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Optimizing Finite Element Integration through Automated Expression Re-writing and Code Specialization

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Abstract goes here

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1. INTRODUCTION

2. PRELIMINARIES

2.1. Quadrature for Finite Element Local Assembly

2.2. Code Generation for Quadrature Representation

Rapid implementation of high performance, robust, and portable code evaluating element matrices using quadrature can be achieved through automated code generation. This has been succesfully proved in the context of the popular FEniCS project [Logg et al. 2012]. The FEniCS Form Compiler (FFC) accepts as input the variational form of a partial differential equation and generates C++ code implementing local assembly routines. The variational form is expressed at high-level by means of the domain-specific Unified Form Language (UFL). Local assembly code must be high performance: as the complexity of a form increases, in terms of number of derivatives, pre-multiplying functions, and polynomial order of the chosen functions, the resulting ker-

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nels evaluating element matrices become more computationally expensive, which impacts significantly the run-time of the overall computation.

Achieving high performance is non-trivial due to the complexity of the mathematical expressions involved in the numerical integration and because of the small sizes of loops and accessed arrays. In [Olgaard and Wells 2010], [Kirby et al. 2005] and [Russell and Kelly 2013], it is shown how automated code generation allows the introduction of powerful optimizations, which a user cannot be expected to write "by hand", as well as the exploration of non-standard integration techniques, based, for instance, on symbolic manipulation. In [**?**] we made one step forward by showing that different problems require distinct set of transformations if close-to-peak performance needs to be obtained, and that low-level, domain-aware code transformations, which are not supported by available compilers, are essential to maximize instruction-level parallelism and register locality.

2.3. Summary of Low-level Optimizations for Quadrature Representation

To neatly distinguish the contributions of this paper from those in [**?**], in this section we summarize the results of our previous work on automated code transformations for quadrature representation.

...TODO...

The work in [**?**] resulted in the development of COFFEE[1], a compiler for the optimization of local assembly kernels relying on quadrature representation.

...TODO...

Temporary arrays can be placed at the right depth in the sorrounding loop nest to store values of sub-expressions that are invariant to one or more loops.

...TODO...

3. A COMPILER FOR OPTIMIZING QUADRATURE-BASED INTEGRATION

In order to generate high performance code, mathematical expressions evaluating the element tensor must be optimized with regards to several interrelated aspects: 1) minimization of floating point operations, 2) instruction-level parallelism, and 3) data locality. In this paper, we tackle these three points building on our previous work ([**?**]).

...We propose a novel structure of how we think a platform-independent, domain-specific optimizing compiler should look like....

...The Expression Re-writer is a software module implemented in COFFEE dealing with the first point...

...The Code Specializer (henceforth CS) is... A key point is that the ER has to perform transformations that do not break the code specializer. Having indirections is really dangerous...

4. EXPRESSION RE-WRITING

As summarized in 2.3, loop-invariant code motion is the key to reduce the computational intensity of a mathematical expression. The Expression Re-writer (henceforth ER) that we have designed and implemented in COFFEE enhances this technique making two steps forward.

Firstly, exploiting arithmetic operations properties like associativity, distributivity, and commutativity, it manipulates the original expression to expose more opportunities to the code hoister. There are many ways an expression can be re-written into, and the search space can become considerably big. Therefore, one problem we solve is finding a sufficiently simple yet systematic way of maximizing the amount of loop-

[1] COFFEE stands for COmpiler For FinitE Element local assembly.

```
for (int i=0; i<I; ++i)
  for (int j=0; j<T; ++j)
    for (int k=0; k<T; ++k)
      A[j][k] += B[i][j]*C[i][k] + (B[i][j]*D[i][k])*F;
```

Fig. 1: Original code

invariant operations in an expression. In Section **??**, we formalize the set of re-writing rules that COFFEE follows to transform an expression.

Secondly, the ER re-structures the loop nest so as to eliminate arithmetic operations over array columns that are statically known to be zero-valued. Zero columns in tabulated basis functions appear, for example, when taking derivatives on a reference element or when using mixed elements. A code transformation eliminating floating point operations on zeros was presented in [Olgaard and Wells 2010]; however, by using indirection arrays in the generated code, it breaks many of the optimizations that can be applied at the Code Specializer level, including SIMD vectorization. In Section 4.2, we show a novel approach to avoiding computation on zeros based on symbolic execution.

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