UNIVERSITÀ di VERONA
Dipartimento di INFORMATICA

Discrete Optimization and Decision Making

Modelling examples in Integer Linear Programming: Vehicle Routing Problems

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Master's degree in Data Science
Academic Year 2021/2022

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The Traveling Salesman Problem

Formulation A 1/2

$$\min \sum_{(i,j)\in A} c_{ij} x_{ij} \tag{1}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \qquad j \in V \tag{2}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \qquad i \in V \tag{3}$$

$$\sum_{(i,j)\in\delta(S)} x_{ij} \geq 1 \qquad S \subset V, |S| \geq 2 \tag{4}$$

$$x_{ij} \geq 0 \quad \text{integer}, \quad (i,j) \in A \tag{5}$$

Formulation A 2/2

- $\delta(S) = \{(i,j) \in A \mid i \in S, j \notin S\}$;
- condition $x_{ij} \leq 1$ is implied by (2) and (3);
- every solution that satisfies the set of constraints (2) and (3) corresponds to a family of disjunct circuits that covers one and only one time all the vertexes in G;
- constraints (4) are called **connectivity constraints**, since they impose connectivity and require that each vertex $t \neq 1$ is reachable from vertex 1.

Formulation B

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{6}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \qquad j \in V \tag{7}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \qquad i \in V \tag{8}$$

$$\sum_{(i,j)\in E(S)} x_{ij} \leq |S| - 1 \qquad S \subset V, S \neq \emptyset \tag{9}$$

$$x_{ij} \geq 0 \quad \text{integer}, \quad (i,j) \in A \tag{10}$$

Constraints (9) are called **subtour elimination constraints**.
$E(S) = \{(i,j) \in A | i,j \in S\}$

Formulation C 1/2

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{11}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \qquad j \in V \tag{12}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \qquad i \in V \tag{13}$$

$$u_1 = 1 \tag{14}$$

$$2 \leq u_i \leq n \qquad i \in V \setminus \{1\} \tag{15}$$

$$u_j \geq u_i + 1 - (n-1)(1 - x_{ij}) \qquad i \in V, j \in V \setminus \{1\}, i \neq j \tag{16}$$

$$x_{ij} \geq 0 \quad \text{integer}, \quad (i,j) \in A \tag{17}$$

$$u_i \geq 0 \qquad i \in V \tag{18}$$

Formulation C 2/2

- Constraints (14)–(16) are known as **Miller-Tucker-Zemlin (MTZ) constraints**;
- u_i represents the position of node i in the cycle;
- Advantages of this formulation? A polynomial number of constraints!
- Disadvantages? The LP relaxation is weaker than the one of Formulation A and B.

The Vehicle Routing Problem

The Vehicle Routing Problem

- A generalization of the TSP: we can have more than one vehicle
- $K = \{1, \ldots, p\}$ set of vehicles
- Assumption 1: each node has to be visited by exactly one vehicle
- Assumption 2: the Triangle Inequality holds

Three-index formulation

$$\min \sum_{(i,j)\in A} c_{ij} \sum_{k\in K} x_{ij}^k \tag{19}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij}^k = y_j^k \qquad j \in V \setminus \{1\}, k \in K \tag{20}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij}^k = y_i^k \qquad i \in V \setminus \{1\}, k \in K \tag{21}$$

$$\sum_{(i,j)\in\delta(S)} x_{ij}^k \geq y_i^k \qquad S \subset V \setminus \{1\}, k \in K \tag{22}$$

$$\sum_{k\in K} y_i^k = 1 \qquad i \in V \setminus \{1\} \tag{23}$$

$$x_{ij}^k \geq 0 \quad \text{integer}, \quad (i,j) \in A, k \in K \tag{24}$$

$$y_i^k \geq 0 \quad \text{integer}, \quad i \in V, k \in K \tag{25}$$

Two-index formulation

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{26}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \qquad j \in V \setminus \{1\} \tag{27}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \qquad i \in V \setminus \{1\} \tag{28}$$

$$\sum_{(i,j)\in\delta(S)} x_{ij} \geq 1 \qquad S \subset V \setminus \{1\}, S \neq \emptyset \tag{29}$$

$$\sum_{(1,j)\in\delta^+(1)} x_{1j} = \sum_{(i,1)\in\delta^-(1)} x_{i1} = |K| \tag{30}$$

$$x_{ij} \geq 0 \quad \text{integer,} \quad (i,j) \in A \tag{31}$$

The Capacitated Vehicle Routing Problem (CVRP)

- Only one variation with respect to the VRP: vehicles have a predefined capacity Q
- This aspect can be exploited to identify a better formulation!

The Capacitated Vehicle Routing Problem: Two-index formulation 1/2

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{32}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \qquad j \in V \setminus \{1\} \tag{33}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \qquad i \in V \setminus \{1\} \tag{34}$$

$$\sum_{(i,j)\in\delta(S)} x_{ij} \geq r(S) = \lceil q(S)/Q \rceil \qquad S \subset V \setminus \{1\}, S \neq \emptyset \tag{35}$$

$$\sum_{(1,j)\in\delta^+(1)} x_{1j} = \sum_{(i,1)\in\delta^-(1)} x_{i1} = |K| \tag{36}$$

$$x_{ij} \geq 0 \quad \text{integer}, \quad (i,j) \in A \tag{37}$$

The Capacitated Vehicle Routing Problem: Two-index formulation 2/2

- Constraints (35) are called **Cut Capacity Constraints**.
- r(S) is a lower bound on the # of vehicles required to serve all nodes in S.
- Constraints (35) can be replaced by the **Generalized Subtour Elimination Constraints**

$$\sum_{(i,j)\in\delta(S)} x_{ij} \leq |S| - r(S) \qquad S \subset V \setminus \{1\}$$

The Capacitated Vehicle Routing Problem: Three-index formulation

$$\min \sum_{(i,j)\in A} c_{ij} \sum_{k\in K} x_{ij}^k \tag{38}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij}^k = y_j^k \qquad j \in V\setminus\{1\}, k \in K \tag{39}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij}^k = y_i^k \qquad i \in V\setminus\{1\}, k \in K \tag{40}$$

$$\sum_{(i,j)\in\delta(S)} x_{ij}^k \geq y_h^k \qquad S \subset V\setminus\{1\}, h \in S, k \in K \tag{41}$$

$$\sum_{i\in V\setminus\{1\}} q_i y_i^k \leq Q \qquad k \in K \tag{42}$$

$$\sum_{k\in K} y_i^k = 1 \qquad i \in V\setminus\{1\} \tag{43}$$

$$x_{ij}^k \geq 0 \quad \text{integer}, \quad (i,j) \in A, k \in K \tag{44}$$

$$y_i^k \geq 0 \quad \text{integer}, \quad i \in V, k \in K \tag{45}$$

CVRP: One-commodity flow formulation 1/2

- Define variable f_{ij}, $(i, j) \in A$, as the amount of load delivered by a vehicle when leaving vertex i.
- $f_{ij} = 0$ when $x_{ij} = 0$.

CVRP: One-commodity flow formulation 2/2

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{46}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \qquad j \in V \setminus \{1\} \tag{47}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \qquad i \in V \setminus \{1\} \tag{48}$$

$$\sum_{(i,j)\in\delta^+(i)} f_{ij} = \sum_{(j,i)\in\delta^-(i)} f_{ji} + q_i \qquad i \in V \setminus \{1\} \tag{49}$$

$$0 \leq f_{ij} \leq Qx_{ij} \qquad (i,j) \in A \tag{50}$$

$$\sum_{(1,j)\in\delta^+(1)} x_{1j} = \sum_{(i,1)\in\delta^-(1)} x_{i1} = |K| \tag{51}$$

$$x_{ij} \geq 0 \quad \text{integer}, \quad (i,j) \in A \tag{52}$$

CVRP: Two-commodity flow formulation 1/3

- For simplicity, we represent the depot as starting node 1 and ending node $n+1$, and use an **undirected** graph.
- There are two different flows:
  - One flow starts at 1 and delivers q_i units when visiting node i, arriving at node $n+1$ with 0 units.
  - The second flow starts at node $n+1$ with mQ units, delivers q_i units when visiting node i, and arrives at node 1 with $mQ - q(V \setminus \{1, n+1\})$ units.
  - If a vehicle traverses (i, j), $f_{ij} + f_{ji} = Q$.

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{53}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \qquad j \in V \setminus \{1\} \tag{54}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \qquad i \in V \setminus \{1\} \tag{55}$$

$$\sum_{(i,j)\in\delta^+(i)} f_{ij} = \sum_{(j,i)\in\delta^-(i)} f_{ji} - 2q_i \qquad i \in V \setminus \{1\} \tag{56}$$

$$\sum_{(1,j)\in\delta^+(1)} f_{1j} = q(V) \tag{57}$$

$$\sum_{(j,1)\in\delta^-(1)} f_{j1} = mQ - q(V) \tag{58}$$

$$\sum_{(n+1,j)\in\delta^+(n+1)} f_{n+1,j} = mQ \tag{59}$$

CVRP: Two-commodity flow formulation 3/3

$$\sum_{(j,n+1)\in\delta^-(n+1)} f_{j,n+1} = 0 \tag{60}$$

$$f_{ij} + f_{ji} = Q(x_{ij} + x_{ji}) \qquad (i,j) \in A \tag{61}$$

$$0 \leq f_{ij} \qquad (i,j) \in A \tag{62}$$

$$\sum_{(1,j)\in\delta^+(1)} x_{1j} = \sum_{(i,n+1)\in\delta^-(n+1)} x_{i,n+1} \tag{63}$$

$$\sum_{(i,1)\in\delta^-(1)} x_{i1} = \sum_{(n+1,j)\in\delta^+(n+1)} x_{n+1,j} = 0 \tag{64}$$

$$x_{ij} \geq 0 \quad \text{integer}, \quad (i,j) \in A \tag{65}$$

CVRP: Multi-commodity flow formulation 1/3

- Starting from the two-index formulation, we add variables representing *one commodity per customer*.
- Variable f_{jl}^{i} represents the load on the vehicle destined for node i when it travels from j to l.

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{66}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \qquad j \in V \setminus \{1\} \tag{67}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \qquad i \in V \setminus \{1\} \tag{68}$$

$$\sum_{(j,i)\in\delta^-(i)} f^i_{ji} - \sum_{(i,j)\in\delta^+(i)} f^i_{ij} = q_i \qquad i \in V \setminus \{1\} \tag{69}$$

$$\sum_{(1,j)\in\delta^+(1)} f^i_{1j} - \sum_{(j,1)\in\delta^-(1)} f^i_{j1} = q_i \qquad i \in V \setminus \{1\} \tag{70}$$

$$\sum_{(j,l)\in\delta^+(j)} f^i_{jl} - \sum_{(l,j)\in\delta^-(j)} f^i_{lj} = 0 \qquad i \in V \setminus \{1\}, j \in V \setminus \{1, i\} \tag{71}$$

$$f^i_{ji} = q_i x_{ji} \qquad i \in V \setminus \{1\}, j \neq i \tag{72}$$

CVRP: Multi-commodity flow formulation 3/3

$$0 \leq f_{jl}^{i} \leq q_i x_{jl} \qquad i \in V \setminus \{1\}, (j,l) \in A \tag{73}$$

$$\sum_{j \in V \setminus \{1,i\}} \sum_{(i,j) \in \delta^+(i)} f_{il}^{j} \leq Q - q_i \qquad i \in V \setminus \{1\} \tag{74}$$

$$\sum_{(1,j) \in \delta^+(1)} x_{1j} = \sum_{(i,1) \in \delta^-(1)} x_{i1} = |K| \tag{75}$$

$$x_{ij} \geq 0 \quad \text{integer}, \quad (i,j) \in A \tag{76}$$

$$\min \sum_{(i,j)\in A} c_{ij} \sum_{k\in K} x_{ij}^k \tag{77}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij}^k = y_j^k \qquad j \in V \setminus \{1\}, k \in K \tag{78}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij}^k = y_i^k \qquad i \in V \setminus \{1\}, k \in K \tag{79}$$

$$x_{ij}^k(w_i^k + s_i + t_{ij} - w_j^k) \leq 0 \qquad k \in K, (i,j) \in A \tag{80}$$

$$a_i \sum_{(i,j)\in\delta^+(i)} x_{ij}^k \leq w_i^k \leq b_i \sum_{(i,j)\in\delta^+(i)} x_{ij}^k \qquad k \in K, i \in V \setminus \{1\} \tag{81}$$

$$\sum_{k\in K} y_i^k = 1 \qquad i \in V \setminus \{1\} \tag{82}$$

$$x_{ij}^k \geq 0 \quad \text{integer}, \quad (i,j) \in A, k \in K \tag{83}$$

$$y_i^k \geq 0 \quad \text{integer}, w_i^k \geq 0 \quad i \in V, k \in K \tag{84}$$

VRP with Time Windows: Three-index formulation 2/2

- Variable w_i^k represents the start of service at node i when serviced by vehicle k.
- Constraints (80) are classic **non-linear** constraints to handle visiting times in VRPTW formulations.
  They can be linearized as follows:

$$w_i^k + s_i + t_{ij} - w_j^k \leq M_{ij}(1 - x_{ij}^k) \qquad k \in K, (i, j) \in A$$

VRP with Time Windows: Two-commodity flow formulation 1/2

$$\min \sum_{(i,j)\in A} c_{ij}x_{ij} \tag{85}$$

$$\sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \qquad j \in V \setminus \{1\} \tag{86}$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \qquad i \in V \setminus \{1\} \tag{87}$$

$$\sum_{(1,j)\in\delta^+(1)} x_{1j} = \sum_{(i,1)\in\delta^-(1)} x_{i1} = |K| \tag{88}$$

$$\sum_{(i,j)\in\delta^+(i)} u_{ij} = \sum_{(j,i)\in\delta^-(i)} v_{ji} \qquad i \in V \setminus \{1\} \tag{89}$$

$$v_{ij} \geq u_{ji} + t_{ij}x_{ij} \qquad i,j \in V \setminus \{1\}, i \neq j \tag{90}$$

$$u_{ij} \geq a_i x_{ij} \qquad i \in V \setminus \{1\}, j \in V, i \neq j \tag{91}$$

$$v_{ij} \leq b_j x_{ij} \qquad i \in V, j \in V \setminus \{1\}, i \neq j \tag{92}$$

$$u_{ij} \leq min\{b_i, b_j - t_{ij}\} x_{ij} \qquad i \in V \setminus \{1\}, j \in V, i \neq j \tag{93}$$

$$v_{ij} \geq max\{a_j, a_i + t_{ij}\} x_{ij} \qquad i \in V, j \in V \setminus \{1\}, i \neq j \tag{94}$$

$$x_{ij} \geq 0 \quad \text{integer}, \quad (i,j) \in A \tag{95}$$

$$u_{ij} \geq 0 \quad v_{ij} \geq 0 \quad (i,j) \in A \tag{96}$$

VRP with Profits — Team Orienteering Problem

$$\max \sum_{i \in V \setminus \{1\}} p_i y_i \tag{97}$$

$$\sum_{(i,j) \in \delta^-(j)} x_{ij} = y_i \qquad j \in V \setminus \{1\} \tag{98}$$

$$\sum_{(i,j) \in \delta^+(i)} x_{ij} = y_i \qquad i \in V \setminus \{1\} \tag{99}$$

$$\sum_{(i,j) \in \delta^+(i)} f_{ij} = \sum_{(j,i) \in \delta^-(i)} f_{ji} + q_i y_i \qquad i \in V \setminus \{1\} \tag{100}$$

$$0 \leq f_{ij} \leq Q x_{ij} \qquad (i,j) \in A \tag{101}$$

$$\sum_{(1,j) \in \delta^+(1)} x_{1j} = \sum_{(i,1) \in \delta^-(1)} x_{i1} = |K| \tag{102}$$

$$x_{ij} \geq 0 \quad \text{integer}, \quad (i,j) \in A \tag{103}$$

$$y_i \in \{0,1\} \qquad i \in V \setminus \{1\} \tag{104}$$

$$\sum_{j \in \mathcal{N}} x_{0j} = m \tag{105}$$

$$\sum_{j \in \mathcal{N}} x_{ij} = 1 \quad \forall i \in \mathcal{N}_0 \tag{106}$$

$$\sum_{i \in \mathcal{N}} x_{ij} = 1 \quad \forall j \in \mathcal{N}_0 \tag{107}$$

$$\sum_{j \in \mathcal{N}} f_{ji} - \sum_{j \in \mathcal{N}} f_{ij} = q_i \quad \forall i \in \mathcal{N}_0 \tag{108}$$

$$q_j x_{ij} \leq f_{ij} \leq (Q - q_i) x_{ij} \quad \forall (i,j) \in \mathcal{A} \tag{109}$$

$$a_i \leq y_i \leq b_i \quad \forall i \in \mathcal{N}_0 \tag{110}$$

$$y_i - y_j + t_i + w_{ij} \leq M_{ij}(1 - x_{ij}) \quad \forall (i,j) \in \mathcal{A} \tag{111}$$

$$y_j + t_j - s_j + w_{j0} \leq L(1 - x_{j0}) \quad \forall j \in \mathcal{N}_0 \tag{112}$$

$$\frac{d_{ij}}{u_{ij}} x_{ij} \leq w_{ij} \leq \frac{d_{ij}}{l_{ij}} x_{ij} \quad \forall (i,j) \in \mathcal{A} \tag{113}$$

$$x_{ij} \in \{0,1\}, f_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A} \tag{114}$$

A better PVRP formulation? 1/2

$$\sum_{j \in \mathcal{N}} x_{0j} = m \tag{115}$$

$$\sum_{j \in \mathcal{N}} x_{ij} = 1 \quad \forall i \in \mathcal{N}_0 \tag{116}$$

$$\sum_{i \in \mathcal{N}} x_{ij} = 1 \quad \forall j \in \mathcal{N}_0 \tag{117}$$

$$\sum_{j \in \mathcal{N}} f_{ji} - \sum_{j \in \mathcal{N}} f_{ij} = q_i \quad \forall i \in \mathcal{N}_0 \tag{118}$$

$$q_j x_{ij} \leq f_{ij} \leq (Q - q_i) x_{ij} \quad \forall (i,j) \in \mathcal{A} \tag{119}$$

$$u_{ij} \geq a_i \quad \forall i \in \mathcal{N}_0, j \in \mathcal{N}, i \neq j \tag{120}$$

$$v_{ij} \leq b_j \quad \forall i \in \mathcal{N}, j \in \mathcal{N}_0, i \neq j \tag{121}$$

$$\sum_{j \in N} u_{ij} = \sum_{j \in N} v_{ji} \quad \forall i \in \mathcal{N}_0 \tag{122}$$

$$v_{ij} \geq u_{ij} + t_i x_{ij} + w_{ij} \quad \forall (i,j) \in A \tag{123}$$

$$u_{ij} \leq \min\{b_i x_{ij}, b_j x_{ij} - w_{ij}\} \quad \forall (i,j) \in A \tag{124}$$

A better PVRP formulation? 2/2

$$v_{ij} \geq \max\{a_j x_{ij}, a_i x_{ij} + w_{ij}\} \quad \forall (i,j) \in A \tag{125}$$

$$\frac{d_{ij}}{u_{ij}} x_{ij} \leq w_{ij} \leq \frac{d_{ij}}{l_{ij}} x_{ij} \quad \forall (i,j) \in \mathcal{A} \tag{126}$$

$$x_{ij} \in \{0,1\}, f_{ij} \geq 0 \quad \forall (i,j) \in \mathcal{A} \tag{127}$$

The Pickup and Delivery Problem with Time Windows 1/2

- n customers
- P is the set of pickup nodes
- D is the set of delivery nodes
- $V = P \cup D \cup \{0, 2n+1\}$ full set of nodes
- Pickup node $i \in P$ is associated with delivery node $n + i \in D$

The Pickup and Delivery Problem with Time Windows 2/2

$$\min \sum_{(i,j)\in A} c_{ij} \sum_{k\in K} x_{ij}^k \tag{128}$$

$$\sum_{k\in K} \sum_{j\in V} x_{ij}^k = 1 \qquad i \in P \tag{129}$$

$$\sum_{j\in V} x_{ij}^k = \sum_{j\in V} x_{n+i,j}^k \qquad i \in P, k \in K \tag{130}$$

$$\sum_{j\in V} x_{ij}^k = \sum_{j\in V} x_{ji}^k \qquad i \in P \cup D, k \in K \tag{131}$$

$$\sum_{j\in V} x_{0j}^k = 1 \qquad k \in K \tag{132}$$

$$\sum_{i\in V} x_{i,2n+1}^k = 1 \qquad k \in K \tag{133}$$

$$w_i^k + s_i + t_{ij} - w_j^k \leq M_{ij}(1 - x_{ij}^k) \qquad k \in K, i, j \in V \tag{134}$$

$$a_i \sum_{j\in V} x_{ij}^k \leq w_i^k \leq b_i \sum_{j\in V} x_{ij}^k \qquad k \in K, i \in P \cup D \tag{135}$$

$$x_{ij}^k \geq 0 \quad \text{integer}, \quad (i,j) \in A, k \in K \tag{136}$$

$$w_i^k \geq 0 \quad i \in V, k \in K \tag{137}$$