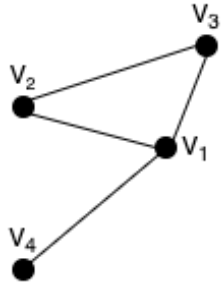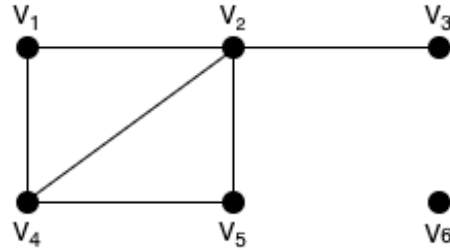# Homework 1 - Theory

Dainese Fabio, 857661

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1 Exercise 1

(a) Undirected graph

(b) Undirected graph

1.1 Answers About Graph 'a'

The degree of each vertex is:

$$d(v_1) = |N(v_1)| - 1 = |\{v_1, v_2, v_3, v_4\}| - 1 = 4 - 1 = 3$$
$$d(v_2) = |N(v_2)| - 1 = |\{v_1, v_2, v_3\}| - 1 = 3 - 1 = 2$$
$$d(v_3) = |N(v_3)| - 1 = |\{v_1, v_2, v_3\}| - 1 = 3 - 1 = 2$$
$$d(v_4) = |N(v_4)| - 1 = |\{v_1, v_4\}| - 1 = 2 - 1 = 1$$

The even vertices are $\{v_2, v_3\}$.

The average degree is:

$$Ad(G) = \frac{1}{|V|} \sum_{v \in V} d(v) = \frac{1}{4} \cdot (3 + 2 + 2 + 1) = \frac{1}{4} \cdot 8 = 2$$

1.2 Answers About Graph 'b'

The degree of each vertex is:

$$d(v_1) = |N(v_1)| - 1 = |\{v_1, v_2, v_4\}| - 1 = 3 - 1 = 2$$
$$d(v_2) = |N(v_2)| - 1 = |\{v_1, v_2, v_3, v_4, v_5\}| - 1 = 5 - 1 = 4$$
$$d(v_3) = |N(v_3)| - 1 = |\{v_2, v_3\}| - 1 = 2 - 1 = 1$$
$$d(v_4) = |N(v_4)| - 1 = |\{v_1, v_2, v_4, v_5\}| - 1 = 4 - 1 = 3$$
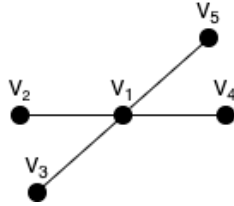$$d(v_5) = |N(v_5)| - 1 = |\{v_2, v_4, v_5\}| - 1 = 3 - 1 = 2$$
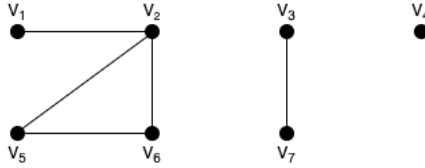$$d(v_6) = 0$$

The even vertices are $\{v_1, v_2, v_5, v_6\}$.

The average degree is:

$$Ad(G) = \frac{1}{|V|} \sum_{v \in V} d(v) = \frac{1}{6} \cdot (2 + 4 + 1 + 3 + 2 + 0) = \frac{1}{6} \cdot 12 = 2$$

2 Exercise 2

(a) Undirected graph   (b) Undirected graph

2.1 Answers About Graph 'a'

There aren't isolated vertices (i.e. vertices with $d(v) = 0$).

The graph is not complete (proof by counterexample: v_2 and v_5 are not adjacent).

A path from v_1 to v_5 is $\{e_1\}$, where $e_1 = \{v_1, v_5\}$.

The graphs is connected since there exists a path from v to w, $\forall v, w \in V$.

The connected components of v_1 and v_3 are:

$$C_{v_1} = C_{v_3} = \{v_1, v_2, v_3, v_4, v_5\}$$

2

The adjacent matrix associated to the graph is:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

2.2 Answers About Graph 'b'

There's only one isolated vertex and that is v_4, since $d(v_4) = 0$.

The graph is not complete (proof by counterexample: v_1 and v_5 are not adjacent).

A path from v_1 to v_5 is $\{e_1, e_2\}$, where $e_1 = \{v_1, v_2\}$ and $e_2 = \{v_2, v_5\}$.

The graphs is not connected since there's not exists a path from v to w, $\forall v, w \in V$, for example from v_2 and v_3.

The connected components of v_1 and v_3 are:

$$C_{v_1} = \{v_1, v_2, v_5, v_6\}$$
$$C_{v_3} = \{v_3, v_7\}$$

The adjacent matrix associated to the graph is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$
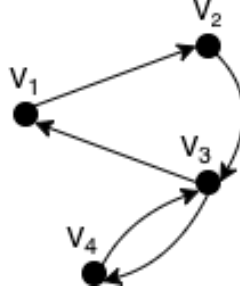
3 Exercise 3

Figure 3: Directed graph

The vertex set of the graph is:

$$V = \{v_1, v_2, v_3, v_4\}$$

The edge set of the graph is:

$$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_3, v_4), (v_4, v_3)\}$$

The adjacent matrix associated to the graph is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The in $(d^+(v))$, out $(d^-(v))$ and total degree $(d(v))$ of each node is:

$$d^+(v_1) = |N^+(v_1)| - 1 = |\{v_1, v_3\}| - 1 = 2 - 1 = 1$$
$$d^-(v_1) = |N^-(v_1)| - 1 = |\{v_1, v_2\}| - 1 = 2 - 1 = 1$$
$$d(v_1) = d^+(v_1) + d^-(v_1) = 1 + 1 = 2$$

$$d^+(v_2) = |N^+(v_2)| - 1 = |\{v_1, v_2\}| - 1 = 2 - 1 = 1$$
$$d^-(v_2) = |N^-(v_2)| - 1 = |\{v_2, v_3\}| - 1 = 2 - 1 = 1$$
$$d(v_2) = d^+(v_2) + d^-(v_2) = 1 + 1 = 2$$

$$d^+(v_3) = |N^+(v_3)| - 1 = |\{v_2, v_3, v_4\}| - 1 = 3 - 1 = 2$$
$$d^-(v_3) = |N^-(v_3)| - 1 = |\{v_1, v_3, v_4\}| - 1 = 3 - 1 = 2$$
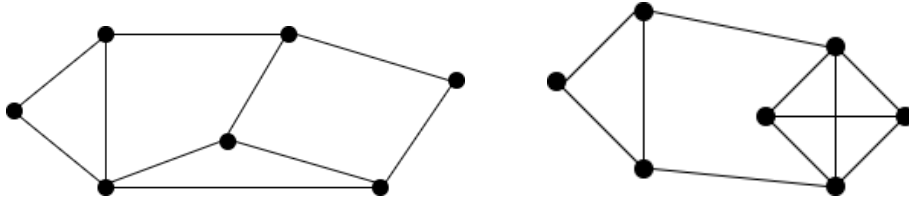$$d(v_3) = d^+(v_3) + d^-(v_3) = 2 + 2 = 4$$

$$d^+(v_4) = |N^+(v_4)| - 1 = |\{v_3, v_4\}| - 1 = 2 - 1 = 1$$
$$d^-(v_4) = |N^-(v_4)| - 1 = |\{v_3, v_4\}| - 1 = 2 - 1 = 1$$
$$d(v_4) = d^+(v_4) + d^-(v_4) = 1 + 1 = 2$$

One of many possible paths that can be found in the graph is for example the one from v_1 to v_3 defined as (e_1, e_2), where $e_1 = (v_1, v_2)$ and $e_2 = (v_2, v_3)$.

4   Exercise 4

These two graphs are not isomorphic. In this case a simple way to prove it is to compare the list of vertex degree of the two given graphs.
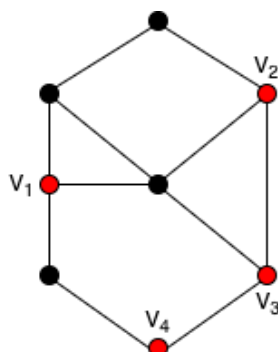
$$G_1 = \{2, 2, 3, 3, 3, 3, 4\}$$
$$G_2 = \{2, 3, 3, 3, 3, 4, 4\}$$

As you can see G_1 and G_2 are not identical, thus the two graphs are not isomorphic.
Another simpler way to prove it is to compare the cardinalities of the two edge sets, i.e. $|E_1| \neq |E_2| \implies$ No isomorphic.

5   Exercise 5

The distances, i.e. the minimum path length between two vertices, of $d(v_1, v_2)$, $d(v_1, v_3)$ and $d(v_1, v_4)$ are all equal to 2.

The diameter of the given graph is equals to 3, recalling that the diameter is the maximum between all the possible distances of the graph's vertices.

The centers of the given graphs are v_1, v_3 and the middle vertex (the 'inside' vertex), recalling that a vertex u is a centre of a graph G if its maximum distance from any other vertex v is minimum.

The radius of this graph is equals to 2 (maximum distance between the centers and the rest of the vertices).
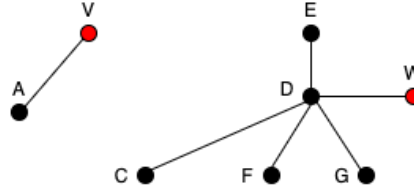
6   Exercise 6
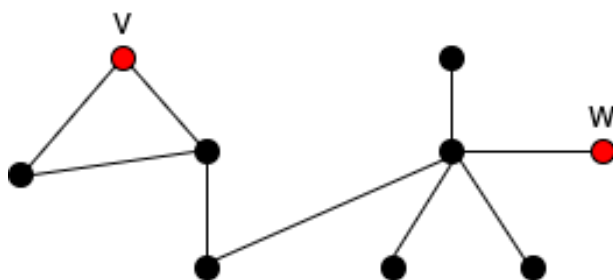
In the given graph there are three cut points as shown in the following images (red vertices):
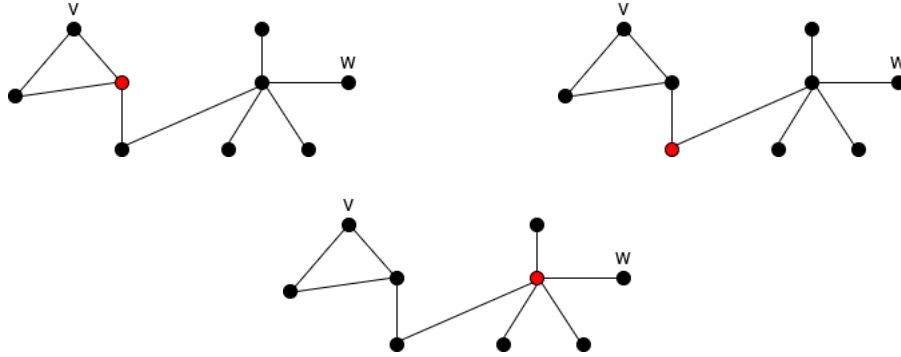
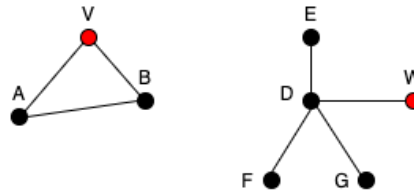By removing the first cut point, we obtain the following situation:

In this configuration the connected component of v and w are:

$$C_v = \{A, V\}$$
$$C_w = \{C, D, E, F, G, W\}$$

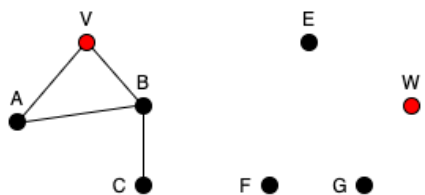Meanwhile if we remove the second cut point, we obtain the following situation:

In this configuration the connected component of v and w are:

$$C_v = \{A, B, V\}$$
$$C_w = \{D, E, F, G, W\}$$

Finally if we remove the third cut point, we obtain the following situation:

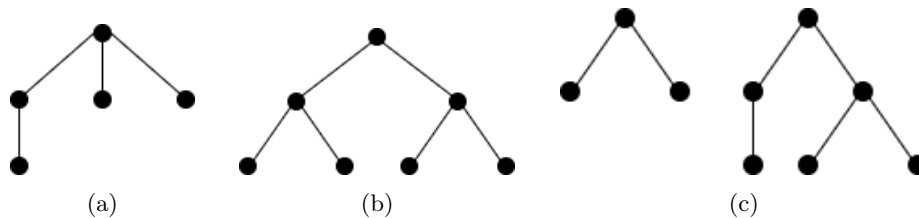In this configuration the connected component of v and w are:

$$C_v = \{A, B, C, V\}$$
$$C_w = \{W\}$$

7 Exercise 7

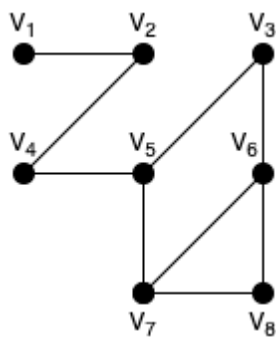(a)   (b)   (c)

The a and b graphs are trees.

The a,b and c graphs are forests.

The b graph is a binary tree. Also a can be a binary three if we consider as a root the middle node.

8 Exercise 8

The adjacent matrix of the graph is:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

The number of chains of length 2 and 4 are:

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 3 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 2 & 3 & 1 & 1 \\ 0 & 0 & 2 & 1 & 0 & 1 & 3 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \end{bmatrix}$$

$$A^4 = A \cdot A \cdot A \cdot A = \begin{bmatrix} 2 & 0 & 1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 0 & 5 & 2 & 0 & 1 \\ 1 & 0 & 10 & 6 & 1 & 3 & 12 & 6 \\ 3 & 0 & 6 & 7 & 0 & 1 & 7 & 2 \\ 0 & 5 & 1 & 0 & 15 & 13 & 3 & 7 \\ 0 & 2 & 3 & 1 & 13 & 15 & 7 & 8 \\ 1 & 0 & 12 & 7 & 3 & 7 & 16 & 8 \\ 0 & 1 & 6 & 2 & 7 & 8 & 8 & 8 \end{bmatrix}$$

Which, by summing all the elements of A^2 gives 44 possible chains of length 2, meanwhile by summing all the elements of A^4 gives 272 possible chains of length 4.

Assuming the given bijection ϕ of the task, the resulting permutation matrix P_ϕ is:

$$P_\phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

As before to calculate the chains of length 2 and 4 of the relabelled graph we need to perform $\widetilde{A^2}$ and $\widetilde{A^4}$, which gives 44 possible chains of length 2 and 272

of length 4 (as before). To recall that now the adjacent matrix is:

$$\widetilde{A} = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

9   Exercise 9

By defining a "Null Graph" as a graph $G = (V, E)$ with n vertices and $E = \emptyset$, we have:

- The number of edges equal to zero ($|E| = 0$);

- Only 1 face (outer face/region);

- A graph that can be drawn on a plane without edges crossing is called planar, meaning that also our "null graph" is a planar graph;

- The value of $|V| - |E| + |F|$ is equals to $n + 1$, since $|V| = n, |E| = 0$ and $|F| = 1$;

- The Euler's formula ($|V| - |E| + |F| = 2$) can't be applied to the "null graph" since it's not connected (unless $|V| = 1$).

10   Exercise 10
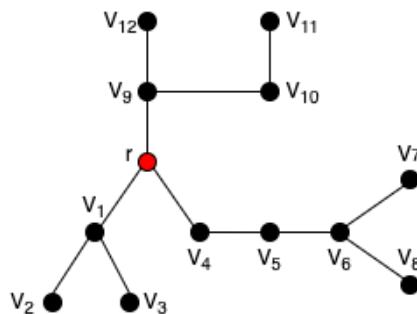
The leaves of the previous graph, with r as root, are $\{v_2, v_3, v_7, v_8, v_{11}, v_{12}\}$.