DIDP model for CVRP-TW

Fabio Giordana

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CVRPTW

In a capacitated vehicle routing problem with time windows (CVRP), we are given a set of locations $N = \{0, \ldots, n-1\}$ where 0 is the depot and $\{1, \ldots, n-1\}$ are customers. We need to pick up commodities from the customers using m vehicles with capacity q, which start from and return to the depot.

By visiting customer i, the load of a vehicle increases by weight d_i. Visiting customer j from i incurs the travel cost c_{ij}. Each customer i must be visited within the time window $[a_i, b_i]$ and has a service time (time to spend at the customer location) s. The objective is to find a set of tours for the vehicles that visit all customers while minimizing the total travel cost and the number of used vehicles.

Model

We construct tours for the vehicles one by one. Let k be the number of used vehicles (including the current vehicle), U be the set of unvisited customers, i∈U be the current location of the vehicle, l be the load of the vehicle, and t be the actual time. Customer j can be visited by the current vehicle if $l + d_j \leq q$ and if the arrival time $\max\{t + c_{ij}, a_j\} \leq b_i$. Otherwise, we need to return to the depot and use a new vehicle to visit j, which is possible only if $k < m$.

Let $V(U, i, l, k, t)$ be the minimum cost to visit customers U from i with the load l using $m - k + 1$ vehicles. We have the following DP model:

$$V(N \setminus \{0\}, 0, 0, 1, 0)$$

$$V(U,i,l,k) = \begin{cases} \min \left\{ \begin{array}{l} \min\limits_{j \in U,\ l+d_j \leq q; t+c_{ij} \leq b_j} (c_{ij} + V(U \setminus \{j\}, j, l+d_i, k, s_j + \max(t+c_{ij}, a_j)), \\ \min\limits_{j \in U} (c_{i0} + c_{0j} + V(U \setminus \{j\}, j, d_i, k+1, s_j + \max(c_{ij}, a_j))) \end{array} \right\}, & \text{if } k < m \\ \min\limits_{j \in U,\ l+d_j \leq q; t+c_{ij} \leq b_j} (c_{ij} + V(U \setminus \{j\}, j, l+d_i, k, s_j + \max(t+c_{ij}, a_j)), & \text{if } k = m \\ c_{i0} + V(U, 0, k, l, t), & \text{if } U = \emptyset \text{ and } i \neq \\ 0, & \text{if } U = \emptyset \text{ and } i = \end{cases} \tag{1}$$

When two states (U,i,l,k,t) and (U,i,l',k',t') have the same set of unvisited customers U and the same location i, if $l \leq l'$ and $k \leq k'$ and $t \leq t'$, then (U,i,l,k,t) leads to a better solution. Therefore:

$$V(U,i,l,k,t) \leq V(U,i,l',k',t') \quad \text{if } l \leq l' \text{ and } k \leq k' \text{ and } t \leq t'$$

The sum of the capacity of the remaining vehicles, $q - l + (m-k)q$, must be greater than or equal to the sum of the weights of the remaining commodities, $\sum_{j \in U} d_j$. Otherwise, the state does not lead to a solution. Therefore:

$$V(U,i,l,k,t) = \infty \quad \text{if } (m-k+1)q - l < \sum_{j \in U} d_j.$$

At any moment, we must be able to return to the depot from the current location without exceeding the deadline of the depot, so:

$$V(U,i,l,k,t) = \infty \quad \text{if } t + c_{i0} \leq b_0.$$

The lowest possible travel cost to visit customer j is:

$$\min_{k \in N \setminus \{j\}} c_{kj}.$$

Because we need to visit all customers in U, the total travel cost is at least:

$$\sum_{j \in U} \min_{k \in N \setminus \{j\}} c_{kj}.$$

Furthermore, if the current location i is not the depot, we need to visit the depot. Therefore:

$$V(U,i,k,l,t) \geq \sum_{j \in (U \cup \{0\}) \setminus \{i\}} \min_{k \in N \setminus \{j\}} c_{kj}.$$

Similarly, we need to depart from each customer in U and the current location i if i is not the depot. Therefore:

$$V(U, i, k, l, t) \geq \sum_{j \in (U \cup \{i\}) \setminus \{0\}} \min_{k \in N \setminus \{j\}} c_{jk}.$$