Dynmac Part 1

(1) Let T be a contraction, i.e. $d(T(x),T(y)) \leq \rho d(x,y)$ for $\rho < 1$

Consider $T^n x, T^m x$ for $n > m$

Then
$$d(T^n x, T^m x) = \rho^m d(x, T^{n-m} x) \to 0 \text{ if } d(x, T^{n-m} x) \text{ is bounded}$$

We have that $d(x, T^{n-m} x)$ is bounded (for high values of n). We have (let $h > 0$)

$$d(x, T^h x) \leq \sum_{l=0}^{h-1} d(T^l x, T^{l+1} x)$$
$$= \sum_{l=0}^{h-1} \rho^l d(x, T(x))$$
$$\leq \frac{1}{1-\rho} d(x, T(x))$$

So the sequence $d(x, T^{n-m})$ is bounded.

(2) We have the map and $Y_t = z_t K_t^\alpha$
$$T(V(x)) = \max_y u(x,y) + \beta V(y)$$

Let $u(x,y) = u(z_t x^\alpha - y)$. We also that $u(x) = \log x$

We show that for $V(k,z) = A + {}^{B}\ln K + C \ln z$ the structure is preserved under the transformation

$$TV(k,z) = \max_{k' \leq z_t k^\alpha} \log(z_t k^\alpha - k') + \beta E[V(k', z') | z]$$

Plugging in $V(k,z) = A + \log k + c \ln z$ yields

$$\max_{k' \leq z_t k^\alpha} \log(z_t k^\alpha - k') + \beta E[A + B \log k + c \ln z' | z]$$
$$= \beta A + C\mu(1-\rho)\beta + C\underline{\rho \log z}\beta + \max_{k' \leq z_t k^\alpha} \log(z_t k^\alpha - k') + \beta B \log k$$

Note incorrect specification in Problem set

Note that $\log(z_t h^\alpha - h') + \beta B \log h'$

is maximized at $h' = \dfrac{B\beta z_t h^\alpha}{1+\beta B} \leq z_t h^\alpha$ as $\beta B > 0$

Hence $\max\limits_{h'} \log(z_t h^\alpha - h') + \beta B \log(h')$

$$= \log\left(1 - \frac{\beta B}{1+\beta B}\right) + \log(z_t h^\alpha) + \beta B \log \frac{\beta B}{1+\beta B}$$

$$+ \log z_t + \alpha \log h$$

So we get that

$$TV(h,z) = \beta A + C_\mu (1-\rho)\beta + \beta C \rho \log z + \log\left(1 - \frac{\beta B}{1+\beta B}\right)$$

$$+ \log z + \alpha \log h + \beta B \log \frac{\beta B}{1+\beta B} + \log z$$

$$+ \alpha \log h$$

Which is of the form $A + B \ln k + C \ln z$ with $B > 0$