Homework 3 - Theory

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1 Exercise 1

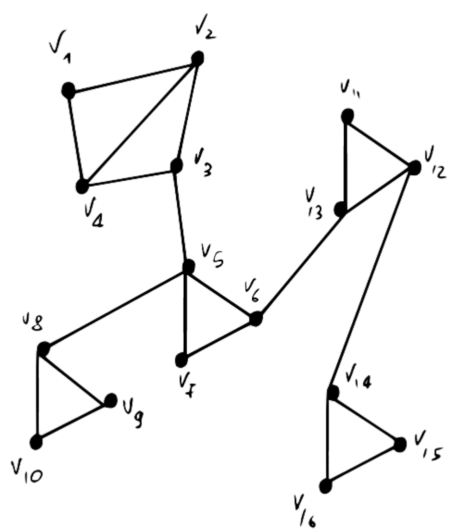

Figure 1: Graph $G = (V, E)$

a) The *clique number* of G is $w(G) = 3$. This also means that the *chromatic number* ($c(G)$) has a lower bound of:

$$c(G) \geq w(G)$$
$$c(G) \geq 3$$

b) The *max degree* of G is $\triangle(G) = max\{d(V)|v \in V\} = 4$. This also means that the *chromatic number* ($c(G)$) has a upper bound of:

$$c(G) \leq \triangle(G) + 1$$
$$c(G) \leq 5$$

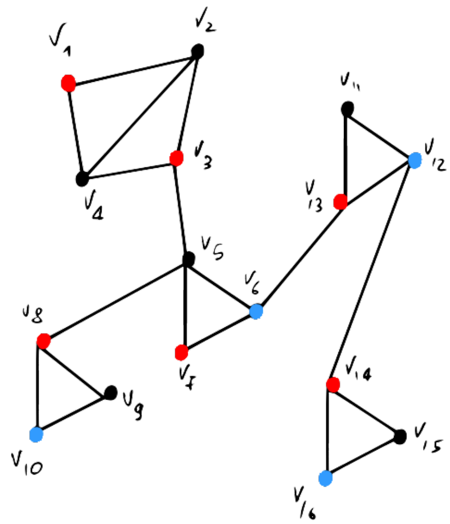

Figure 2: Graph $G = (V, E)$

c) The coloring illustrated in the 'Figure 2' is not a proper 3-colouring of G, since the vertices v_2 and v_4 are adjacent and with the same color.

c) The partitions of the vertex set induced by the provided colouring are:

$$V_{black} = \{v_2, v_4, v_5, v_9, v_{11}, v_{15}, \}$$
$$V_{red} = \{v_1, v_3, v_7, v_8, v_{13}, v_{14}, \}$$
$$V_{blue} = \{v_6, v_{10}, v_{12}, v_{16}, \}$$

2 Exercise 2

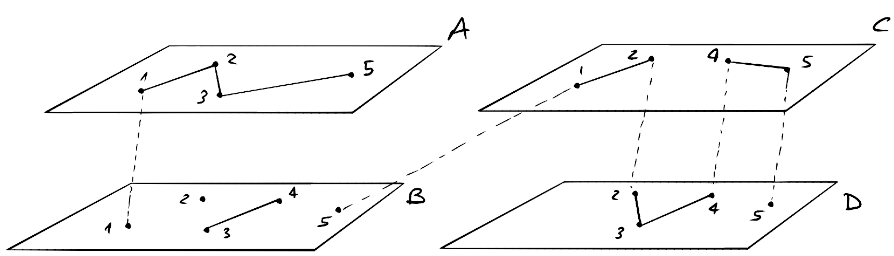

Figure 3: Multi-layer network

The multi-layer network pictured in the 'Figure 3' has its vertex and edge sets equals to:

$$
\begin{aligned}
V_M = \{&v_1 = \{1,A\}, v_2 = \{2,A\}, v_3 = \{3,A\}, v_4 = \{5,A\}, v_5 = \{1,B\}, v_6 = \{2,B\},\\
&v_7 = \{3,B\}, v_8 = \{4,B\}, v_9 = \{5,B\}, v_{10} = \{1,C\}, v_{11} = \{2,C\}, v_{12} = \{4,C\},\\
&v_{13} = \{5,C\}, v_{14} = \{2,D\}, v_{15} = \{3,D\}, v_{16} = \{4,D\}, v_{17} = \{5,D\}\}
\end{aligned}
$$

$$
\begin{aligned}
E_M = \{&\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\},\{v_1,v_5\},\{v_7,v_8\},\{v_9,v_{10}\},\{v_{10},v_{11}\},\{v_{12},v_{13}\},\\
&\{v_{11},v_{14}\},\{v_{12},v_{16}\},\{v_{13},v_{17}\},\{v_{14},v_{15}\},\{v_{15},v_{16}\}\}
\end{aligned}
$$

With that said, it also has the following characteristics:

a) Intra-layer edge sets:

$$
\begin{aligned}
E_{A,A} &= \{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\}\}\\
E_{A,B} &= \{\{v_7,v_8\}\}\\
E_{A,C} &= \{\{v_{10},v_{11}\},\{v_{12},v_{13}\}\}\\
E_{A,D} &= \{\{v_{14},v_{15}\},\{v_{15},v_{16}\}\}
\end{aligned}
$$

$$
\begin{aligned}
E_A &= E_{A,A} \cup E_{A,B} \cup E_{A,C} \cup E_{A,D}\\
&= \{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\},\{v_7,v_8\},\\
&\quad \{v_{10},v_{11}\},\{v_{12},v_{13}\},\{v_{14},v_{15}\},\{v_{15},v_{16}\}\}
\end{aligned}
$$

Inter-layer edge sets:

$$
\begin{aligned}
E_{C,A,B} &= \{\{v_1,v_5\}\}\\
E_{C,B,C} &= \{\{v_9,v_{10}\}\}\\
E_{C,C,D} &= \{\{v_{11},v_{14}\},\{v_{12},v_{16}\},\{v_{13},v_{17}\}\}
\end{aligned}
$$

$$
\begin{aligned}
E_C &= E_{C,A,B} \cup E_{C,B,C} \cup E_{C,C,D} = E_M \setminus E_A\\
&= \{\{v_1,v_5\},\{v_9,v_{10}\},\{v_{11},v_{14}\},\{v_{12},v_{16}\},\{v_{13},v_{17}\}\}
\end{aligned}
$$

Coupling edge set:

$$
\begin{aligned}
E_{\widetilde{C},A,B} &= \{\{v_1,v_5\}\}\\
E_{\widetilde{C},C,D} &= \{\{v_{11},v_{14}\},\{v_{12},v_{16}\},\{v_{13},v_{17}\}\}
\end{aligned}
$$

$$
\begin{aligned}
E_{\widetilde{C}} &= E_{\widetilde{C},A,B} \cup E_{\widetilde{C},C,D}\\
&= \{\{v_1,v_5\},\{v_{11},v_{14}\},\{v_{12},v_{16}\},\{v_{13},v_{17}\}\}
\end{aligned}
$$

b) The provided network is not *fully interconnected* since not all the layers contain all the nodes.

c) The tensor representation of the network is:

$$A_{::,A,A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A_{::,B,B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{::,C,C} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_{::,D,D} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{::,A,B} = A_{::,B,A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{::,B,C} = A_{::,C,B} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{::,C,D} = A_{::,D,C} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Just to clarify, the rest of the tensors combinations has not been reported for brevity, since they all are equals to the null matrix.

3 Exercise 3

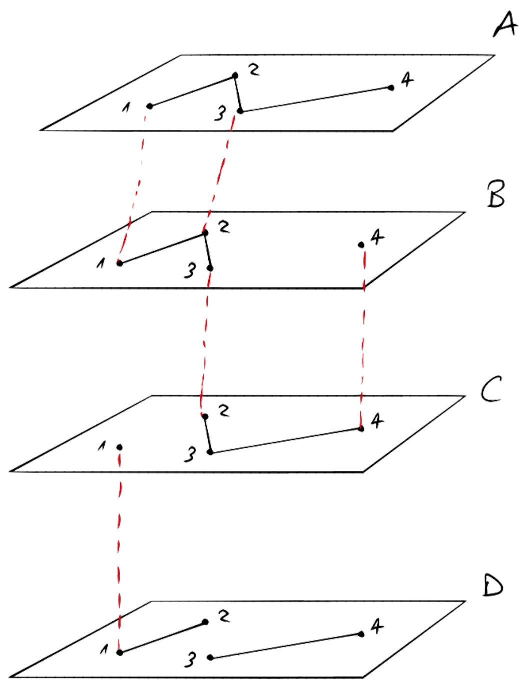

Figure 4: Multi-layer graph

a) The multi-layer graph pictured in the 'Figure 4' has the following tensor representation:

$$A_{::,A,A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_{::,B,B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{::,C,C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_{::,D,D} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A_{::,A,B} = A_{::,B,A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_{::,B,C} = A_{::,B,C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_{::,C,D} = A_{::,D,C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

As before, the rest of the tensors combinations has not been reported for brevity, since they all are equals to the null matrix.

b) The degree of each node are:

$$d_1 = (1, 1, 0, 1)$$
$$d_2 = (2, 2, 1, 1)$$
$$d_3 = (2, 1, 2, 1)$$
$$d_4 = (1, 0, 1, 1)$$

Meanwhile the overlapping degree are:

$$\begin{aligned} o_1 &= 3 \\ o_2 &= 6 \\ o_3 &= 6 \\ o_4 &= 3 \end{aligned}$$

Finally by applying the uniform-vector-like eigenvector centrality we obtain:

$$\tilde{A} = A^T_{::,A,A} + A^T_{::,B,B} + A^T_{::,C,C} + A^T_{::,D,D} = \begin{bmatrix} 0 & 3 & 0 & 0 \\ 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

In which every column, and also every row since it's a symmetric matrix, identifies the eigenvector of the relative node (e.g. first row = first column = eigenvector of v_1).