\forall$xP(x) depends on the domain!

CounterExample

- Let Q(x) be the statement "$x < 2$." What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

Solution: Q(x) is not true for every real number x, because, for instance, Q(3) is false. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$. Thus, $\forall x Q(x)$ is false.

- Suppose that P(x) is x2 > 0." To show that the statement $\forall x P(x)$ is false where the universe of discourse consists of all integers.

Solution:

What does the statement $\forall x N(x)$ mean if $N(x)$ is "Computer x is connected to the network" and the domain consists of all computers on campus?

The statement $\forall x N(x)$ means that for every computer x on campus, that computer x is connected to the network. This statement can be expressed in English as "Every computer on campus is connected to the network."

Existential Quantification

- ***"There exists an element x in the domain such that P(x)."***
- Mathematical statements assert that there is an element with a certain property . Such statements are expressed using existential quantification.
- A proposition is true only if and only if *P(x)* is true for at least one value of x in the domain.

EXAMPLE:

- If P(x) denotes "$x > 0$" and U is the integers, then $\exists x\, P(x)$ is true. It is also true if U is the positive integers.
- If P(x) denotes "$x < 0$" and U is the positive integers, then $\exists x\, P(x)$ is false.
- If P(x) denotes "x is even" and U is the integers, then $\exists x\, P(x)$ is true.

- **Let Q(x) denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?**

Because Q(x) is false for every real number x, the existential quantification of Q(x), which is $\exists x Q(x)$, is false.

Uniqueness Quantifier

- $\exists\, !x\, P(x)$ means that $P(x)$ is true for one and only one x in the universe of discourse.
  - Examples:

1. If $P(x)$ denotes "$x - 1 = 0$" and U is the integers, then $\exists\, !x\, P(x)$ is true.
2. But if $P(x)$ denotes "$x > 0$," then $\exists\, !x\, P(x)$ is false.

Properties of Quantifiers

- The truth value of $\exists x\, P(x)$ and $\forall x\, P(x)$ depend on both the propositional function $P(x)$ and on the domain U.
- **Examples**:
  1. If U is the positive integers and $P(x)$ is the statement "$x < 2$", then $\exists x\, P(x)$ is true, but $\forall x\, P(x)$ is false.
  2. If U is the negative integers and $P(x)$ is the statement "$x < 2$", then both $\exists x\, P(x)$ and $\forall x\, P(x)$ are true.
  3. If U consists of 3, 4, and 5, and $P(x)$ is the statement "$x > 2$",then ?

Precedence of Quantifiers

- *Higher than all other logical operators.*
- *Which one is correct and means this* $\forall x\, P(x) \vee Q(x)$
    - $(\forall x\, P(x)) \vee Q(x)$
    - $\forall x\, (P(x) \vee Q(x))$

Translating from English to Logic

Translate:

"Every student in this class has taken a course in web dev"

Solution1:

Translate:

"Some students in this class has taken a course in web dev"

Solution2:

Translating into quantified statement

For each real number x, x2 > 0

The square of every real number is greater than 0

The square of a real number is greater than 0

If x $\in$ R ,then x2 > 0

Quantifier as Conjunctions and Disjunctions

- If the domain is finite, a universally quantified proposition is equivalent to a conjunction of propositions without quantifiers and
- an existentially quantified proposition is equivalent to a disjunction of propositions without quantifiers.

Negating Quantified Expressions

"Every student in this class has taken a course in web dev"

Negation: "It is not the case that every student in t class has taken a course in web dev"

- **(∀x∈U)[P(x)]is false is equivalent to ?**
-

$$\neg(\forall x \in \mathbb{R})(x^3 \geq x^2) \equiv (\exists x \in \mathbb{R})\neg(x^3 \geq x^2).$$

Rewrite without negation

Equivalences in Predicate Logic

- Statements involving predicates and quantifiers are *logically equivalent* if and only if they have the same truth value
- **Example:** $\forall x \neg\neg S(x) \equiv \forall x\, S(x)$

Translation from English to Logic

- U = {fleegles, snurds, thingama}
  $F(x)$: x is a fleegle
  $S(x)$: x is a snurd
  $T(x)$: x is a thingama
  Translate "Everything is a fleegle"

  Solution: $\forall x\, F(x)$

Translation from English to Logic

- U = {fleegles, snurds, thingama}
  F(x): *x* is a fleegle
  Sx): *x* is a snurd
  T(x): x is a thingama
  “Nothing is a snurd.”

Solution ?

Translation from English to Logic

- U = {fleegles, snurds, thingama}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingama

"All fleegles are snurds."

Solution ?

Translation from English to Logic

- U = {fleegles, snurds, thingama}

  F(x): *x* is a fleegle

  S(x): *x* is a snurd

  T(x): *x* is a thingama

"If any fleegle is a snurd then it is also a thingama"."

Solution ?

1. "Some student in this class has visited Mexico."
2. Every student in this class has visited Canada or Mexico

Examples Of FOL

1. Not all cars have carburetors
2. Some people are either religious or pious
3. No dogs are intelligent
4. All babies are illogical
5. Every number is either negative or has a square root
6. Some numbers are not real
7. Every connected and circuit-free graph is a tree
8. Not every graph is connected
9. All that glitters is not gold
10. Not all that glitters is gold
11. There is a barber who shaves all men in the town who do not shave themselves
12. There is no business like show business

System Specification Example : home task

Lewis Carroll Example

Home task

Nested Quantifier

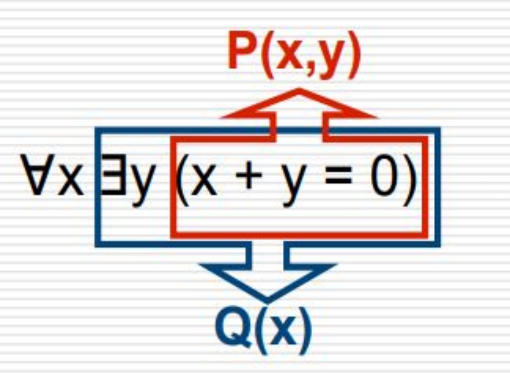
A possible statement with multiple quantifiers is nested quantifier OR

Two quantifiers are nested if one is within the scope of the other.

We can have different combinations: $\forall x, \forall y$, $\exists x, \forall y$, or $\exists x, \exists y$.

- **(Important)** We can **switch the order** of the quantifiers if they are the same: **$\forall x, \forall y$ is logically equivalent to $\forall y, \forall x$** and **$\exists x, \exists y$ is logically equivalent to $\exists y, \exists x$**
- **Be careful**: $\forall x \exists y$ is not logically equivalent to $\exists y \forall x$

Example 1



x Q(x)

Q(x) is ∃y P(x,y)

P(x,y) is (x + y = 0)

- Nested Loops

a. To see if $\forall x \forall y P(x,y)$ is true, loop through the values of x :
   - At each step, loop through the values for y.
   - If for some pair of x and y, $P(x,y)$ is false, then $\forall x \forall y P(x,y)$ is false and both the outer and inner loop terminate.

   $\forall x \forall y P(x,y)$ is true if the outer loop ends after stepping through each x.

b. To see if $\forall x \exists y P(x,y)$ is true, loop through the values of x:
   - At each step, loop through the values for y.
   - The inner loop ends when a pair x and y is found such that $P(x, y)$ is true.
   - If no y is found such that $P(x, y)$ is true the outer loop terminates as $\forall x \exists y P(x,y)$ has been shown to be false.

   $\forall x \exists y P(x,y)$ is true if the outer loop ends after stepping through each x.

- If the domains of the variables are infinite, then this process can not actually be carried out.

Examples

1. For all real numbers x and y, $x + y = y + x$
2. For every real number x, there is a real number y such that $x = - y$
3. For every real numbers x and y, if x is positive and y is negative then xy is negative.
4. The product of a positive real number and a negative real number is always a negative real number
5. For every pair of real numbers x, y, $xy = yx$
6. There are integers x and y such that $x+y=5$
7. Every real number except zero has a multiplicative inverse.

Quantifications of Two Variables

Statement	True When	False When
$\forall x \forall y P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is at least one pair, x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x, there is a y for which $P(x, y)$ is true.	There is an x for which $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x, there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$	There is at least one pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Table: Truth Values of 2-variate Quantifiers

Find truth values

1. $\forall x \exists y(x\ 2 = y)$:
2. $\forall x \exists y(x = y\ 2\ )$:

Order of Nested Quantifiers

The order of nested **universal** quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.

The order of nested **existential** quantifiers in a statement without other quantifiers can be changed without changing the meaning of the quantified statement.

Find truth values

- **Assume P(x,y) is (x + y = 10) domain: real numbers**
  - **$\forall x \exists y P(x,y)$**
  - **$\exists y \forall x P(x,y)$**

Translating Nested Quantifiers into English

$\forall x\, (C(x) \lor \exists y\, (C(y) \land F(x,y)))$

C(x): x has a computer.

F(x,y): x and y are friends

Domain of x and y: all students

Examples

Example: Use quantifiers to express the statement "There is a woman who has taken a flight on every airline in the world."

Solution:

1. Let $P(w,f)$ be "w has taken f " and $Q(f,a)$ be "f is a flight on a ."
2. The domain of w is all women, the domain of f is all flights, and the domain of a is all airlines.
3. Then the statement can be expressed as:

$$\exists w \, \forall a \, \exists f \, (P(w,f) \wedge Q(f,a))$$

Negating Nested Quantifiers

Example 1: "There is a woman who has taken a flight on every airline in the world."

$$\exists w \forall a \exists f\ (P(w,f) \wedge Q(f,a))$$

Part 1: Use quantifiers to express the statement that "There does not exist a woman who has taken a flight on every airline in the world."

Solution: $\neg \exists w \forall a \exists f\ (P(w,f) \wedge Q(f,a))$

Part 2: Now use De Morgan's Laws to move the negation as far inwards as possible.

Solution:

1. $\neg \exists w \forall a \exists f\ (P(w,f) \wedge Q(f,a))$
2. $\forall w \neg \forall a \exists f\ (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
3. $\forall w \exists a \neg \exists f\ (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\forall$
4. $\forall w \exists a \forall f \neg (P(w,f) \wedge Q(f,a))$ by De Morgan's for $\exists$
5. $\forall w \exists a \forall f (\neg P(w,f) \vee \neg Q(f,a))$ by De Morgan's for $\wedge$.

Part 3: Can you translate the result back into English?

Solution:

"For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline"

$\neg \exists y(\forall x \exists z T(x, y, z) \vee \exists x \forall z U(x, y, z))$: