Master in Control and Robotics

AMORO Lab Report

Kinematic and Dynamic Models of a Biglide Mechanism

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1 Introduction

The main objective of the present lab is to compute the geometric, kinematic and dynamic models of a Biglide mechanism and to compare them with the results obtained with GAZEBO. Then, a controller will be designed to track a trajectory in simulation. In the following sections will explain step by step what the reasoning was for computing all the geometric kinematic and dynamic models required.

2 Model

The kinematic architecture of the five-bar mechanism is shown in Fig.**??**. For the GAZEBO model, the geometric parameters are:

- $d = 0.4$ m
- $l_{A1C} = 0.3606$ m
- $l_{A2C} = 0.3606$ m

The two prismatic joints are actuated.

The base dynamic parameters are:

- $m_p = 3$ kg the mass of the end-effector
- $m_f = 1$ kg the mass of each foot

All other dynamic parameters are neglected meaning that the links connecting the actuated feet to the end effector will be considered as null masses.

3 Geometric models

3.1 Direct geometric model

The direct geometric model gives the position of the end-effector (x, y) as a function of the active joints coordinates (q_1, q_2) and the assembly mode. The final objective will be obtaining the vector $\overrightarrow{OC}$ as a function of those variables. The plan is to build up the vector as a sum of others which depend on the active joints coordinates. Using Fig.**??**, the direct geometric model can be computed according to the following method. The geometric model seen from the **left leg** of the glide robot can be summarized as the following vectorial sum:

$$\overrightarrow{OC} = -\overrightarrow{OA_1'} + \overrightarrow{A_1'A_1} + \overrightarrow{A_1'H} + \overrightarrow{HC} \tag{1}$$

Where: A_1': represents the projection on the x_0 axis of point A_1.
H: represents the mid-point between the A_1 and A_2.

The vectors $\overrightarrow{OA_1'}$ and $\overrightarrow{A_1'A_1}$ can be easily computed just by looking at the robot's schematic. The first one is described as a constant distance between the origin frame and the A_1' point : $\overrightarrow{OA_1'} = -d/2$
The second one can be assumed to be directly equal to the q_1 variable:

$$\overrightarrow{A_1'A_1} = q_1 \tag{2}$$

As far as the $\overrightarrow{A_1'H}$ and $\overrightarrow{HC}$ vectors are concerned, the computation is more complicated. Starting from the definition of point H as the mid-point between A_1 and A_2, we first have to compute the vector connecting the two points and then take its half. The computation was carried out as follows:

$$\overrightarrow{A_1H} = 1/2\overrightarrow{A_1A_2} \tag{3}$$

With:

$$\overrightarrow{A_1A_2} = -\overrightarrow{OA_1} + \overrightarrow{OA_2} \tag{4}$$

Where:

$$\overrightarrow{OA_1} = -\overrightarrow{OA'_1} + \overrightarrow{A'_1A_1} = -d/2 + q_1 \tag{5}$$

$$\overrightarrow{OA_2} = \overrightarrow{OA'_2} + \overrightarrow{A'_2A_2} = d/2 + q_2 \tag{6}$$

Then, in a matrix form we obtain:

$$\overrightarrow{A_1H} = 1/2\overrightarrow{A_1A_2} == 1/2 \begin{bmatrix} d/2 + d/2 \\ q_1 + q_2 \end{bmatrix} = \begin{bmatrix} d/2 \\ \frac{q_1 + q_2}{2} \end{bmatrix} \tag{7}$$

There are two solutions for the final component $\overrightarrow{HC}$, relative to the two assembly mode of the robot. They are obtained by the 90° rotation (clockwise and counter-clockwise) of the vector $\overrightarrow{A_1H}$ scaled to the length h, giving:

$$\overrightarrow{HC} = \gamma\frac{h}{a} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \overrightarrow{A_1A_2} \tag{8}$$

With $a = \|\overrightarrow{A_1H}\|$ and $h = \sqrt{l^2 - a^2}$.

This last step concludes the calculation of the geometric model since all the component of eq: (1) have be found with q_1 and q_2 as the only dependencies.

3.2 Passive joints geometric model

The computation of the passive joint coordinate Φ_1 can be done using the expression of the end-effector position as a function of all the left arm joints.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{d}{2} + lcos(\Phi_1) \\ q_1 + lsin(\Phi_1) \end{bmatrix} \tag{9}$$

This formulation implies that:

$$\Phi_1 = arcos\left[(x - \frac{d}{2})\frac{1}{l}\right] \tag{10}$$

And,

$$\Phi_1 = arcsin\left[(y - q_1)\frac{1}{l}\right] \tag{11}$$

We can then retrieve the value of Φ_1 as follows:

$$\Phi_1 = arctan\left[\frac{y - q_1}{x + \frac{d}{2}}]\right] \tag{12}$$

Similarly for Φ_2:

$$\Phi_2 = arctan\left[\frac{y - q_2}{x - \frac{d}{2}}\right] \tag{13}$$

Obtaining the final expressions for Φ_1 and Φ_2 as function of q_1 and q_2 respectively.

3.3 Inverse geometric model

The IGM of a robot with a closed chain structure gives the active joint variables as a function of the location of the end-effector. To compute the inverse geometric model for the left arm, first we have to determine the direct geometric model to identify the x and y variables. Secondly, we have to solve the geometric constraint equations of the loop to compute the passive joint variables belonging to this path in terms of the active joint variables. In the case of the bi-glide robot (Fig.??), the geometric constraint equations can be identified as:

$$\overrightarrow{A_1C} = l \tag{14}$$

$$\overrightarrow{A_2C} = l \tag{15}$$

Which lead to the equations:

$$\begin{cases} ||\overrightarrow{A_1C}||^2 = \overrightarrow{OC}^2 - \overrightarrow{OA_1}^2 = l^2 \\ ||\overrightarrow{A_2C}||^2 = \overrightarrow{OC}^2 - \overrightarrow{OA_2}^2 = l^2 \end{cases} . \tag{16}$$

Since we have already obtained every member of these equations as a function of (q_1, q_2), it is just a matter of finding (q_1, q_2) as a function of (x, y). The final equations being:

$$\begin{cases} q_1 = y \pm \sqrt{-x^2 - \dfrac{d^2}{4} - dx + l^2} \\ q_2 = y \pm \sqrt{-x^2 - \dfrac{d^2}{4} + dx + l^2} \end{cases} . \tag{17}$$

Meaning that the IGM has a total of 4 solutions.

4 First order kinematic models

4.1 Forward and inverse kinematic model

To compute the first order kinematic model, we start with the computation of the position of the end-effector using two vector equations, one for each leg.

$$\xi = -\frac{d}{2}x_0 + q_1 y_0 - l\overrightarrow{u_1} \tag{18}$$

$$\xi = +\frac{d}{2}x_0 + q_2 y_0 - l\overrightarrow{u_2} \tag{19}$$

Where:

$$\overrightarrow{u_1} = \begin{bmatrix} cos(\Phi_1) \\ sin(\Phi_1) \end{bmatrix} \quad \overrightarrow{u_2} = \begin{bmatrix} cos(\Phi_2) \\ sin(\Phi_2) \end{bmatrix} \quad \overrightarrow{\xi} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{20}$$

It is known that the derivative of a unit rotating vector $\overrightarrow{u}$ can be addressed as:

$$\frac{d\boldsymbol{u}}{dt} = \dot{\Phi} \begin{bmatrix} -sin(\phi) \\ cos(\phi) \end{bmatrix} = \dot{\Phi}\boldsymbol{v} = \dot{\Phi}E\boldsymbol{u} \tag{21}$$

With:

$$\boldsymbol{v} = rot_2(\frac{\pi}{2})\boldsymbol{u} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \boldsymbol{u} = E\boldsymbol{u} \tag{22}$$

Taking this fact into account, we can calculate the time derivative of the ξ equation as:

$$\dot{\xi} = \dot{q_1} y_0 + \dot{\Phi_1} l \overrightarrow{v_1} \tag{23}$$

$$\dot{\xi} = \dot{q_2} y_0 + \dot{\Phi_2} l \overrightarrow{v_2} \tag{24}$$

We have here 4 equations (each line contains two equations) to express the the end-effector velocity. Since we do not know the value of the passive joint velocities $\dot{\Phi}_1$ and $\dot{\Phi}_2$, we need to find the vector whose dot product with the kinematic equations of the legs, will cancel the terms containing the passive joint velocities. The solution is obtained by applying the dot product of equation 4.1 and 4.1 with $\overrightarrow{u_1}$ and $\overrightarrow{u_2}$, as they will respectively cancel $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. Namely:

$$\boldsymbol{u_1^T}\dot{\xi} = \boldsymbol{u_1^T}\dot{q}_1 y_0 + \boldsymbol{u_1^T}\dot{\Phi}_1 l\overrightarrow{v_1} \tag{25}$$

$$\boldsymbol{u_2^T}\dot{\xi} = \boldsymbol{u_2^T}\dot{q}_1 y_0 + \boldsymbol{u_i^T}\dot{\Phi}_1 l\overrightarrow{v_2} \tag{26}$$

In matrix form:

$$\begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} u_1^T y_0 & 0 \\ 0 & u_2^T y_0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} \tag{27}$$

Where:

$$\begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} = \boldsymbol{A} \quad \begin{bmatrix} u_1^T y_0 & 0 \\ 0 & u_2^T y_0 \end{bmatrix} = \boldsymbol{B} \quad \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \dot{q} \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{\xi} \tag{28}$$

Which leads to the much more simpler form:

$$\boldsymbol{A\dot{\xi}} = \boldsymbol{B\dot{q}} \tag{29}$$

The forward and inverse kinematic model are obtained by inverting either A or B.

4.2 Passive joints kinematic model

just like we did for obtaining the velocities of the end-effector as a function of the active joint velocity, we can again use equation 4.1 and 4.1 to compute the passive joint velocities. For expressing the end effector velocities as a function of the two passive joint velocities some of the terms have to be canceled out of the equation in order to lose the dependencies on other variables. A wise approach is again to make a pre-multiplication with a vector capable of erasing those variables. For the current case the vector would be x_0 since it will null the dependencies on the $(\dot{q}_1, \dot{q}_2)$ vectors, leading to:

$$\boldsymbol{x_0^T}\dot{\xi} = \boldsymbol{x_0^T}\dot{q}_1 y_0 + \boldsymbol{x_0^T}\dot{\Phi}_1 l\overrightarrow{v_1} \tag{30}$$

$$\boldsymbol{x_0^T}\dot{\xi} = \boldsymbol{x_0^T}\dot{q}_1 y_0 + \boldsymbol{x_0^T}\dot{\Phi}_1 l\overrightarrow{v_2} \tag{31}$$

In matrix form:

$$\begin{bmatrix} i^T \\ i^T \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = l \begin{bmatrix} \boldsymbol{x_0^T} v_1 & 0 \\ 0 & \boldsymbol{x_0^T} v_2 \end{bmatrix} \begin{bmatrix} \dot{\Phi}_1 \\ \dot{\Phi}_2 \end{bmatrix} \tag{32}$$

Where:

$$\begin{bmatrix} \boldsymbol{x_0^T} \\ \boldsymbol{x_0^T} \end{bmatrix} = \boldsymbol{A} \quad \begin{bmatrix} \boldsymbol{x_0^T}\overrightarrow{v_1} & 0 \\ 0 & \boldsymbol{x_0^T}\overrightarrow{v_2} \end{bmatrix} = \boldsymbol{B} \quad \begin{bmatrix} \dot{\Phi}_1 \\ \dot{\Phi}_2 \end{bmatrix} = \dot{\Phi} \quad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \dot{\xi} \tag{33}$$

5 Second order kinematic models

5.1 Forward and inverse kinematic model

To compute the second order kinematic model, we use the derivative of equation 4.1 and 4.1. Namely:

$$\ddot{\xi} = \ddot{q}_1 y_0 + l\ddot{\Phi}_1\overrightarrow{v_1} - l\dot{\Phi}_1{}^2\overrightarrow{u_1} \tag{34}$$

$$\ddot{\xi} = \ddot{q}_2 y_0 + l\ddot{\Phi}_2\overrightarrow{v_2} - l\dot{\Phi}_2{}^2\overrightarrow{u_2} \tag{35}$$

Again, we can apply the dot product on both equations 5.1 and 5.1 with $\overrightarrow{u_1}$ and the second line with $\overrightarrow{u_2}$, as they will cancel respectively $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. Then, we obtain:

$$\overrightarrow{u_1^T}\ddot{\xi} = \ddot{q}_1\overrightarrow{u_1^T} y_0 + l\ddot{\Phi}_1\overrightarrow{u_1^T}\overrightarrow{v_1} - l\dot{\Phi}_1{}^2 \tag{36}$$

$$\overrightarrow{u_2^T}\ddot{\xi} = \overrightarrow{u_1^T}\ddot{q}_2 y_0 + l\ddot{\Phi}_2\overrightarrow{u_1^T}\overrightarrow{v_2} - l\dot{\Phi}_2{}^2 \tag{37}$$

or, as a matrix expression:

$$\begin{bmatrix} u_1^T \\ u_2^T \end{bmatrix} \ddot{\xi} = \begin{bmatrix} \overrightarrow{u_1}y_0 & 0 \\ 0 & \overrightarrow{u_2}y_0 \end{bmatrix} \ddot{q} - l \begin{bmatrix} \dot{\Phi}_1^2 \\ \dot{\Phi}_2^2 \end{bmatrix} \tag{38}$$

This second order kinematic model is under the form:

$$\boldsymbol{A} = \boldsymbol{B}\ddot{q} - \boldsymbol{d} \tag{39}$$

5.2 Passive joints kinematic model

For the second order passive joints kinematic model the procedure is pretty similar to the active one. Just like in the first order case, the vector $[x_0^T, x_0^T]$ will pre-multiply both sides of the equation giving up the q dependent terms. Then, we obtain in ma matrix form:

$$\begin{bmatrix} x_0^T \\ x_0^T \end{bmatrix} \ddot{\xi} = l \begin{bmatrix} x_0^T\overrightarrow{v_1} & 0 \\ 0 & x_0^T\overrightarrow{v_2} \end{bmatrix} \ddot{\Phi} - l x_0 \begin{bmatrix} \overrightarrow{u_1}\dot{\Phi}_1^2 \\ \overrightarrow{u_1}\dot{\Phi}_2^2 \end{bmatrix} \tag{40}$$

6 Dynamic model

The dynamic model of the Bi-glide is based only on the following dynamic parameters:

- $m_p = 3$ kg the mass of the end-effector
- $m_f = 1$ kg the mass of each foot

Since no other information is given regarding the dynamic parameters of the structure, we can assume that the mass of the diagonal links is null. Moreover, their kinetic energy will not be considered in the calculation of the energy of the entire structure.
Due to the closed loop architecture of the biglide mechanism, we can virtually cut it at the end effector joint and deal with a virtual tree structure and virtually free-moving platform. For the computation of the dynamic model of the structure, we started by finding the **Lagrangian equation**. Afterwards, the Lagrangian's derivatives will be calculated to retrieve the components of the vector of generalized forces and moments $\boldsymbol{\tau}$. The Lagrangian equation is defined as follows:

$$\boldsymbol{L} = \boldsymbol{E} - \boldsymbol{U} \tag{41}$$

Where, **E**: Kinetic energy of the system and **U**: Potential energy. Since it is a planar mechanism, the gravity effect can be neglected. The potential energy of our system will then equate to 0. The Lagrange's equation will then be considered as equal to the Kinetic energy of the system. Meaning:

$$\boldsymbol{L} = \boldsymbol{E} \tag{42}$$

The total kinetic energy of the system will be expressed as the sum of all energies of the bodies components. In the current case the only bodies with kinetic energy are the two feet and the end effector. Therefore, The total energy will equate to:

$$\boldsymbol{E_{tot}} = \boldsymbol{E_{foot1}} + \boldsymbol{E_{foot2}} + \boldsymbol{E_{EE}} \tag{43}$$

The dynamic model of the Bi-Glide mechanism is obtained by computing first the dynamic model of the tree structure without the end effector. Since the formulation of the kinetic energy for a generic rigid-body is the following:

$$E_i = \frac{1}{2}(m_i v_i^T v_i + \omega_i^T I_{O_i}\omega_i + 2^i m s_i^T(v_i\omega_i)) \tag{44}$$

And the feet action is only transnational and not rotational, the equation for both bodies simplifies to:

$$E_1 = \frac{1}{2} m_1 v_1^T v_1 \tag{45}$$

$$E_2 = \frac{1}{2}m_2v_2^Tv_2 \tag{46}$$

The Lagrangian of the tree structure is given by:

$$E_{feet} = \frac{1}{2}m_1v_1^Tv_1 + \frac{1}{2}m_2v_2^Tv_2 \tag{47}$$

The computation of the Lagrangian of the end effector can be simply computed as the kinetic energy of a point mass moving on the plane. Since we can easily compute the velocity of point C with the First order kinematic model, the Lagrangian turns out to be:

$$E_{EE} = L_{EE} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)m_{EE} \tag{48}$$

By summing up 47 and 48, we obtain the total Lagrangian's equation as:

$$L = L_{EE} + L_{feet} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)m_{EE} + \frac{1}{2}m_1v_1^Tv_1 + \frac{1}{2}m_2v_2^Tv_2 \tag{49}$$

Then using the Lagrange formalism, we can calculate $\boldsymbol{\tau}_a$ (virtual input torques of the actuated robot's joints), $\boldsymbol{\tau}_d$ (virtual input torques of the passive robot's joints),w_p (force and moments applied by the end effector), namely:

$$\begin{aligned} \boldsymbol{\tau}_{a_1} &= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_1}\right] - \frac{\partial L}{\partial q_1} \\ \boldsymbol{\tau}_{a_2} &= \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_2}\right] - \frac{\partial L}{\partial q_2} \end{aligned} \tag{50}$$

Obtaining:

$$\boldsymbol{\tau}_t = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} \quad ; \quad \boldsymbol{\tau}_d = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad ; \quad \boldsymbol{w}_p = m_e\begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} = m_e\ddot{\xi} \tag{51}$$

Thanks to the Lagrangian multipliers, we know that the total vector of generalized forces is turns out to be:

$$\boldsymbol{\tau}_{tot} = \boldsymbol{\tau}_t + J^T\boldsymbol{w}_p \tag{52}$$

Where:

$$J = A^{-1}B \tag{53}$$

Since we also know that $\ddot{\xi} = A^{-1}(B\ddot{q}_a + d)$ we can conclude that:

$$\begin{aligned} \boldsymbol{\tau}_{tot} &= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + (A^{-1}B)^T m_e\ddot{\xi} = \\ &= \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + (A^{-1}B)^T m_e A^{-1}(B\ddot{q}_a + d) = \\ &= \left(\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + (A^{-1}B)^T m_e A^{-1}B\right)\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + (A^{-1}B)^T m_e A^{-1}d = \\ &= \left(\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + J^T m_e J\right)\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + J^T m_e A^{-1}d = \end{aligned} \tag{54}$$

Where:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + J^T m_e J = M \qquad and \qquad J^T m_e A^{-1}d = c \tag{55}$$

obtaining the expression of the dynamic model as a function of the active joint acceleration under the form:

$$\boldsymbol{\tau}_{tot} = \boldsymbol{M}\ddot{\boldsymbol{q}} + \boldsymbol{c} \tag{56}$$

where:

- **M** is the inertia matrix and it is positive definite.
- **c** is the vector of Coriolis and centrifugal effects.