ADVANCED ECONOMETRICS

Homework 1

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1 Exercise 1

Consider two independent random variables, X and Z, both distributed as standard normal, $N(0,1)$. Define $Y = X^2 + 2Z$.

1.1 $E[Y|X]$

Let's start by defining the expected value of X and Z such that $E[X] = E[Z] = 0$ by assumption. This is true since both Random Variables are Gaussian with mean zero and variance 1. Now we can define also $E[X|Z] = 0$, it comes out the assumption of independence of the two Random Variables. Then we can define $E[Y|X]$ as:

$$E[Y|X] = E[X^2 + 2Z|X] = E[X^2|X] + E[2Z|X] = E[X^2|X] + 2E[Z|X]$$

Now we can deduce that $E[X^2|X] = X^2$. Final solution is:

$$E[Y|X] = E[X^2|X] + 2E[Z|X] = X^2 + 2E[Z|X] = X^2 + 2 \cdot 0 = X^2$$

1.2 $E[Y]$ and $E[XY]$

1.2.1 $E[Y]$

From previous results we know that $E[Z] = 0$. In addiction we know that variance is define as $VAR[X] = E[X^2] - E[X]^2$, then if we solve for second moment we obtain $E[X^2] = VAR[X] + E[X]^2$. finally we can compute $E[Y]$:

$$E[Y] = E[X^2 + 2Z] = VAR[X] + E[X]^2 + 2E[Z] = 1 + 0 + 2 \cdot 0 = 1$$

1.2.2 $E[XY]$

since X and Z are normally distributed also Y is normally distributed and this comes from the previous results. We can define $E[XY] = E[XE[Y|X]]$, then:

$$E[XY] = E[XE[Y|X]] = E[X \cdot X^2] = E[X^3] = 0$$

we obtained $E[XE[Y|X]]$ from the law of iterated expectation. Final result is zero since $E[X^3]$ is the skewness (third moment) and it has zero mean.

1.3 COV[Y,X]

We know that the Covariance formula is $COV[Y, X] = E[XY] - E[X]E[Y]$.Then from the previous results comes from that:

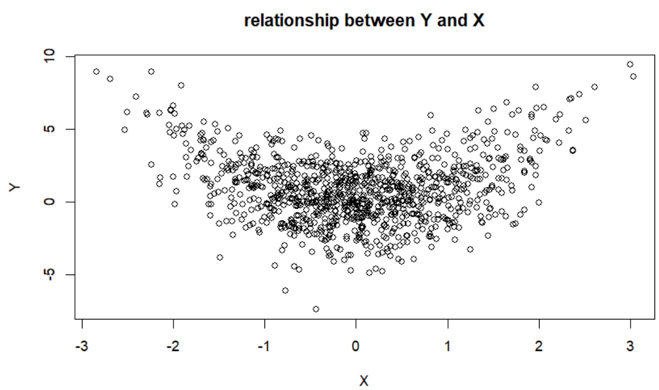
$$COV[Y, X] = E[XY] - E[X]E[Y] = 0 - 0 \cdot 1 = 0$$

and this is true since we have two Random Variables, in case in which we have X and Z not normally distributed the result in which $COV[Y, X] = 0$ does not implies that $E[Z|X] = 0$ in general.

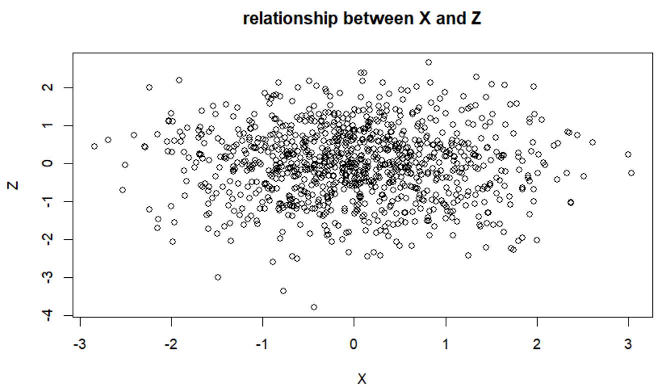
2 EXERCISE 2

This exercise is in the R script but we can comment anyway.

2.1 relationship between Y and X



we can see in the picture that the relation between X and Y is not linear, but we expected it since X is not linear. at the same time we are interested to check the relation between X and Z.



no realtion since they are independent.

2.2 $E[Y]$ and $E[XY]$

We can see that $E[Y]$ is close to 1 and $E[XY]$ close to zero. then the previous result are correct.

2.3 $COV[YX]$

As we can check in the exercise 1.3 covariance is zero (close).

3 EXERCISE 3

We generate 100 observation from a linear model. u_i is the squared of sum of 3 Random Variables with mean 0 and variance 1 minus 3 in this case, then mean is zero.

4 EXERCISE 4

Formula for $\hat{\beta}_2$is:

$$\hat{\beta}_2 = \frac{COV[XY]}{VAR[X]} = \frac{0.9861547}{3.318227} = 0.2971933$$

Formula for $\hat{\beta}_1$is (intercept):

$$\hat{\beta}_1 = E[Y] - E[X]\hat{\beta}_2 = 2.105709 - 3.092044 \cdot 0.2971933 = 1.186774$$

5 EXERCISE 5

We can see that with the lm command both coefficients are equal to the beta

Table 1:

	Dependent variable:
	yi
xi	0.297*
	(0.153)
Constant	1.187**
	(0.548)
Observations	100
R^2	0.037
Adjusted R^2	0.027
Residual Std. Error	2.772 (df = 98)
F Statistic	3.777* (df = 1; 98)
Note:	*p<0.1; **p<0.05; ***p<0.01

estimates in the previous exercise, if there are small differences in decimals this is due to the distribution of the residuals.

6 EXERCISE 6

We have to assume that expected value of u_i is zero, $E[u_i] = 0$ and that $E[u_i|x_i] = 0$. Empirically this can be seen in the correlation of u_i and x_i and it is zero. This assumption is crucial, if we suppose this and the full rank of $E[x^T x]$ matrix (invertible), then we can say that β is consistent through the weak law of large numbers.

6.1 Is OLS unbiased for β_1?

OLS is unbiased for β_1 since the expected value of the residuals given x is zero $E[u_i|x_i] = 0 = E[u_i]$, and this hold for every sample size. This comes from the fact that u_i has 3 degrees of freedom minus 3 then has mean zero and is independent from x_i by assumption.

6.2 Is OLS unbiased for β_2?

Ols is unbiased for β_2 for the same reason of for β_1

6.3 Is the model conditionally homoskedastic?

Model is homoskedastic since variance of u_i is independent from x_i, then $VAR[u_i|x_i] = \sigma^2$. then variance of u_i $VAR[u_i] = 2 \cdot k = 2 \cdot 3 = 6$ (variance of chi-square distribution have variance 2k with "k" degrees of freedom) is homogeneous with respect to x_i.

6.4 is $\hat{\beta}_2$ estimating the partial effect?

Partial effect is defined as:

$$\beta_2 = \frac{\partial E[y|x]}{\partial x}$$

7 EXERCISE 7

We start from a vector V composed by two random variables (v_1, v_2), this vector has mean 0 and variance 1 by construction:

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

We know the variance covariance matrix of AV and we denote it with: σ:

$$\sigma = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

Then variance of AV is defined by $A \cdot var(V) \cdot A^T$, since $var(V)$ is the identity matrix 2×2:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Given that we have variances in diagonal and covariances in off diagonal, then this matrix is true since by definition covariance are zero and comes from that random variables are orthogonal and they have variances one.
Then variance of AV is composed by $A \cdot A^T$, finally by Cholesky A^T is the Cholesky decomposition of sigma and this gives us A.