

Optimal Finite Element Integration

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We tackle the problem of automatically generating optimal finite element integration routines for arbitrary multilinear forms. Optimality is defined in terms of floating point operations required to compute a result. We establish a model under which optimality can systematically be reached and comment on the general applicability of this model. The model is integrated in a compiler such that, given a representation of an integration routine, a loop nest in optimal form can automatically be derived. Extensive performance evaluation proves the robustness of the model, showing systematic performance improvements over a number of alternative code generation systems. The effect of low-level optimization is also discussed.

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1. INTRODUCTION

The need for rapidly implementing high performance, robust, and portable finite element methods has led to approaches based on automated code generation. This has been proved successful in the context of the FEniCS ([Logg et al. 2012]) and Firedrake ([Firedrake contributors 2014]) projects, which have become increasingly popular over the last years. In these frameworks, the weak variational form of a problem is expressed at high-level by means of a domain-specific language. The mathematical specification is manipulated by a form compiler that generates a representation of assembly operators. By applying these operators to an element in the discretized domain, a local matrix and a local vector, which represent the contributions of that element to the equation solution, are computed. The code for assembly operators should be high performance: as the complexity of a variational form increases, in terms of number

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of derivatives, pre-multiplying functions, or polynomial order of the chosen function spaces, the operation count increases, thus making assembly covering a significant fraction of the overall runtime.

As demonstrated across a great deal of works, automating the generation of such high performance implementations poses several challenges. This is a result of the complexity inherent to the mathematical expressions involved in the numerical integration, which varies from problem to problem, and the particular structure of the loop nests enclosing the integrals. General-purpose compilers, such as *GNU's* and *Intel's*, fail at exploiting the structure inherent the expressions, thus producing sub-optimal code (i.e., code which performs more floating-point operations, or “flops”, than necessary). Research compilers, for instance those based on polyhedral analysis of loop nests such as PLUTO ([Bondhugula et al. 2008]), focus on aggressive loop optimizations for cache locality, so they are not particularly helpful in our context. The lack of suitable third-party tools has led to the development of a number of domain-specific code transformation systems or synthesizers. In Olgaard and Wells [2010], it is shown how automated code generation can be leveraged to introduce optimizations that a user should not be expected to write “by hand”. In Kirby et al. [2005] and Russell and Kelly [2013], mathematical reformulations of finite element integration are studied with the aim of minimizing the operation count. In Luporini et al. [2015], the effects and the interplay of generalized code motion and a set of low-level optimizations are analysed. It is also worth mentioning the on-going effort to produce a novel form compiler in FEniCS, called UFLACS ([?]), which adds to the abundant set of code generation (or transformation) systems for assembly operators.

However, in spite of such a considerable research effort, still there is no answer to one fundamental question: can we automatically generate an implementation of a form which is optimal in the number of flops executed? With this paper, we aim to make a first step towards solving this problem.

We summarize the contributions of this research as follows

- We characterize flop-optimality in a loop nest and we instantiate this concept to finite element integration. As part of this construction, we establish the notion of sharing and demonstrate that sharing can always be eradicated from the loop nests we are interested in.
- We provide an incremental, constructive proof that optimal implementations can be synthesized by analyzing each monomial appearing in a form
- We implement the theory in a compiler, COFFEE¹, which is the default optimization system in the Firedrake framework. Our implementation is general: we can apply our results to any forms expressible in Firedrake (and, theoretically, in FEniCS too).
- We provide an unprecedented (for its extension) performance evaluation comparing several code generation or transformation systems in a range of forms of increasing complexity. This is essential to demonstrate our optimality claim.
- We comment on the cases in which our characterization of optimality becomes weaker, and intuitions about how we handle them
- We enrich the set of low-level transformations described in Luporini et al. [2015]

2. PRELIMINARIES

We review finite element integration and possible implementations using notation and examples adopted in [Olgaard and Wells 2010] and [Russell and Kelly 2013].

¹COFFEE stands for COMpiler For Fast Expression Evaluation. The compiler is open-source and available at <https://github.com/coneoproject/COFFEE>

We consider the weak formulation of a linear variational problem

$$\begin{aligned} &\text{Find } u \in U \text{ such that} \\ &a(u, v) = L(v), \forall v \in V \end{aligned} \quad (1)$$

where a and L are, respectively, a bilinear and a linear form. The set of *trial* functions U and the set of *test* functions V are discrete function spaces. For simplicity, we assume $U = V$ and $\{\phi_i\}$ be the set of basis functions spanning U . The unknown solution u can be approximated as a linear combination of the basis functions $\{\phi_i\}$. From the solution of the following linear system it is possible to determine a set of coefficients to express u

$$A\mathbf{u} = b \quad (2)$$

in which A and b discretize a and L respectively:

$$\begin{aligned} A_{ij} &= a(\phi_i(x), \phi_j(x)) \\ b_i &= L(\phi_i(x)) \end{aligned} \quad (3)$$

The matrix A and the vector b are “assembled” and subsequently used to solve the linear system through (typically) an iterative method.

We focus on the assembly phase, which is often characterized as a two-step procedure: *local* and *global* assembly. Local assembly is the subject of this article. It consists of computing the contributions of a single element in the discretized domain to the equation solution. In global assembly, such local contributions are “coupled” by suitably inserting them into A and b .

We illustrate local assembly in a concrete example, the evaluation of the local element matrix for a Laplacian operator. Consider the weighted Laplace equation

$$-\nabla \cdot (w \nabla u) = 0 \quad (4)$$

in which u is unknown, while w is prescribed. The bilinear form associated with the weak variational form of the equation is:

$$a(v, u) = \int_{\Omega} w \nabla v \cdot \nabla u \, dx \quad (5)$$

The domain Ω of the equation is partitioned into a set of cells (elements) T such that $\bigcup T = \Omega$ and $\bigcap T = \emptyset$. By defining $\{\phi_i^K\}$ as the set of local basis functions spanning U on the element K , we can express the local element matrix as

$$A_{ij}^K = \int_K w \nabla \phi_i^K \cdot \nabla \phi_j^K \, dx \quad (6)$$

The local element vector L can be determined in an analogous way.

2.1. Quadrature Mode

Quadrature schemes are conveniently used to numerically evaluate A_{ij}^K . For convenience, a reference element K_0 and an affine mapping $F_K : K_0 \rightarrow K$ to any element $K \in T$ are introduced. This implies a change of variables from reference coordinates X_0 to real coordinates $x = F_K(X_0)$ is necessary any time a new element is evaluated. The numerical integration routine based on quadrature over an element K can be expressed as follows

$$A_{ij}^K = \sum_{q=1}^N \sum_{\alpha_3=1}^n \phi_{\alpha_3}(X^q) w_{\alpha_3} \sum_{\alpha_1=1}^d \sum_{\alpha_2=1}^d \sum_{\beta=1}^d \frac{\partial X_{\alpha_1}}{\partial x_{\beta}} \frac{\partial \phi_i^K(X^q)}{\partial X_{\alpha_1}} \frac{\partial X_{\alpha_2}}{\partial x_{\beta}} \frac{\partial \phi_j^K(X^q)}{\partial X_{\alpha_2}} \det F'_K W^q \quad (7)$$

where N is the number of integration points, W^q the quadrature weight at the integration point X^q , d is the dimension of Ω , n the number of degrees of freedom associated to the local basis functions, and \det the determinant of the Jacobian matrix used for the aforementioned change of coordinates.

2.2. Tensor Contraction Mode

Starting from Equation 7, exploiting linearity, associativity and distributivity of the involved mathematical operators, we can rewrite the expression as

$$A_{ij}^K = \sum_{\alpha_1=1}^d \sum_{\alpha_2=1}^d \sum_{\alpha_3=1}^n \det F_K^l w_{\alpha_3} \sum_{\beta=1}^d \frac{X_{\alpha_1}}{\partial x_{\beta}} \frac{\partial X_{\alpha_2}}{\partial x_{\beta}} \int_{K_0} \phi_{\alpha_3} \frac{\partial \phi_{i_1}}{\partial X_{\alpha_1}} \frac{\partial \phi_{i_2}}{\partial X_{\alpha_2}} dX. \quad (8)$$

A generalization of this transformation has been proposed in [Kirby and Logg 2007]. By only involving reference element terms, the integral in the equation can be pre-evaluated and stored in temporary variables. The evaluation of the local tensor can then be abstracted as

$$A_{ij}^K = \sum_{\alpha} A_{i_1 i_2 \alpha}^0 G_K^{\alpha} \quad (9)$$

in which the pre-evaluated “reference tensor” $A_{i_1 i_2 \alpha}$ and the cell-dependent “geometry tensor” G_K^{α} are exposed.

2.3. Qualitative Comparison

Depending on form and discretization, the relative performance of the two modes, in terms of the operation count, can vary even quite dramatically. The presence of derivatives or coefficient functions in the input form tends to increase the size of the geometry tensor, making the traditional quadrature mode more indicate for “complex” forms. On the other hand, speed ups from adopting tensor mode can be significant in a wide class of forms in which the geometry tensor remains “sufficiently small”. The discretization, particularly the relative polynomial order of trial, test, and coefficient functions, also plays a key role in the resulting operation count.

These two modes have been implemented in the FEniCS Form Compiler ([Kirby and Logg 2006]). In this compiler, a heuristic is used to choose the most suitable mode for a given form. It consists of analysing each monomial in the form, counting the number of derivatives and coefficient functions, and checking if this number is greater than a constant found empirically ([Logg et al. 2012]). We will later comment on the efficacy of this approach (Section 4). For the moment, we just recall that one of the goals of this research is to produce an intelligent system that is capable of reasoning at the monomials level – not at the form level – without affecting the code generation cost.

3. OPTIMALITY OF LOOP NESTS

In this section, we provide a characterization of optimality for loop nests of various nature. In order to make the document self-contained, we start with reviewing basic compiler terminology.

Definition 1 (Perfect and imperfect loop nests). *A loop nest is said to be perfect when non-loop statements appear only in the body of the innermost loop. If this condition does not hold, a loop nest is said imperfect.*

Definition 2 (Independent basic block). *A sequence of statements that present no data dependencies compose an independent basic block*

We focus on perfect nests whose innermost loop body is an independent basic block. A straightforward property of this class is that hoisting invariant expressions from the innermost to any of the outer loops or the preheader (i.e., the block that precedes the entry point of the nest) is always safe, as long as the obvious loop dependencies are preserved. We will make use of this property. The results of this section could also be generalized to larger classes of loop nests, in which basic block independence does not hold, although this would require a refinement that goes beyond the scope of this paper.

We find convenient abstracting mathematical properties at the level of loops

Definition 3 (Linear loop). *A loop L defining the iteration space I through the iteration variable i , or simply L_i , is linear if all expressions appearing in the body of L that use i to access some memory locations are linear functions over I .*

In particular, we are interested in the following class since it naturally arises from the math of finite element integration, reviewed in Section 2.

Definition 4 (Perfect multilinear loop nest). *A perfect multilinear loop nest of arity n is a perfect nest composed of n loops, in which all of the expressions appearing in the body of the innermost loop are linear in each loop L_i separately.*

Note that perfect multilinear loop nests could actually be rooted in deeper, possibly imperfect loop nests. In fact, we will focus on this particular structure.

We introduce the fundamental notion of sharing.

Definition 5 (Sharing). *A loop L_i presents sharing if it contains at least two expressions depending on i that are symbolically identical.*

Figure 1 shows an example of a trivial multilinear loop nest of arity $n = 2$ with sharing along dimension j .

```
for (j = 0; j < 0; j++)
  for (i = 0; i < N; i++)
    A[i,j] += B[j]*C[i]*b + B[j]*D[i]*c
```

Fig. 1: Multilinear loop nest with sharing

We now have the ingredients to formulate a simple yet fundamental result.

Proposition 1. *Assume $LN = [L_{i_0}, L_{i_1}, \dots, L_{i_{n-1}}]$ is a perfect multilinear loop nest with the body of $L_{i_{n-1}}$ being an independent basic block. Then sharing can always be eliminated from LN .*

Proof. The demonstration is by construction and exploits linearity. We want to transform LN into LN' such that there is no sharing in any $L_i \in LN'$. Starting from the innermost loop $L_{i_{n-1}}$, the expressions are "flattened" by expanding all products involving terms depending on $L_{i_{n-1}}$. Being now on the same level of the expression tree, such terms can be factorized. Due to linearity, each factored product only has one term depending on $L_{i_{n-1}}$, and such term is now unique in the expression. The other terms, independent of $L_{i_{n-1}}$, are, by definition, loop-invariant, and as such can be hoisted at the level of $L_{i_{n-2}}$. This procedure can be applied recursively up to L_{i_0} : multilinearity allows factorization at each level; perfectness ensures hoisting is safe. \square

This result will be essential to prove that optimality, characterized as follows, can always be reached in this class of loops.

Definition 6 (Optimality of a multilinear loop nest). *The synthesis of a multilinear loop nest is optimal if the amount of operations performed in the innermost loop is minimum.*

In other words, optimality implies there is no other synthesis able to further decrease the number of operations in the innermost loop. Note that this definition does not take into account memory requirements. If the loop nest were memory-bound – the ratio of operations to bytes transferred from memory to the CPU being too low – then speaking of “optimality” would clearly make no sense. In the following, we assume to operate in a CPU-bound regime, in which arithmetic-intensive expressions need be evaluated. This suits the context of finite element integration.

A second result follows.

Proposition 2. *A perfect multilinear loop nest LN with basic block independence can always be reduced to optimal form.*

Proof. By construction. Loop-dependent terms are logically grouped into n disjoint sets S_i , each S_i containing all terms depending on L_i . These sets are sorted in descending order based on their cardinality. We produce a new loop permutation that reflects the S_i ordering; that is, loops are placed such that, as going down the nest, L_i is characterized by less unique terms than L_{i-1} . Note the loop permutation is semantically correct because of perfectness. Starting from $L_{i_{n-1}}$, we can apply the sharing-removal procedure described in Proposition 1. This renders $L_{i_{n-1}}$ optimal. In particular, the number of operations is equal to $LN_{ops} = \#S_{i_{n-1}} + (\#S_{i_{n-1}} - 1)$, in which the second term represents the cost of summing the $\#S_{i_{n-1}}$ terms. \square

For example, by applying the construction described in Proposition 2 to the code in Figure 1, we obtain the optimal form in Figure 2.

```

for (i = 0; i < N; i++)
  T = C[i]*b + D[i]*c
  for (j = 0; j < O; j++)
    A[i,j] += B[j]*T

```

Fig. 2: Multilinear loop nest in optimal form

We now consider a wider class of loop nests in which perfectness and multilinearity apply to a sub-nest only. Consider the example in Figure 3.

```

for (e = 0; e < L; e++)
  ...
  for (i = 0; i < M; i++)
    ..
    for (j = 0; j < N; j++)
      for (k = 0; k < O; k++)
        A[e,j,k] += F(...)

```

Fig. 3: Loop nest example

The imperfect nest $LN = [L_e, L_i, L_j, L_k]$ comprises a reduction loop L_i and a perfect doubly nested loop $[L_j, L_k]$, which we assume to be multilinear. The right hand side of the statement computing the multidimensional array A is a generic expression F including standard arithmetic operations such as addition and multiplication. We observe that F might contain sub-expressions that depend on a subset of loops only and, as such, hoistable outside of L_k as long as data dependencies are preserved (note

that LN is imperfect). One could think of searching for sub-expressions for which the reduction could be pre-evaluated, thus obtaining a decrease proportional to M in the operation count. However, finding or exposing such sub-expressions requires, in general, a full exploration of the expression tree transformation space. This is obviously challenging, and so is the problem of synthesizing an optimal LN' starting from LN . Even though we assumed we could generate LN' through a suitable transformation algorithm, the following issues should be addressed

- as opposed to what happens with hoisting in perfect multilinear loop nests, the temporary variable size would be proportional to the number of non-reduction loops crossed (in the example, $N \cdot O$ for sub-expressions depending on $[L_i, L_j, L_k]$ and $L \cdot N \cdot O$ for those depending on $[L_e, L_i, L_j, L_k]$). This might shift the loop nest from a CPU-bound to a memory-bound regime, which might be counter-productive for actual run-time
- the transformations exposing invariant sub-expressions could require expansion and factorization. In terms of number of operations, the save originating from hoisting could be overwhelmed by the increase in arithmetic complexity within L_k (expansion can increase the operation count, e.g. $A(B + C) = AB + AC$).

We shall then refine our definition of optimality for generic loop nests as follows

Definition 7 (Optimality of a loop nest). *The synthesis of a loop nest is optimal if, under a set of memory constraints C , the total amount of operations performed in all innermost loops is minimum.*

Note how the definition contemplates the possibility for a nest to have multiple innermost loops. In fact, multiple sub-nests could be rooted in the outermost loop, either because part of the input or as a result of some transformations.

4. SYNTHESIS OF OPTIMAL LOOP NESTS IN FINITE ELEMENT INTEGRATION

In this section, we instantiate Definition 7 in our domain of interest, finite element integration. This will require reasoning at two different levels of abstraction: the math, in terms of the multilinear forms arising from the weak variational formulation of a problem, which we reviewed in Section 2; and the (partly multilinear) loop nests implementing such forms.

Our point of departure is the example loop nest in Figure 3. This loop nest is actually a simplified view of a typical bilinear form implementation. L_e represents iteration over the elements of a mesh; L_i derives from using numerical quadrature; $[L_j, L_k]$ implement the computation over test and trial functions. We deliberately omitted useless portions of code to not hinder readability (e.g. matrix insertion) and to avoid tying our discussion to specific forms (e.g. F is unspecified).

We make the following observations. 1) $L \gg \gg M, N, O$; that is, the number of elements L is typically order of magnitude larger than both quadrature points (M) and degrees of freedom (N and O for test and trial functions); 2) $[L_j, L_k]$ (or simply L_j with a linear form) is perfect and multilinear; this naturally descends from the translation of Equation 7 into a loop nest.

4.1. Memory constraints

The fact that the iteration space of L_e is so larger than that of other loops suggests we should be cautious when hoisting out of LN . Imagine a time stepping loop L_t wraps LN . One could think of identifying time-invariant sub-expressions that access both geometry and reference element terms, and pre-evaluate them within L_t . Unless adopting complex engineering solutions (e.g. aggressive blocking), which are practically difficult to devise and maintain, this kind of code motion increases the working set by a

factor L . It is our opinion that the drop in number of operations would be overwhelmed, from the run-time viewpoint, by a much larger memory pressure.

A second, more general observation is that, for certain forms and discretizations, aggressive hoisting can make the working set exceed the size of “some level of local memory” (e.g. the last level of private cache on a conventional CPU, the shared memory on a GPU). We will provide precise details about this in the following sections. For the moment, and just as an example, note that applying tensor contraction mode (see Section 2), which essentially means lifting code outside of L_e , requires a set of temporary arrays of size $N \cdot O$; with some discretizations, this can break the local memory threshold.

Based upon these considerations, we add two constraints to C (see Definition 7)

- (1) The size of a temporary due to code motion cannot be bigger than the size of the multilinear loop nest iteration space.
- (2) The total size of the hoisted temporaries cannot exceed a threshold T_H

A corollary of C_1 is that hoisting expressions involving geometry terms outside of L_e is now forbidden. Consequently, the search space for optimality becomes smaller.

4.2. Minimizing the Operation Cost

Definition 7 states that a necessary condition for a loop nest synthesis to be optimal is that the sum of the operation cost in the innermost loops is minimum. We now discuss how we can systematically achieve this.

Eliminating sharing from the inner multilinear loops does not suffice. In fact, as suggested in Section 7, we wonder whether, and under what transformations, the reduction imposed by L_i could be pre-evaluated, thus reducing the operation count.

To answer this question, we make use of a result – the foundation of tensor contraction mode – from Kirby and Logg [2007]. Essentially, multilinear forms can be seen as sums of monomials, each monomial being an integral over the equation domain of products (of derivatives) of functions from discrete spaces; such monomials can always be reduced to a product of two tensors (see Section 2). We interpret this result at the loop nest level: with the input as in Figure 3, we can always dissect F into distinct sub-expressions (the monomials). Each sub-expression is then factorized so as to split constant from $[L_i, L_j, L_k]$ -dependent terms, the latter ones are hoisted outside of LN , and finally pre-evaluated into temporaries. As part of this pre-evaluation, the reduction induced by L_i vanishes. In the following, we simply refer to this special sort of code hoisting as “*pre-evaluation*”.

The challenge is to understand when, and for which monomials, pre-evaluation is profitable. We propose an algorithm and a discussion of its optimality. The intuition of the main algorithm for reducing a loop nest to optimal form is shown in Figure 4.

The point of departure consists of understanding the impact, as number of operations saved or introduced, of pre-evaluation. This is studied “locally”; that is, for each monomial, in isolation. If we estimate that, for a given monomial, pre-evaluation will decrease the operation count, then the corresponding sub-expression is extracted, a sequence of transformation steps – involving expansion, factorization, code motion – takes place (details in Section 5), and the evaluation eventually performed. The result is a set of n -dimensional tables (these can be seen as “slices” of the reference tensor at the math level), n being the arity of the multilinear form. Identical tables are mapped to the same temporary. Eventually, sharing is removed from the resulting expressions by applying a procedure as described in Proposition 2. The transformed loop nest is as in Figure 5.

Before elaborating on the profitability of pre-evaluation, we need to discuss under which conditions this approach, based on a “local analysis” of monomials, is optimal.


```

dissect the input expression into monomials
for each monomial M:
     $\theta_w$  = estimate operation count with pre-evaluation
     $\theta_{wo}$  = estimate operation count without pre-evaluation
    if  $\theta_w < \theta_{wo}$  and memory constraints satisfied:
        mark M as candidate for pre-evaluation
for each monomial M:
    if M does not share terms with M', an unmarked monomial:
        extract M into a separate loop nest
        apply pre-evaluation to M
for each expression:
    remove sharing

```

Fig. 4: Intuition of the main algorithm

```

for (e = 0; e < L; e++)
    ...
    // Pre-evaluated tables
    ...
    // Loop nests for each dissected monomial
    for (j = 0; j < N; j++)
        for (k = 0; k < O; k++)
            A[e,j,k] += F(...)
        for (k = 0; k < O; k++)
            A[e,j,k] += G(...)
    ...
    // Loop nest for monomials for which run-time
    // integration (/i/ loop) is preferable
    for (i = 0; i < M; i++)
        ...
        for (j = 0; j < N; j++)
            for (k = 0; k < O; k++)
                A[e,j,k] += H(...)

```

Fig. 5: Optimized Loop nest example

Proposition 3. Consider an expression comprising a set of monomials M . Let P be the set of pre-evaluated monomials, determined as described in Figure 4, and be $Z = M \setminus P$. Assume that:

- (1) the cost function employed is optimal; that is, it predicts correctly whether pre-evaluation is profitable or not for a monomial
- (2) pre-evaluating distinct monomials does not produce identical tables
- (3) monomials in P do not share terms

Then, the loop nest LN is optimal under memory constraints C , once sharing is removed.

Proof. We first comment on the assumptions. 1) The creation of an optimal cost function is discussed in Section 4.3. 2) Identical pre-evaluated tables from distinct monomials could be the result of symmetries in tabulated basis functions. This is however a pathological case. 3) In complex forms with several monomials, different pre-evaluation candidates could actually share terms. For simplicity, we do not handle this case, which otherwise would require a “global” analysis of the monomials in the form. Further, even though, in practice, assumptions 2) and 3) did not hold, the resulting loop nest would still be nearly-optimal.

We distinguish two classes of loop nests rooted in LN : $[L_e, L_j, L_k]$, for the pre-evaluated monomials in P , and $[L_e, L_i, L_j, L_k]$, enclosing the remaining monomials in Z . Since they only differ for the presence of L_i , we relieve the notation by omitting the shared loops when discussing operation counts. The operation count of what we are proving to be the optimal LN synthesis is $LN_{ops} = LN_{ops_1} + LN_{ops_2} = \sum_{\alpha}^{#P} p_{\alpha} + I \sum_{\beta}^{#Z} z_{\beta}$, where p_{α} and z_{β} represent the operation cost of monomials in P and Z , respectively, while I is the iteration space size of L_i . This can also be inferred from Figure 5.

We note that, as explained in Section 4.1, C imposes constraints on hoisting. This narrows the proof to demonstrating the following: A) pre-evaluating any $Z_P : Z_P \subseteq Z$ would increase LN_{ops} ; B) not pre-evaluating any $P_Z : P_Z \subseteq P$ would increase LN_{ops} .

A) We prove that $LN'_{ops} = LN'_{ops_1} + LN'_{ops_2} > LN_{ops}$. It is rather obvious that $LN'_{ops_1} \geq LN_{ops_1}$ (it is equal only if, trivially, $Z_P = \emptyset$). We note that if monomials in Z_P share terms with $\bar{Z} = Z \setminus Z_P$, then we have $LN'_{ops_2} = LN_{ops_2}$, so our statement is true. If, on the other hand, at least one monomial does not share any terms, we obtain $LN'_{ops_2} < LN_{ops_2}$ or, equivalently, $LN'_{ops_2} = LN_{ops_2} - I \cdot \delta$. What we have to show now is that even by exposing more pre-evaluations, $LN'_{ops_1} - LN_{ops_1} = \gamma \geq I \cdot \delta$ holds. Because of assumption 2), the new pre-evaluations cannot expose further sharing. Therefore, the optimality of the cost function (assumption 1) ensures our claim holds.

B) In absence of sharing, the statement is trivially true since we would have $LN'_{ops_2} > LN_{ops_2}$, being the cost function optimal due to assumption 1). Assumption 3) guarantees there can be no sharing within P_Z , which avoids subtle cases wherein pre-evaluation would result sub-optimal due to destroying sharing-removal opportunities. The last case we have to consider is when $p \in P_Z$ shares at least one term with $z \in Z$. This situation cannot actually occur by construction: all candidates for pre-evaluation sharing terms with monomials in Z are “de-classified” from P to Z : the rationale is that we would have to pay anyway the presence of the shared terms in the innermost loop due to z , so adding p to Z does not augment the operation count. \square

4.3. A-Priori Operation Counting

It remains to tie one loose hand: the construction of the pre-evaluation cost function. We introduce the cost function $\theta : M \rightarrow \mathbb{N} \times \mathbb{N}$ that, given a monomial, returns two natural numbers representing the optimal operation count without (θ_{wo}) and with (θ_w) pre-evaluation. Since θ is expected to be used by a compiler to drive the transformation process, requirements are simplicity and velocity.

We can easily predict θ_{wo} thanks to our key property, linearity. As explained in Proposition 2, a simple analysis suffices to obtain the cost of a sharing-free multilinear loop nest, namely MLN_{ops} . Then, assuming I to be the size of the L_i iteration space, we simply have that $\theta_{wo} = MLN_{ops} \cdot I$.

For θ_w , things are more complicated. We first need to account for the presence of (derivatives of) coefficients to estimate the *increase factor* ι . This number captures the increase in arithmetic complexity due to the transformations enabling pre-evaluation. To contextualize, consider the example in Figure 6.

```

for (i = 0; i < M; i++)
  for (j = 0; j < N; j++)
    for (k = 0; k < 0; k++)
      A[j,k] += B[i,j]*B[i,k]*(f[0]*B[i,0]+f[1]*B[i,1]+f[2]*B[i,2])

```

Fig. 6: Optimized Loop nest example

One can think of this as the (simplified) loop nest originating from the assembly of a pre-multiplied mass matrix. The sub-expression $f[0]*B[i,0]+f[1]*B[i,1]+f[2]*B[i,2]$ represents the field f over (tabulated) basis functions B . In order to apply pre-evaluation, the expression needs to be transformed to separate f from the integration-dependent (i.e., L_i -dependent) quantities. By expanding the product we observe an increase in the number of L_k -dependent operations of a factor 3 (the local degrees of freedom for the coefficient). Intuitively, ι captures such a growth in non-hoistable operation.

With just a single coefficient, ι directly descends from the cost of expansion. In general, however, the calculation is less straightforward. Consider, for example, the case in which multiple coefficients originate from the same function space. Expansion would then lead to identical terms (i.e., identical pre-evaluated tables). Therefore, for a precise estimate of ι , we instead need to calculate the k -combinations with repetitions of n elements, with k being the number of coefficient-dependent terms appearing in a product (in the example, there is only f , so $k = 1$) and n the cardinality of the set of symbols involved in the coefficient expansion (in the example, $B[i,0]$, $B[i,1]$, and $B[i,2]$, so $n = 3$; note that we are considering sets, so duplicates would be counted as 1).

If $\iota \geq I$ we can immediately say that pre-evaluation will not be profitable. This is indeed a necessary condition that, intuitively, tells us that if we add to the innermost loop more operations than we actually save from eliminating L_i , then for sure $\theta_{wo} < \theta_w$. This observation can speed up the compilation time by decreasing the analysis cost.

If, on the other hand, $\iota < I$, a further step is necessary to estimate θ_w . In particular, we need to calculate the number terms ρ such that $\theta_w = \rho \cdot \iota$. Consider again Figure 6. In the case of the mass matrix, the body of L_k is simply characterized by the dot product of test and trial functions, $B[j]*B[k]$, so trivially $\rho = 1$. In general, ρ varies with the discretization and the differential operators employed. For example, in the case of the bi-dimensional Poisson equation, after a suitable factorization, we have $\rho = 3$. There are several ways of determining ρ . The fastest would be to extract it from high-level analysis of the operators in the form; for convenience, in our implementation we instead project (by analysis of the expression tree) the output of expansion and factorization in the monomial.

5. CODE GENERATION

The analysis described in Section 4 has been fully automated in COFFEE, the optimization system for local assembly used in Firedrake. In this section, we describe the structure of the code generation system and we also comment on a set of low-level optimizations.

5.1. Automation through the COFFEE Language

As opposed to what happens in FFC with quadrature and tensor modes, there are no separate trunks in COFFEE handling pre-evaluation, sharing, and code motion in general: all optimizations are expressed as composition of parametric “building-block” operations. This has several advantages. Firstly, extendibility: novel transformations – for instance, sum-factorization in spectral methods – could ideally be expressed using the existing operators, or with small effort building on what is already available. Secondly, generality: other domains sharing properties similar to that of finite element integration (e.g., multilinear loop nests) could be optimized through the same compiler. Thirdly, robustness: the same building-block operations are exploited, and therefore stressed, by different optimization pipelines.

A non-exhaustive list of such operations includes expansion, factorization, re-association, generalized code motion. These are parametric operations. For example, one could ask to factorize only constant or only linear terms, while hoisting can be lim-

ited by imposing a constraint on the size of the temporaries. Each of these operations is implemented by suitable manipulation of the abstract syntax tree representing the integration routine.

5.1.1. Heuristic Optimization of Integration-dependent Expressions. As a proof-of-concept of our generality claim, we briefly discuss our optimization strategy for integration-dependent expressions. These are expressions that should logically be placed within L_i . They can originate, for example, from the extensive use of tensor algebra in the derivation of the weak variational form or from the use of a non-affine reference-to-physical element mapping, which Jacobian needs be re-evaluated at every quadrature point. For some complex monomials (e.g., see Olgaard and Wells [2010] and for coarser discretizations, the operation count within L_i could be comparable or, in some circumstances, outweigh that in the multilinear loop nest. In these cases, our definition of optimality becomes weaker, since the underlying assumption is that the bulk of the computation is carried out in innermost loops.

Despite the fact that we are not characterizing optimality for this wider class of loops, we can still heuristically apply the same reasoning of Sections 3 and 4, particularly for removing sharing, to minimize the operational cost. This is straightforward in our code generation system by composing rewrite operators. Our strategy is as follows.

COFFEE is agnostic with respect to the high level form compiler, so the first step consists of removing redundant sub-expressions. This is because a form compiler abstracting from optimization will translate expressions as in Equation 7 directly to trivial code without performing any sort of transformation. Eliminating redundant sub-expressions is usually helpful to relieve the arithmetic pressure inherent to L_i . We then synthesize an optimal loop nest as described in the previous sections. This may in turn expose a set of L_i -dependent statements evaluating some temporaries. For each of this statements, we try to remove sharing by greedily applying factorization and code motion. In the COFFEE language, this process is expressed by simply concatenating five operators (i.e., five function calls named “factorize”, “expand”, etc.).

5.2. Low-level Optimization

We comment on a set of low-level optimizations. These are essential to 1) achieve machine peak performance (Sections 5.2.1 and 5.2.2) and 2) make COFFEE independent of the high-level form compiler (Section 5.2.3). As we will see, there are interplays among different transformations. For completeness, we present all of the transformations available in the compiler, although we will only use a subset of them for a fair performance evaluation.

5.2.1. Review of Existing Optimizations. We start with briefly reviewing the low-level optimizations presented in Luporini et al. [2015].

Padding and data alignment. All of the arrays involved in the evaluation of the local element matrix or vector are padded to a multiple of the vector register length. This is a simple yet powerful transformation that maximizes the effectiveness of vectorization. Padding, and then loop bounds rounding, enable data alignment and avoid the introduction of scalar remainder loops.

Vector-register Tiling. Blocking (or tiling) at the level of vector registers improves data locality beyond traditional unroll-and-jam transformations. The blocking strategy consists of evaluating outer products by just using two vector register without ever spilling to cache.

Expression Splitting. When the number of basis functions arrays (or, equivalently, temporaries introduced by code motion) and constants is large, the chances of spilling

to cache are high in architectures with a relatively low number of logical registers (e.g. 16/32). By exploiting sums associativity, an expression can be fissioned so that the resulting sub-expressions can be computed in a separate loop nest.

5.2.2. Vector-promotion of Integration-dependent Expressions. Integration-dependent expressions are inherently executed as scalar code because vectorization (unless employing special hand-written schemes) occurs along a single loop, typically the innermost. In our loop nest, L_i is clearly an outer loop. In situations such as in Section 5.1.1, we may want to vectorize also along L_i . One way to achieve this is vector-promotion. This implies creating a clone loop of L_i in the preheader of the main loop nest, in which vector temporaries, instead of scalar ones, are evaluated.

5.2.3. Handling Block-sparse Tables. Consider a tabulated set of basis functions having quadrature points along the rows and functions along the columns. For example, $A[i, j]$ provides the value of the j -th basis function at quadrature point i . Unless using a smart form compiler (and one of our goals is precisely to relieve it from the burden of smart code generation), there are several circumstances in which the tables can be block-sparse. Zero-valued columns arise when taking derivatives on a reference element or when employing vector-valued elements. Zero-valued rows can result from using non-standard functions spaces, like Raviart-Thomas. Zero-valued blocks can appear in pre-evaluated temporaries. Our objective is a general system that avoids useless iteration over these zero-entries while preserving the effectiveness of the other low-level optimizations, especially vectorization.

In Olgaard and Wells [2010], a technique to avoid iteration over zero-valued columns based on the use of indirection arrays (e.g. $A[B[i]]$, in which A is a tabulated basis function and B a map from loop iterations to non-zero columns in A) was proposed. Our approach, which will be compared to this pioneering work, aims to free the generated code from any indirection arrays. This is because we want to avoid non-contiguous memory loads and stores, which can nullify the benefits of vectorization.

The idea is that if the dimension along which vectorization is performed (typically the innermost) has a contiguous slice (rows, columns) of zero-valued entries, and if that slice is smaller than the vector length, that we do nothing (i.e., the loop nest is not transformed). Otherwise, we restructure the loop nest. This has several complicated implications, including loop fission and a memory offsetting usually dangerous for padding and data alignment, which are, however, beyond the scope of this paper. The implementation is based on symbolic execution: the loop nests are traversed and for each statement the location of zeros in each of the involved symbols is tracked. Arithmetic operators have a different impact on the tracking. For example, for each loop dimension, multiplication requires computing the set intersection of the zero-valued slices, whereas addition requires computing the set union.

6. PERFORMANCE EVALUATION

6.1. Experimental Setup

Experiments were run on a single core of an Intel I7-2600 (Sandy Bridge) CPU, running at 3.4GHz, 32KB L1 cache and 256KB L2 cache. The Intel Turbo Boost and Intel Speed Step technologies were disabled. The Intel icc 15.2 compiler was used. The compilation flags used were `-O3`, `-xHost`, `-ip`, which can trigger AVX autovectorization.

We analyze the runtime performance in four real-world bilinear forms of increasing complexity, which comprise the differential operators that are most common in finite element methods. In particular, we study the mass matrix (“Mass”) and the bilinear forms arising in a Helmholtz equation (“Helmholtz”), in an elastic model

(“Elasticity”), and in a hyperelastic model (“Hyperelasticity”). The complete specification of these forms is made publicly available ([ufl 2014]).

We evaluate the speed ups achieved by a wide variety of transformation systems over the original code (i.e., no optimizations applied) as returned by the FEniCS Form Compiler. We analyze the impact of

- FEniCS Form Compiler’s built-in optimization system: optimized quadrature mode (work presented in Olgaard and Wells [2010]). Referred to as `quad`
- FEniCS Form Compiler’s built-in optimization system: tensor mode (work presented in Kirby and Logg [2006]). Referred to as `tens`
- FEniCS Form Compiler’s built-in optimization system: automatic mode (choice between `tens` and `quad` driven by heuristic, discussed in Logg et al. [2012]). Referred to as `auto`
- UFLACS: a novel back-end for the FEniCS Form Compiler (whose primary goals are improved code generation time and runtime). Referred to as `ufls`
- COFFEE: generalized loop-invariant code motion and padding (work presented in Luporini et al. [2015]). Referred to as `cf01`
- COFFEE: optimal multilinear loop nest synthesis, padding and symbolic execution (captures the contributions of this paper). Referred to as `cf02`

The values that we report are the average of three runs with “warm cache” (no code generation time, no compilation time). They include the cost of local assembly as well as the cost of matrix insertion. However, the unstructured mesh used to run the simulations was chosen small enough to fit the L3 cache of the CPU, so as to minimize the “noise” due to operations outside of the element matrix evaluation.

For a fair comparison, small patches (publicly available) were written to be able to run all simulations through Firedrake: this means the cost of matrix insertion and mesh iteration is exactly the same in all variants. UFLACS and the FEniCS Form Compiler’s optimization systems generate code suitable for FEniCS, which employs a data storage layout different than Firedrake. Our patches fix this problem by producing code with a data storage layout as expected by Firedrake.

Since we are running sequential programs, with respect to the memory constraint C_2 introduced in Section 4.1, we set $T_H = \infty$; that is, temporaries can occupy an arbitrary amount of memory. This is because a single process that has exclusive access to all of the bandwidth available in the system can count on pre-fetching to minimize cache misses. If, on the other hand, a parallel program were run, then a more sensible value for T_H would be the size of the last level of private cache; this would avoid memory bank conflicts and, therefore, potential drops in runtime.

Following the methodology adopted in Olgaard and Wells [2010], we increasingly vary the complexity of each form. In particular:

- the polynomial order of test and trial functions, $q \in \{1, 2, 3, 4\}$. Test and trial functions always have same degree
- the polynomial order of coefficient (or “pre-multiplying”) functions, $p \in \{1, 2, 3, 4\}$
- the number of coefficient functions $nf \in \{0, 1, 2, 3\}$

Constants of our analysis are instead

- the space of test, trial, and coefficient functions, Lagrange
- the mesh, tetrahedral with a total of 4374 elements

The upcoming Figures (7, 8, 9, and 10) are “nested” plots. We refer to the outer plot as the “grid”, while the inner are the actual “plots”. In a grid, p varies along the horizontal axis and q varies along the vertical axis. The top-left plot in a grid shows the speed up over unmodified code for $[q = 1, p = 1]$; the plot on its right for $[q = 1, p = 2]$, and

so on. The diagonal of the grid provides the behaviour when test, trial and coefficient functions have same polynomial order, that is $q = p$. A grid can therefore be looked at from different perspectives, which allows us to make structured considerations on the performance achieved. In each plot there are six groups of bars, each group referring to a particular code variant (quad, tens, ...). There are four bars per group: the leftmost bar corresponds to the case $nf = 0$, the one on its right to the case $nf = 1$, and so on.

6.2. Performance of Forms

The first observation is that our optimality model does not imply minimum execution time. In particular, in the 0.03% of the test cases (we are not counting marginal differences), cf02 does not result in the best runtime. There are two reasons for this. The former is that low-level optimization can have a significant impact. Compare, as an example, tens and cf02 in Mass and Helmholtz when $q = 1$. With $p \in [2, 3]$, tens tends to slightly outperform cf02. With $p = 4$, tens is quite faster than cf02. The motivation is the aggressive unrolling performed in tens, which 1) allows to eliminate all operations involving zero-valued entries and 2) can drastically decrease both working set and register pressure by avoiding storing entire temporaries. In COFFEE we never unroll loops to maximize the chances of autovectorization, which usually results in considerable speed ups throughout the majority of test cases. The latter reason is simply inherent to our optimality model. As explained in Section 5.1.1, we use heuristics to optimize the integration loop. The reason we are studying an hyperelastic model is indeed to assess their effectiveness; the systematic performance improvements observed over all other variants are, however, extremely encouraging.

A second observation concerns the “triggering” of pre-evaluation in cf02. The fact that a bar in cf02 matches the corresponding bar in cf01 usually suggests that pre-evaluation was found unprofitable. The trend – although there are quite a few exceptions – is that as nf increases, pre-evaluation tends not to be performed. This is a consequence of a larger increase factor, as discussed in Section 4.3.

Also note some other minor things.

- The lack of improvements in all code variants when $[q = 1, p = 1]$ is due to the cost of matrix insertion, which outweighs that of local assembly.
- The slow downs experienced with quad are imputable to the presence of indirection arrays in the generated code; especially with dense tables, trading vectorization for avoiding iteration over all zero-valued columns may be counter-productive (see also Section 5.2.3).
- A missing bar in tens or auto means that the code generation system failed because of either exceeding the memory limit or being unable to manipulate the math characterizing the form (this happens in the case of the hyperelastic model).
- The bars for $nf = 0$ do not change as p increases: this is expected since varying p when there are no coefficients should not make any difference.

Mass. From analysis of Figure 7, we can see that cf02 represents the best variant among the ones tested, apart from the few points previously discussed. This suggests that the optimality model holds in these simple bilinear forms; that is, cf02 generates loop nests that are optimal from both a theoretical and practical viewpoints. It is worthwhile noting how auto is not capable of selecting the proper synthesis in many of the test cases, especially as $nf \geq 2$.

Helmholtz. Figure 8 shows that the performance improvements achieved by various code variants over a non-optimized implementation can be significant. We appreciate the fact that the relative order of the employed function spaces plays a key role in

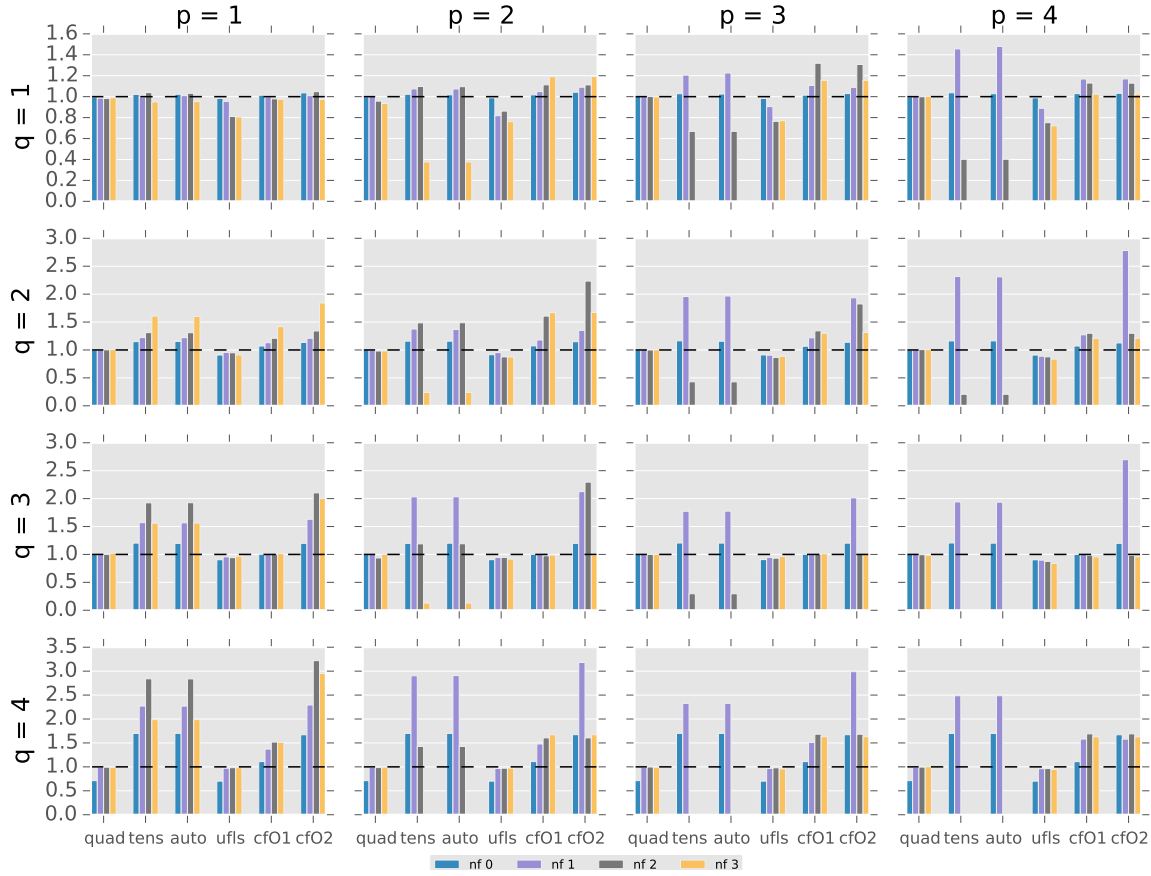


Fig. 7: Performance evaluation for the mass matrix .

the code synthesis choice. Take, for example, the case $[q = 3, p = 1, nf = 2]$. The adoption of pre-evaluation makes a tremendous speed up be achievable by *cf02* over the competitors. On the other hand, *auto* triggers *quad* (since one monomial in the form has two derivatives and two coefficients, which exceeds the threshold within which *tens* is selected), which, just like the other quadrature-based variants *ufls* and *cf01*, is sub-optimal.

Elasticity. The performance results for the elastic model are displayed in Figure 9. We note how the fact that *auto* opts for *tens* when $nf = 1$ leads, generally, to sub-optimal execution times. This is different than in the case of the Helmholtz equation, in which *tens* (and, generally, pre-evaluation through *cf02*), tends to be the best alternative when a single coefficient is present.

Hyperelasticity. In the hyperelastic model experiments, pre-evaluation is never triggered by *cf02*. This is a consequence of the form complexity, which makes the increase factor overwhelm any potential flop reduction. Another distinguishing aspect is the use of vector-valued function spaces, which requires a technique as in Section 5.2.3 to avoid wasteful operations over zero-valued entries; *quad*, *tens*, *ufls* and *cf02* employ different techniques. Results are displayed in Figure 10. *cf02* is the best alter-

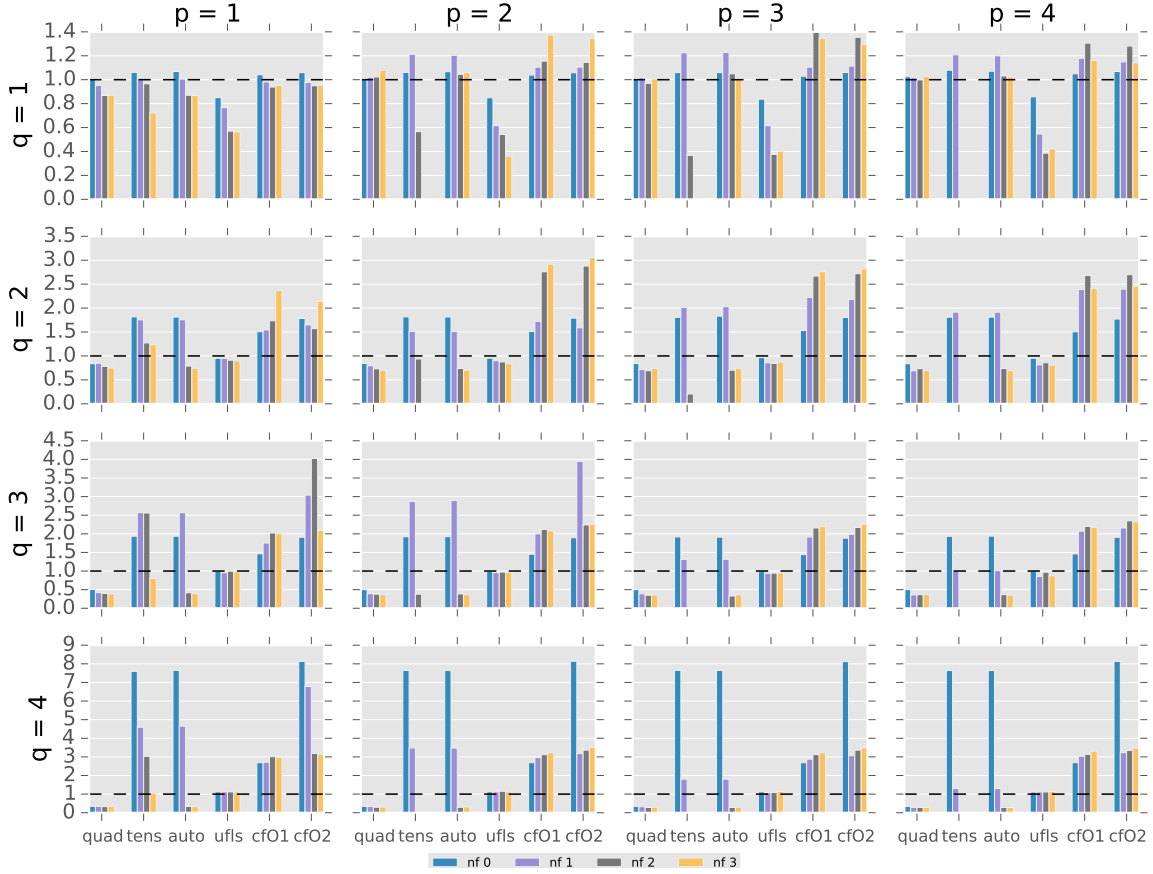


Fig. 8: Performance evaluation for the bilinear form of a Helmholtz equation.

native due to removing sharing from the multilinear loop nest and optimization of integration-dependent expressions. ufls performs generally well and exhibits significant speed ups over non-optimized syntheses; this is a result of the effort in optimizing integration-dependent expressions.

7. CONCLUSIONS

With this research we have made a first step towards producing a theory and an automated system for the optimal synthesis of loop nests arising in finite element integration. The results are extremely encouraging, suggesting our model applies to a variety of contexts. We have discussed the conditions under which the optimality model becomes weaker. An open problem is understanding how to refine this model to include outer loops. This will probably require exploiting properties of differential operators. A second open problem is extending our methodology to classes of loops arising in spectral methods; here, the interaction with low level optimization will probably become stronger due to the typically larger working sets deriving from the use of high order function spaces.

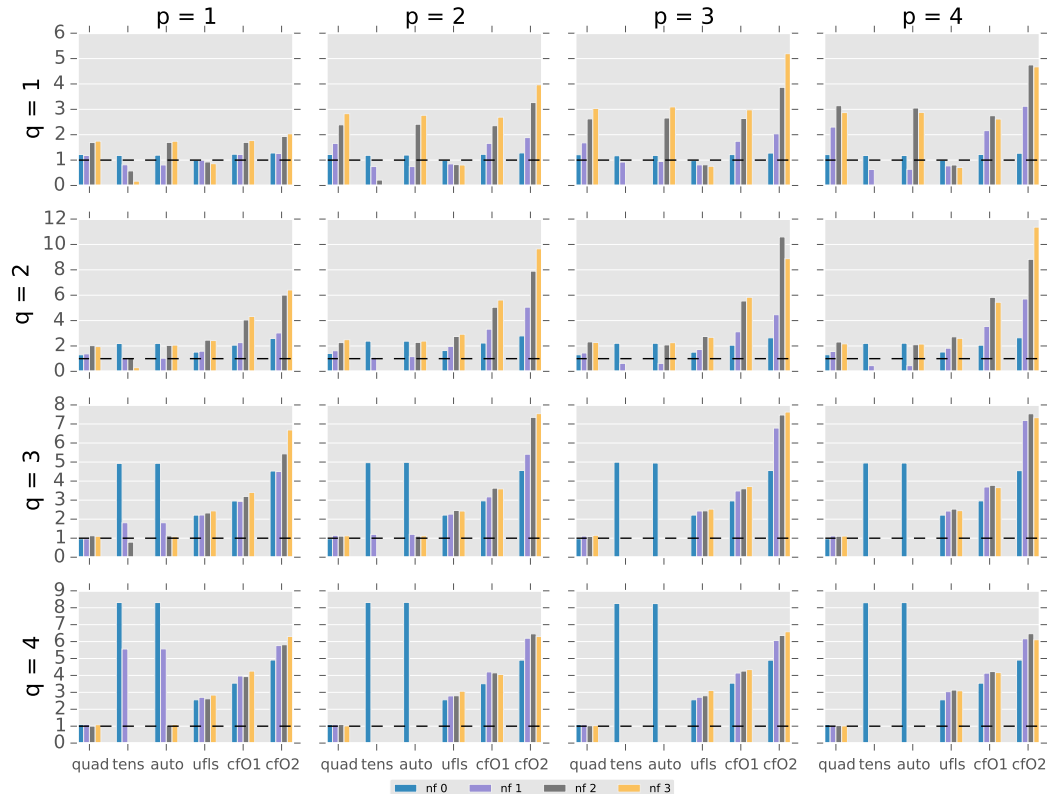


Fig. 9: Performance evaluation for the bilinear form arising in a elastic model.

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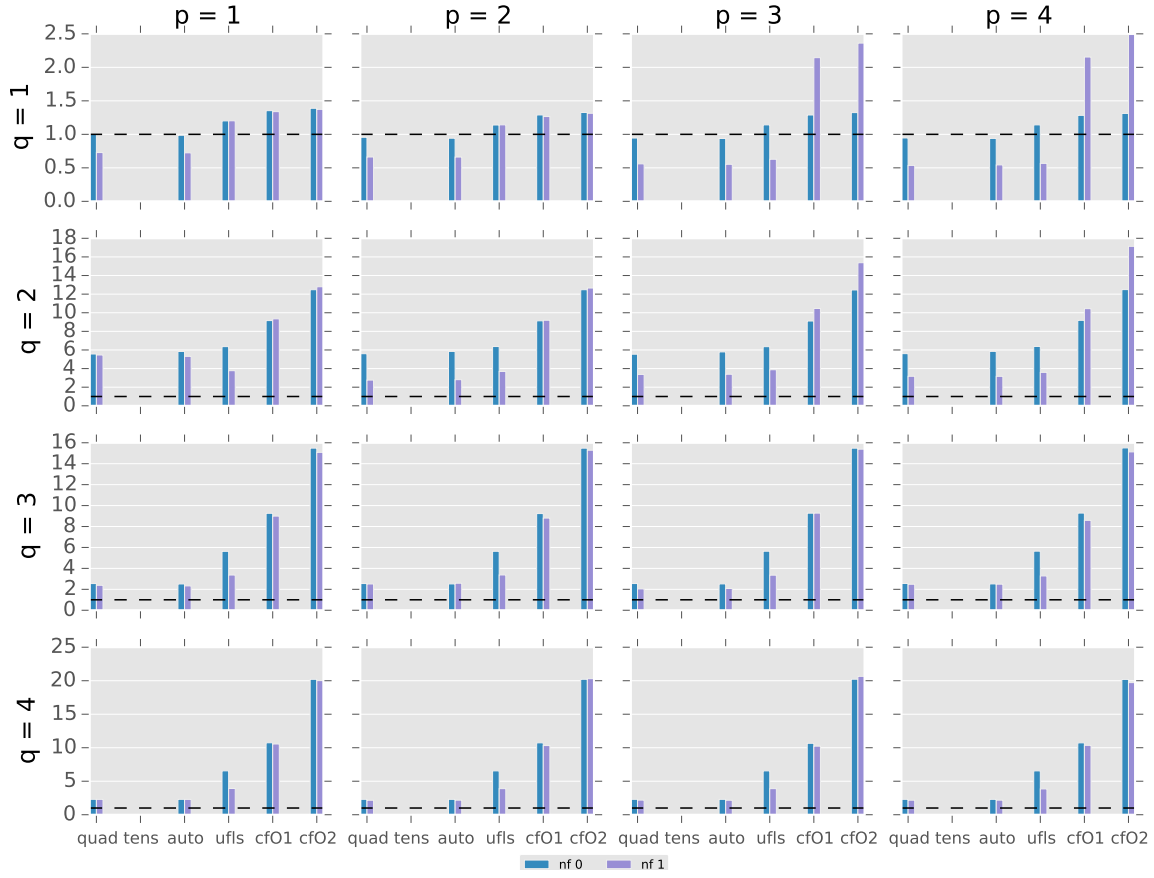


Fig. 10: Performance evaluation for the bilinear form arising in a hyperelastic model

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