Punto 3

3.1) Integrar la serie de Fourier de $f(t)=t^2$ en el intervalo $-\pi \leq t \leq \pi$ y $f(t+2\pi)=f(t)$

$$a_0=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\,dt=\frac{2}{2\pi}\int_{-\pi}^{\pi} t^2 dt=\frac{1}{\pi}\int_{-\pi}^{\pi} t^2 dt=\frac{1}{\pi}\left.\frac{t^3}{3}\right|_{-\pi}^{\pi}=\frac{1}{\pi}\left(\frac{\pi^3}{3}-\frac{(-\pi^3)}{3}\right)$$

$a_0=\frac{2}{3}\pi^2$; note que $\omega_0=2\pi/T$ y $T=2\pi$ Por ende $\omega_0=1$

$$a_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(nt\omega_0)\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi} t^2\cos(nt)\,dt \qquad \begin{array}{ll} u=t^2 & dV=\cos(nt) \\ du=2t\,dt & V=\frac{\operatorname{Sen}(nt)}{n}\end{array}$$

$$a_n=\frac{1}{\pi}\left.\frac{t^2\operatorname{Sen}(nt)}{n}\right|_{-\pi}^{\pi}-\frac{2}{\pi}\int_{-\pi}^{\pi}\frac{t}{n}\operatorname{Sen}(nt)\,dt \qquad \begin{array}{ll} u=t & dV=\operatorname{Sen}(nt)\,dt \\ du=dt & V=-\frac{\cos(nt)}{n}\end{array}$$

$$a_n=\frac{1}{\pi}\left(\frac{\pi^2\operatorname{sen}(n\pi)}{n}+\frac{\pi^2\operatorname{Sen}(n\pi)}{n}\right)+\left.\frac{2t\cos(nt)}{\pi n^2}\right|_{-\pi}^{\pi}-\int_{-\pi}^{\pi}\frac{\cos(nt)}{n^2}\,dt$$

$$a_n=\frac{2\pi\cos(n\pi)}{n^2\pi}-\left(\frac{2(-\pi)\cos(-\pi n)}{n^2\pi}\right)-\left(\left.\frac{\operatorname{Sen}(nt)}{n^3}\right|_{-\pi}^{\pi}\right)$$

de modo que: $a_n=\frac{4}{n^2}\cos(n\pi)$

$$b_n=\frac{2}{T}\int_{-T/2}^{T/2} f(t)\operatorname{Sen}(nt\omega_0)\,dt=\frac{2}{2\pi}\int_{-\pi}^{\pi} t^2\operatorname{Sen}(nt)\,dt \qquad \begin{array}{ll} u=t^2 & dV=\operatorname{Sen}(nt)\,dt \\ du=2t\,dt & V=-\frac{\cos(nt)}{n}\end{array}$$

$$b_n=\left.\frac{-1}{\pi}t^2\frac{\cos(nt)}{n}\right|_{-\pi}^{\pi}+\int_{-\pi}^{\pi}\frac{2t\cos(nt)}{n}\,dt \qquad \begin{array}{ll} u=t & dV=\cos(nt)\,dt \\ du=dt & V=\operatorname{Sen}(nt)/n\end{array}$$

$$b_n=-\frac{\pi^2}{\pi}\frac{\cos(n\pi)}{n}+\frac{\pi^2}{\pi}\frac{\cos(n\pi)}{n}+\left.\frac{2t\operatorname{Sen}(nt)}{n^2}\right|_{-\pi}^{\pi}-2\int_{-\pi}^{\pi}\frac{\operatorname{Sen}(nt)}{n^2}\,dt$$

$$b_n=-2\left.\left(\frac{-\cos(nt)}{n^3}\right)\right|_{-\pi}^{\pi}=\frac{2\cos(n\pi)}{n^3}-\frac{2\cos(n\pi)}{n^3}=0$$

de modo que: $b_n=0$ dando así:

$$a_0=\frac{2\pi^2}{3}\ ;\ a_n=\frac{4\cos(n\pi)}{n^2}\ ;\ b_n=0$$

donde la integral estaría dada Como:

$$\int_{-\pi}^{\pi} \frac{\pi^2}{3} + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{4 \cos(nt)}{n^2}\, dt = \frac{\pi^2}{3} t \Big|_{-\pi}^{\pi} + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} \frac{4(-1)^{n-1} \cos(nt)\,dt}{n^2}$$

$$\int_{-\pi}^{\pi} \hat{f}(t)\,dt = \frac{2\pi^3}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n-1}}{n^3} \operatorname{Sen}(nt) \Big|_{-\pi}^{\pi} = \frac{2\pi^3}{3}$$

b.)

note que se tiene:

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{\cos(n\pi)}{n^2} \cos(nt) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nt)$$

note la relación de Parseval indica

$$\frac{1}{T}\int_{-T/2}^{T/2} |f(x)|^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$ de modo que:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} t^4\, dt = \left(\frac{\pi^2}{3}\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} \left(\frac{(-1)^n 4}{n^2}\right)^2$$

entonces se tiene:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} t^4\, dt = \frac{\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{1}{2\pi}\frac{2\pi^5}{5} = \frac{\pi^4}{5} = \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4}$$ note que $\sum_{n=1}^{\infty}\frac{1}{n^4}$ es $\zeta(4)$ de modo que: $$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{9\pi^4 - 5\pi^4}{45}\frac{1}{8} = \frac{4\pi^4}{8 \cdot 45} = \frac{\pi^4}{90}$$

de modo que: $\zeta(4) = \frac{\pi^4}{90}$

note ahora que Para $f(t)$ se tendía:

$f(t) = t^3$ donde su transformada sería:

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi} t^3 dt = \frac{1}{\pi}\left.\frac{t^4}{4}\right|_{-\pi}^{\pi} = 0$$

ahora se tiene:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^3 \cos(nt)\, dt = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} t^3 \operatorname{sen}(nt)\, dt = \frac{(12\pi n - 2\pi^3 n^3)}{\pi n^4}\cos(\pi n) = (-1)^n\left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right)$$

Se empleo un Programa Para Calcular las integral. de modo que:

$$t^3 = \sum_{n=1}^{\infty}(-1)^n\left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right)\cos(nt) \rightarrow \text{Parseval se tiene:}$$

$$\frac{1}{2\pi}\int_{\pi}^{\pi} t^6\, dt = \sum_{n=1}^{\infty}(-1)^{2n}\left(\frac{12 - 2\pi^2 n^2}{n^3}\right)^2 = \sum_{n=1}^{\infty}\frac{144 - 48\pi^2 n^2 + 4\pi^4 n^4}{n^6}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} t^6\, dt = \sum_{n=1}^{\infty}\frac{144}{n^6} - 48\pi^2\sum_{n=1}^{\infty}\frac{1}{n^4} + 4\pi^2\sum_{n=1}^{\infty}\frac{1}{n^2}$$

note anteriormente se encontro $\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$ y $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

$$\frac{\pi^6}{7} = 144\sum_{n=1}^{\infty}\frac{1}{n^6} - 48\pi^2\left(\frac{\pi^4}{90}\right) + 4\pi^4\left(\frac{\pi^2}{6}\right)$$

$$\sum_{n=1}^{\infty}\frac{1}{n^6} = \frac{\pi^6}{945}$$