A toolkit for solving overlapping generations models with family-linked bequest and intergenerational skill transmission

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June 24, 2021

Abstract

Some papers incorporate family-linked bequest and intergenerational transmission of skills in life-cycle models to match empirical wealth distributions. However, to the best of our knowledge, none of these papers offer a comprehensive guide on their solution methods. Thus, the contribution of this paper is to develop a detailed toolkit on how to solve, simulate, and estimate an overlapping generations model with family-linked accidental and voluntary bequest, intergenerational transmission of skills, persistent idiosyncratic income risk, permanent income heterogeneity, and retirement. This model reasonably matches empirical measures of wealth inequality. Extending our model by a public sector would constitute a suitable framework to study the effects of policy reforms on wealth inequality.

JEL: E21, J14, C63

KEYWORDS: Wealth inequality, family-linked bequest, computational techniques

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1 Introduction

A body of literature uses overlapping generations (OLG) models to study the determinants and effects of income and wealth inequality. A subset of this literature explores the significance of bequest in generating a heavily right-skewed wealth distribution, see De Nardi and Fella (2017) for a review. The majority for OLG models taking into account bequest, assume that leftover savings are distributed evenly across the entire population (e.g. Platzer and Peruffo (2020)). Yet other papers introduce bequest based on skill type. For instance, Straub (2019) builds a model with intergenerational skill transmission where children receive bequest based on the expected skill type of their parents. Both methods are reductionist, as there is no direct family link. One consequence of the missing family link is that the wealth of parents and the random timing of bequest does not influence the choices and welfare of children. A paper formally linking parents to their children in terms of bequest and skill is De Nardi and Yang (2014). Their proposed mechanisms for bequest and intergenerational skill transmission are conceptually straightforward. However, they do not offer a comprehensible computational guide on how to solve and simulate the model. Thus, we develop and provide a toolkit allowing scholars and students to implement linked bequest and intergenerational transmission of skill in OLG models more easily.[1] To this end, we build a model with family-linked accidental and voluntary bequest, intergenerational transmission of skill, persistent idiosyncratic income risk, permanent income heterogeneity, and retirement. With this outline, the paper proceeds as follows. First, Section 2 describes the model, its underlying assumptions, and how to solve it. Next, Section 3 discusses the calibration of the exogenous model parameters. Section 4 describes the simulation procedure. Section 5 explains the estimation strategy. Section 6 illustrates some simulation results for the estimated and calibrated parameters. Analytical and computational details are found in the appendices.

[1] The code is available on Github.

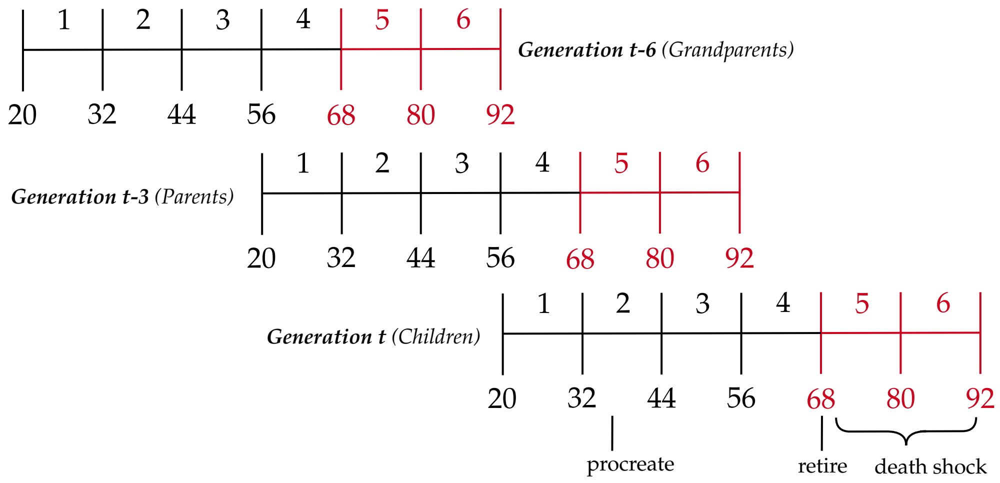


Figure 1: Demographic structure

2 Model

The model is a discrete-time, overlapping generations model with exogenous prices. At each date $t = 1, 2, ..., \infty$ a generation of heterogeneous agents is born. Agents differ by being born with different permanent skill levels that are imperfectly transmitted from their parents. Individuals live for up to T periods, but face mortality risk only after entering retirement at age $\mathcal{R}$. When parents die, any unclaimed savings are transferred to their biological children as bequests. Importantly, the model incorporates both accidental and voluntary bequests. Before retirement, individuals work and face a persistent productivity shock process. Combined with the permanent skill level, this determines individual labour income, out of which individuals either consume or save. The following section provides technical details for each of these model features.[2]

2.1 Demographics, skill transmission and labour productivity

To keep our model tractable, we impose a demographic structure with only two simultaneous generations of a family. Introducing more simultaneous generations would increase the state space. For instance, with three simultaneous generations the choices and welfare of children not only depend on the wealth of their parents, but also that of their grandparents. To make the model easier to solve, we make some additional assumptions: (1) Children cannot die before their parents, (2) individuals only face mortality risk in retirement, and finally (3) when adult children

[2] See an overview of the model features in Appendix C.

enter the economy, their parents enter retirement and face no more income risk. Agents enter the economy and the workforce at age 20 ($t = 1$) and live for $T = 6$ periods. At age 36 agents' children are born. Agents enter retirement at age 68 ($t = 5$). When entering retirement, and in every period thereafter, agents face mortality risk. Formally, we denote the likelihood of surviving to period t conditional on having survived to $t - 1$ as ψ_t. Figure 1 summarizes the demographic structure.

The productivity $y_{i,t}$ of individual i at time t is given by

$$y_{i,t} = h_i \cdot l_t \cdot e^{z_{i,t}},$$

which has three components. First, l_t is a deterministic term which changes with age and is chosen to reflect the empirically observed hump-shaped labour income profiles over the life cycle. Second, $z_{i,t}$ is a persistent productivity shock, which follows an autoregressive process:

$$z_{i,t} = \alpha z_{i,t-1} + \epsilon_{i,t},$$

where $\epsilon_{i,t} \sim \mathcal{N}\left(0, \sigma_{\epsilon}^2\right)$. In the simulation, we draw initial levels of y from the ergodic distribution.[3] Third, h_i is a permanent productivity term that agents inherit from their parents via a Markov-process with transition matrix, $P_{parents|children}$. In the model, every permanent productivity type faces idiosyncratic income risk, but the expected lifetime income is permanently greater for higher h_i agents. For comparison, De Nardi and Yang (2014) do not model permanent income heterogeneity, but allow for intergenerational transmission of productivity governed by an auxiliary autoregressive process of order one.

2.2 Preferences

Preferences are time separable, and instantaneous utilities are discounted exponentially with a discount factor, β. The instantaneous utility function of consumption, c, is defined as:

$$u\left(c\right) = \frac{c^{1-\rho}}{1-\rho}$$

where $\rho > 0$ is a risk-aversion parameter.[4] To model a warm-glow bequest motive, we assume that deceased individuals derive utility from leftover savings, a, passed

[3] See more details on the ergodic distribution in appendix D.

[4] Due to the bequest motive, this cannot generally be interpreted as the inverse elasticity of intertemporal substitution.

on to children in the form of bequest. The instantaneous utility function from bequest is given by:

$$\tilde{u}(a) = \kappa \frac{(a + \underline{a})^{1-\gamma}}{1-\gamma}$$

where γ is again a risk-aversion parameter. Here, κ measures the strength of the bequest motive, while $\underline{a}$ is a shift parameter that makes warm-glow bequest a luxury good. Importantly, this setup could deviate from homothetic consumption-saving behaviour in several ways. First, if $\gamma < \rho$, the expansion path of consumption and savings/bequest becomes non-linear even when there is no warm-glow, i.e. $\kappa = 0$. In this case, the marginal propensity to save increases in cash-on-hand, leading to more inequality than in the homothetic case. Second, if $\underline{a} > 0$ and $\kappa > 0$, the expansion path is non-differentiable and non-homothetic even when $\gamma = \rho$. In that case, low-income individuals save for consumption-smoothing purposes, but not out of a bequest motive. On the contrary, people with sizeable resources are able to attain levels of consumption where the marginal utility from leaving bequest is larger than that from consumption, and therefore, begin to feel the warm-glow.

2.3 The household's recursive problem

The Bellman equation of the agents' problem reads as follows:

$$V_{t,h}\left(m_t, z_t, \phi_t, m_t^p\right) = \max_{c_t} \left\{u\left(c_t\right) + \beta\psi_{t+1}\mathbb{E}\left[V_{t+1,h}\left(m_{t+1}, z_{t+1}, \phi_{t+1}, m_{t+1}^p\right)\right] + \left(1-\psi_{t+1}\right)\tilde{u}\left(a_t\right)\right\}$$

$$\begin{aligned}
&s.t. \\
m_{t+1} &= whl_{t+1}e^{z_{t+1}} + R\left(m_t - c_t\right) + b_{t+1} \\
m_{t+1}^p &= R\left(m_t^p - c_t^p\right) \\
b_{t+1} &= \begin{cases} R\left[m_t^p - c_t^p\left(m_t^p\right)\right] & \text{if } \phi_{t+1} = 1, \phi_t = 0 \\ 0 & \text{else} \end{cases}
\end{aligned}$$

Here, $a > 0$ is savings, m is cash-on-hand, c is consumption, b is bequest, R is the gross interest rate, and w is the wage. The parameter ϕ indicates whether or not the individual has already received bequest. Superscript p indicates parents' variables. In retirement, households no longer face any income shocks.[5] Furthermore, because of the chosen demographic structure, retired agents have already received bequest from their dead parents. Thus, the Bellmann equation of these "deterministic" peri-

[5] In the code, this is handled by setting $l_t = 0 \quad \forall t > \mathcal{R}$.

ods of life reduces to:

$$V_t^D(m_t) = \max_{c_t} \left\{ u(c_t) + \beta \psi_{t+1} V_{t+1}^D(m_{t+1}) + (1-\psi_{t+1}) \tilde{u}(a_t) \right\}$$
$$s.t.$$
$$m_{t+1} = R(m_t - c_t)$$

The entire model can be solved by either backward induction using value functions or by the endogenous grid method. We apply the latter to improve speed and accuracy. For a detailed description of the solution and simulation algorithms, see the computational Appendix A.

3 Calibration

Before we can solve and simulate the model, we must take a stance on the calibration of model parameters. In this context, Table 1 lists the exogenously determined parameter values and their source of origin. The remaining two paramenters κ and $\underline{a}$ are determined by estimation as described in Section 5.

Table 1: Calibration of exogenous parameters

Parameter	Description	Value	Origin
Preferences			
β	Discount factor	0.85	
ρ	Risk aversion para. of consumption	2	Literature standard
γ	Risk aversion para. of bequest	2	Literature standard
Demographics			
ψ^j	Survival prob. betw. age j and $j+1$	$\{1,1,1,1,0.83,0.58,0\}$	SSA life tables
Labour productivity			
$\{m_i\}$	Population shares by skill type	$\{0.2,0.2,0.2,0.2,0.2\}$	Quintiles
$\{s_i\}$	Labor income shares $\{s_i\}$	$\{0.07,0.11,0.14,0.20,0.48\}$	CBO 2016
$\mathcal{H}=\{h_i\}$	Skill of agent i	$\{0.38,0.53,0.72,1.01,2.36\}$	Matches labour inc. shares
$\{l_t\}$	Age-depend. labour prod. pattern $\{l_t\}$	$\{0.74,1.12,1.18,0.96,0,0\}$	Survey of Cons. Finance
$P_{parents\|children}$	Skill transmission matrix	See Table 2	Chetty et al. (2014)
α	AR(1) coef. of prod. process	0.85	De Nardi and Yang (2014)
σ_ϵ^2	Variance of productivity shock	0.3	De Nardi and Yang (2014)
Prices			
w	Prices of efficient labour unit	1	Normalization
R	Gross interest rate	1.18	$1/\beta$
Timing			
T	Maximum length of life	6	
$\mathcal{R}$	Retire. age / years in the labour force	4	
ch	Parent age when child enters work force	4	

Notes: SSA is the abreviation for The United States Social Security Administration. CBO is the abrevation for Congressional Budget Office.

With regard to skill transmission, the inheritance probability of parental skill is governed by an empirical income rank transmission matrix by Chetty et al. (2014).[6] In our model, we operate with 5 uniformly distributed skill types, corresponding to quintiles of the income or skill distribution. Parents transmit their skill through the 5-state Markov process with a doubly stochastic transition matrix $P^{h}_{parents|children}$ in Table 2.

Table 2: National quintile transition matrix $P_{parents|children}$

Child quintile	Parent quintile 1	2	3	4	5
1	0.337	0.242	0.178	0.134	0.109
2	0.280	0.242	0.198	0.160	0.120
3	0.184	0.217	0.221	0.208	0.170
4	0.124	0.176	0.22	0.244	0.236
5	0.075	0.123	0.183	0.254	0.365

Notes: This table is directly taken form Chetty et al. (2014). Each cell reports the share of children with family income in the quintile given by the row conditional on having parents with family income in the quintile given by the column for the 9,867,736 children in the core sample (1980-1982 birth cohorts).

4 Simulation

For given parameters, we simulate $N = 100000$ families until the distribution of wealth converges to stationarity. To asses convergence, we inspect whether the first and second moments have both converged to stationary processes.[7] A similar approach is found in De Nardi and Yang (2014). To initialize this experiment, we assume that $m_t^{i,p} = 0\ \forall$i,t. That is, the initial generation of parents have no wealth to pass on. Next, the simulation follows the algorithm outlined in Appendix B. In less formal terms, for each iteration, we draw bequest shocks, that is deaths, for parents, ϕ_t^i , and productivity shocks, ϵ_t^i, for children. From the income and bequest shocks, we compute productivity, total income, and cash-on-hand for children. Next, we identify optimal consumption by means of three-dimensional interpolation and iterate forward in time cash-on-hand, m_{t+1}^i, by drawing another income and bequest shock. The process repeats until the hypothetical end of life, $t = T$. After the final period, we let children become parents, by writing forward the wealth distribution, $m_t^{i,p} = m_t^i$. The resulting distribution of parents' cash-on-hand serves as the distribution of potential bequest for the next generation. We continue to roll over generations of parents and children until the distribution of wealth and, thus, po-

[6] Straub (2019) uses a similar approach to model and calibrate permanent income heterogeneity but with different skill groups and shares.

[7] A meaningful stopping rule would depend on parameters.

tential bequest converges. As shown in Figure 3 in Appendix B for a roll-over of 10 generations, both moments converge to stationarity after just a few generations.

The simulation method illustrated here shows that our partial equilibrium model can easily be extended to a computable general equilibrium model. Such an extension requires endogenous factor prices, R, and, w, determined by firm profit maximization given some production function. The general equilibrium simulation algorithm would proceed as follows: (1) Guess factor prices consistent with the production function. (2) Solve the model given these factor prices using our solution method. (3) Simulate the whole population using our partial-equilibrium simulation method and calculate aggregate capital determined by aggregate savings in the economy. (4) Calculate the factor prices associated with this capital level. (5) Check whether these new factor prices are equal to the initial guess. (6) Continue the algorithm until convergence of factor prices.

5 Estimation

We estimate the model over two free-floating parameters, κ and $\underline{a}$, to match empirical moments. Specifically, we estimate parameters to match the empirical Gini coefficient for wealth at retirement, and the share of wealth owned by the 60th-80th percentile of the general population. As shown in Table 3, we obtain both moments from the PSID. Whereas the wealth Gini at retirement is 0.62, the wealth share of the 60th-80th percentile of the entire population is 16.3%. In practice, we minimize an objective function:

$$Loss\left(\theta\right) = W^T\left(M_0 - M\left(\theta\right)\right)W : \ \mathbb{R}^2 \mapsto \ \mathbb{R}$$

over a 2×1 vector of estimated parameters, θ, to match a 2×1 vector of model moments, $M\left(\theta\right)$, to a vector of desired moments, M_0, given some weighing scheme, W.

Table 3: Estimation of parameters

Moments to be matched	Data	Model
Wealth Gini at retirement	0.62	0.61
Wealth share of 60th-80th Percentile	0.163	0.162
Parameter	**Description**	**Value**
κ	Strength of bequest motive	4.68
$\underline{a}$	Bequest shifter	12.49

Notes: For more details on the PSID moments, we refer to De Nardi and Yang (2014).

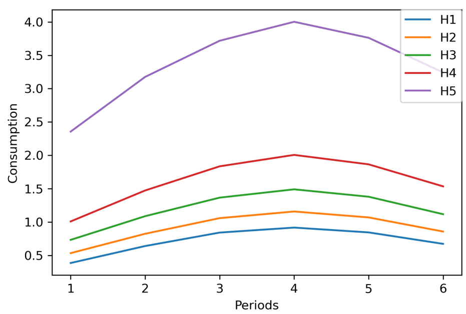
Using this method, we match the desired moments well for the solution: $\kappa = 4.68$ and $\underline{a} = 12.49$. It is worth to notice, this is not far from comparable parameter values found in Platzer and Peruffo (2020).

6 Results

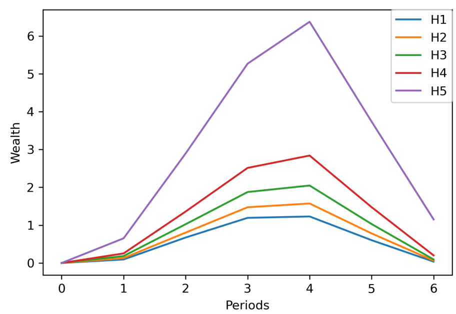
Based on the simulation algorithm shown in Appendix B and outlined in Section 4, we are able to produce results. As it is infeasible to go over all facets of the results, we focus on the variables and distributions in Figure 2. First, we examine the average consumption path for every skill type as depicted in Panel (a). Due to the precautionary savings motive before retirement, consumption increases over time, only to decline because of mortality risk and the desire to pass on bequest. Next, we depict the average wealth paths for the different skill groups in Panel (b). This reveals a compelling life-cycle pattern, and shows how bequest is, in fact, a luxury good. Third, we consider the distribution of wealth at retirement as shown in Panel (c). The wealth distribution is significantly right-skewed, and somewhat resembles a power law.

Figure 2: Consumption and wealth

(a) Consumption path



(b) Wealth path



(c) Wealth distribution at retirement

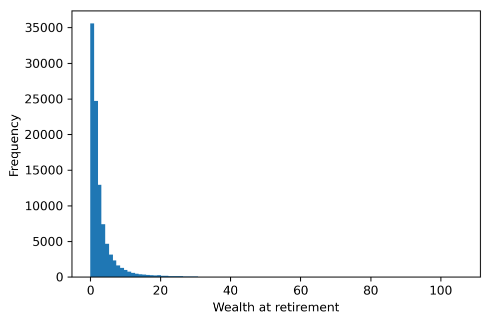


Table 4: Wealth at retirement

	Gini	Percentile (%)						
		0-40	40-60	60-80	80-90	90-95	95-99	99-100
PSID	0.62	6.2	10.2	18.8	17.0	13.8	18.3	15.7
Full model	0.61	7.7	9.8	17.8	16.2	13.3	20.4	14.8
No bequest motive	0.57	8.8	11.0	19.6	17.2	13.5	18.4	11.6
Random skill transmission	0.60	7.9	9.9	17.9	16.2	13.3	20.2	14.5

In Table 4, we compare the wealth distribution at the beginning of retirement to the Panel Study of Income Dynamics (PSID). Looking at this, the model seems to be doing well across most of the wealth distribution in terms of wealth shares as well as overall inequality (Gini). Only at the far end of the distribution, the model has difficulties matching wealth shares. This is a common challenge within the literature (see e.g. De Nardi and Fella (2017); Benhabib and Bisin (2018)).

Accordingly, a number of papers have sought to modify models to better match top wealth inequality. Whereas some papers, including Guvenen et al. (2015, 2018) and Castaneda et al. (2003), modify the shock processes to include additional skewness, others look to more systematic drivers of fat-tailed income distributions such as wealth- and/or type-dependent returns on financial investment (see e.g. Fagereng et al. (2019); Gabaix et al. (2016)).

To asses how the intergenerational transmission of skill and a linked bequest mechanism each affects the stationary wealth distribution, we conduct two separate sensitivity studies. Specifically, we simulate two watered-down versions of the model; one with no bequest motive, $\kappa = 0$, and one with uncorrelated skill transmission, $P^{h,j}_{parents|children} = 0.2 \; \forall \; (h,j) \in \mathcal{H} \times \mathcal{H}$. We then compare the results from the two special cases to the full model and to actual PSID data as listed in Table 4. Based on this, it is clear that both mechanism matter for the wealth distribution. In the first experiment, the bequest motive is seen to increase overall inequality, wealth concentration at the top, and to make the wealth distribution more right-skewed in general. Correlated transmission of skills will have qualitatively similar implications, but at a smaller scale.

7 Conclusion

Among others, De Nardi and Yang (2014) show that linked bequest and the intergenerational transmission of skills are important to understand wealth inequality.

Models incorporating these features are complex and difficult to solve. The contribution of this paper is to provide a detailed guide on how to solve, simulate, and estimate such models. Specifically, we develop an overlapping generations model with family-linked accidental and voluntary bequest, intergenerational transmission of skills, persistent idiosyncratic income risk, permanent income heterogeneity, and retirement. Besides offering a detailed analytical and computational description, we find that the model is suited for explaining empirical wealth distributions and their inequality measures. An interesting extension of the model would be to incorporate a public sector to study the effects of policy reforms on wealth inequality. We leave this for future work.

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A Computational appendix

A.1 Solving the final period T

General description of the solution method

In the final period households die for sure and the Bellman equations reduces to:

$$
\begin{aligned}
V_T^D\left(m_T\right) &= \max_{c_T}\left\{u\left(c_T\right)+\tilde{u}\left(a_T\right)\right\} \\
&s.t. \\
a_T &= m_T - c_T
\end{aligned}
$$

FOC:

$$
\begin{aligned}
u'\left(c_T\right) &= \tilde{u}'\left(a_T\right) \\
c_t^{-\rho} &= \kappa\left(a_t+\underline{a}\right)^{-\gamma}
\end{aligned}
$$

With $a_t = m_t - c_t$ the following loss function $L_T\left(c,m\right)$ can be minimized by a solver solving for optimal consumption c given m:

$$
L_T\left(c,m\right) = \left[c_t^{-\rho} - \kappa\left(m-c+\underline{a}\right)^{-\gamma}\right]^2 \tag{1}
$$

In the homothetic case of $\rho=\gamma$ we have a closed-form solution for c (we do not use it in the code):

$$
\begin{aligned}
c &= \kappa^{-1/\rho}\left(m-c+\underline{a}\right) \\
c[1+\kappa^{-1/\rho}] &= \kappa^{-1/\rho}\left(m+\underline{a}\right) \\
c &= \frac{\kappa^{-1/\rho}}{1+\kappa^{-1/\rho}}\left(m+\underline{a}\right)
\end{aligned}
$$

Specific description of the solution method

Algorithm 1: Solving the final period $T = 6$

Input : $\mathcal{G}_T^m = \{m_1 = 0, m_2, \ldots, m_\#\}$ (exogenous grid of cash-on-hand m)

Output: $C_T^\star$ (optimal choice given m)

for $i_m = 1$ **to** $\#_m$ **do**

  $m = \mathcal{G}_T^m[i_m]$;

  $C_T^\star[i_m] =$ solver.minimize $L_T(c, m)$

end

A.2 Solving the rest of the deterministic periods of life

General description of the solution method

The Bellman equation in the deterministic periods of life reads as follows:

$$\begin{aligned} V_t^D(m_t) &= \max_{c_t} \left\{ u(c_t) + \beta \psi_{t+1} V_{t+1}^D(m_{t+1}) + (1-\psi_{t+1}) \tilde{u}(a_t) \right\} \\ & s.t. \\ m_{t+1} &= R(m_t - c_t) \end{aligned}$$

Rewrite in terms of savings leads to:

$$V_t^D(m_t) = \max_{a_t} \left\{ u(m_t - a_t) + \beta \psi_{t+1} V_{t+1}^D(Ra_t) + (1-\psi_{t+1}) \tilde{u}(a_t) \right\}$$

Substituting in V_{t+1}^D gives:

$$V_t^D(m_t) = \max_{a_t} \left\{ u(m_t - a_t) + \beta \psi_{t+1} \left\{ u(c_{t+1}(m_{t+1})) + \beta \psi_{t+2} V_{t+2}^D(Ra_{t+1}) + (1-\psi_{t+2}) \tilde{u}(a_t) \right\} + (1-\psi_{t+1}) \tilde{u}(a_t) \right\}$$

FOC:

$$\begin{aligned} u'(c_t) &= \beta R \cdot \psi_{t+1} u'\left(c_{t+1}^*(m_{t+1})\right) + (1-\psi_{t+1}) \tilde{u}'(m_t - c_t) \\ m_{t+1} &= R(m_t - c_t) \end{aligned}$$

which is

$$
\begin{aligned}
(c_t)^{-\rho} &= \beta R \cdot \psi_{t+1} \left(c^*_{t+1}\left(R\left(m_t - c_t\right)\right)\right)^{-\rho} + \left(1-\psi_{t+1}\right)\kappa\left(m_t - c_t + \underline{a}\right)^{-\gamma} \\
m_{t+1} &= R\left(m_t - c_t\right)
\end{aligned}
$$

We use the endogenous grid method to solve for consumption in these determinist periods. Thus, for every a_t in the grid, $\mathcal{G}_a = [0; a_{max}]$ we solve for c:

$$
c_t = \left[\beta R \cdot \psi_{t+1}\left(c^*_{t+1}\left(R a_t\right)\right)^{-\rho} + \left(1-\psi_{t+1}\right)\kappa\left(a_t + \underline{a}\right)^{-\gamma}\right]^{-\frac{1}{\rho}}
$$

With optimal consumption c^*_t in hand we can find the corresponding m_t using $a_t = m_t - c_t \Leftrightarrow m_t = a_t + c_t$.

Specific description of the solution method

Algorithm 2: Solving the remaining deterministic periods (EGM)

Input : $C^\star_T$, (optimal choice in T given m),
$\mathcal{G}^m_T = \{m_1 = 0, m_2, \ldots, m_\#\}$, (exogenous grid of cash-on-hand m)
$\mathcal{G}^a_D = \{a_1 = 0, a_2, \ldots, a_\#\}$ (exogenous grids of savings a)
Output: $\mathcal{G}^m_{T-1}$,$\mathcal{G}^m_{T-2}$(endo. grids),$C^\star_{T-1}$,$C^\star_{T-2}$
for $t = T-1$ **to** $T-2$ **do**
  for $i_a = 1$ **to** $\#_a$ **do**
    $a = \mathcal{G}^a_D[i_a]$;
    $m_{t+1} = aR$; (cash-on-hand tomorrow)
    $c_{t+1} = \text{interp}(\mathcal{G}^m_T, C^\star_T, m_{t+1})$;
    $C^\star_t\left[i_a\right] = \left[\beta R \cdot \psi_{t+1}\left(c^*_{t+1}\left(R a_t\right)\right)^{-\rho} + \left(1-\psi_{t+1}\right)\kappa\left(a + \underline{a}\right)^{-\gamma}\right]^{-\frac{1}{\rho}}$; (optimal choice)
    $\mathcal{G}^m_t\left[i_a\right] = a + C^\star_t\left[i_a\right]$ (endogenous grid)
  end
end

A.3 Solving the period with income risk

General description of the solution method

We know that people survive for sure in periods with income risk. Thus, the Bellman equation reduces to:

$$
\begin{aligned}
V_{t,h}\left(m_t, z_t, \phi_t, m_t^p\right) &= \max_{c_t}\left\{u\left(c_t\right) + \beta \mathbb{E}\left[V_{t+1,h}\left(m_{t+1}, z_{t+1}, \phi_{t+1}, m_{t+1}^p\right)\right]\right\} \\
&s.t. \\
m_{t+1} &= whl_{t+1}e^{z_{t+1}} + R\left(m_t - c_t\right) + b_{t+1} \\
m_{t+1}^p &= R\left(m_t^p - c_t^p\right) \\
b_{t+1} &= \begin{cases} R\left[m_t^p - c_t^p(m_t^p)\right] & \text{if } \phi_{t+1} = 1, \phi_t = 0 \\ 0 & \text{else} \end{cases}
\end{aligned}
$$

To solve this problem we use the endogenous grid method using backwards induction. Thus, for every period t, we loop over the skill levels $\mathcal{H}$, the grid of productivity $\mathcal{G}_z$, cash-on-hands of parents m^p and over the grid of savings $\mathcal{G}_a$ and then apply the following steps:

1. Find consumption using Euler equation

$$
c_t(a_t, z_t, \phi_t, m^p) = \mathbb{E}_t\left[\psi_t \beta R\left(C_{t+1}^*(m_{t+1}, \phi_{t+1}, m_{t+1}^p)\right)^{-\rho}\right]^{-\frac{1}{\rho}} \tag{2}
$$

$$
with \tag{3}
$$

$$
m_{t+1} = whl_t e^{z_t} + R\left(m_t - c_t\right) + b_{t+1} \tag{4}
$$

$$
m_{t+1}^p = R\left(m_t^p - c_t^p\right) \tag{5}
$$

$$
b_{t+1} = \begin{cases} R\left[m_t^p - c_t^p(m_t^p)\right] & \text{if } \phi_{t+1} = 1, \phi_t = 0 \\ 0 & \text{else} \end{cases} \tag{6}
$$

2. Find endogenous state: $a_i = m_i - c_i \Leftrightarrow m_i = a_i + c_i$

Specific description of the solution method

Algorithm 3: Solving the periods with income risk (EGM)

Input : $C^\star_T, C^\star_{T-1}, C^\star_{T-2}$, (optimal choice in det. periods given m)
$\mathcal{G}^m_T, \mathcal{G}^m_{T-1}\mathcal{G}^m_{T-2}$, (grid of cash-on-hand m in det. periods)
$\mathcal{G}^z = \{z_1 = -2.576\sigma_z, z_2, \ldots, z_\# = 2.576\sigma_z\}$, (exogenous grid of productivity shock z)
$[x, \omega]$ (Vectors of Gauss-Hermite $S = 7$ nodes and rescaled weights)

Output : $\{\mathcal{G}^m_{h,T-3}[z, \phi, m^p], \mathcal{G}^m_{h,T-4}[\cdot], \mathcal{G}^m_{h,T-5}[\cdot]\}^{[h_1, h_2, \ldots, h_\#]}$, (endo. grids given z, ϕ and m^p)
$\{C^\star_{h,T-3}[m, z, \phi, m^p], C^\star_{h,T-4}[\cdot], C^\star_{h,T-5}[\cdot]\}^{[h_1, h_2, \ldots, h_\#]}$ (optimal choice given z, ϕ, m^p)

for $i_h = 1$ **to** $\#_h$ **do** // Loop over skills $\mathcal{H} = \{h_1, h_2, \ldots, h_{\#=5}\}$
  $h = \mathcal{H}[i_h]$;
  for $i_z = 1$ **to** $\#_z$ **do** // Loop over last period productivity level
    $z = \mathcal{G}^z[i_z]$;
    $e^{z_{t+1}} = e^{(\sigma_z \sqrt{2}x + \alpha z)}$; $(1 \times S)$-vector of prod. shocks
    for $t = T - ch + 1$ **to** 1 **do** // Loop backwards over time periods
      $\mathcal{G}^a_{temp} = \text{linspace}(\min = 0, \max = \text{increasing in } z_t \text{ and } h, \#_a)$; Temporary grid of savings a
      for $i_{mp} = 1$ **to** $\#_{mp}$ **do** // Loop over parents cash-on-hand
        $m^p_t = \mathcal{G}^m_{t+ch-1}$; parents' wealth with $\mathcal{G}^{mp}_t = \mathcal{G}^m_t \forall t > \mathcal{R} - 1$
        $c^p = C^\star_{t+ch-1}[i_{mp}]$;
        $m^p_{t+1} = b_{t+1} = R(m^p_t - c^p)$; parents' cash-on-hand tomorrow = bequest b_{t+1}
        if $\phi = 0$ **then** // Parents are alive $\phi = 0$
          for $i_a = 1$ **to** $\#_a$ **do** // Loop over children's a in $\mathcal{G}^a_t$
            $a = \mathcal{G}^a_{temp}[i_a]$;
            $m^{b>0}_{t+1} = whl_{t+1}e^{z_{t+1}} + Ra + b_{t+1}$; $(1 \times S)$-vector
            $m^{b=0}_{t+1} = whl_{t+1}e^{z_{t+1}} + Ra$; $(1 \times S)$-vector
            if $t = T - ch$ **then** // parents will die next period
              $c_{t+1} = \text{interp}(\mathcal{G}^m_{t+1}, C^\star_{t+1}, m^{b>0}_{t+1})$; $(1 \times S)$-vector
              $C^\star_{h,t}\left[i_z, i_a, \phi = 0, i_{mp}\right] = \psi_t \beta R \left(\omega * (c_{t+1})^{-\rho}\right)^{-\frac{1}{\rho}}$; Scalar
            else // parents may not die next period
              $c^{b=0}_{t+1} = \text{interp}(\mathcal{G}^m_{h,t+1}[i_z, \phi, i_{mp}], C^\star_{h,t+1}[i_z, \phi, i_{mp}], m^{b=0}_{t+1})$; $(1 \times S)$-vector
              $c^{b>0}_{t+1} = \text{interp}(\mathcal{G}^m_{h,t+1}[i_z, \phi, i_{mp}], C^\star_{h,t+1}[i_z, \phi, i_{mp}], m^{b>0}_{t+1})$; $(1 \times S)$-vector
              $C^\star_{h,t}\left[i_z, i_a, \phi = 0, i_{mp}\right] = \psi_t \beta R \left(\psi^p_t \omega * (c^{b=0}_{t+1})^{-\rho} + [1 - \psi^p_t]\omega * (c^{b=0}_{t+1})^{-\rho}\right)^{-\frac{1}{\rho}}$;
              Scalar
            end
          end
        else // Parents are dead $\phi = 1$ and bequest has been received
          if $t = T - ch$ **then** // Use grids from determinist periods
            $c^{b=0}_{t+1} = \text{interp}(\mathcal{G}^m_{t+1}, C^\star_{t+1}, m^{b=0}_{t+1})$; $(1 \times S)$-vector
          else // Use grids form stochastic periods
            $c^{b=0}_{t+1} = \text{interp}(\mathcal{G}^m_{h,t+1}[i_z, \phi, i_{mp}], C^\star_{h,t+1}[i_z, \phi, i_{mp}], m^{b=0}_{t+1})$; $(1 \times S)$-vector
          end
          $C^\star_{h,t}\left[i_z, i_a, \phi = 1, i_{mp}\right] = \psi_t \beta R \left(\omega * (c^{b=0}_{t+1})^{-\rho}\right)^{-\frac{1}{\rho}}$; Scalar
        end
        $\mathcal{G}^m_{h,t}\left[i_z, \phi, i_{mp}\right] = a + C^\star_{h,t}\left[i_z, i_a, \phi, i_{mp}\right]$ Endogenous grid
      end
    end
  end
end

B Simulation

Algorithm 4: Simulation

Input : $\{C^\star_T, C^\star_{T-1}, C^\star_{T-2}\mathcal{G}^m_T, \mathcal{G}^m_{T-1}\mathcal{G}^m_{T-2}\}$,(grids and optimal choice of deter. periods)
#simulated families N, Initial skill vector H_{child},
M_{guess}, (Guess of wealth distribution $(T+1, N)$-Matrix)
$\{\mathcal{G}^m_{h,T-3}[z,\phi,m^p], \mathcal{G}^m_{h,T-4}[\cdot], \mathcal{G}^m_{h,T-5}[\cdot]\}^{[h_1,h_2,..,h_\#]}$, (endo. grids given z, ϕ and m^p)
$\{C^\star_{h,T-3}[m,z,\phi,m^p], C^\star_{h,T-4}[\cdot], C^\star_{h,T-5}[\cdot]\}^{[h_1,h_2,..,h_\#]}$ (optimal choice given z, ϕ, m^p)

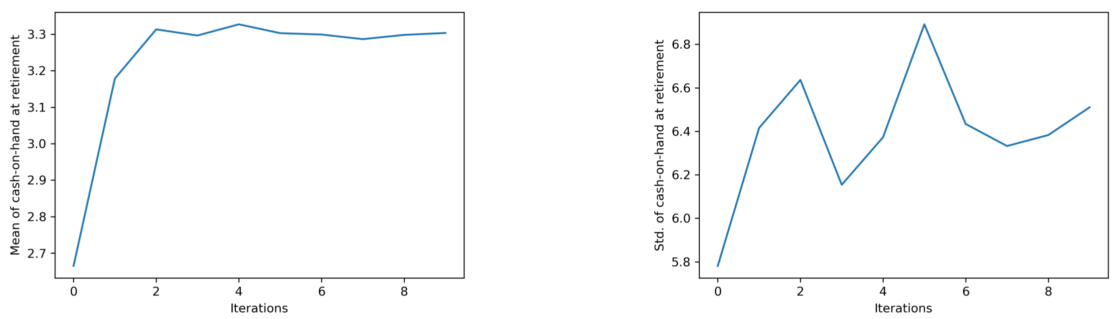
Output: Model simulation (Consumption $c_{i,t}$, and Cash-on-hand $m_{i,t}$ for all t, i)

for 1 **to** *20* **do** // `Iterate 20 times and verify convergence ex post`
- $H_p = H_{child}$; Children become parents
- H_{child} = MarkovChain($P_{parents|children}$); Simulate skill transmission
- $M_{justdead}, M_{dead}$ =sim_mortality(binominal, size = $[N, T]$, $p = 1-\psi$) ; $(N \times T)$ - Mortality matrice, $M_{justdead}$ equals 1 in period of parents' death, M_{dead} equals 1 in period of parents' death and thereafter
- sim$z[1, 1:N]$ = random.normal(0,σ_z,N); Initiate productivity parameter z from ergodic distr.
- sim$y[1, 1:N]$ = $l[1]H_{child} \cdot e^{simZ[0,1:N]}$; Initial total productivity y
- sim$m[1, 1:N]$ = $w \cdot$sim$y[1, 1:N]$; Intial cash-on-hand m
- **for** $t = 1$ **to** T **do**
  - sim$z[t+1, 1:N]$ = random.normal(mean = α sim$z[t-1, 1:N]$,std=σ_ϵ,size= N); Iterate z forward using draws from conditional distr.
  - sim$y[t+1, 1:N]$ = $l[t]H_{child} \cdot e^{simZ[t,1:N]}$; Iterate y forward
  - **for** $i_n = 1$ **to** N **do** // `Loop over parent-child relation`
    - $h = H_{child}[i_n]$;
    - $\phi_{i_n} = M_{dead}[i_n, t+ch-1]$; Look up if parents are dead or alive
    - **if** $t \leq T - ch$ **then** // `parents could still be alive next period`
      - $m^p = M_{guess}[i_n, t+ch-1]$; Parents' cash-on-hand
      - i_{mp} = binary_search(0, $\#_m$, $\mathcal{G}^m_t$, m^p); Find the corresponding grid point
      - i_z = binary_search(0, $\#_m$, $\mathcal{G}^z$,sim$z[t, i_n]$); Find the corresponding grid point
      - sim$c[t, i_n]$ = interp($\mathcal{G}^m_{h,t+1}[i_z, \phi_{i_n}, i_{mp}], C^\star_{h,t+1}[\cdot, z, \phi_{i_n}, m^p]$, sim$m[t, i_n]$); Children's consumption given m; this is a simplification; in the code we use 3D-interpolation
      - c^p = interp($\mathcal{G}^m_{t+1}, C^\star_{t+1}, m^p$); Parents' c given m^p
      - $b = M_{justdead}[i_n, t+ch]$; Bequest
    - **else** // `Parents are dead for sure`
      - sim$c[t, i_n]$ = interp($\mathcal{G}^m_{t+1}, C^\star_{t+1}$, sim$m[t, i_n]$); Children's consumption given m
      - $b = 0$; No bequest if parents have already died
    - **end**
    - sim$m[t+1, i_n] = w \cdot$ sim$y[t+1, i_n] + R($sim$m[t, i_n] -$ sim$c[t+1, i_n]) + b$; Iterate m forward
  - **end**
- **end**
- M_{guess} = sim$m$; Update guess

end

Figure 3: Convergence plots

(a) Mean of wealth at retirement (b) Std. of wealth at retirement



C Overview of model features

Model features	
Overlapping generations	✓
Persistent idiosyncratic income risk	✓
Permanent income inequality	✓
Transmission of skill	✓
Linked accidental bequest	✓
Linked voluntary luxury bequest	✓
Borrowing constraint	✓

D Details of the productivity shock process

Write the AR(1) as an MA(∞) using the lag-operator:

$$z_t = \alpha z_{t-1} + \epsilon_t = \frac{\epsilon_t}{1 - \alpha L} = \sum_{s=1}^{\infty} \alpha^s \epsilon_{t-s}$$

Hence, $\mathbb{E}[z_t]$ and $V(z_t)$ read:

$$\mathbb{E}[z_t] = \mathbb{E}\left[\sum_{s=1}^{\infty} \alpha^s \epsilon_{t-s}\right] = \sum_{s=1}^{\infty} \alpha^s \mathbb{E}[\epsilon_{t-s}] = 0$$

$$V(z_t) = \sum_{s=1}^{\infty} V(\alpha^s \epsilon_{t-s}) = \sum_{s=1}^{\infty} \alpha^{2s} V(\epsilon_{t-s}) = \sigma^2 \sum_{s=1}^{\infty} \alpha^{2s} = \frac{\sigma^2}{1 - \alpha^2}$$

Ergodic distribution of $z \sim N\left(0, \dfrac{\sigma^2}{1-\alpha^2}\right)$
Ergodic distribution of y $\sim \mathit{Lognormal}\left(0, \dfrac{\sigma^2}{1-\alpha^2}\right)$ with expected value

$$\mathbb{E}\left[y_t\right] = \mathbb{E}\left[e^{z_t}\right] = e^{\frac{\sigma^2}{2(1-\alpha^2)}}$$