Syllabus.

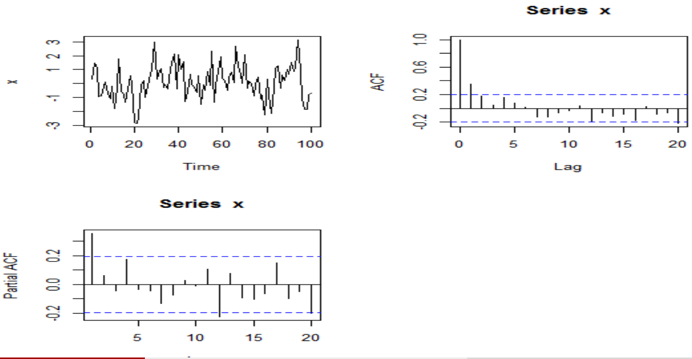
- Chapter 1. Concepts in time series.
- Chapter 2. Univariate ARIMA models.
- Chapter 3. Model fitting and checking.
- Chapter 4. Prediction and model selection.
- Chapter 5. Outliers and influential observations.
- Chapter 6. Heterocedastic models.
- Chapter 7. Multivariate time series.

Chapter 2. Univariate ARIMA models.

- 2.1.Summary of stationarity.
- 2.2.Introduction to ARIMA(p,d,q) models
- 2.3.AutoRegressive AR(p) Models.
- 2.4.Moving Average MA(q) Models
- 2.5.Non seasonal ARMA(p,q) models.

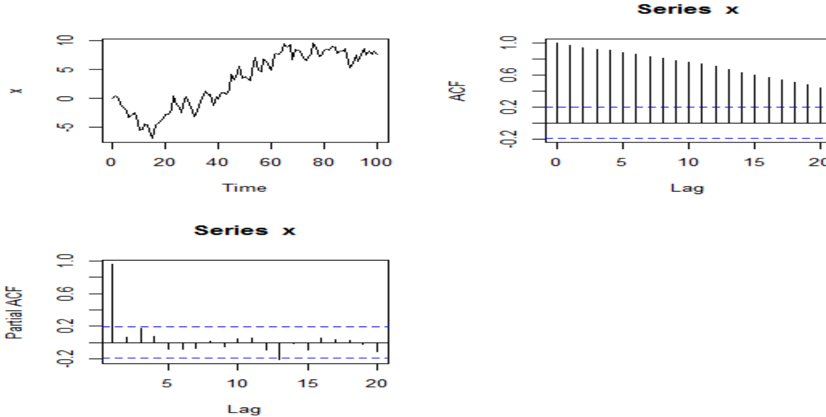
Weak stationarity

- Unconditional mean and variance are constant. Dependence structure only depends on lag (and not on time). Short memory. No need of differencing. The null in Box-Ljung test, can be rejected if there are significant correlation coefficients.



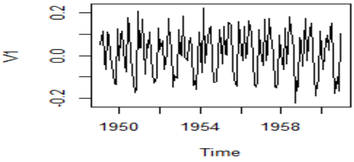
Non stationarity in trend

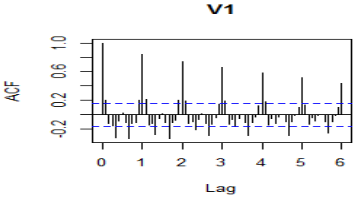
- Unconditional mean is not constant. Long memory. Need of regular difference to achieve stationarity.Reject the null in Box-Ljung test.

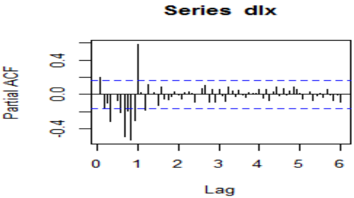


Non stationarity in seasonality

- Unconditional mean is not constant. Long memory. Need of seasonal difference to achieve stationarity.Reject the null in Box-Ljung test







ARIMA models.

- ARIMA stands for Autoregressive Integrated Moving Average models.
- ARIMA models usually use the notation (p,d,q), where p is the order of the autoregression, d is the number of regular differences needed to render the series stationary and q is the order of the moving average part.
- So, from now on, we will assume stationarity and talk about ARMA(p,q) models.

ARMA models.

- All the AR(p), MA(q) and ARMA(p,q) models are based on the assumption that the time series can be seen as:

$$X_t = \mu_t + a_t.$$

Where μ_t is the conditional mean of the series:

$$\mu_t = E[X_t \mid X_{t-1}, X_{t-2}, \ldots]$$

and a_t is a disturbance term assumed to be white noise.

ARMA models.

- Weak stationarity implies that the series have a constant marginal expectation μ (no long-term memory).
- But, still, the conditional mean μ_t can be non-constant and time dependent. Short-term memory.
- The ARMA (p,q) processes are built on the following notion:

$$\mu_t = f(X_{t-1}, \ldots, X_{t-p}, a_t, a_{t-1}, \ldots, a_{t-q}).$$

That is, the conditional mean is a (linear) function of past instances of the series as well as past innovations.

White noise process.

- The most basic stochastic process is the discrete white noise.
- A time series X_t is said to be white noise if:

$$var(X_t) = \sigma^2$$

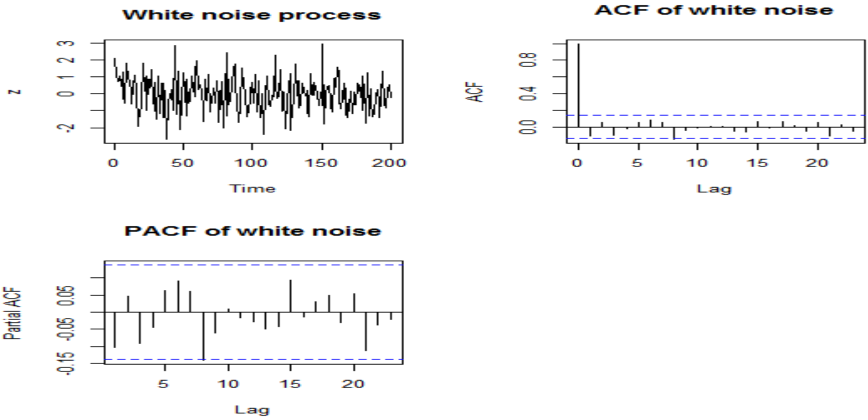
(identical constant variance for all t), and

$$\rho(k) = \pi(k) = 0$$

for all lags k (no dependence)

- If the variables follow a Gaussian distribution, i.e. $X_t \sim N(0, \sigma^2)$, the series is called Gaussian White Noise (wn).

White noise process.



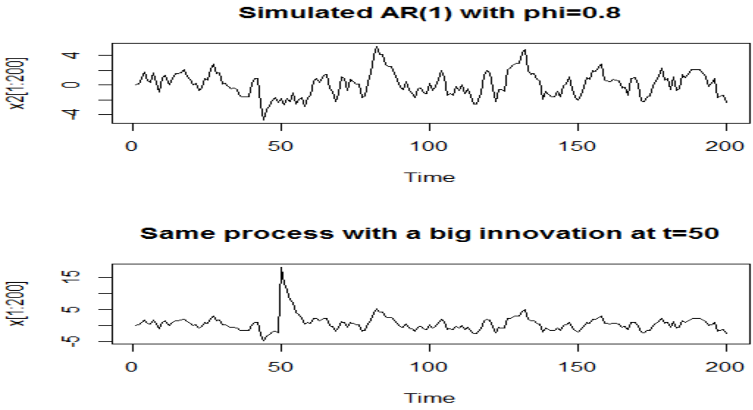
Definition and properties of AR(p) models.

- An AR(p) model is based on a linear combination of past observations:

$$X_t = c + \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + a_t$$

Where c is the constant term, a_t is white noise and can also be called innovation- in the sense that is stochastically independent of $X_{t-1}, X_{t-2}, \ldots, X_{t-p}$ but it has the power to drive the series into a new direction, meaning that it is strong enough so that it can overplay the dependence of the series from its own past.

Definition and properties of AR(p) models.



Definition and properties of AR(p) models.

- An alternative notation for an AR(p) model is possible with the backshift operator:

$$(1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p) X_t = c + a_t,$$

or, in short, $\Phi(B) X_t = c + a_t$ where $\Phi(B)$ is called the characteristic polynomial and it determines all the relevant properties of the process.

Definition and properties. Stationarity of an AR(p) model.

- In order to be stationary, the unconditional expectation of the AR(p) process should be constant,

$$E[X_t] = \mu, \quad for \quad all \quad t$$

- Taking expectations in both sides, we obtain:

$$\mu = E[X_t] = E[c + \phi_1 X_{t-1} + \ldots + \phi_p X_{t-p} + a_t] = c + (\phi_1 + \ldots + \phi_p)\mu + 0,$$

- Hence $\mu = c/(1 - \phi_1 - \ldots - \phi_p)$ and ,if $c = 0$, the AR(p) process will have a global mean of zero.
- Be aware that the conditional mean is time dependent and generally different from zero:

$$\mu_t = E[X_t \mid X_{t-1}, \ldots, X_{t-p}] = c + \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p}$$

Definition and properties. Stationarity of an AR(p) model.

- In order to be stationary, the unconditional variance of the AR(p) process should be constant,

$$Var[X_t] = \sigma_x^2, \quad for \quad all \quad t$$

- Taking ,for illustrative purpose, the case of an AR(1),

$$\sigma_x^2 = Var(X_t) = Var(\phi_1 X_{t-1} + a_t) = \phi_1^2 \sigma_x^2 + \sigma_a^2, \quad \sigma_x^2 = \frac{\sigma_a^2}{1 - \phi_1^2}$$

  and, therefore, an AR(1) can only be stationary if $\mid \phi_1 \mid < 1$. That limitation means that the dependence from the series past must not be too strong, so that the memory fades out.

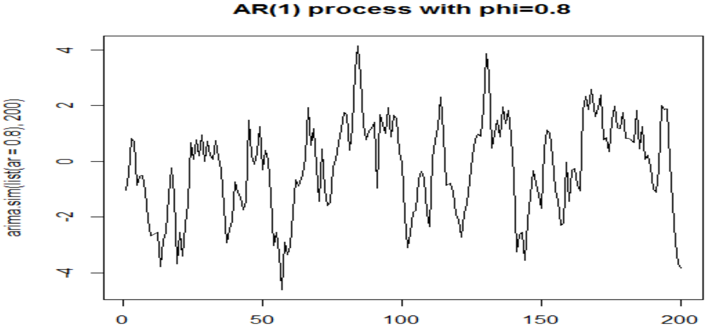
- The (potentially complex) roots of the characteristic polynomial $\Phi(B)$ must all exceed 1 in absolute value for an AR(p) process to be stationary.

Definition and properties. Correlation in an AR(1) model.

- Consider, first, the AR(1) model given by,

$$X_t = \phi_1 X_{t-1}$$

with a_t Gaussian white noise. If the series takes a large value, then the next one is determined as 0,8 times that large plus the innovation.



Definition and properties. Correlation in an AR(1) model.

- The autocorrelation at lag 1 is, therefore, positive and can be computed as:

$$Cor(X_t, X_{t-1}) = Cor(\phi_1 X_{t-1} + a_t, X_{t-1}) = \phi_1.$$

  Thus,in an AR(1), $\rho(1) = \phi_1$ (In the example $\rho(1) = 0{,}8$)
- The correlation for higher lags can be determined by repeated plug-in of the model equation,for example for $k = 2$,

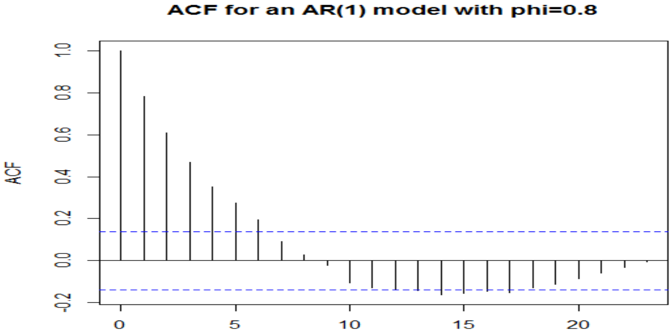
$$\begin{aligned} Cor(X_{t+1}, X_{t-1}) &= Cor(\phi_1 X_t + a_{t+1}, X_{t-1}) \\ &= Cor(\phi_1(\phi_1 X_{t-1} + a_t) + a_{t+1}, X_{t-1}) = \phi_1^2 \end{aligned}$$

Definition and properties. Correlation in an AR(1) model.

- And, in general, for AR(1) models we have that

$$\rho(k) = \phi_1 \rho(k-1) = \phi_1^k.$$

which implies that, for stationary AR(1) models, we have an exponentially decay of the autocorrelation coefficients.

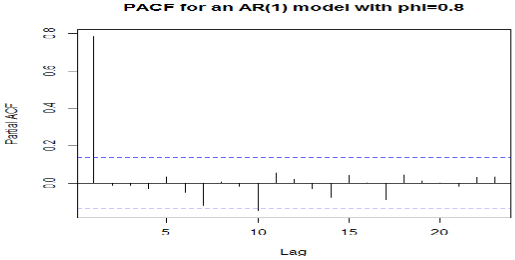


Definition and properties. Partial Correlation in an AR(1) model.

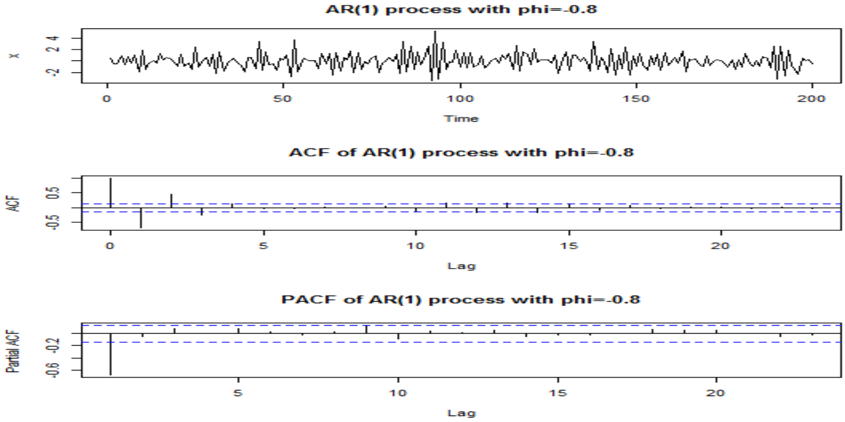
- The PACF, $\pi(k)$ is a measure of the linear relation among observations k-periods apart, independently of the intermediate values. Similar to the k-coefficient in the following regression:

$$X_t = \beta_0 + \beta_1 X_{t-1} + \ldots + \beta_k X_{t-k}.$$

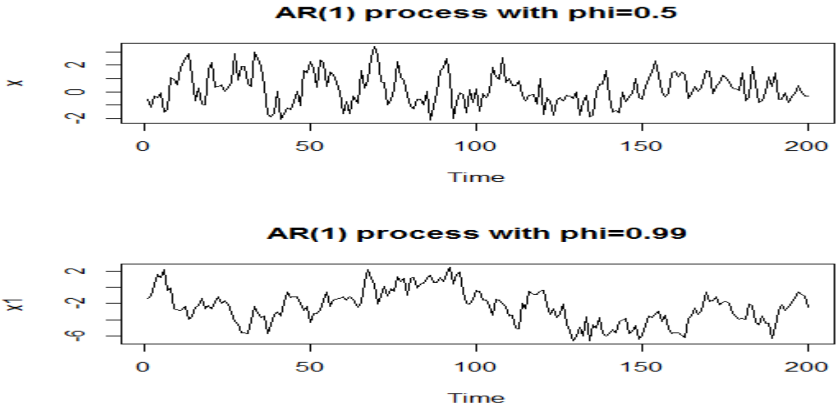
- For a stationary AR(1) model, $\pi(k) = \phi_1$ for $k = 1$ and $\pi(k) = 0$ for $k > 1$.



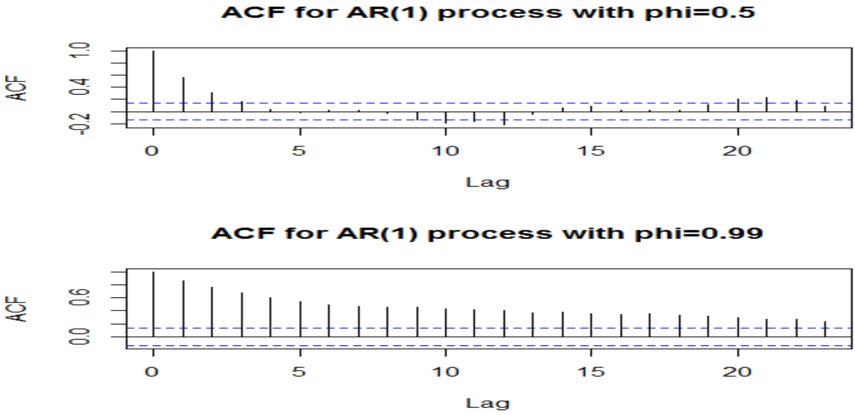
AR(1) model with $\phi_1 = -0{,}8$.



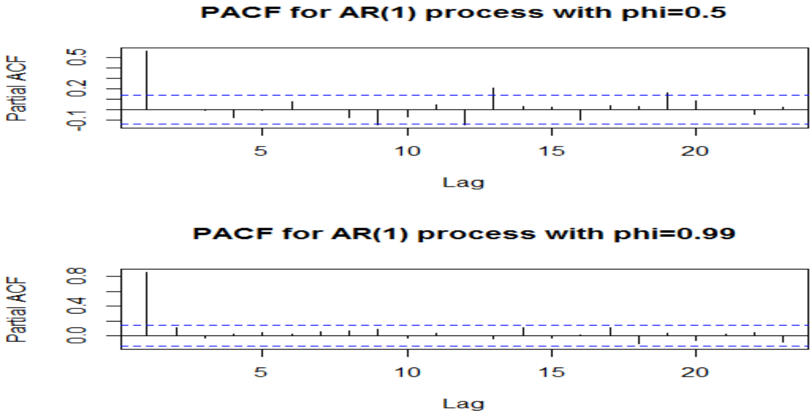
Comparing AR(1) model with $\phi_1 = -0{,}5$ and $\phi = {,}99$.



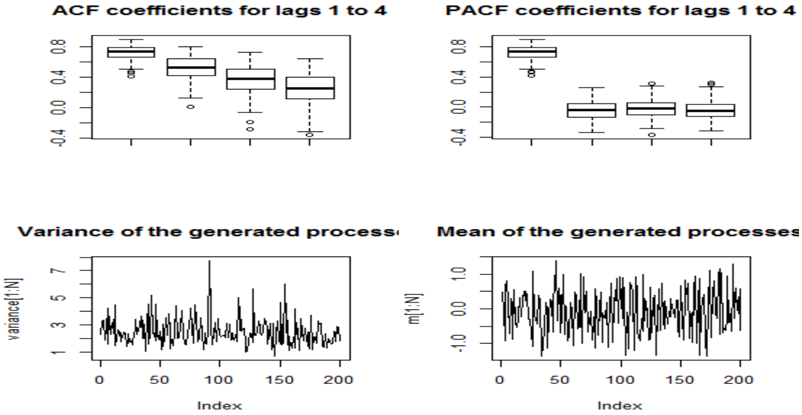
Comparing AR(1) model with $\phi_1 = 0{,}5$ and $\phi = {,}99$.



Comparing AR(1) model with $\phi_1 = 0{,}5$ and $\phi = {,}99$.



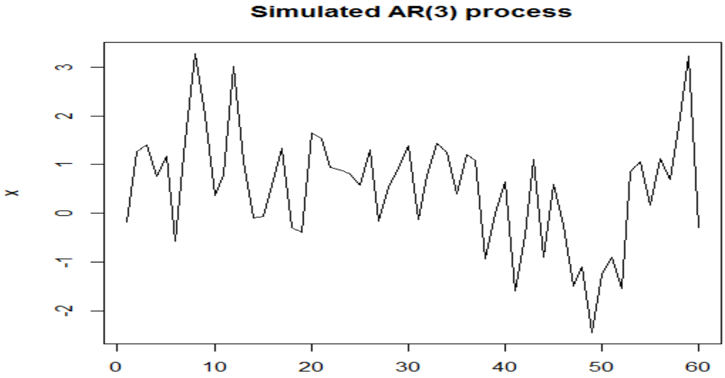
200 simulated AR(1) processes of 60 observations with $\phi_1 = 0{,}8$, $\sigma_a = 1$ and constant $c = 0$.



Definition and properties. Correlation in an AR(3) model.

- Consider now the AR(3) process,

$$(1 - 0{,}4B + 0{,}2B^2 - 0{,}3B^3)X_t = a_t$$



Definition and properties. Correlation in an AR(3) model.

- Applying the standard trick of plugging-in the model equation, we obtain that for a general AR(p):

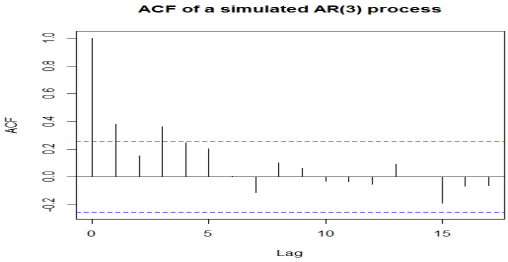
$$\begin{aligned}\rho(k) &= Cor(X_{t+k}, X_t) \\ &= Cor(\phi_1 X_{t+k-1} + \ldots + \phi_p X_{t+k-p}, X_t) \\ &= \phi_1 \rho(k-1) + \ldots + \phi_p \rho(k-p)\end{aligned}$$

With $\rho(0) = 1$. For $k = 1, \ldots, p$ this results is a *pxp* linear system called Yule-Walker equations.It can be solved to obtain the ACF correlations coefficients which can finally be propagated for $k = p+1, p+2, \ldots$

Definition and properties. Correlation in an AR(3) model.

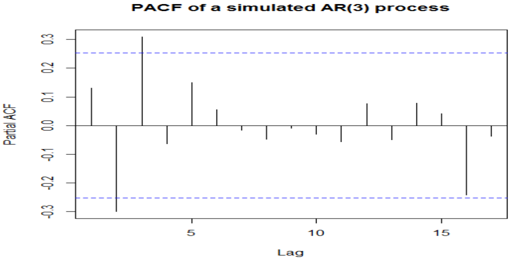
- For our AR(3) process

$$\rho(1) = \phi_1(1) = \phi_1$$
$$\rho(2) = \phi_1(\phi_1) + \phi_2(1) = \phi_1^2 + \phi_2$$
$$\rho(3) = \phi_1(\phi_1^2 + \phi_2) + \phi_2(\phi_1) + \phi_3(1)$$
$$\ldots$$



Definition and properties. Partial Correlation in an AR(3) model.

- For a stationary AR(p) model, $\pi(k) = \phi_k$ for $1 \leq k \leq p$ and $\pi(k) = 0$ for $k > 1$. In other words, $\pi(k)$ vanishes for $k > p$.



Definition and properties of MA(q) models.

- An MA(q) model is based on a linear combination of current and past innovations:

$$X_t = a_t + \theta_1 a_{t-1} + \theta_2 a_{t-2} + \ldots + \theta_q a_{t-q}$$

  Where, for simplicity, we have assume $c = 0$ and a_t is white noise.

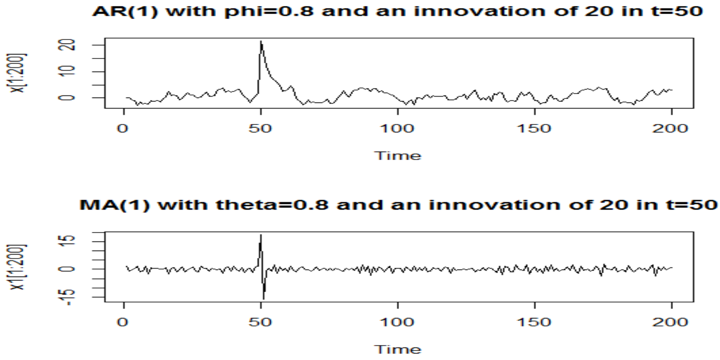
- An alternative notation for an MA(q) model is possible with the backshift operator:

$$X_t = (1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q) a_t$$

  or, in short, $X_t = \Theta(B) a_t$ where $\Theta(B)$ is called the characteristic polynomial of the MA(q) and it determines all the relevant properties of the process.

Rationale for MA(q) processes.

- Time series may be affected by a variety of random events that will not only have an inmediate effect on the series, but may also affect its value in several of the consecutive periods.



Definition and properties. Stationarity of an MA(q) model.

- In order to be stationary, the unconditional mean and variance of the MA(q) process should be constant. Considering first the unconditional mean μ, we have that,

$$\mu = E[X_t] = E[c + a_t + \theta_1 a_{t-1} + \ldots + \theta_q a_{t-q}] = c + (\theta_1 + \ldots + \theta_q) \cdot 0,$$

  and hence $\mu = c$ for any MA(q).
- The unconditional variance of the MA(q) process,

$$Var[X_t] = \sigma_x^2 = Var(1 + \theta_1 + \ldots + \theta_q a_{t-q}) = (1 + \sum_{i=1}^{q} \theta_i^2)\sigma_a^2$$

  and hence $\sigma_x^2 = \textit{constant}$ for any MA(q) process.

Stationarity of an MA(q) model. Correlation.

- Therefore, by proving that the autovariance in MA(q) processes is independent of time t, we prove stationarity. Let us consider an MA(1),

$$\gamma(1) = Cov(X_t, X_{t-1}) = Cov(a_t + \theta_1 a_{t-1}, a_{t-1} + \theta_1 a_{t-2}) = \theta_1 \sigma_a^2.$$

For any lag k exceeding the order $q = 1$ we use the plugging-in the model equation and obtain:

$$\gamma(k) = Cov(X_t, X_{t-k}) = 0, \quad for \quad all \quad k > 1$$

Thus, there is no more unconditional serial dependence in lags < 1.

Stationarity of an MA(q) model. Correlation.

- For the autocorrelation of a MA(1) process, we have,

$$\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\theta_1}{1+\theta_1^2} \quad \textit{and} \quad \rho(k) = 0 \quad \textit{for} \quad k < 1$$

which only depends on k and hence, any MA(1) process is stationary (the stationarity does not depend on the choice of the parameter θ_1).

- For a general MA(q) process, using $\theta_0 = 1$,

$$\rho(k) = \left\{ \begin{array}{lll} \sum_{j=0}^{q-k} \theta_j \theta_{j+k} / \sum_{j=0}^{q} \theta_j^2 & \textit{for} & k = 1, \ldots q \\ 0 & \textit{for} & k > q \end{array} \right\}$$

Hence, $\rho(k)$ is independent of time for any MA(q) process, irrespective of the order q.

Simulated MA(1) with $\theta = 0{,}8$.