Syllabus.

- Chapter 1. Concepts in time series.
- Chapter 2. Univariate ARIMA models.
- Chapter 3. Model fitting and checking.
- Chapter 4. Prediction and model selection.
- Chapter 5. Outliers and influential observations.
- Chapter 6. Heterocedastic models.
- Chapter 7. Multivariate time series.

Chapter 6. Heterocedastic models.

- 6.1.Motivation and example.
- 6.2.ARCH models.
- 6.3. GARCH models.

Outlook to Non-Linear Models

- What are linear models?Models which can be written as a linear combination of X_t. This includes all invertible AR, MA and ARMA models.
- What are non-linear models? Everything else, e.g. non-linear combinations of X_t, terms like X_t^2 in the linear combinations,etc
- Motivation for non-linear models?
  - cyclic behavior with quicker increase and decrease (natural sciencies, chaos theory).
  - volatility models with conditional heterocedasticity (the ones we are going to discuss.)

Volatility.

- In option trading and in the foreign exchange rate market, volatility plays an important role.
- Here volatility means conditional variance of the underlying asset return.
- The models we are going to discuss describe the evolution of volatility over time.
- Although volatility is not directly measurable, it has some properties that we may want to replicate in the models:
  - Volatility clusters.
  - Soft evolution, no volatily jumps.
  - Volatility does not diverge to infinity.
  - Different reactions to positive and negative big returns.

Assumptions and notation. Conditional Heteroskedasticity.

Let X_t be the return series of an asset, specifically,

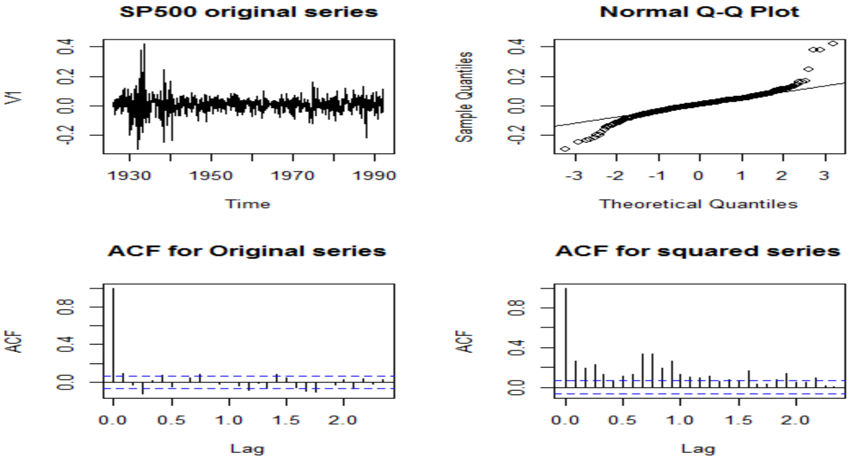
$$X_t = ln(p_t) - ln(p_{t-1})$$

It is informative to consider the conditional mean and variance:

$$\mu_t = E(X_t/x_1, \ldots, x_n) \quad and \quad \sigma_t^2 = Var(X_t/x_1, \ldots, x_n)$$

All volatility models we will mention are models for σ_t^2.

SP500 example.



SP500 example. Stylized facts.

- periods of higher and lower volatility.
- long-tailed appearance in the Normal plot.
- no significant autocorrelation in the series.
- significant autocorrelation if the series is squared.

Despite not showing direct autocorrelation, these series are not white noise. There is dependency which can be exploited by using conditional heterocesdasticity models (ARCH/GARCH).

ARCH(r) models. Engle (1982)

The basic idea in an ARCH(r) model is that the time series is serially uncorrelated but dependent. The dependency can be described with a cuadratic function of order r as follows:

$$X_t = \sigma_t a_t, \quad with \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 +, \ldots, \alpha_r X_{t-r}^2$$

where a_t are white noise and $\alpha_0 > 0, \alpha_i \geq 0$ for $i = 1, \ldots, r$.

Interpretation: Large past square returns imply a large conditional variance for the return. Consequently, the return tends to assume a large value. Under the ARCH framework, large returns tend to be followed by another large return. This is similar to the volatility clustering property.

ARCH(1) model. Properties.

Consider the ARCH(1) model,

$$X_t = \sigma_t a_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$$

- The conditional mean remains zero: $E[X_t/x1, \ldots, x_n] = 0$
- The unconditional variance can be shown to be,

$$Var(X_t) = \alpha_0/(1 - \alpha_1)$$

and, because variances must be positive, we need

$$0 \leq \alpha_1 < 1$$

.

ARCH models. Properties and weakness.

- Under the ARCH assumptions, the tail distribution of X_t is heavier than that of a normal distribution. Therefore, the probility of outliers is higher.
- This is in agreement with the empirical finding that outliers appear more often in asset returns than that implied by an iid sequence of normal random variates.

- Weakness of ARCH models:
  - treats positive and negative returns in the same way.
  - is very restrictive in parameters.
  - often over-predicts the volatility.

Building ARCH models.

- An ARIMA model is built for the observed series to remove any serial correlation in the data.
- Examine the squared residuals to check for conditional heterocedasticity.
- Use the PACF of squared residuals to determine the ARCH order and to perform the MLE of the specified model.

GARCH(r,s) models. Bollerslev(1986).

ARCH(r) models often require too many parameters to describe the evolution of volatility. An alternative is the GARCH(r,s) model,

$$X_t = \sigma_t a_t, \quad with \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 +, \ldots, \alpha_r X_{t-r}^2 + \beta_1 \sigma_{t-1}^2 + \ldots + \beta_s \sigma_{t-s}^2$$

where a_t are white noise, $\alpha_0 > 0$,

$$\alpha_i \geq 0 \quad for \quad i = 1, \ldots, r$$

$$\beta_j \geq 0 \quad for \quad j = 1, \ldots, s$$

and

$$\sum_{i=1}^{max(r,s)} (\alpha_i + \beta_i) < 0$$

.

GARCH models properties.

- The GARCH model implies that the unconditional variance of X_t is finite whereas its conditional variance, σ_t^2 evolves over time.Let

$$\eta_t = X_t^2 - \sigma_t^2 \quad \Rightarrow \quad \sigma_t^2 = X_t^2 - \eta_t$$

- We can rewrite the GARCH(r,s) model as:

$$X_t^2 = \alpha_0 + \sum_{i=1}^{max(r,s)} (\alpha_i + \beta_i) X_{t-i}^2 + \eta - \sum_{j=1}^{s} \beta_j \eta_{t-j}$$

The equation before is an ARMA form of the squared series , X_t^2. From which we can obtain that the unconditional variance is,

$$Var(X_t) = \frac{\alpha_0}{1 - \sum_{i=1}^{max(r,s)} (\alpha_i + \beta_i)}$$

and, therefore, the unconditional variance is finite whereas the conditional one evolves over time.

GARCH models properties.

- Recreates the clustering behavior.
- Heavier tail than that of a normal distribution.
- Simple parametric form that can be used to describe the evolution of volatility but not understanding.
- Do not reflect asymmetric behavior.
- Tails are still too short.

SP500 example.

```
> fit<-garch(sp500,order=c(1,1))

> summary(fit)

Call:
garch(x = sp500, order = c(1, 1))

Model:
GARCH(1,1)

Residuals:
    Min      1Q  Median      3Q     Max
-4.4839 -0.4489  0.1587  0.7715  2.9447

Coefficient(s):
    Estimate  Std. Error  t value Pr(>|t|)
a0 7.785e-05   2.500e-05    3.115  0.00184 **
a1 1.155e-01   1.958e-02    5.900 3.64e-09 ***
b1 8.617e-01   1.967e-02   43.808  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostic Tests:
        Jarque Bera Test

data:  Residuals
X-squared = 78.144, df = 2, p-value < 2.2e-16


        Box-Ljung test

data:  Squared.Residuals
X-squared = 1.3434, df = 1, p-value = 0.2464
```

SP500 example.