Cellular Automata

Definition 1 (Cellular Automaton – CA). *A (finite) Cellular Automaton (CA) is defined by a six-tuple* $\mathcal{A} = \langle L, Q, \mathcal{N}, \delta, \varphi, q_0 \rangle$ *where:*

1. $L = \bigtimes_{\ell \in \mathbb{Z}_n} \mathbb{Z}_{m_\ell} \subset \mathbb{N}^n$*, with* $n \in \mathbb{N}$ *and* $\forall \ell \in \mathbb{Z}_n : m_\ell \in \mathbb{N}$*, is an* n*-dimensional (finite)* lattice of cells $\mathbf{c}_i = \langle i_k \rangle_{k \in \mathbb{Z}_n} \in L$*, with* $i = \sum_{k \in \mathbb{Z}_n} i_k \prod_{\ell \in \mathbb{Z}_k} m_\ell \in \mathbb{Z}_N$*, and a total of* $N = \prod_{\ell \in \mathbb{Z}_n} m_\ell \in \mathbb{N}$ *cells.*
2. $Q \cong \mathbb{Z}_s$ *is the (finite)* set of states *of dimension* $s \in \mathbb{N}$*.*
3. $\mathcal{N} : L \to \bigcup_{d \in \mathbb{Z}_{N+1}} L^d$ *is the* neighbourhood function*, which maps every cell* $\mathbf{c}_i \in L$ *to its neighbour cells, defined as the tuple* $\mathcal{N}(\mathbf{c}_i) = \langle \mathbf{c}_{i,k} \rangle_{k \in \mathbb{Z}_{d_i}} \in L^{d_i}$*, consisting of* $d_i = |\mathcal{N}(\mathbf{c}_i)| \in \mathbb{N}$ *distinct cells.*
4. $\delta : L \times \bigcup_{d \in \mathbb{Z}_{N+1}} Q^d \to Q$ *is the* local transition function*, which is used to update the state of every cell according to the current states of its neighbour cells.*
5. $\varphi : L \times \mathbb{N} \to Q$ *is the* local evolution function*, which maps every cell* $\mathbf{c}_i \in L$ *to a state* $q_i^{(t)} \in Q$ *at each unit of time* $t \in \mathbb{N}$*.*
6. $q_0 : L \to Q$ *is the* initial state function*, which assigns an initial state* $q_i^{(0)} \in Q$ *to every cell* $\mathbf{c}_i \in L$ *at time* $t = 0$*, that is* $\forall \mathbf{c}_i \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) \in Q$*.*

Example 1 (Wolfram's Rule 30 CA).

1. $L = \mathbb{Z}_N$, $\mathbf{c}_i = \langle i \rangle \in L$.
2. $Q = \mathbb{Z}_2$.
3. $\mathcal{N} : L \to L^3$, $\forall \mathbf{c}_i = \langle i \rangle \in L : \mathcal{N}(\langle i \rangle) = \langle \langle i \ominus_N 1 \rangle, \langle i \rangle, \langle i \oplus_N 1 \rangle \rangle$.
4. $\delta : L \times Q^3 \to Q$, $\forall \mathbf{c}_i \in L : \delta(\mathbf{c}_i, \langle q_l, q, q_r \rangle) = q \oplus_2 q_l \oplus_2 q_r \oplus_2 q \odot_2 q_r$.
5. $\forall \langle \mathbf{c}_i, t \rangle \in L \times \mathbb{N}^+ : \varphi(\mathbf{c}_i, t) = q_i^{(t)} = \delta(\mathbf{c}_i, \langle \varphi(\mathbf{c}_{i,l}, t-1), \varphi(\mathbf{c}_i, t-1), \varphi(\mathbf{c}_{i,r}, t-1) \rangle)$, $\langle \mathbf{c}_{i,l}, \mathbf{c}_i, \mathbf{c}_{i,r} \rangle = \mathcal{N}(\mathbf{c}_i)$.
6. $\forall \mathbf{c}_i \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) = \begin{cases} 1 & \text{if } i = \left\lfloor \frac{N}{2} \right\rfloor, \\ 0 & \text{otherwise.} \end{cases}$ $\diamondsuit$

Example 2 (Conway's Game of Life CA – GoL).

1. $L = \mathbb{Z}_c \times \mathbb{Z}_r$, $\mathbf{c}_i = \langle x, y \rangle \in L$, $i = x + c \cdot y \in \mathbb{Z}_N$, $N = c \cdot r$.
2. $Q = \mathbb{Z}_2$.
3. $\mathcal{N} : L \to L^9$, $\forall \mathbf{c}_i = \langle x, y \rangle \in L : \mathcal{N}(\langle x, y \rangle) = \langle \langle x \oplus_c j, y \oplus_r k \rangle \rangle_{\langle j,k \rangle \in \{-1,0,1\}^2}$.
4. $\delta : L \times Q^9 \to Q$,
   $\forall \mathbf{c}_i \in L : \delta\left(\mathbf{c}_i, \langle q_{j,k} \rangle_{\langle j,k \rangle \in \{-1,0,1\}^2}\right) = \begin{cases} 1 & \text{if } S = 3 \vee (S = 4 \wedge q_{0,0} = 1), \\ 0 & \text{otherwise.} \end{cases}$,
   $S = \sum_{\langle j,k \rangle \in \{-1,0,1\}^2} q_{j,k}$.
5. $\forall \langle \mathbf{c}_i, t \rangle \in L \times \mathbb{N}^+ : \varphi(\mathbf{c}_i, t) = q_i^{(t)} = \delta\left(\mathbf{c}_i, \langle \varphi(\mathbf{c}_{i,k}, t-1) \rangle_{k \in \mathbb{Z}_9}\right)$, $\langle \mathbf{c}_{i,k} \rangle_{k \in \mathbb{Z}_9} = \mathcal{N}(\mathbf{c}_i)$.
6. $\forall \mathbf{c}_i \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) = t$, $t \xleftarrow{\$} \mathbb{Z}_2$. $\diamondsuit$

Example 3 (Brians's Brain CA).

1. $L = \mathbb{Z}_c \times \mathbb{Z}_r$, $\mathbf{c}_i = \langle x, y \rangle \in L$, $i = x + c \cdot y \in \mathbb{Z}_N$, $N = c \cdot r$.
2. $Q = \mathbb{Z}_3$.
3. $\mathcal{N} : L \to L^9$, $\forall \mathbf{c}_i = \langle x, y \rangle \in L : \mathcal{N}(\langle x, y \rangle) = \langle \langle x \oplus_c j, y \oplus_r k \rangle \rangle_{\langle j,k \rangle \in \{-1,0,1\}^2}$.
4. $\delta : L \times Q^9 \to Q$, $\forall \mathbf{c}_i \in L : \delta\left(\mathbf{c}_i, \langle q_{j,k} \rangle_{\langle j,k \rangle \in \{-1,0,1\}^2}\right) = \begin{cases} 2 & \text{if } q_{0,0} = 0 \wedge S = 2, \\ 1 & \text{if } q_{0,0} = 2, \\ 0 & \text{otherwise.} \end{cases}$,
   $S = \left| \left\{ q \in \langle q_{j,k} \rangle_{\langle j,k \rangle \in \{-1,0,1\}^2 \setminus \{\langle 0,0 \rangle\}} \mid q = 2 \right\} \right|$.
5. $\forall \langle \mathbf{c}_i, t \rangle \in L \times \mathbb{N}^+ : \varphi(\mathbf{c}_i, t) = q_i^{(t)} = \delta\left(\mathbf{c}_i, \langle \varphi(\mathbf{c}_{i,k}, t-1) \rangle_{k \in \mathbb{Z}_9}\right)$, $\langle \mathbf{c}_{i,k} \rangle_{k \in \mathbb{Z}_9} = \mathcal{N}(\mathbf{c}_i)$.
6. $\forall \mathbf{c}_i \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) = t$, $t \xleftarrow{\$} \mathbb{Z}_3$. $\diamondsuit$

Example 4 (Langton's Ant CA).

1. $L = \mathbb{Z}_c \times \mathbb{Z}_r$, $\mathbf{c}_i = \langle x, y\rangle \in L$, $i = x + c \cdot y \in \mathbb{Z}_N$, $N = c \cdot r$.
2. $Q = \{w, b, N_w, E_w, S_w, W_w, N_b, E_b, S_b, W_b\}$.
3. $\mathcal{N} : L \to L^9$, $\forall \mathbf{c}_i = \langle x, y\rangle \in L : \mathcal{N}(\langle x, y\rangle) = \langle\langle x \oplus_c j, y \oplus_r k\rangle\rangle_{\langle j,k\rangle \in \{-1,0,1\}^2, |j|+|k| \leq 1}$.
4. $\delta : L \times Q^9 \to Q$, $\forall \mathbf{c}_i \in L$:
$$\delta\left(\mathbf{c}_i, \langle q_S, q_W, q, q_E, q_N\rangle\right) = \begin{cases} N_c & \text{if } q = c \wedge q_S \in \{W_w, E_b\}, \\ E_c & \text{if } q = c \wedge q_W \in \{N_w, S_b\}, \\ S_c & \text{if } q = c \wedge q_N \in \{E_w, W_b\}, \\ W_c & \text{if } q = c \wedge q_E \in \{S_w, N_b\}, \\ w & \text{if } q = D_b, \\ b & \text{if } q = D_w, \\ c & \text{otherwise.} \end{cases},$$
$c \in \{w, b\}$, $D \in \{N, E, S, W\}$.
5. $\forall \langle \mathbf{c}_i, t\rangle \in L \times \mathbb{N}^+ : \varphi(\mathbf{c}_i, t) = q_i^{(t)} = \delta\left(\mathbf{c}_i, \langle \varphi(\mathbf{c}_{i,k}, t-1)\rangle_{k \in \mathbb{Z}_9}\right)$, $\langle \mathbf{c}_{i,k}\rangle_{k\in\mathbb{Z}_9} = \mathcal{N}(\mathbf{c}_i)$.
6. $\forall \mathbf{c}_i = \langle x, y\rangle \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) = \begin{cases} 1 & \text{if } x = \left\lfloor \frac{c}{2} \right\rfloor \wedge y = \left\lfloor \frac{r}{2} \right\rfloor, \\ 0 & \text{otherwise.} \end{cases}$. $\diamondsuit$

Example 5 (Hexagonal Lattice Wolfram's Snowflake CA).

1. $L = \mathbb{Z}_c \times \mathbb{Z}_r$, $\mathbf{c}_i = \langle x, y\rangle \in L$, $i = x + c \cdot y \in \mathbb{Z}_N$, $N = c \cdot r$.
2. $Q = \mathbb{Z}_2$.
3. $\mathcal{N} : L \to L^7, \forall \mathbf{c}_i = \langle x, y\rangle \in L : \mathcal{N}(\langle x, y\rangle) = \langle\langle x \oplus_c j, y \oplus_r k\rangle\rangle_{\langle j,k\rangle \in \{-1,0,1\}^2 \setminus \{\langle -1,-1\rangle, \langle 1,1\rangle\}}$.
4. $\delta : L \times Q^7 \to Q$, $\forall \mathbf{c}_i \in L$:
$$\delta\left(\mathbf{c}_i, \langle q_{j,k}\rangle_{\langle j,k\rangle \in \{-1,0,1\}^2 \setminus \{\langle -1,-1\rangle, \langle 1,1\rangle\}}\right) = \begin{cases} 1 & \text{if } q = 1 \vee q = 0 \wedge S = 1, \\ 0 & \text{otherwise.} \end{cases},$$
$S = \sum_{\langle j,k\rangle \in \{-1,0,1\}^2 \setminus \{\langle -1,-1\rangle, \langle 1,1\rangle\}} q_{j,k}$.
5. $\forall \langle \mathbf{c}_i, t\rangle \in L \times \mathbb{N}^+ : \varphi(\mathbf{c}_i, t) = q_i^{(t)} = \delta\left(\mathbf{c}_i, \langle \varphi(\mathbf{c}_{i,k}, t-1)\rangle_{k \in \mathbb{Z}_7}\right)$, $\langle \mathbf{c}_{i,k}\rangle_{k\in\mathbb{Z}_7} = \mathcal{N}(\mathbf{c}_i)$.
6. $\forall \mathbf{c}_i = \langle x, y\rangle \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) = \begin{cases} 1 & \text{if } x = \left\lfloor \frac{c}{2} \right\rfloor \wedge y = \left\lfloor \frac{r}{2} \right\rfloor, \\ 0 & \text{otherwise.} \end{cases}$. $\diamondsuit$

Example 6 (Rock/Paper/Scissor CA).

1. $L = \mathbb{Z}_c \times \mathbb{Z}_r$, $\mathbf{c}_i = \langle x, y \rangle \in L$, $i = x + c \cdot y \in \mathbb{Z}_N$, $N = c \cdot r$.
2. $Q = \mathbb{Z}_3$.
3. $\mathcal{N} : L \to L^2$, $\forall \mathbf{c}_i = \langle x, y \rangle \in L : \mathcal{N}(\mathbf{c}_i) = \langle \mathbf{c}_i, \mathbf{c}'_i \rangle$,
   $\mathbf{c}'_i \xleftarrow{\$} \{\langle x \oplus_c j, y \oplus_r k \rangle\}_{\langle j,k \rangle \in \{-1,0,1\}^2 \setminus \{\langle 0,0 \rangle\}}$.
4. $\delta : L \times Q^2 \to Q$, $\forall \mathbf{c}_i \in L : \delta(\mathbf{c}_i, \langle q, q' \rangle) = \begin{cases} q' & \text{if } q = q' \ominus_3 1, \\ q & \text{otherwise.} \end{cases}$.
5. $\forall \langle \mathbf{c}_i, t \rangle \in L \times \mathbb{N}^+ : \varphi(\mathbf{c}_i, t) = q_i^{(t)} = \delta(\mathbf{c}_i, \langle \varphi(\mathbf{c}_i, t-1), \varphi(\mathbf{c}'_i, t-1) \rangle)$, $\langle \mathbf{c}_i, \mathbf{c}'_i \rangle = \mathcal{N}(\mathbf{c}_i)$.
6. $\forall \mathbf{c}_i \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) = t$, $t \xleftarrow{\$} \mathbb{Z}_3$. $\diamondsuit$

Example 7 (Rules 90 and 150 Hybrid CA (HCA)).

1. $L = \mathbb{Z}_N$, $\mathbf{c}_i = \langle i \rangle \in L$.
2. $Q = \mathbb{Z}_2$.
3. $\mathcal{N} : L \to L^3$, $\forall \mathbf{c}_i = \langle i \rangle \in L : \mathcal{N}(\langle i \rangle) = \langle \langle i \ominus_N 1 \rangle, \langle i \rangle, \langle i \oplus_N 1 \rangle \rangle$.
4. $\delta : L \times Q^3 \to Q$, $\forall \mathbf{c}_i \in L : \delta(\mathbf{c}_i, \langle q_l, q, q_r \rangle) = q_l \oplus_2 d_i \odot_2 q \oplus_2 q_r$.
5. $\forall \langle \mathbf{c}_i, t \rangle \in L \times \mathbb{N}^+ : \varphi(\mathbf{c}_i, t) = q_i^{(t)} = \delta(\mathbf{c}_i, \langle \varphi(\mathbf{c}_i, t-1), \varphi(\mathbf{c}_{i,r}, t-1) \rangle)$,
   $\langle \mathbf{c}_i, \mathbf{c}_{i,r} \rangle = \mathcal{N}(\mathbf{c}_i)$.
6. $\forall \mathbf{c}_i \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) = \begin{cases} 1 & \text{if } i = \left\lfloor \frac{N}{2} \right\rfloor, \\ 0 & \text{otherwise.} \end{cases}$.

The values $\langle d_i \rangle_{i \in \mathbb{Z}_N}$ are obtained once and for all uniformly at random from $\mathbb{Z}_2$, that is:

$$\forall i \in \mathbb{Z}_N : d_i \xleftarrow{\$} \mathbb{Z}_2.$$

$\diamondsuit$

Example 8 (Rule 150 Reversible Second-Order CA (RCA^2)).

1. $L = \mathbb{Z}_N$, $\mathbf{c}_i = \langle i \rangle \in L$.
2. $Q = \mathbb{Z}_2$.
3. $\mathcal{N} : L \to L^3$, $\forall \mathbf{c}_i = \langle i \rangle \in L : \mathcal{N}(\langle i \rangle) = \langle \langle i \ominus_N 1 \rangle, \langle i \rangle, \langle i \oplus_N 1 \rangle \rangle$.
4. $\delta : L \times Q^3 \to Q$, $\forall \mathbf{c}_i \in L : \delta(\mathbf{c}_i, \langle q_l, q, q_r \rangle) = q_l \oplus_2 q \oplus_2 q_r$.
5. $\forall \langle \mathbf{c}_i, t \rangle \in L \times \mathbb{N}_{>1} : \varphi(\mathbf{c}_i, t) = q_i^{(t)} = \delta(\mathbf{c}_i, \langle \varphi(\mathbf{c}_{i,l}, t-1), \varphi(\mathbf{c}_i, t-1), \varphi(\mathbf{c}_{i,r}, t-1) \rangle) \ominus_2 \varphi(\mathbf{c}_i, t-2)$,
   $\forall \mathbf{c}_i \in L : \varphi(\mathbf{c}_i, 1) = q_i^{(1)} = \delta(\mathbf{c}_i, \langle \varphi(\mathbf{c}_{i,l}, 0), \varphi(\mathbf{c}_i, 0), \varphi(\mathbf{c}_{i,r}, 0) \rangle)$, $\langle \mathbf{c}_{i,l}, \mathbf{c}_i, \mathbf{c}_{i,r} \rangle = \mathcal{N}(\mathbf{c}_i)$.
6. $\forall \mathbf{c}_i \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) = \begin{cases} 1 & \text{if } i = \left\lfloor \frac{N}{2} \right\rfloor, \\ 0 & \text{otherwise.} \end{cases}$. ♢

Example 9 (Cyclically Asynchronous Rule 30 CA).

1. $L = \mathbb{Z}_N$, $\mathbf{c}_i = \langle i \rangle \in L$.
2. $Q = \mathbb{Z}_2$.
3. $\mathcal{N} : L \to L^3$, $\forall \mathbf{c}_i = \langle i \rangle \in L : \mathcal{N}(\langle i \rangle) = \langle \langle i \ominus_N 1 \rangle, \langle i \rangle, \langle i \oplus_N 1 \rangle \rangle$.
4. $\delta : L \times Q^3 \to Q$, $\forall \mathbf{c}_i \in L : \delta(\mathbf{c}_i, \langle q_l, q, q_r \rangle) = q \oplus_2 q_l \oplus_2 q_r \oplus_2 q \odot_2 q_r$.
5. $\forall \langle \mathbf{c}_i, t \rangle \in L \times \mathbb{N}^+ : \varphi(\mathbf{c}_i, t) = q_i^{(t)} =$
   $$= \begin{cases} \delta(\mathbf{c}_i, \langle \varphi(\mathbf{c}_{i,l}, t-1), \varphi(\mathbf{c}_i, t-1), \varphi(\mathbf{c}_{i,r}, t-1) \rangle) & \text{if } t = \pi([i \bmod N]), \\ \varphi(\mathbf{c}_i, t-1) & \text{otherwise.} \end{cases},$$
   $\langle \mathbf{c}_{i,l}, \mathbf{c}_i, \mathbf{c}_{i,r} \rangle = \mathcal{N}(\mathbf{c}_i)$, $\pi : \mathbb{Z}_N \to \mathbb{Z}_N$, π bijective.
6. $\forall \mathbf{c}_i \in L : \varphi(\mathbf{c}_i, 0) = q_i^{(0)} = q_0(\mathbf{c}_i) = t$, $t \xleftarrow{\$} \mathbb{Z}_2$. ♢