Modern Tools in Imaging

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0. Introduction

Discrete Images. In the following, we consider the domain $[0,1]^d$. According to a cell size $h = \frac{1}{N}$ with $N \in \mathbb{N}$, we divide the d-dimensional unit cube into a set

$$\mathscr{C}_h = \{C_\alpha \ : \ \alpha \in \{1,\dots,N\}^d\}$$

of $|\mathscr{C}_h|$ disjoint d-dimensional cubes C_α, which we call cells. Note that these cells can be combinations of e.g. open, half-open, and closed cubes. Then a discrete d-dimensional (gray-value) image is a function $u_h \colon [0,1]^d \to [0,1]$ with

$$u_h(x) = \sum_{\alpha \in \{1,\dots,N\}^d} u_\alpha \, \chi_{C_\alpha}(x) \,. \tag{1}$$

Thus, u_h is a step function w.r.t. $\mathscr{C}_h$ and can be represented by an array $(u_\alpha)_{\alpha \in \{1,\dots,N\}^d}$ of size $|\mathscr{C}_h|$. Here, the value $u = 0$ corresponds to black and the value $u = 1$ to white. For the most common case $d = 2$, we call the cells C_α pixels. Also the case $d = 3$ often appears in reality, where the cells C_α are called voxels. Note that a generalization to rectangular domains $[0, L_1] \times \ldots \times [0, L_d]$ and coordinate-depending cell sizes $h_1, \dots, h_d$ is straight-forward.

Continuous Images. More generally, we will study continuous (gray-value) images as functions $u \colon \Omega \to [0,1]$, where $\Omega \subset \mathbb{R}^d$ is a compact domain with Lipschitz boundary. This will allow us to define appropriate models for imaging applications by taking into account the regularity of the images. By definition of a discrete image as a step function, a natural image space is $L^p(\Omega)$ for some $p \in [1,\infty]$. Note that $L^p(\Omega)$ includes e.g. noisy images. However, we desire a notation of smooth images. The Sobolev spaces $W^{1,p}(\Omega)$ are not suitable, since e.g. characteristic functions are not contained in $W^{1,p}$ (see Example 0.3), but smooth objects are represented as characteristic functions. Now, this leads us to the space of functions of bounded variation $\mathrm{BV}(\Omega)$ as a natural space for smooth images. For an application oriented introduction to the space BV, we refer to overview article [Cha+10]. A general introduction is given in the textbook [AFP00].

Repetition 0.1 (Radon measures). Recall that a Radon measure is a Borel measure, which is finite on compact domain. Since we only consider a compact domain Ω, in our case, a Radon measure is just a scaled Borel probability measure. Also on compact domains, we have that $C(\Omega)' = \mathscr{M}(\Omega,\mathbb{R})$. More precisely, the topological dual of the space of continuous functions $C(\Omega)$ endowed with the L^∞-norm $\|f\|_{L^\infty} := \sup_{x\in\Omega} |f(x)|$ is the space of signed Radon measures $\mathscr{M}(\Omega,\mathbb{R})$. The duality pairing for $f \in C(\Omega)$ and $\mu \in \mathscr{M}(\Omega,\mathbb{R})$ is given by

$$\langle \mu, f \rangle = \int_\Omega f \,\mathrm{d}\mu \,.$$

Hence, a sequence of Radon measures $(\mu_n)_{n\in\mathbb{N}} \subset \mathscr{M}(\Omega,\mathbb{R})$ converges weakly-* to $\mu \in \mathscr{M}(\Omega,\mathbb{R})$ if

$$\int_\Omega f \,\mathrm{d}\mu_n \to \int_\Omega f \,\mathrm{d}\mu \quad \forall f \in C(\Omega) \,.$$

Furthermore, since $C(\Omega)$ is a separable space, every bounded sequence $(\mu_n)_{n\in\mathbb{N}} \subset \mathscr{M}(\Omega,\mathbb{R})$ of signed Radon measures has a weakly-* converging subsequence.

Definition 0.2 (Functions of Bounded Variation)**.**

(1) We denote by $\mathscr{B}(\Omega)$ the Borel σ-algebra. Let $\mu\colon \mathscr{B}(\Omega) \to \mathbb{R}^m$ be a vectorial measure. Then the total variation $|\mu|_{TV}$ for $E \in \mathscr{B}(\Omega)$ is given by

$$|\mu|_{TV}(E) := \sup\left\{ \sum_{n\in\mathbb{N}} \|\mu(E_n)\| \;:\; E = \bigcup_{n\in\mathbb{N}} E_n \text{ for } E_n \in \mathscr{B}(\Omega) \text{ pairwise disjoint} \right\}$$

and defines a positive and finite measure.

(2) The space of functions of bounded variation is defined by

$$\mathrm{BV}(\Omega) := \{u \in L^1(\Omega) \;:\; Du \in \mathscr{M}(\Omega, \mathbb{R}^d) \text{ for the distributional gradient}\}\,.$$

More precisely, $Du = (D_i u)_{i=1,\dots,d}$ is a vectorial Radon measure with

$$\int_\Omega u \partial_i \phi \, \mathrm{d}x = -\int_\Omega \phi \, \mathrm{d}D_i u(x) \quad \forall \phi \in C_c^\infty(\Omega)\,.$$

(3) For $u \in \mathrm{BV}(\Omega)$, we define the norm $\|u\|_{\mathrm{BV}(\Omega)} := \|u\|_{L^1(\Omega)} + |Du|_{TV}(\Omega)$.

(4) For a sequence $u_k \in \mathrm{BV}(\Omega)$ and $u \in \mathrm{BV}(\Omega)$, we say that u_k converges weak-* to u in BV if $u_k \to u$ strongly in $L^1(\Omega)$ and $Du_k \overset{*}{\rightharpoonup} Du$ in $\mathscr{M}(\Omega, \mathbb{R}^d)$.

Example 0.3 (BV vs $W^{1,1}$)**.** Consider the sequence

$$u_k(x) = \begin{cases} -1 & \text{if } x < 1/k \\ kx & \text{if } x \in [-1/k, 1/k] \\ 1 & \text{if } x > 1/k \end{cases}$$

Then $u_k \in W^{1,1}([-1,1])$ with

$$\|u_k\|_{W^{1,1}} = \int_{[-1,1]} |u_k(x)| \, \mathrm{d}x + \int_{[-1,1]} |Du_k(x)| \, \mathrm{d}x \leqslant 2 + 2 = 4\,.$$

But the point-wise limit

$$u(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases}$$

is not in $W^{1,1}$, since the distributional derivative is given by

$$Du(\phi) = -\int_{[-1,1]} u D\phi \, \mathrm{d}x = -\int_{[-1,0]} -D\phi \, \mathrm{d}x - \int_{[0,1]} D\phi \, \mathrm{d}x = 2\phi(0)\,.$$

Hence, $Du = 2\delta_0 \notin L^1$. But $2\delta_0 \in \mathscr{M}([-1,1])$ and $u \in \mathrm{BV}([-1,1])$.

Example 0.4 (BV vs L^p)**.** For the set $S = \bigcup_{k\in\mathbb{N}_+} \left(\frac{1}{2k}, \frac{1}{2k-1}\right)$ we define

$$u(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

Then u has infintely many jumps. Hence $u \notin \mathrm{BV}$, but $u \in L^\infty$.

Example 0.5 (Connection to Perimeter)**.** Let $E \subset \Omega$ be a smooth subset. Then $\chi_E \in \mathrm{BV}(\Omega)$ and

$$\mathscr{H}^{d-1}(\partial E) = |D\chi_E|_{TV}(\Omega)$$

More generally, for $\chi \in \mathrm{BV}(\Omega, \{0,1\})$, we define $\mathrm{Per}_\Omega(\chi) := |D\chi|_{TV}(\Omega)$

Example 0.6 (BV Norm of Discrete Image)**.** Let $u_h\colon [0,1]^d \to [0,1]$ be a discrete image as in (1). Then the set of interior interfaces is given by

$$\mathscr{I}_h := \left\{ I_{\alpha,\beta} = \overline{C_\alpha} \cap \overline{C_\beta} \;:\; \alpha \neq \beta \right\} .$$

Thus,

$$|Du_h|_{TV}([0,1]^d) = \sum_{I_{\alpha,\beta}\in\mathscr{I}_h} |u_\alpha - u_\beta| \mathscr{H}^{d-1}(E) = \sum_{I_{\alpha,\beta}\in\mathscr{I}_h} h^{d-1}|u_\alpha - u_\beta| .$$

Proposition 0.7 (Properties of BV functions)**.**

(1) Let $u \in \mathrm{BV}(\Omega)$. Then

$$|Du|_{TV}(\Omega) = \sup\left\{ \int_\Omega u \operatorname{div}\phi \,\mathrm{d}x \;:\; \phi \in C_c^1(\Omega, \mathbb{R}^d)\,,\ \|\phi\|_\infty \leqslant 1 \right\} . \tag{2}$$

(2) Density:
Let $u \in \mathrm{BV}(\Omega)$. Then there exists a sequence $u_k \in \mathrm{BV}(\Omega) \cap C^\infty(\Omega)$ s.t. $u_k \to u$ in L^1 and $|Du_k|_{TV}(\Omega) \to |Du|_{TV}(\Omega)$ for $k \to \infty$.

(3) Compactness in $\mathrm{BV}(\Omega)$:
Let $(u_k) \subset \mathrm{BV}(\Omega)$ with $\|u_k\| \leqslant C$. Then there exists a subsequence with $u_k \overset{*}{\rightharpoonup} u \in \mathrm{BV}(\Omega)$.

(4) Lower semi-continuity of BV-norm:
Let $u_k \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega)$. Then $|Du|_{TV}(\Omega) \leqslant \liminf_{k\to\infty} |Du_k|_{TV}(\Omega)$.

(5) $\mathrm{BV}(\Omega)$ is a Banach space.

(6) Embedding of BV:
Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary and let $1 \leqslant p < \frac{d}{d-1}$. Then the embedding $\mathrm{id}\colon \mathrm{BV}(\Omega) \to L^p(\Omega)$ is continuous and compact.

Proof (sketch).

(1) By duality of Radon measures, we have that

$$|Du|_{TV}(\Omega) = \sup\left\{ \int_\Omega \phi \,\mathrm{d}Du(x) \;:\; \phi \in C_c^\infty(\Omega, \mathbb{R}^d)\,,\ \|\phi\|_\infty \leqslant 1 \right\}$$

Since for all $\phi \in C_c^\infty(\Omega)$, we have by definition that $\int_\Omega u\partial_i\phi \,\mathrm{d}x = -\int_\Omega \phi \,\mathrm{d}D_i u(x)$, it follows by summation that $\int_\Omega u \operatorname{div}\phi \,\mathrm{d}x = -\int_\Omega \phi \,\mathrm{d}Du(x)$. By a smoothing argument, this holds for all $\phi \in C_c^1(\Omega)$.

(2) Use an exhaustion of Ω and standard mollifier.

(3) By density, for each u_k, there exists $g_k \in C^\infty(\Omega)$ with $\|u_k - g_k\|_{L^1} \leqslant \frac{1}{k}$ and $\|Dg_k\|_{L^1} = |Dg_k|_{TV}(\Omega) \leqslant |Du_k|_{TV}(\Omega) + 1$. Hence, by Rellich's Theorem in $W^{1,1}$, $g_k \to g$ in L^1 for a subsequence. But then, we must have $u_k \to g$ in L^1. Futhermore, by compactness of Radon measures, there exists $\mu \in \mathscr{M}(\Omega, \mathbb{R}^d)$ s.t. $Du_k \xrightharpoonup{*} \mu$.

(4) Follows from the representation (2).

(5) Completeness follows by compactness and lower semicontinuity.

(6) We take a smooth approximation $(u_k) \subset W^{1,1}$ of u and apply for each u_k the compact embedding theorem in $W^{1,1}$ and pass to the limit.

□

[Lecture 01 – 04/20/2020]

[Lecture 02 – 04/27/2020]

1. Convex Optimization

1.1. Convex Analysis

For a more detailed introduction into convex analysis we refer to the textbooks [BC17] and [ET99]. We start with some basic definitions.

Definition 1.1. Let U be a Banach space. We say that a functional $E\colon U \to \mathbb{R} \cup \{\infty\}$ is

(1) proper if $\operatorname{dom}(E) := \{u \in U \ : \ E(u) < \infty\} \neq \emptyset$,

(2) convex if $E(\lambda u_1 + (1-\lambda)u_2) \leqslant \lambda E(u_1) + (1-\lambda)E(u_2)$ for all $u_1, u_2 \in U$, $\lambda \in [0,1]$, and

(3) lower semi-continuous (lsc) if $E(u) \leqslant \liminf_{k\to\infty} E(u_k)$ for all $u_k \to u$.

We denote by $\Gamma_0(U)$ the set of all proper, convex and lower semi-continuous functionals on U.

In the following, we are interested in minimizing a functional $E \in \Gamma_0(U)$. In particular, E is in general nonsmooth. Here, our goal is to characterize minimizers of E. Therefore, we will frequently make use the so-called epi-graph of the functional E.

Definition 1.2. The epi-graph of a functional $E\colon U \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\operatorname{epi} E := \{(u,t) \in U \times \mathbb{R} \ : \ E(u) \leqslant t\} .$$

Lemma 1.3. *Let U be a Banach space and $E \in \Gamma_0(U)$. Then E is weakly lsc, i.e. for all $u_k \rightharpoonup u$ we have $E(u) \leqslant \liminf_{k\to\infty} E(u_k)$.*

Proof. We consider the epi-graph $\operatorname{epi} E$.

By convexity of E, $\operatorname{epi} E \subset U \times \mathbb{R}$ is a convex set, since for $(u_1,t_1),(u_2,t_2) \in \operatorname{epi} E$ and $\lambda \in [0,1]$ we have $E(\lambda u_1 + (1-\lambda)u_2) \leqslant \lambda E(u_1) + (1-\lambda)E(u_2) \leqslant \lambda t_1 + (1-\lambda)t_2$, thus $\lambda(u_1,t_1) + (1-\lambda)(u_2,t_2) \in \operatorname{epi} E$.

By lsc of E, $\operatorname{epi} E$ is a closed set, since for a sequence $(u_k,t_k) \in \operatorname{epi}$ with $(u_k,t_k) \to (u,t)$ we have $E(u) \leqslant \liminf E(u_k) \leqslant \liminf t_k = t$. By Mazur's Lemma, every closed convex set is weakly closed. Hence, $\operatorname{epi} E$ is weakly closed in $U \times \mathbb{R}$.

Now, let $u_k \rightharpoonup u$. We take a subsequence (u_{k_l}) s.t. $t := \lim E(u_{k_l}) = \liminf E(u_k)$. Then $(u_{k_l}, E(u_{k_l})) \in \operatorname{epi} E$, but $(u_{k_l}, E(u_{k_l})) \rightharpoonup (u,t)$. Since $\operatorname{epi} E$ is weakly closed, we have $(u,t) \in \operatorname{epi} E$, hence $E(u) \leqslant t = \liminf E(u_k)$. $\square$

Theorem 1.4 (Existence of minimizers)**.** *Let U be a reflexive Banach space. Let $E \in \Gamma_0(U)$ with growth condition $\lim_{\|u\|\to\infty} E(u) = \infty$. Then there exists a minimizer of E in U.*

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be a minmizing sequence, i.e. $\lim_{k\to\infty} E(u_k) = \inf E$. Because of the growth condition $\lim_{\|u\|\to\infty} E(u) = \infty$, we can assume $\|u_k\| \leqslant C$ for all $k \in \mathbb{N}$. Hence, since U is reflexive, there exists a weakly converging subsequence $u_k \rightharpoonup u \in U$. By Lemma 1.3, E is also weakly lsc, i.e.

$$E(u) \leqslant \liminf_{k\to\infty} E(u_k) = \inf E .$$

Thus, u is a minimizer of E. $\square$

Example 1.5 (Nonexistence of minimizers)**.**

(1) In Theorem 1.4, the condition that E is lsc is necessary. E.g. we consider the function $E\colon \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ with

$$E(x) = \begin{cases} \infty & \text{if } x < 0 \\ c & \text{if } x = 0 \\ x & \text{if } x > 0 \end{cases}$$

for some constant $c \in \mathbb{R} \cup \{\infty\}$. Then E is convex for $c \in [0, \infty]$, and E is lsc for $c \in (-\infty, 0]$. The infimum is only attained for $(-\infty, 0]$.

(2) In Theorem 1.4, the condition $\lim_{\|u\|\to\infty} E(u) = \infty$ is necessary. E.g., a linear function $E\colon \mathbb{R} \to \mathbb{R}$ does not have a minimizer.

(3) In Theorem 1.4, the condition that U is reflexive is necessary. For $E\colon W^{1,1}([-1,1]) \to \mathbb{R}$ we define the functional

$$E(u) = \int_{[-1,0]} |u+1| \,\mathrm{d}x + \int_{[0,1]} |u-1| \,\mathrm{d}x\,.$$

The infimum is given by 0 (e.g. take the sequence $(u_k)_{k\in\mathbb{N}}$ as considered in Example 0.3). But the minimizer is given by $u = -\chi_{[-1,0]} + \chi_{[0,1]} \notin W^{1,1}$.

Note that the space BV is not reflexive. However, as we will discuss later in the applications, we can apply the compactness result Proposition 0.7(3) to obtain a weakly-* converging subsequence.

Now, to minimize a smooth functional $E \in \Gamma_0(U)$, we typically would apply a gradient descent method to solve the necessary optimality condition $DE = 0$. Since E is in general nonsmooth, we have to introduce a generalized version of differentiability.

Definition 1.6 (Subdifferential)**.** Let $E\colon U \to \mathbb{R} \cup \{\infty\}$ be proper. Then the subdifferential of E in $u \in U$ is defined by

$$\partial E(u) = \{u' \in U' \;:\; \langle u', \hat{u} - u\rangle \leqslant E(\hat{u}) - E(u) \;\forall \hat{u} \in U\}\,.$$

We call E subdifferentiable at u if $\partial E(u) \neq \emptyset$, and we call E subdifferentiable if $\partial E(u) \neq \emptyset$ for all $u \in U$.

Lemma 1.7 (Properties of the Subdifferential)**.**

(1) Characterization of Minimizers:
Let $E\colon U \to \mathbb{R} \cup \{\infty\}$ be proper. Then

$$\begin{aligned} \operatorname{arg\,min} E &:= \{u \in U \;:\; E(u) \leqslant E(\hat{u}) \;\forall \hat{u} \in U\} \\ &= \{u \in U \;:\; 0 \in \partial E(u)\} =: \operatorname{zero} \partial E\,. \end{aligned}$$

(2) Let $E\colon U \to \mathbb{R}$ be convex and differentiable at $u \in U$, then $\partial E(u) = \{DE(u)\}$.

Thus, the subdifferential admits a generalization of the necessary condition $DE(u) = 0$ for a minimizer u of a smooth function E. Note that Lemma 1.7(2) does in general not hold if E is nonconvex.

Example 1.8 (Subdifferentiable Functions)**.**

(1) The absolute value function $|\cdot|\colon \mathbb{R} \to \mathbb{R}$ is subdifferentiable with

$$\partial|x| = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1,1] & \text{if } x = 0 \\ \{1\} & \text{if } x > 0\,. \end{cases}$$

(2) More generally, the Euclidean norm $\|\cdot\|_2\colon \mathbb{R}^N \to \mathbb{R}$ is subdifferentiable with

$$\partial\|x\|_2 = \begin{cases} \dfrac{x}{\|x\|_2} & \text{if } x \neq 0 \\ \left\{x' \in \mathbb{R}^N \ : \ \langle x', \hat{x}\rangle \leqslant \|\hat{x}\|_2 \ \forall \hat{x} \in \mathbb{R}^N\right\} = B_1(0) & \text{if } x = 0 \end{cases}$$

(3) Let $C \subset U$ be a nonempty, closed, and convex set and $\mathcal{I}_C$ be its indicator function given by

$$\mathcal{I}_C(u) = \begin{cases} 0 & \text{if } u \in C\,, \\ \infty & \text{if } u \notin C\,. \end{cases}$$

Then

$$\partial\,\mathcal{I}_C(u) = \begin{cases} \varnothing & \text{if } u \notin C \\ \{u' \in U' \ : \ \langle u', \widehat{u} - u\rangle \leqslant 0 \ \forall \widehat{u} \in C\} & \text{if } u \in \partial C \\ \{0\} & \text{if } u \in C^\circ\,. \end{cases}$$

The set $N_C = \{u' \in U' \ : \ \langle u', \widehat{u} - u\rangle \leqslant 0 \ \forall \widehat{u} \in C\}$ is called the normal cone.

Definition 1.9 (Convex Conjugate)**.** Let $E : U \to \mathbb{R} \cup \{\infty\}$ be proper. We define

(1) its convex conjugate $E^* : U' \to \mathbb{R} \cup \{\infty\}$ by

$$E^*(u') = \sup_{u \in U} \langle u', u\rangle - E(u)\,.$$

(2) its convex biconjugate $E^{**} : U \to \mathbb{R} \cup \{\infty\}$ by

$$E^{**}(u) = \sup_{u' \in U'} \langle u', u\rangle - E^*(u')\,.$$

Lemma 1.10 (Properties of the Convex Biconjugate)**.** *Let $E : U \to \mathbb{R} \cup \{\infty\}$ be proper.*

(1) Let U be reflexive. Then $(E^*)^* = E^{**}$

(2) The convex biconjugate is the convex relaxation of E, i.e.

$$E^{**}(u) = \operatorname{conv} E(u) := \sup_{\beta \in \mathbb{R}, u' \in U'} \left\{\langle u', u\rangle - \beta \ : \ \langle u', \widehat{u}\rangle - \beta \leqslant E(\widehat{u}) \ \forall \widehat{u} \in U\right\}\,.$$

(3) $E^{**} \in \Gamma_0(U)$ and $E^{**} \leqslant E$.

(4) Let $E \in \Gamma_0(U)$. Then $E = E^{**}$.

Proof.

(1) By definition, U is reflexive if the embedding $J_U \colon U \to U''$, $u \mapsto (u' \to \langle u', u\rangle)$ is an isomorphism. Hence,

$$(E^*)^*(J_U u) = \sup_{u' \in U'} \langle J_U u, u'\rangle - E^*(u') = \sup_{u' \in U'} \langle u', u\rangle - E^*(u') = E^{**}(u)$$

(2) This follows since $\langle u', \hat{u}\rangle - \beta \leqslant E(\hat{u})\ \forall \hat{u} \in U \Leftrightarrow \beta \geqslant E^*(u')$.

(3) Easy to check by using the representation in (2).

(4) Since $E \in \Gamma_0(U)$, it is given as pointwise supremum over affine functions. Indeed, using Lemma 1.3, $\operatorname{epi} E$ is a closed and convex set, thus the Hahn–Banach theorem guarantees the existence of a separating hyperplane. Hence, E coincides with the representation in (2).

$\square$

Lemma 1.11 (Properties of the convex conjugate)**.** *Let $E : U \to \mathbb{R} \cup \{\infty\}$ be proper.*

(1) $E^* \in \Gamma_0(U')$.

(2) Fenchel–Young inequality:
$E(u) + E^*(u') \geqslant \langle u', u\rangle$ *for all $u \in U$, $u' \in U'$*

(3) $E(u) + E^*(u') = \langle u', u\rangle \Leftrightarrow u' \in \partial E(u)$

(4) Let U be reflexive. Then $u' \in \partial E(u) \Rightarrow u \in \partial E^*(u')$.

(5) Let $E \in \Gamma_0(U)$. Then $u' \in \partial E(u) \Leftrightarrow u \in \partial E^*(u')$. In particuar, $0 \in \partial E(u) \Leftrightarrow u \in \partial E^*(0)$.

Proof.

(1) E^* is proper, since E is proper. It is easy to check that E^* is convex. For given $u \in U$ with $E(u) < \infty$, the map $h_u \colon u' \mapsto \langle u', u\rangle - E(u)$ is convex and lsc. Thus, E^* is a supremum over convex and lsc functions. As in the proof of Lemma 1.3, by lsc, $\operatorname{epi} h_u$ is closed. Now, $\operatorname{epi} E^* = \bigcap_{u \in U} \operatorname{epi} h_u$ is closed, hence E^* is lsc.

(2) $E(u) + E^*(u') = E(u) + \sup_{\hat{u} \in U} \langle u', \hat{u}\rangle - E(\hat{u}) \geqslant \langle u', u\rangle$

(3) We have

$$\begin{aligned} u' \in \partial E(u) \quad &\Leftrightarrow \quad \langle u', \hat{u}\rangle - E(\hat{u}) \leqslant \langle u', u\rangle - E(u) \quad \forall \hat{u} \in U \\ &\Leftrightarrow \quad E^*(u') = \langle u', u\rangle - E(u)\,. \end{aligned}$$

(4) First, by (2), we have

$$E^{**}(u) + E^*(u') \geqslant \langle u', u\rangle\,.$$

Second, since $E^{**} \leqslant E$, by Lemma 1.10(3), we observe by using (3) that

$$E^{**}(u) + E^*(u') \leqslant E(u) + E^*(u') = \langle u', u\rangle\,.$$

Combining both inequalities implies

$$E^{**}(u) + E^*(u') = \langle u', u\rangle.$$

Now, because U is reflexive, by Lemma 1.10(1) we have $E^{**} = (E^*)^*$. Then, using (3) for E^*, we obtain $u \in \partial E^*(u')$.

(5) Let $u \in \partial E^*(u')$. By Lemma 1.10(4), $E^{**} = E$. Using (3), we get

$$E(u) + E^*(u) = E^{**}(u) + E^*(u) = \langle u', u \rangle$$

Again using (3), this implies $u' \in \partial E(u)$.

□

Example 1.12 (L^p norm)**.** Let $U = L^p(\Omega)$ for $p \in (1, \infty)$. Note that $U' = L^{p'}(\Omega)$ for $\frac{1}{p} + \frac{1}{p'} = 1$. We consider $E(u) = \frac{1}{p} \int_\Omega \|u\|^p \, \mathrm{d}x$. Then $E^*(u') = \frac{1}{p'} \int_\Omega \|u'\|^{p'} \, \mathrm{d}x$.

Example 1.13 (Absolute Value)**.** For the absolute value $E(x) = |x|$ we have that $E^*(x') = \mathcal{I}_{[-1,1]}(x')$.

Example 1.14 (Indicator Function)**.** Let $A \subset U$ and $\mathcal{I}_A$ be its indicator function. Then

$$\mathcal{I}_A^*(u') = \sup_{u \in A} \langle u', u \rangle$$

is the so-called support function of A. Furthermore, $\mathcal{I}_A^{**} = \mathcal{I}_{\operatorname{conv} A}$ is the indicator function of the convex hull of A.

[Lecture 02 – 04/27/2020]

[Lecture 03 – 05/04/2020]

Later in our applications, a direct minimization of E will turn out to be complicated. In the following, we will consider a specific class of problems, where E can be splitted into two functions F and G with the intention that minimizers of F and G are easier to compute separately.

Problem 1.15: Minimization of Splitted Function

Let U, V be Banach spaces, $L\colon U \to V$ a continuous linear operator. Let $F\colon V \to \mathbb{R} \cup \{\infty\}$ and $G\colon U \to \mathbb{R} \cup \{\infty\}$ be proper. Then

$$\text{minimize} \quad E(u) := F(Lu) + G(u) \qquad \text{over all } u \in U\,.$$

Theorem 1.16 (Convex Duality)**.** *Consider the minimization problem 1.15.*

(1) weak duality:

$$\sup_{v' \in V'} -F^*(v') - G^*(-L^*v') \leqslant \inf_{u \in U} F(Lu) + G(u) \tag{1.1}$$

(2) strong duality:
Let $F \in \Gamma_0(U)$, $G \in \Gamma_0(V)$. Assume that there exists $u_0 \in U$ s.t. $E(u_0) < \infty$ and F is continuous at $v_0 = Lu_0$. Then

$$\sup_{v' \in V'} -F^*(v') - G^*(-L^*v') = \inf_{u \in U} F(Lu) + G(u)$$

Proof.

(1) Let $v' \in V'$. By definition of F^* and G^* we have for all $v \in V$ and all $u \in U$

$$F^*(v') \geqslant \langle v', v\rangle - F(v)\,,$$
$$G^*(-L^*v') \geqslant \langle -L^*v', u\rangle - G(u)\,.$$

Choosing $v = Lu$, we obtain

$$-F^*(v') - G^*(-L^*v') \leqslant -\langle v', Lu\rangle + F(Lu) - \langle -L^*v', u\rangle + G(u) = F(Lu) + G(u)\,.$$

(2) If $\inf E = -\infty$, the statement is trivial. Since $\inf E \leqslant E(u_0) < \infty$, we can assume $\inf E \in \mathbb{R}$. Now, we define sets

$$\begin{aligned} C_1 &:= \{(v_1, t_1) \in V \times \mathbb{R} \;:\; F(v) \leqslant \inf E - t_1\} \\ C_2 &:= \{(u_2, t_2) \in U \times \mathbb{R} \;:\; G(u_2) < t_2\} \\ C &:= \{(Lu_2 - v_1, t_2 - t_1) \;:\; (v_1, t_1) \in C_1,\ (u_2, t_2) \in C_2\} \end{aligned}$$

Note that C_1, C_2 and thus C are convex sets. By definition of C_1 and C_2, we obtain that $(0, 0) \notin C$. By continuity of F at Lu_0, the interior of C_1 is nonempty. Moreover, since $G(u_0) < \infty$, the interior of C is nonempty. Thus, by the Hahn–Banach separation theorem, there exists $(v', t') \in (V' \times \mathbb{R})\backslash(0, 0)$, s.t.

$$\langle v', Lu_2 - v_1\rangle + t'(t_2 - t_1) \geqslant 0 \quad \forall (Lu_2 - v_1, t_2 - t_1) \in C\,.$$

We observe that $t' > 0$, since for $t' < 0$ we could choose $t_2 \to \infty$ and $t_1 \to -\infty$ leading to a contradiction, and for $t' = 0$ we could choose $u_2 = u_0$, $v_1 = Lu_0 - v_\varepsilon$ for some $v_\varepsilon \in V$ with $\langle v', v_\varepsilon\rangle \geqslant 0$ (existence of such v_ε is guaranteed by continuity of F). Now, we can define $v_0' := \frac{1}{t'}v'$ and get

$$\langle v_0', Lu_2 - v_1\rangle + (t_2 - t_1) \geqslant 0 \quad \forall (Lu_2 - v_1, t_2 - t_1) \in C\,.$$

Thus, by definition of C_1, C_2, for all $u_2 \in U$, $v_1 \in V$, and $\varepsilon > 0$, we can choose $t_1 = \inf E - F(v_1)$ and $t_2 = G(u_2) + \varepsilon$ and obtain

$$\inf E \leqslant F(v_1) + G(u_2) + \langle v_0', Lu_2 - v_1\rangle \quad \forall u_2 \in U\ \forall v_1 \in V\,.$$

Hence,

$$\begin{aligned} \inf E &\leqslant \inf_{u \in U, v \in V} F(v) - \langle v_0', v\rangle + G(u) + \langle L^*v_0', u\rangle \\ &\leqslant \sup_{v' \in V'} \inf_{u \in U, v \in V} F(v) - \langle v', v\rangle + G(u) + \langle L^*v', u\rangle = \sup_{v' \in V'} -F^*(v') - G^*(-L^*v')\,. \end{aligned}$$

□

Remark 1.17 (on strong duality). Formally we have

$$\begin{aligned} \inf_{u \in U} E(u) &= \inf_{u \in U} (F(Lu) + G(u)) && \text{(primal problem)} \\ &= \inf_{u \in U} \sup_{v' \in V'} (\langle v', Lu\rangle - F^*(v') + G(u)) && \text{(saddle point problem)} \\ &= \sup_{v' \in V'} \left(-F^*(v') - \sup_{u \in U} (-\langle L^*v', u\rangle - G(u)) \right) \\ &= \sup_{v' \in V'} (-F^*(v') - G^*(-L^*v')) && \text{(dual problem)} \end{aligned}$$

For the second equality, we have used $F^{**} = F$, which holds for $F \in \Gamma_0(V)$. For the third equality, we have assumed that $\inf\sup = \sup\inf$.

1.2. Proximal Splitting Algorithm

Now, we will derive a concrete algorithm for the minimization problem above. Therefore, we restrict to finite dimensional spaces U_h, V_h. However, we remark that most of the here presented tools can be generalized to Hilbert spaces.

Definition 1.18 (Proximal Mapping)**.** For $E_h \in \Gamma_0(U_h)$, the proximal mapping is defined as

$$\mathrm{prox}_{E_h}(u_h) := \operatorname*{arg\,min}_{\widehat{u_h}\in U_h} \frac{1}{2}\|u_h - \widehat{u_h}\|^2 + E_h(\widehat{u_h})\,.$$

Note that for $E_h \in \Gamma_0(U_h)$, the function $\widehat{u_h} \mapsto \frac{1}{2}\|u_h - \widehat{u_h}\|^2 + E_h(\widehat{u_h})$ is uniformly convex, and thus has a unique minimizer.

Proposition 1.19 (Relation between Proximal Mapping and Subdifferential)**.** *Let $E_h \in \Gamma_0(U_h)$. Then for $u_h, p_h \in U_h$ and $\lambda > 0$ we have the relation*

$$p_h = \mathrm{prox}_{\lambda E_h}(u_h) \Leftrightarrow \frac{1}{\lambda}(u_h - p_h) \in \partial E_h(p)\,.$$

Proof. We consider the function $g(\widehat{u_h}) = \frac{1}{2}\|u_h - \widehat{u_h}\|^2 + \lambda E_h(\widehat{u_h})$. By Lemma 1.7 and the definition of the proximal mapping, $p_h = \mathrm{prox}_{\lambda E_h}(u_h)$ is equivalent to $0 \in \partial g(p_h) = \{p_h - u_h\} + \lambda \partial E_h(p_h)$, which is equivalent to $\frac{1}{\lambda}(u_h - p_h) \in \partial E_h(p_h)$. □

Example 1.20 (Proximal Map of Indicator Function)**.** Let $C \subset U_h$ be a closed and convex set. Then

$$\mathrm{prox}_{\mathcal{I}_C}(u_h) = \operatorname*{arg\,min}_{\widehat{u_h}\in C} \frac{1}{2}\|u_h - \widehat{u_h}\|^2 = \mathrm{proj}_C(u_h)\,,$$

where proj_C denotes the orthogonal projection on C w.r.t. the norm $\|\cdot\|$ on U_h.

Example 1.21 (Proximal mapping of absolute value)**.** We consider the absolute value $x \mapsto \lambda|x|$ for some factor $\lambda > 0$. Then $p = \mathrm{prox}_{\lambda|\cdot|}(x)$ must satisfy

$$0 \in \partial\left(\frac{1}{2}(x-p)^2 + \lambda|p|\right) = \begin{cases} \{p - x - \lambda\} & \text{if } p < 0 \\ \{x' - x \;:\; x' \in [-\lambda, \lambda]\} & \text{if } p = 0 \\ \{p - x + \lambda\} & \text{if } p > 0 \end{cases}$$

Hence

$$\mathrm{prox}_{\lambda|\cdot|}(x) = \begin{cases} x + \lambda & \text{if } x < -\lambda \\ 0 & \text{if } x \in [-\lambda, \lambda] \\ x - \lambda & \text{if } x > \lambda \end{cases} = \mathrm{sgn}(x)\max(|x| - \lambda, 0)$$

Now, similar to a gradient descent method, proximal point algorithms iteratively perform proximal operators to obtain a sequence, which converges to a minimizer of E_h. In many applications, a closed-form expression of the proximal operator of E_h is not available, but E_h admits a splitting into functions F_h and G_h as in 1.15, s.t. the proximal operators of F_h and G_h can be computed explicitly. Then, in the optimization scheme, these proximal operators of F_h and G_h are applied alternatingly, where specific step sizes are given according to an appropriate fixed point map. Furthermore, the algorithm developed by Chambolle and Pock [CP11] makes use of the convex dual formulation of the actual minimization problem.

Algorithm 1.1 Primal-Dual proximal splitting algorithm.

function PRIMALDUALPROXSPLITTING(u_h^0, v_h^0, $z_h^0 = u_h^0$)
  for $n = 1, \ldots, N$ **do**

$$
\begin{aligned}
v_h^{n+1} &= \operatorname{prox}_{\sigma F_h^*}\left(v_h^n + \sigma L_h z_h^n\right) \\
u_h^{n+1} &= \operatorname{prox}_{\tau G_h}\left(u_h^n - \tau L_h^* v_h^{n+1}\right) \\
z_h^{n+1} &= u_h^{n+1} + (u_h^{n+1} - u_h^n)
\end{aligned}
$$

  end for
end function

Theorem 1.22 (Convergence of Proximal Splitting Algorithm)**.** *Let $F_h \in \Gamma_0(V_h)$ and $G_h \in \Gamma_0(U_h)$. Let $(u_h^0, v_h^0) \in U_h \times V_h$ be two initial values and set $z_h^0 = u_h^0$. Furthermore, let $\tau, \sigma > 0$ s.t. $\tau\sigma\|L_h\|^2 < 1$. Consider the sequences $(v_h^n)_{n\in\mathbb{N}}$ and $(u_h^n)_{n\in\mathbb{N}}$ generated by Algorithm 1.1. Assume that there exists a minimizer u_h of the primal problem and a solution v_h of the dual problem. Then we have convergence $u_h^n \to u_h$ and $v_h^n \to v_h$ for $n \to \infty$.*

Proof (sketch). **Step 1: A fixed point of iteration solves the primal and dual problem.** Assume that $(\bar{u}_h, \bar{v}_h, \bar{z}_h = \bar{u}_h)$ is a fixpoint of the iteration, i.e.

$$
\begin{aligned}
\bar{v}_h &= \operatorname{prox}_{\sigma F_h^*}\left(\bar{v}_h + \sigma L_h \bar{u}_h\right) , \\
\bar{u}_h &= \operatorname{prox}_{\tau G_h}\left(\bar{u}_h - \tau L_h^* \bar{v}_h\right) .
\end{aligned}
$$

From Proposition 1.19 we get

$$
\begin{aligned}
\frac{1}{\sigma}\left(\bar{v}_h + \sigma L_h \bar{u}_h - \bar{v}_h\right) = L_h \bar{u}_h \in \partial F_h^*(\bar{v}_h) , \\
\frac{1}{\tau}\left(\bar{u}_h - \tau L_h^* \bar{u}_h - \bar{u}_h\right) = -L_h^* \bar{v}_h \in \partial G_h(\bar{u}_h) .
\end{aligned}
$$

Thus, for all $(\hat{u}_h, \hat{v}_h) \in U_h \times V_h$, by definition of the subdifferential we have

$$
\begin{aligned}
\langle \hat{v}_h - \bar{v}_h, L_h \bar{u}_h \rangle \leqslant F_h^*(\hat{v}_h) - F_h^*(\bar{v}_h) , \\
\langle \hat{u}_h - \bar{u}_h, -L_h^* \bar{v}_h \rangle \leqslant G_h(\hat{u}_h) - G_h(\bar{u}_h) .
\end{aligned}
$$

Summing up leads to

$$
0 \leqslant F_h^*(\hat{v}_h) - F_h^*(\bar{v}_h) - \langle v_h - \bar{v}_h, L_h \bar{u}_h \rangle + G_h(\hat{u}_h) - G_h(\bar{u}_h) + \langle L_h(\hat{u}_h - \bar{u}_h), \bar{v}_h \rangle
$$

and thus

$$
\langle \hat{v}_h, L_h \bar{u}_h \rangle - F_h^*(\hat{v}_h) + G_h(\bar{u}_h) \leqslant \langle \bar{v}_h, L_h \hat{u}_h \rangle - F_h^*(\bar{v}_h) + G_h(\hat{u}_h) .
$$

This allows to compare the values of the primal problem at $\bar{u}_h$ with the dual problem at $\bar{v}_h$:

$$\begin{aligned}
F_h(L_h\bar{u}_h) + G_h(\bar{u}_h) &= F_h^{**}(L_h\bar{u}_h) + G_h(\bar{u}_h) \\
&= \sup_{\hat{v}_h}\langle \hat{v}_h, L_h\bar{u}_h\rangle - F_h^*(\hat{v}_h) + G_h(\bar{u}_h) \\
&\leqslant \inf_{\hat{u}_h}\langle \bar{v}_h, L_h\hat{u}_h\rangle - F_h^*(\bar{v}_h) + G_h(\hat{u}_h) \\
&= -F_h^*(\bar{v}_h) - \sup_{\hat{u}_h}\langle -L_h^*\bar{v}_h, \hat{u}_h\rangle - G_h(\hat{u}_h) \\
&= -F_h^*(\bar{v}_h) - G_h^*(-L_h^*\bar{v}_h)\,.
\end{aligned}$$

By weak duality (1.1), $\bar{u}_h$ must be a solution of the primal problem, and $\bar{v}_h$ must be a solution of the dual problem.

Step 2: A priori error estimate
From Proposition 1.19 we get

$$\begin{aligned}
\frac{v_h^n - v_h^{n+1}}{\sigma} + L_h z_h^n \in \partial F_h^*(v_h^{n+1})\,, \\
\frac{u_h^n - u_h^{n+1}}{\tau} - L_h^* v_h^{n+1} \in \partial G_h(u_h^{n+1})\,.
\end{aligned}$$

Using similar arguments as in Step 1 and summation over n, we observe

$$\frac{\|v_h^n - v_h\|^2}{2\sigma} + \frac{\|u_h^n - u_h\|^2}{2\tau} \leqslant \frac{1}{1-\sigma\tau\|L_h\|^2}\left(\frac{\|v_h^0 - v_h\|^2}{2\sigma} + \frac{\|u_h^0 - u_h\|^2}{2\tau}\right)\,,$$

where u_h is the solution of the primal problem and v_h is the solution of the dual problem. For details, we refer to [CP11][Theorem 1]. □

Remark 1.23. Assuming that at least one of the functionals F_h or G_h is uniformly convex, there is an accelerated version ([CP11][Algorithm 2]), which uses variable step sizes τ^n, σ^n.

At first glance, computing $\mathrm{prox}_{E_h^*}$ might be easier to compute prox_{E_h} or vice-versa, but the following theorem allows computing one of these expressions if the other one is known.

Theorem 1.24 (Moreau Decomposition). *For $E_h \in \Gamma_0(U_h)$ and $\lambda > 0$, we have the following identity*

$$\mathrm{prox}_{\lambda E_h}(u_h) + \lambda\,\mathrm{prox}_{\frac{1}{\lambda}E_h^*}\left(\frac{u_h}{\lambda}\right) = u_h\,.$$

Proof. We have

$$\begin{aligned}
& p_h = \mathrm{prox}_{\lambda E_h}(u_h) \\
\Leftrightarrow\quad & \frac{1}{\lambda}(u_h - p_h) \in \partial E_h(p_h) && \text{by Proposition 1.19} \\
\Leftrightarrow\quad & p_h = \frac{1}{\frac{1}{\lambda}}\left(\frac{u_h}{\lambda} - \frac{u_h - p_h}{\lambda}\right) \in \partial E_h^*(\frac{u_h - p_h}{\lambda}) && \text{by Lemma 1.11} \\
\Leftrightarrow\quad & \frac{u_h - p_h}{\lambda} = \mathrm{prox}_{\frac{1}{\lambda}E_h^*}(\frac{u_h}{\lambda}) && \text{by Proposition 1.19} \\
\Leftrightarrow\quad & u_h = \mathrm{prox}_{\lambda E_h}(u_h) + \lambda\,\mathrm{prox}_{\frac{1}{\lambda}E_h^*}\left(\frac{u_h}{\lambda}\right)
\end{aligned}$$

□

1.3. Denoising

The following model for image denoising goes back to Rudin, Osher, and Fatemi [ROF92].

Problem 1.25: ROF-Model

Given $u^{\text{input}} \in L^2(\Omega, [0,1])$. Then, we aim to minimize the function

$$E^{\text{ROF}}(u) := \frac{1}{2}\int_\Omega (u - u^{\text{input}})^2 \, \mathrm{d}x + \eta |Du|_{TV}(\Omega)$$

over all $u \in \mathrm{BV}(\Omega, [0,1]) \cap L^2(\Omega, [0,1])$.

Since $\mathrm{BV}(\Omega)$ is not a reflexive space, we cannot apply Theorem 1.4 to prove existence of minimizers. However, by using properties of BV, we obtain the following result.

Theorem 1.26 (Existence of Minimizer for ROF)**.** *There exists a minimizer $u \in \mathrm{BV}(\Omega, [0,1]) \cap L^2(\Omega, [0,1])$ of E^{ROF}.*

Proof (not discussed in the lecture). We use the direct method.

boundedness:
We can choose $u = 0$, hence $\inf_{u \in \mathrm{BV}} E^{\text{ROF}}(u) < \frac{1}{2}\int_\Omega |u^{\text{input}}|^2 < \infty$.

compactness:
Let $(u_k)_{k\in\mathbb{N}} \subset \mathrm{BV}(\Omega, [0,1]) \cap L^2(\Omega, [0,1])$ be a minimizing sequence of E^{ROF}. By boundedness, we can assume $E^{\text{ROF}}(u_k) \leqslant C < \infty$ for all $k \in \mathbb{N}$. Then

$$\begin{aligned}
\|u_k\|_{\mathrm{BV}(\Omega)} &= \|u_k\|_{L^1(\Omega)} + |Du_k|_{TV}(\Omega) \\
&\leqslant C(\|u_k - u^{\text{input}}\|_{L^2(\Omega)} + \|u^{\text{input}}\|_{L^2(\Omega)}) + |Du_k|_{TV}(\Omega) \\
&\leqslant C\sqrt{E^{\text{ROF}}(u_k)} + C + E^{\text{ROF}}(u_k) < \infty\,.
\end{aligned}$$

By Proposition 0.7(3) (compactness), there exists a subsequence $u_k \overset{*}{\rightharpoonup} u \in \mathrm{BV}$. Since $\|u_k\|_{L^2} \leqslant \|u_k - u^{\text{input}}\|_{L^2} + \|u^{\text{input}}\|_{L^2} \leqslant \sqrt{2E^{\text{ROF}}(u_k)} + C < \infty$, there exists a further subsequence $u_k \rightharpoonup u \in L^2$.

lower semicontinuity:
Let $u_k \overset{*}{\rightharpoonup} u \in \mathrm{BV}$. Recall from Definition 0.2 that this means $u_k \to u$ in L^1 and $Du_k \overset{*}{\rightharpoonup} Du$ in $\mathscr{M}(\Omega, \mathbb{R}^d)$. By Proposition 0.7(4) (lsc), we obtain $|Du|_{TV}(\Omega) \leqslant \liminf_{k\to\infty} |Du_k|_{TV}(\Omega)$. Let $u_k \rightharpoonup u \in L^2$. Then $\|u - u^{\text{input}}\|_{L^2}^2 \leqslant \liminf_{k\to\infty} \|u_k - u^{\text{input}}\|_{L^2}^2$. Together, for a sequence $(u_k)_{k\in\mathbb{N}}$ with $u_k \overset{*}{\rightharpoonup} u \in \mathrm{BV}$ and $u_k \rightharpoonup u \in L^2$, we have

$$E^{\text{ROF}}(u) \leqslant \liminf_{k\to\infty} E^{\text{ROF}}(u_k)\,.$$

□

Note that in space dimension $d = 2$, the embedding $\mathrm{id}\colon \mathrm{BV}(\Omega) \to L^2(\Omega)$ is continuous.

Application of Proximal Splitting Algorithm. Now, we want to apply the algorithm 1.1 to solve a discrete version of the minimization problem 1.25. Formally, for continuous images, we can choose $G(u) = \frac{1}{2}\int_\Omega |u - u^{\text{input}}|^2 \, dx$, $F(v) = \eta |v|_{TV}(\Omega)$, and $L(u) = Du$.

In the discrete setting, we choose $U_h = \mathbb{R}^{|\mathscr{C}_h|}$ and $V_h = \mathbb{R}^{|\mathscr{I}_h|}$ with inner products

$$\langle u_h, \hat{u}_h \rangle_{U_h} = \sum_{C_\alpha \in \mathscr{C}_h} h^d u_\alpha \hat{u}_\alpha \,,$$

$$\langle v_h, \hat{v}_h \rangle_{V_h} = \sum_{I_{\alpha,\beta} \in \mathscr{I}_h} h^{d-1} v_{\alpha,\beta} \hat{v}_{\alpha,\beta} \,.$$

Then we define

$$F_h(v_h) = \eta \sum_{I_{\alpha,\beta} \in \mathscr{I}_h} h^{d-1} |v_{\alpha,\beta}| \,,$$

$$G_h(u_h) = \frac{1}{2} \|u_h - u^{\text{input}}\|_{U_h}^2 = \sum_{C_\alpha \in \mathscr{C}_h} \frac{h^d}{2} (u_\alpha - u_\alpha^{\text{input}})^2 \,.$$

Furthermore, we define $L_h = \nabla_h \colon U_h \to V_h$ by

$$\nabla_h (u_h)_{\alpha,\beta} = u_\alpha - u_\beta \,.$$

Then we denote the adjoint operator by

$$\operatorname{div}_h \coloneqq L_h^* \colon V_h \to U_h \,.$$

Finally, by using Example 1.21, we compute

$$\begin{aligned}
\operatorname{prox}_{\tau G_h}(u_h) &= \operatorname*{arg\,min}_{\hat{u}_h \in U_h} \frac{h^d}{2} \sum_{C_\alpha \in \mathscr{C}_h} (\hat{u}_\alpha - u_\alpha)^2 + \frac{\tau h^d}{2} \sum_{C_\alpha \in \mathscr{C}_h} (\hat{u}_\alpha - u_\alpha^{\text{input}})^2 \\
&= \frac{u_h + \tau u^{\text{input}}}{1 + \tau} \\
\operatorname{prox}_{\sigma F_h}(v_h) &= \operatorname*{arg\,min}_{\hat{v}_h \in V_h} \frac{h^{d-1}}{2} \sum_{I_{\alpha,\beta} \in \mathscr{I}_h} (\hat{v}_{\alpha,\beta} - v_{\alpha,\beta})^2 + \frac{\eta \sigma h^{d-1}}{2} \sum_{I_{\alpha,\beta} \in \mathscr{I}_h} |\hat{v}_{\alpha,\beta}| \\
&= \begin{cases} v_{\alpha,\beta} + \eta\sigma & \text{if } v_{\alpha,\beta} < -\eta\sigma \\ 0 & \text{if } |v_{\alpha,\beta}| \leqslant \eta\sigma \\ v_{\alpha,\beta} - \eta\sigma & \text{if } v_{\alpha,\beta} > \eta\sigma \end{cases}
\end{aligned}$$

Then, by applying Moreau's Decomposition Theorem 13 1.24, we obtain

$$\begin{aligned}
\operatorname{prox}_{\sigma F_h^*}(v_h) &= v_h - \sigma \operatorname{prox}_{\frac{1}{\sigma} F_h}(\frac{1}{\sigma} v_h) \\
&= v_{\alpha,\beta} - \sigma \begin{cases} \frac{1}{\sigma}(v_{\alpha,\beta} + \eta) & \text{if } \frac{1}{\sigma} v_{\alpha,\beta} < -\frac{\eta}{\sigma} \\ 0 & \text{if } \frac{1}{\sigma} |v_{\alpha,\beta}| \leqslant \frac{\eta}{\sigma} \\ \frac{1}{\sigma}(v_{\alpha,\beta} - \eta) & \text{if } \frac{1}{\sigma} v_{\alpha,\beta} > \frac{\eta}{\sigma} \end{cases} = v_{\alpha,\beta} - \begin{cases} v_{\alpha,\beta} + \eta & \text{if } v_{\alpha,\beta} < -\eta \\ 0 & \text{if } |v_{\alpha,\beta}| \leqslant \eta \\ v_{\alpha,\beta} - \eta & \text{if } v_{\alpha,\beta} > \eta \end{cases} \\
&= \operatorname{proj}_{[-\eta,\eta]}(v_{\alpha,\beta})
\end{aligned}$$

Thus, we arrive at the following algorithm:

Algorithm 1.2 Proximal Splitting algorithm for denoising with ROF model.

function ROF-DENOISING(u_h^{input}, u_h^0, v_h^0, $z_h^0 = u_h^0$)

for $n = 1, \dots, N$ **do**

$$
\begin{aligned}
v_h^{n+1} &= \operatorname{proj}_{[-\eta,\eta]}\left(v_h^{n+1} + \sigma \nabla_h z_h^n\right) \\
u_h^{n+1} &= \frac{u_h^n - \tau \operatorname{div}_h v_h^{n+1} + \tau u^{\text{input}}}{1+\tau} \\
z_h^{n+1} &= u_h^{n+1} + (u_h^{n+1} - u_h^n)
\end{aligned}
$$

end for

end function

[Lecture 03 – 05/04/2020]

[Lecture 04 – 05/11/2020]

Remark 1.27.

(1) In Algorithm 1.2, no constraint on the values of u_h is enforced. However, if we want that $u_\alpha \in [0,1]$ for all α, we can define $G_h(u_h) = \frac{1}{2}\|u_h - u^{\text{input}}\|_{U_h}^2 + \mathcal{I}_{[0,1]^{|\mathscr{C}_h|}}(u_h)$. Since the computation of $\operatorname{prox}_{\tau G_h}(u_h)$ decouples for every cell, we simply have

$$
\operatorname{prox}_{\tau G_h}(u_h)_\alpha = \operatorname{proj}_{[0,1]}\left(\frac{u_\alpha + \tau u_\alpha^{\text{input}}}{1+\tau}\right).
$$

(2) From a theretical point of view, for continuous images, we require a factor $\eta > 0$ s.t. the solution is in BV. In the discrete setting, for every discrete image u_h, the value $F_h(\nabla_h u_h)$ is finite, but it becomes very large for noisy images. Thus, the choice of η allows to control how regular the denoised image should be.

(3) Alternatively, we could consider the dual representation (2) of the total variation norm. Therefore, we define a discrete version of $\eta|Du|_{TV}(\Omega)$ by

$$
\sup\left\{\sum_{C_\alpha \in \mathscr{C}_h} h^d u_\alpha (\operatorname{div}_h \phi_h)_\alpha \;:\; \phi_h \in V_h = \mathbb{R}^{|\mathscr{I}_h|} \text{ s.t. } |\phi_{\alpha,\beta}| \leqslant \eta\right\}.
$$

Then, a discrete version of the ROF problem is given by

$$
\begin{aligned}
&\inf_{u_h \in U_h} \frac{h^d}{2} \sum_{C_\alpha \in \mathscr{C}_h} (u_\alpha - u_\alpha^{\text{input}})^2 + \sup_{\phi_h \in V_h : |\phi_{\alpha,\beta}| \leqslant \eta} h^d \sum_{C_\alpha \in \mathscr{C}_h} u_\alpha (\operatorname{div}_h \phi_h)_\alpha \\
&= \sup_{\phi_h \in V_h} \inf_{u_h \in U_h} h^d \sum_{C_\alpha \in \mathscr{C}_h} \frac{1}{2}(u_\alpha - u_\alpha^{\text{input}})^2 + u_\alpha (\operatorname{div}_h \phi_h)_\alpha - \mathcal{I}_{|\phi_{\alpha,\beta}| \leqslant \eta}(\phi_h) \\
&= \sup_{\phi_h \in V_h} h^d \sum_{C_\alpha \in \mathscr{C}_h} -\frac{1}{2}(\operatorname{div}_h \phi_h)_\alpha^2 + u_\alpha^{\text{input}} (\operatorname{div}_h \phi_h)_\alpha - \mathcal{I}_{|\phi_{\alpha,\beta}| \leqslant \eta}(\phi_h)
\end{aligned}
$$

In fact, this maximization problem is of type $\sup_{\phi_h \in V_h} -F_h^*(\phi_h) - G_h^*(-L_h^* \phi_h)$ with $F_h^* = \mathcal{I}_{|\phi_{\alpha,\beta}| \leqslant \eta}$ and $G_h^*(u_h') = h^d \sum_{C_\alpha \in \mathscr{C}_h} \frac{1}{2}(u_\alpha')^2 + u_\alpha' u_\alpha^{\text{input}}$.

1.4. Inpainting

In general, inpainting is a method to restore a damaged image. Here, we consider a damage on a subdomain $D \subset \Omega$ and propose the following model.

Problem 1.28: Inpainting

Let $D \subset \Omega$ and $u^{\text{input}} \in \mathrm{BV}(\Omega \backslash D, [0,1]) \cap L^2(\Omega \backslash D, [0,1])$. We define a set of admissible images

$$\mathcal{A} := \left\{ u \in \mathrm{BV}(\Omega, [0,1]) \cap L^2(\Omega, [0,1]) \;:\; u = u^{\text{input}} \text{ in } \Omega \backslash D \right\} .$$

Then, we aim to minimize the function

$$E^{\text{inpaint}}(u) = \mathcal{I}_{\mathcal{A}}(u) + \eta |Du|_{TV}(\Omega)$$

over all $u \in \mathrm{BV}(\Omega, [0,1]) \cap L^2(\Omega, [0,1])$.

Now, we directly discuss the discrete model and the corresponding algorithm.

Application of Proximal Splitting Algorithm. As for the ROF-model, we take into account discrete spaces $U_h = \mathbb{R}^{|\mathscr{C}_h|}$, $V_h = \mathbb{R}^{|\mathscr{I}_h|}$. Furthermore, we assume that $D_h \subset \mathscr{C}_h$ and define

$$\mathcal{A}_h := \{ u_h \in U_h \;:\; u_h(C_\alpha) = u_h^{\text{input}}(C_\alpha) \; \forall C_\alpha \in \mathscr{C}_h \backslash D_h \} .$$

Then we define $G_h(u_h) = \mathcal{I}_{\mathcal{A}_h}(u_h)$ and compute

$$\mathrm{prox}_{\sigma G_h}(u_h)_\alpha = \mathrm{proj}_{\mathcal{I}_{\mathcal{A}_h}}(u_h)_\alpha = \begin{cases} u_\alpha & \text{if } C_\alpha \in D_h \\ u_\alpha^{\text{input}} & \text{if } C_\alpha \in \mathscr{C}_h \backslash D_h \end{cases}$$

With F_h and L_h as for the ROF model, we obtain the following algorithm:

Algorithm 1.3 Proximal splitting algorithm for inpainting.

function TV-INPAINTING($u_h^{\text{input}}, u_h^0, v_h^0, z_h^0 = u_h^0$)
  for $n = 1, \ldots, N$ **do**

$$\begin{aligned} v_h^{n+1} &= \mathrm{proj}_{[-\eta,\eta]}\left(v_h^n + \sigma \nabla_h z_h^n \right) \\ u_\alpha^{n+1} &= \begin{cases} u_\alpha^n - \tau \,\mathrm{div}_h\, v_\alpha^{n+1} & \text{if } C_\alpha \in D_h \\ u_\alpha^{\text{input}} & \text{if } C_\alpha \in \mathscr{C}_h \backslash D_h \end{cases} \\ z_h^{n+1} &= u_h^{n+1} + (u_h^{n+1} - u_h^n) \end{aligned}$$

  end for
end function

2. Convex Lifting

2.1. Depth Map from Stereo

A so called stereo camera consists of two parallel cameras C^A, C^B in the (w.l.o.g.) x_1-direction (see Figure 2.1). Here, we directly restrict to the case $\Omega = [0,1]^2$. Given two images u^A and u^B of the the two cameras, our aim is to reconstruct the depth of the three-dimensional object. Therefore, we take into account the so-called disparity map $w\colon [0,1]^2 \to [0,1]$, which describes for a given point (x_1, x_2) in u^A the displacement in u^B, i.e.

$$u^B(x_1, x_2 + w(x_1, x_2)) = u^A(x_1, x_2)\,. \tag{2.1}$$

Now, taking into account the distance c between the two cameras, which are assumed to have the same focal length f, we get the following geometric relation

$$\frac{c}{c - w(x)} = \frac{d(x)}{d(x) - f}\,.$$

Hence, the disparity map is given by

$$w(x_1, x_2) = \frac{c \cdot f}{d(x_1, x_2)}\,,$$

or vice-versa, if the disparity map w is known, we can recover the depth of the three-dimensional object by

$$d(x_1, x_2) = \frac{c \cdot f}{w(x_1, x_2)}\,,$$

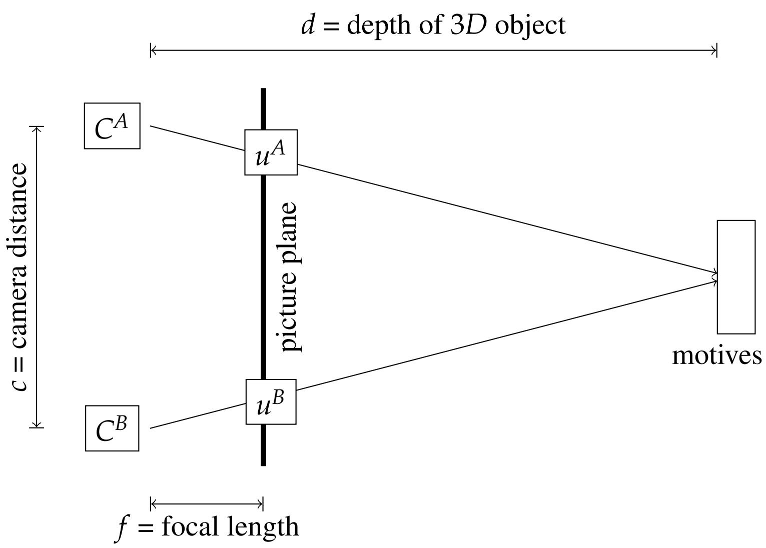

Figure 2.1: Two images u^A, u^B of the same object are taken from two parallel cameras C_A and C_B.

For real images, due to usual artefacts, it is in general not possible to satisfy the exact constraint (2.1) for every pixel. Therefore, we use a penalty in the L^2-norm. Furthermore, we assume that the disparity map is in the space $\mathrm{BV}(\Omega, [0,1])$. This motivates to define the following functional to compute the disparity map.

Problem 2.1: Depth Map from Stereo

Let $u^A, u^B \in L^2(\Omega, [0,1])$ be two images taken from two parallel cameras in x_1-direction with distance c and focal length f. Then, minimize the function

$$E^{\text{Stereo}}(w) = \frac{1}{2}\int_\Omega |u^B(x_1, x_2 + w(x_1,x_2)) - u^A(x_1,x_2)|^2 \,\mathrm{d}x + \eta |Dw|_{TV}(\Omega)$$

over all $w \in \mathrm{BV}(\Omega)$.

Thus, $E^{\text{Stereo}}\colon \mathrm{BV}(\Omega, \mathbb{R})$ is of type

$$E^{\text{Stereo}}(w) = \int_\Omega g(x, w(x))\,\mathrm{d}x + \eta |Dw|_{TV}(\Omega)\,,$$

where $g(x = (x_1,x_2), t) = \frac{1}{2}|u^B(x_1, x_2 + t) - u^A(x_1,x_2)|^2$.

Now, the idea of convex lifting is to define a convex functional on a higher dimensional space, s.t. the solution of the nonconvex optimization problem can be recovered from the solution of the convex optimization problem. Therefore, we use the following function.

Lemma 2.2. *Let $g\colon \Omega \times \mathbb{R} \to \mathbb{R}$ We define $h\colon \Omega \times \mathbb{R} \times (\mathbb{R}^d \times \mathbb{R})$ by*

$$h(x,t,(p^x,p^t)) := \begin{cases} |p^t| g(x,t) + \eta |p^x| & \text{if } p^t \leqslant 0\,,\\ \infty & \text{if } p^t > 0\,.\end{cases}$$

Then $h(x,t,\cdot)$ is convex and lsc. Furthermore, h is positively 1-homogeneous, i.e. for all $\lambda \geqslant 0$ we have $h(x,t,\lambda(p^x,p^t)) = \lambda h(x,t,(p^x,p^t))$.

Proof.
First, we show that h is convex.
For a convex combination $\lambda(p_1^x, p_1^t) + (1-\lambda)(p_2^x,p_2^t)$ we can distinguish the following cases:

- If $p_1^t \leqslant 0$ and $p_2^t \leqslant 0$, we must have $\lambda p_1^t + (1-\lambda)p_2^t \leqslant 0$.
- If $p_1^t > 0$ and $p_2^t > 0$, we must have $\lambda p_1^t + (1-\lambda)p_2^t > 0$.
- If w.l.o.g. $p_1^t > 0$ and $p_2^t \leqslant 0$, we already have $\lambda h(p_1^x, p_1^t) = \infty$.

Now, we show that h is lsc. For a sequence $(p_k^x, p_k^t) \to (p^x, p^t)$, we can distinguish two cases:

- If $p^t > 0$, there is $K \in \mathbb{N}$ s.t. $p_k^t > 0$ for all $k \geqslant K$. Thus, $\liminf_{k\to\infty} h(x,t,(p_k^x,p_k^t)) = \infty$.
- If $p^t \leqslant 0$, we can assume $p_k^t \leqslant 0$. In this case, $h(x,t,\cdot)$ is continuous.

Finally, we show that h is positively 1-homogeneous. For $\lambda > 0$, we have that

$$h(x,t,\lambda(p^x,p^t)) = \begin{cases} |\lambda p^t| g(x,t) + \eta|\lambda p^x| & \text{if } p^t \leqslant 0 \\ \infty & \text{if } p^t > 0 \end{cases} = \lambda h(x,t,(p^x,p^t))$$

and for $\lambda = 0$, we have that $h(x,t,(0,0)) = 0$. □

In the following, we study minimizers of the functional

$$E^{\text{lift}}(v) := \int_{\Omega\times[0,1]} h(x,t,Dv)\, \mathrm{d}(x,t)\,.$$

Theorem 2.3 (Lifted Functional)**.** *Assume that $w\colon \Omega \to [0,1]$. On the lifted domain $\Omega \times [0,1]$ we consider the characteristic function*

$$\chi_{\operatorname{epi} w}(x,t) = \begin{cases} 1 & \textit{if } w(x) \leqslant t\,, \\ 0 & \textit{if } w(x) > t\,. \end{cases}$$

Furthermore, we define the interface

$$\Gamma_w := \{(x,t) \;:\; w(x) = t\}$$

and the corresponding normal

$$\nu_{\Gamma_w}(x,w(x)) = \frac{(Dw,-1)}{\sqrt{\|Dw\|^2+1}}\,.$$

Then, we have that $E^{Stereo}(w) = E^{lift}(D\chi_{\operatorname{epi} w})$.

Proof. We compute

$$\begin{aligned}
&E^{\text{Stereo}}(w) \\
&= \int_\Omega g(x,w(x))\, \mathrm{d}x + |Dw|_{TV}(\Omega) \\
&= \int_\Omega g(x,w(x)) + \eta|Dw|\, \mathrm{d}x && \text{since } w \text{ is smooth} \\
&= \int_\Omega h(x,w(x),(Dw,-1))\, \mathrm{d}x && \text{by definition of } h \\
&= \int_\Omega h(x,w(x),\nu_{\Gamma_w}(x,w(x)))\sqrt{\|Dw\|^2+1}\, \mathrm{d}x && \text{by 1-homogeneity of } h \\
&= \int_{\Gamma_w} h(x,t,\nu_{\Gamma_w}(x,t))\, \mathrm{d}\mathscr{H}^d(x,t) && \text{by integral transformation } x \mapsto (x,w(x))
\end{aligned}$$

Now, since $D\chi_{\operatorname{epi} w}(x,t) = \nu_{\Gamma_w}(x,t)\mathscr{H}^d\big|_{\Gamma_w}$, we can formally write

$$E^{\text{Stereo}}(w) = \int_{\Omega\times[0,1]} h(x,t,D\chi_{\operatorname{epi} w}(x,t))\, \mathrm{d}(x,t)\,,$$

□

Thus, a convex relaxation of E^{Stereo} is given by minimizing the functional E^{lift} over all $v \in \text{BV}(\Omega \times [0,1],[0,1])$ with $v(x,0) = 0$ and $v(x,1) = 1$. Since the function h is convex and lsc in the last argument, this is indeed a convex optimization problem. Furthermore, the 1-homogeneity allows to rigorously define $h(x,t,\mu)$ for a Radon measure μ. Finally, one can prove a coarea formula

$$E^{\text{lift}}(v) = \int_0^1 E^{\text{lift}}(\chi_{v>t})\,\mathrm{d}t\,,$$

which shows that for a minimizer v of E^{lift} and for any $t > 0$, the characteristic fucntion $\chi_{v>t}$ is also a minimizer of E^{lift}. Thus, v already represents an epi-graph of a function $u \in \text{BV}(\Omega,[0,1])$, which minimizes the original functional E^{Stereo}.

For the numerical algorithm, we will make use of the following dual representation.

Theorem 2.4 (Dual representation)**.** *For $v \in \text{BV}(\Omega \times [0,1],[0,1])$ we define*

$$\tilde{E}^{lift}(v) = \sup_{\phi \in K} \int_{\Omega \times [0,1]} \phi(x,t)\,\mathrm{d}Dv(x,t)\,,$$

where the set K is given by

$$K := \left\{ \phi = (\phi^x, \phi^t) \in C(\Omega \times \mathbb{R}, \mathbb{R}^{d+1}) \;:\; \phi^t(x,t) + g(x,t) \geqslant 0,\; \|\phi^x(x,t)\| \leqslant \eta \right\}.$$

Let w be smooth. Then $E^{Stereo}(w) = \tilde{E}^{lift}(\chi_{\text{epi}\,w})$.

Proof.
Step 1: $\tilde{E}^{\text{lift}}(\chi_{\text{epi}\,w}) \leqslant E^{\text{Stereo}}(w)$
For $\phi \in C(\Omega \times \mathbb{R}, \mathbb{R}^{d+1})$, we have

$$\begin{aligned}\int_{\Omega \times [0,1]} \phi(x,t) D\chi_{\text{epi}\,w}(x,t)\,\mathrm{d}(x,t) &= \int_{\Gamma_w} \phi(x,t) \cdot \frac{(Dw,-1)}{\sqrt{|Dw|^2+1}}\,\mathrm{d}\mathscr{H}^d(x,t) \\ &= \int_\Omega \phi^x(x,w(x)) \cdot Dw(x) - \phi^t(x,w(x))\,\mathrm{d}x\end{aligned}$$

Hence, taking into account the set K, we have that

$$\begin{aligned}&\sup_{\phi \in K} \int_{\Omega \times [0,1]} \phi(x,t) D\chi_{\text{epi}\,w}(x,t)\,\mathrm{d}(x,t) \\ &\leqslant \sup_{\phi^x \in C(\Omega \times [0,1],\mathbb{R}^d)\,:\,\|\phi^x(x,t)\| \leqslant \eta} \int_\Omega \phi^x(x,w(x)) Dw(x) + g(x,w(x))\,\mathrm{d}x \\ &= \int_\Omega g(x,w(x))\,\mathrm{d}x + \eta |Dw|_{TV}(\Omega) = E^{\text{Stereo}}(w)\,.\end{aligned}$$

Step 2: $E^{\text{Stereo}}(w) = \tilde{E}^{\text{lift}}(\chi_{\text{epi}\,w})$
To get equality, we have to construct $\phi \in K$. If w and g are smooth, we can choose $\phi^x(x,w(x)) = \frac{Dw}{|Dw|}$ and $\phi^t(x,w(x)) = -g(x,w(x))$. For the general case, we have to approximate ϕ.

□

This motivates to study the follwing convex optimization problem.

Problem 2.5: Depth Map from Stereo: Convex Lifting

Let $u^A, u^B \in L^2(\Omega, [0,1])$ be two images taken from two parallel cameras in x_1-direction with distance c and focal length f. We define $g(x_1, x_2, t) = \frac{1}{2}|u^B(x_1, x_2 + t) - u^A(x_1, x_2)|^2$ and

$$K := \left\{ \phi = (\phi^x, \phi^t) \in C(\Omega \times \mathbb{R}, \mathbb{R}^{d+1}) \ : \ \phi^t(x,t) + g(x,t) \geqslant 0, \ \|\phi^x(x,t)\| \leqslant \eta \right\}.$$

Then, minimize the function

$$\tilde{E}^{\text{lift}}(v) = \sup_{\phi \in K} \int_{\Omega \times [0,1]} \phi(x,t) \, \mathrm{d}Dv(x,t)$$

over all $v \in \mathrm{BV}(\Omega \times [0,1])$ with $v(x,0) = 0$ and $v(x,1) = 1$.

Application of Proximal Splitting Algorithm. First, we define the set of admissible functions by

$$C = \{ v \in \mathrm{BV}(\Omega \times [0,1], [0,1]) \ : \ v(\cdot, 0) = 0, \ v(\cdot, 1) = 1 \}.$$

Then, we have to solve the saddle point problem

$$\begin{aligned} &\inf_{v \in C} \sup_{\phi \in K} \int_{\Omega \times [0,1]} \phi \cdot Dv \, \mathrm{d}(x,t) \\ &= \inf_{v \in \mathrm{BV}(\Omega \times [0,1],[0,1])} \ \sup_{\phi \in C(\Omega \times [0,1], \mathbb{R}^d \times \mathbb{R})} \int_{\Omega \times [0,1]} \phi \cdot Dv \, \mathrm{d}(x,t) - \mathcal{I}_K(\phi) + \mathcal{I}_C(v) \end{aligned}$$

Thus, we can formally set

$$\begin{aligned} L(v) &= Dv \\ F^*(\phi) &= \mathcal{I}_K(\phi) \\ G(v) &= \mathcal{I}_C(v) \end{aligned}$$

(cf. with the saddle point formuation in the formal derivation of the strong duality in Remark 1.17).

For a numerical discretization, we take into account three dimensional cells C_α. Note for discrete images u_h^A, u_h^B defined on two-dimensional cells, we can rigorously define $g_h(x,t) = |u_h^B(x_1, x_2 + t) - u_h^A(x)|^2$ on three dimensional cells. Therefore, we define

$$K_h = \{ \phi_h = (\phi_h^x, \phi_h^t) \in \mathbb{R}^{|\mathscr{C}_h|^2} \times \mathbb{R}^{|\mathscr{C}_h|} \ : \ \phi_\alpha^t + g_\alpha \geqslant 0 \,, \ |\phi_\alpha^x| \leqslant \eta \}.$$

Furthermore, since we want to evaluate the product $\phi_h \cdot \nabla_h v_h$ pointwise, we choose

$$C_h = \{ v_h \in C^0([0,1]^3, [0,1]) \ : \ v_h \text{ multilinear on each cell}, v_h(x,0) = 0 \,, \ v_h(x,1) = 1 \}$$

and define $\nabla_h \colon C_h \to \mathbb{R}^{|\mathscr{C}_h|^2} \times \mathbb{R}^{|\mathscr{C}_h|}$ by

$$\nabla_h(v_h)_\alpha = \sum_{q \in Q} \omega(q) D v_h(q) \,,$$

Algorithm 2.4 Proximal splitting algorithm for depth map from stereo.

function DEPTHMAPFROMSTEREO(u_h^A,u_h^B, v^0, ϕ^0, $z_h^0 = v^0$)
  for $n = 1, \dots, N$ **do**

$$\begin{aligned} \phi_h^{n+1} &= \operatorname{proj}_{K_h}\left(\phi_h^n + \sigma \nabla_h z_h^n\right) \\ v_h^{n+1} &= \operatorname{proj}_{C_h}(v^n - \tau \operatorname{div}_h \phi^{n+1}) \\ z_h^{n+1} &= v^{n+1} + (v^{n+1} - v^n) \end{aligned}$$

  end for
end function

where e.g. Q is a tensor-product simpson quadrature.

For the the speed of computation, it is crucial that the projection in the proximal splttting algorithm can be performed pointwise.

Remark 2.6 (Different Discretization). Instead of the discretization of C with multilinear finite elments as above, we could also define v_h on interfaces. In this case, we also consider boundary interfaces to define boundary values. In fact, this are precisely the degrees of freedom for a quadrilateral Raviart-Thomas finite element.

[Lecture 04 – 05/11/2020]

[Lecture 05 – 05/18/2020]

2.2. Binary Segmentation

The following model to segment an image into an a priori unknown number of components was proposed by Mumford and Shah [MS89].

Problem 2.7: Mumford–Shah functional for general segmentation

Given $u^{\text{input}} \in L^2(\Omega, [0,1])$ and $\mu, \eta > 0$. Then minimize

$$E^{\text{MS}}(u, S) = \frac{1}{2}\int_\Omega (u - u^{\text{input}})^2 \,\mathrm{d}x + \frac{\mu}{2}\int_{\Omega\setminus S} |Du|^2 \,\mathrm{d}x + \eta \mathscr{H}^{d-1}(S) \tag{2.2}$$

over all pairs (u, S), where $S \subset \Omega$ is a set with $\mathscr{H}^{d-1}(S) < \infty$ and $u \colon \Omega \to [0,1]$ is a function which is piecewise smooth (i.e. $W^{1,2}$) w.r.t. the set S.

Note that the $W^{1,2}$ semi-norm on $\Omega\setminus S$ guarantees the smoothness of the function u on the components. However, the Mumford–Shah functional (2.2) is nonconvex.

Example 2.8 (Mumford–Shah functional is nonconvex)**.** We consider $u_1 = \chi_{[0,1]\times[0,1]}$, $u_2 = \chi_{[1,2]\times[0,1]}$. and the convex combination $u = \frac{1}{2}u_1 + \frac{1}{2}u_2$. Then for $u^{\text{input}} = 0$, we compute

$$\begin{aligned}
E^{\text{MS}}(u_1, S_1) &= \frac{1}{2}\int_{[0,1]^2} \,\mathrm{d}x + \frac{\mu}{2}\int_{\Omega\setminus S} 0 \,\mathrm{d}x + \eta\mathscr{H}^1([0,1]^2) = \frac{1}{2} + 4\eta \\
E^{\text{MS}}(u_2, S_2) &= \frac{1}{2}\int_{[1,2]\times[0,1]} \,\mathrm{d}x + \frac{\mu}{2}\int_{\Omega\setminus S} 0 \,\mathrm{d}x + \eta\mathscr{H}^1([1,2]\times[0,1]) = \frac{1}{2} + 4\eta \\
E^{\text{MS}}(\frac{1}{2}u_1 + \frac{1}{2}u_2) &= \frac{1}{2}\int_{[0,2]\times[0,1]} \frac{1}{4} \,\mathrm{d}x + \frac{\mu}{2}\int_{\Omega\setminus S} 0 \,\mathrm{d}x + \eta\mathscr{H}^1([0,2]\times[0,1]) = \frac{1}{4} + 6\eta
\end{aligned}$$

Hence $\frac{1}{4} + 6\eta = E^{\text{MS}}(\frac{1}{2}u_1 + \frac{1}{2}u_2) > \frac{1}{2}E^{\text{MS}}(u_1, S_1) + \frac{1}{2}E^{\text{MS}}(u_2, S_2) = \frac{1}{2} + 4\eta$ for $\eta > \frac{1}{8}$.

For a better understanding of the Mumford–Shah model, we first discuss the so-called binary case, where the number of components is fixed by two and the segmentation function is assumed to be constant on each of the two components. More precisely, we make the assumption that

$$u = c^A\chi + c^B(1 - \chi)$$

for two constants $c^A, c^B \in [0,1]$ and a characteristic function $\chi \in \text{BV}(\Omega, \{0,1\})$. Thus, the set S is given by

$$S = \{x \in \Omega \;:\; D\chi(x) \neq 0\}\,.$$

Note that $\mathscr{H}^{d-1}(S) = |D\chi|_{TV}(\Omega)$. By inserting u and S in the energy (2.2), we obtain

$$\begin{aligned}
E^{\text{MS}}(u, S) &= \frac{1}{2}\int_\Omega (u - u^{\text{input}})^2 \,\mathrm{d}x + \frac{\mu}{2}\int_{\Omega\setminus S} |Du|^2 \,\mathrm{d}x + \eta\mathscr{H}^{d-1}(S) \\
&= \frac{1}{2}\int_\Omega (c^A\chi + c^B(1-\chi) - u^{\text{input}})^2 \,\mathrm{d}x + 0 + \eta|D\chi|_{TV}(\Omega) \\
&= \frac{1}{2}\int_\Omega (c^A - u^{\text{input}})^2\chi + (c^B - u^{\text{input}})^2(1-\chi) \,\mathrm{d}x + \eta|D\chi|_{TV}(\Omega) \\
&=: E^{\text{bMS}}(\chi, c^A, c^B)\,.
\end{aligned}$$

Then, the optimality conditions in (c^A, c^B) lead to

$$c^A = \fint_{\chi=1} u^{\text{input}}\, \mathrm{d}x\,, \quad c^B = \fint_{\chi=0} u^{\text{input}}\, \mathrm{d}x\,. \tag{2.3}$$

Moreover, the formal optimality condition in χ leads to

$$\eta\,\mathrm{div}\left(\frac{D\chi}{|D\chi|}\right) = (c^A - u^{\text{input}})^2 - (c^B - u^{\text{input}})^2\,. \tag{2.4}$$

In [CV00], similar to a proximal splitting algorithm, an iteration scheme by alternatingly solving (2.3) for fixed χ and (2.4) for fixed c^A, c^B was proposed. However, note that in general convex combinations of triples c^A, c^B, χ cannot again be represented by such a triple.

Theorem 2.9 (Existence of Minimizers for the binary Mumford–Shah functional)**.** *There exists a minimizer (χ, c^A, c^B) of E^{bMS}.*

Proof. First, we observe that $E^{\text{bMS}}(0,0,0) = \frac{1}{2}\int_\Omega (u^{\text{input}})^2\,\mathrm{d}x < \infty$. Hence, there exists a minimizing sequence $(\chi_k, c_k^A, c_k^B)_{k\in\mathbb{N}}$ with $E^{\text{bMS}}(\chi_k, c_k^A, c_k^B) \leqslant C < \infty$. Furthermore, we may assume that the values for c_k^A, c_k^B are already optimal w.r.t. χ_k, i.e. they are given by the expression in (2.3). To point out the optimality, we denote these values by $c_k^A(\chi_k), c_k^B(\chi_k)$. Now, since $|D\chi_k|_{TV}(\Omega) < C$, by Proposition 0.7(3), there exists a subsequence $\chi_k \overset{*}{\rightharpoonup} \chi \in \mathrm{BV}(\Omega, \{0,1\})$. (Note that the range of χ is given by the set $\{0,1\}$, since the weak-* convergence in BV implies the strong convergence L^1.) Moreover, by the strong convergence in L^1, under the assumption that $\fint_\Omega \chi \in (0,1)$, we obtain convergence $c^A(\chi_k) \to c^A(\chi)$ and $c^B(\chi_k) \to c^B(\chi)$. In the case that $\chi = 0$ a.e., we set $c^A(0) = 0$ and in the case that $\chi = 1$ a.e., we set $c^B(1) = 0$. Again, by the strong convergence in L^1, there exists a subsequence of (χ_k), which converges pointwise a.e.. Hence, by Fatou's lemma and the lower semi-continuity Proposition 0.7(4), we observe that

$$\begin{aligned}
&E^{\text{bMS}}(\chi, c^A(\chi), c^B(\chi)) \\
&= \frac{1}{2}\int_\Omega (c^A(\chi) - u^{\text{input}})^2\chi + (c^B(\chi) - u^{\text{input}})^2(1-\chi)\,\mathrm{d}x + \eta|D\chi|_{TV}(\Omega) \\
&\leqslant \liminf_{k\to\infty} \frac{1}{2}\int_\Omega (c^A(\chi_k) - u^{\text{input}})^2\chi_k + (c_k^B(\chi_k) - u^{\text{input}})^2(1-\chi_k)\,\mathrm{d}x + \eta|D\chi_k|_{TV}(\Omega) \\
&= E^{\text{bMS}}(\chi_k, c^A(\chi_k), c^B(\chi_k)) = \inf E^{\text{bMS}}\,.
\end{aligned}$$

Hence, $(\chi, c^A(\chi), c^B(\chi))$ is a minimizer of E^{bMS}. ◻

The last proof also motivates to consider a functional $E^{\text{bMS}}\colon \mathrm{BV}(\Omega, \{0,1\}) \to \mathbb{R}$ defined by

$$E^{\text{bMS}}(\chi) := E^{\text{bMS}}(\chi, c^A(\chi), c^B(\chi))\,.$$

i.e. we directly choose the optimal values for c^A and c^B.

In the following, we assume that the values c^A and c^B are fixed. Then we obtain the following problem.

Problem 2.10: Binary Segmentation with fixed values

Let $c^A, c^B \in [0,1]$ be fixed and $u^{\text{input}} \in L^2(\Omega, [0,1])$. Then minimize

$$E^{\text{bMS}}(\chi) = \int_\Omega \left((c^A - u^{\text{input}}(x))^2 - (c^B - u^{\text{input}}(x))^2\right) \chi(x) \,\mathrm{d}x + \eta |D\chi|_{TV}(\Omega)$$

over all $\chi \in \mathrm{BV}(\Omega, \{0,1\})$.

Thus, the binary Mumford–Shah functional with fixed values c^A, c^B is of type

$$E^{\text{bMS}}(\chi) = \int_\Omega g(x, \chi(x)) \,\mathrm{d}x + \eta |D\chi|_{TV}(\Omega)$$

with $g(x,t) = \left((c^A - u^{\text{input}}(x))^2 - (c^B - u^{\text{input}}(x))^2\right) t$.
Note that we have neglegted the additional term $\int_\Omega (c^B - u^{\text{input}}(x))^2 \,\mathrm{d}x$, since this constant term (in the variable χ) does not change the minimizer χ.

Compared to problem 2.1, here we ask for minimizers $\chi \in \mathrm{BV}(\Omega, \{0,1\})$. To enforce the constraint $\chi(x) \in \{0,1\}$ is computationally quite challenging. Therefore, we are interested in a functional, which can be minimized over all $u \in \mathrm{BV}(\Omega, [0,1])$ and to recover the solution χ from the solution of the relaxed problem. In the following, we will provide a general result, which we can apply to the binary Mumford–Shah model.

First, we recall the coarea formula.

Definition 2.11 (Perimeter)**.** For $E \subset \Omega$ measurable, we define

$$\mathrm{Per}_\Omega(E) := \sup \left\{ \int_E \operatorname{div} \phi \,\mathrm{d}x \;:\; \phi \in C^1(\Omega)^d \,,\; \|\phi\|_{L^\infty} \leqslant 1 \right\}$$

Theorem 2.12 (Coarea formula)**.**

(1) Let $f\colon \Omega \to \mathbb{R}$ be Lipschitz continuous. Then

$$\int_\Omega |Df(x)| \,\mathrm{d}x = \int_{\mathbb{R}} \int_{f^{-1}(s)} \mathrm{d}\mathcal{H}^{d-1}(x) \,\mathrm{d}s = \int_{\mathbb{R}} Per_\Omega(\{x \in \Omega \;:\; f(x) > s\}) \,\mathrm{d}s\,.$$

(2) More generally, for $f\colon \Omega \to \mathbb{R}$ Lipschitz continuous and $g \in L^1(\Omega)$, we have

$$\int_\Omega |Df(x)| g(x) \,\mathrm{d}x = \int_{\mathbb{R}} \int_{f^{-1}(s)} g(x) \,\mathrm{d}\mathcal{H}^{d-1}(x) \,\mathrm{d}s\,.$$

(3) For $u \in \mathrm{BV}(\Omega)$ we have

$$|Du|_{TV}(\Omega) = \int_{\mathbb{R}} Per_\Omega(\{x \in \Omega \;:\; u(x) > s\}) \,\mathrm{d}s = \int_{\mathbb{R}} |D\chi_{\{x \in \Omega \,:\, u(x) > s\}}|_{TV}(\Omega)\,.$$

Lemma 2.13 (Monotonicty of Minimizers). *For $g \in L^1(\Omega)$, we define $E_g \colon \mathrm{BV}(\Omega, \{0,1\}) \to \mathbb{R}$ by*

$$E_g(\chi) = \int_\Omega g(x)\chi(x)\,\mathrm{d}x + \eta |D\chi|_{TV}(\Omega)\,.$$

Let $g_A, g_B \in L^1(\Omega)$ with $g_A(x) < g_B(x)$ for a.e. $x \in \Omega$.
Let χ_A be a minimizer of E_{g_A} and χ_B be a minimizer of E_{g_B}.
Then $\chi_A(x) \geqslant \chi_B(x)$ for a.e. $x \in \Omega$.

Proof.
Step 1: Perimeter inequality
For sets $A, B \subset \Omega$ with finite Perimeter we have

$$\mathrm{Per}_\Omega(A \cup B) + \mathrm{Per}_\Omega(A \cap B) \leqslant \mathrm{Per}_\Omega(A) + \mathrm{Per}_\Omega(B)\,. \tag{2.5}$$

Indeed, we can approximate $\mathrm{Per}_\Omega(A)$ by a sequence $\phi_h^A \in C^\infty(\Omega, [0,1])$ s.t. $\phi_h^A \to \chi_A$ in L^1 and $\int_\Omega |D\phi_n^A|\,\mathrm{d}x \to \mathrm{Per}(A, \Omega)$. In analogy, we can approximate $\mathrm{Per}(B, \Omega)$ by a sequence $\phi_h^B \in C^\infty(\Omega, [0,1])$. Then $\phi_n^A\phi_n^B \to \chi_{A\cap B}$ and $(\phi_n^A + \phi_n^B - \phi_n^A\phi_n^B) \to \chi_{A\cup B}$. Furthermore, we can estimate

$$\begin{aligned}
&\int_\Omega |D(\phi_n^A\phi_n^B)|\,\mathrm{d}x + \int_\Omega |D(\phi_n^A + \phi_n^B - \phi_n^A\phi_n^B)|\,\mathrm{d}x \\
&\leqslant \int_\Omega \phi_n^A|D\phi_n^B| + \phi_n^B|D\phi_n^A|\,\mathrm{d}x + \int_\Omega (1-\phi_n^A)|D\phi_n^B| + (1-\phi_n^B)|D\phi_n^B|\,\mathrm{d}x \\
&= \int_\Omega |D\phi_n^A|\,\mathrm{d}x + \int_\Omega |D\phi_n^B|\,\mathrm{d}x\,.
\end{aligned}$$

Passing to the limit gives the inequality (2.5).

Step 2:
Recall that $|D\chi_E|_{TV}(\Omega) = \mathrm{Per}_\Omega(E)$ for all $E \subset \Omega$ measurable. Now, since χ_A is a minimizer of E_{g_A}, we have that

$$\int_\Omega g_A\chi_A\,\mathrm{d}x + \eta\mathrm{Per}_\Omega(A) \leqslant \int_\Omega g_A\chi_{A\cup B}\,\mathrm{d}x + \eta\mathrm{Per}_\Omega(A\cup B)$$

and since χ_B is a minimizer of E_{g_B}, we have that

$$\int_\Omega g_B\chi_B\,\mathrm{d}x + \eta\mathrm{Per}_\Omega(B) \leqslant \int_\Omega g_B\chi_{A\cap B}\,\mathrm{d}x + \eta\mathrm{Per}_\Omega(A\cap B)\,.$$

Together, using the inequality (2.5), we obtain

$$\int_\Omega g_A\chi_A\,\mathrm{d}x + \int_\Omega g_B\chi_B\,\mathrm{d}x \leqslant \int_\Omega g_A\chi_{A\cup B}\,\mathrm{d}x + \int_\Omega g_B\chi_{A\cap B}\,\mathrm{d}x$$

Then we observe

$$\int_\Omega g_B\chi_{B\setminus A}\,\mathrm{d}x = \int_\Omega g_B(\chi_B - \chi_{A\cap B})\,\mathrm{d}x \leqslant \int_\Omega g_A(\chi_{A\cup B} - \chi_A)\,\mathrm{d}x = \int_\Omega g_A\chi_{B\setminus A}\,\mathrm{d}x\,.$$

Thus,

$$0 \leqslant \int_\Omega (g_A - g_B)\chi_{B\setminus A}\,\mathrm{d}x\,,$$

which implies $\chi_{B\setminus A} = 0$ a.e., since $g_A - g_B < 0$ a.e. by assumption. □

Example 2.14 (Perimeter inequality)**.** The inequality in (2.5) can be strict. Consider for example $A = [0,1] \times [0,1]$ and $B = [1,2] \times [0,1]$. Then $7 = 6 + 1 = \mathrm{Per}_\Omega(A \cup B) + \mathrm{Per}_\Omega(A \cap B) < \mathrm{Per}_\Omega(A) + \mathrm{Per}_\Omega(B) = 4 + 4 = 8$.

Now, we arrive at the theorem, which allows us to consider a relaxed function defined on $\mathrm{BV}(\Omega, [0,1])$.

Theorem 2.15 (Thresholding of Relaxed Functional)**.** *Let* $\Psi \colon \Omega \times \mathbb{R} \to \mathbb{R}$ *s.t.*

- $\Psi(\cdot, t) \in L^1(\Omega)$ *for all* t*,*
- $\Psi(x, \cdot) \in C^1$*, stricly convex for a.e.* $x \in \Omega$*, and*
- $\Psi(x,t) \geqslant c_1|t| - c_2$ *for some* $c_1, c_2 > 0$.

Let $u \in \mathrm{BV}(\Omega, [0,1])$ *be a minimizer of the functional*

$$E_\Psi(u) \coloneqq \int_\Omega \Psi(x, u(x)) \,\mathrm{d}x + \eta |Du|_{TV}(\Omega)\,.$$

Now, for a fixed $t \in \mathbb{R}$*, we define* $g_t(x) = \partial_t \Psi(x,t)$*. Then the characteristic function* $\chi_{[u>t]}$ *is a minimizer of the functional*

$$E_{g_t}(\chi) \coloneqq \int_\Omega g_t(x)\chi(x) \,\mathrm{d}x + \eta |D\chi|_{TV}(\Omega)\,.$$

over all $\chi \in \mathrm{BV}(\Omega, \{0,1\})$.

Proof.
Step 0: Existence of minimizer of E_Ψ.
Let $(u_k)_{k\in\mathbb{N}}$ be a minimizing sequence. W.l.o.g. $E_\Psi(u_k) \leqslant C < \infty$ for all $k \in \mathbb{N}$. Then $\eta|Du_k|_{TV}(\Omega) \leqslant E_\Psi(u_k) < \infty$. Moreover, since $\Psi(x,t) \geqslant c_1|t| - c_2$, we observe that $\|u_k\|_{L^1} \leqslant c_2 + \frac{1}{c_1}E(u) \leqslant C$. Hence, $\|u_k\|_{\mathrm{BV}} \leqslant C$. By compactness in BV, there exists a subsequence $u_k \overset{*}{\rightharpoonup} u$ in BV. Since $u_k \to u$ in L^1, there is a further subsequence, s.t. $u_k(x) \to u(x)$ for a.e. $x \in \Omega$. Furthermore, since $\Psi(x, \cdot)$ is continuous, we have that $\Psi(x, u_k(x)) \to \Psi(x, u(x))$ for a.e. $x \in \Omega$. Thus, by lsc in BV and Fatou's lemma, we obtain that $E_\Psi(u) \leqslant \liminf E_\Psi(u_k)$.

Step 1: Compare E_Ψ with E_{g_t}
First, we apply Fubini's Theorem on the set $\{(x,t) \in \Omega \times \mathbb{R} \;:\; 0 \leqslant t \leqslant u(x)\}$:

$$\begin{aligned}
\int_\Omega \Psi(x, u(x)) \,\mathrm{d}x &= \int_\Omega \Psi(x, 0) \,\mathrm{d}x + \int_\Omega \int_0^{u(x)} \partial_t \Psi(x,t) \,\mathrm{d}t \,\mathrm{d}x \\
&= C_\Psi + \int_\Omega \int_\mathbb{R} \partial_t \Psi(x,t) \chi_{[0,u(x)]}(t) \,\mathrm{d}t \,\mathrm{d}x \\
&= C_\Psi + \int_0^\infty \int_\Omega \partial_t \Psi(x,t) \chi_{[u>t]}(x) \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}$$

Next, by the coarea formula Theorem 2.12, we have

$$|Du|_{TV}(\Omega) = \int_\mathbb{R} |D\chi_{[u>t]}|_{TV}(\Omega) \,\mathrm{d}t\,.$$

Together, we observe

$$
\begin{aligned}
E_\Psi(u) &= C_\Psi + \int_{\mathbb{R}} \left(\int_\Omega \partial_t \Psi(x,t) \chi_{[u>t]}(x) \,\mathrm{d}x + \eta |D\chi_{[u>t]}|_{TV}(\Omega) \right) \mathrm{d}t \\
&= C_\Psi + \int_{\mathbb{R}} E_{g_t}(\chi_{[u>t]}) \,\mathrm{d}t \\
&\geqslant C_\Psi + \int_{\mathbb{R}} E_{g_t}(\chi_t) \,\mathrm{d}t \,,
\end{aligned}
$$

where $\chi_t \in \mathrm{BV}(\Omega, \{0,1\})$ is a minimizer of E_{g_t}.

Step 2: Construct minimizer of E_Ψ from $(\chi_t)_t$
We define

$$
u(x) := \sup \{ t \in \mathbb{R} \;:\; \chi_t(x) = 1 \} \,.
$$

Because $\Psi(x,\cdot)$ is stricly convex, $\partial_t \Psi(x,\cdot)$ is stricly monotone. Thus, by Lemma 2.13, for $t_1 \leqslant t_2$, we obtain $\chi_{t_2}(x) \leqslant \chi_{t_1}(x)$ for a.e. x. Therefore, $\chi_t = \chi_{[u>t]}$. Now, using the same arguments as in Step 1, we obtain that for every $\widehat{u} \in \mathrm{BV}(\Omega, [0,1])$

$$
E_\Psi(\widehat{u}) \geqslant C_\Psi + \int_{\mathbb{R}} E_{g_t}(\chi_t) \,\mathrm{d}t = C_\Psi + \int_{\mathbb{R}} E_{g_t}(\chi_{[u*>t]}) \,\mathrm{d}t = E_\Psi(u) \,.
$$

Hence, u is a minimizer of E_Ψ and $E_{g_t}(\chi_{[u>t]}) = E_{g_t}(\chi_t)$ for a.e. t.
Step 3: statement for every $t \in [0,1)$.
To prove the statement for every t, we can choose a monotone decreasing sequence $t_n \to t$ s.t. $E_{g_{t_n}}(\chi_{[u>t_n]}) = E_{g_{t_n}}(\chi_{t_n})$ for all $n \in \mathbb{N}$. Then, by monotonicity, optimality of χ_{t_n}, and dominated convergence, we obtain for all $\widehat{\chi} \in \mathrm{BV}(\Omega, \{0,1\})$

$$
E_{g_t}(\chi_{[u>t]}) = \liminf E_{g_{t_n}}(\chi_{[u>t_n]}) \leqslant \liminf E_{g_{t_n}}(\widehat{\chi}) = E_{g_t}(\widehat{\chi}) \,,
$$

which shows that $\chi_{[u>t]}$ is a minimizer of E_{g_t}. □

Thus, applying Theorem 2.15 to the binary Mumford–Shah functional, we finally arrive at the following minimization problem.

[Lecture 05 – 05/18/2020]

[Lecture 06 – 05/25/2020]

Corollary 2.16 (Thresholding the relaxed Binary Mumford–Shah functional)**.** *We define*

$$
\Psi(x,t) := \frac{1}{2} \left(t - (c^B - u^{input}(x))^2 + (c^A - u^{input}(x))^2 \right)^2 \,.
$$

Let u be a miminizer of

$$
E_\Psi(u) = \int_\Omega \Psi(x, u(x)) \,\mathrm{d}x + \eta |Du|_{TV}(\Omega) \,.
$$

over all $u \in \mathrm{BV}(\Omega, [0,1])$. Then $\chi_{[u>0]}$ is a minimizer of E^{bMS} over all $\chi \in \mathrm{BV}(\Omega, \{0,1\})$.

Proof. We simply apply Theorem 2.15 and compute

$$\partial_t \Psi(x,t) = t - (c^B - u^{\text{input}}(x))^2 + (c^A - u^{\text{input}}(x))^2 \,.$$

Hence, for $t = 0$, we have

$$\partial_t \Psi(x,0) = (c^A - u^{\text{input}}(x))^2 - (c^B - u^{\text{input}}(x))^2 \,.$$

Finally, we check the conditions on Ψ:
Since $u^{\text{input}} \in L^\infty$, we observe that $\Psi(\cdot,t) \in L^1$. Now, $\partial^2_{tt}\Psi(x,t) = 1$, hence $\Psi(x,\cdot)$ is strictly convex. Finally, since Ψ is quadratic in t and u^{input} is bounded, we can estimate $\Psi(x,t) \geqslant c_1|t| - c_2$. □

Thus, defining $\theta(x) = (c^B - u^{\text{input}}(x))^2 - (c^A - u^{\text{input}}(x))^2$, we arrive the following minimization problem:

$$E(u) = \int \frac{1}{2}(u-\theta)^2 \,\mathrm{d}x + \eta |Du|_{TV}(\Omega) \,.$$

This is precisely the ROF model for the input θ. Therefore, we can derive the following algorithm for binary segmentation.

Algorithm 2.5 Binary Segmentation by thresholding ROF.

function BINARYSEGMENTATION(u_h^{input}, c^A, c^B, u_h^0, v_h^0, $z_h^0 = u_h^0$)
  set $\theta_\alpha = (c^B - u_\alpha^{\text{input}})^2 - (c^A - u_\alpha^{\text{input}})^2$
  u_h = ROF-DENOISING(θ_h, u_h^0, v_h^0, $z_h^0 = u_h^0$)
  return $\chi_{u_h > 0}$
end function

Remark 2.17. The choice of Ψ is non-unique. E.g., we can also consider $\Psi(x,t) := t^2(c^A - u^{\text{input}}(x))^2 + (1-t)^2(c^B - u^{\text{input}}(x))^2$ and the threshold $t = \frac{1}{2}$.

2.3. The General Mumford–Shah model

Now, we consider the general Mumford–Shah functional as defined in problem 2.7. The convex lifting for the general problem was proposed in [Poc+09]. Similar to the approach for the depth map from stereo images, we will rewrite the Mumford–Shah functional in terms of the epigraph. Therefore, we will introduce the space SBV of special functions of bounded variation, which allows to rigorously define a normal on the boundary.

Special functions of bounded variation.

Definition 2.18 (Jump set)**.** Let $u \in \mathrm{BV}(\Omega)$. Then we define

(1) the upper and lower approximate limits of u by

$$u^+(x) := \inf\left\{t \in \mathbb{R} \;:\; \lim_{r\to 0} \frac{|B_r(x) \cap [u > t]|}{|B_r(x)|} = 0\right\} ,$$
$$u^-(x) := \sup\left\{t \in \mathbb{R} \;:\; \lim_{r\to 0} \frac{|B_r(x) \cap [u < t]|}{|B_r(x)|} = 0\right\} ,$$

(2) the jump set of u by

$$J_u := \{x \in \Omega \;:\; u^-(x) < u^+(x)\}$$

Lemma 2.19 (Decomposition of BV functions)**.** *Let $u \in \mathrm{BV}(\Omega)$. Then, the Radon measure Du has a decomposition*

$$Du = [Du]_a + [Du]_j + [Du]_c\,,$$

where $[Du]_a$ is absolutely continuous w.r.t. the Lebesgue measure on Ω and $[Du]_j$ is absolutely continuous w.r.t. the jump set J_u. Moreover, for $\mathscr{H}^{d-1}$-a.e. $x \in J_u$ there exists a normal $\nu_u(x) \in S^{d-1}$ s.t. $[Du]_j\big|_{J_u} = (u^+ - u^-)\nu_u\big|_{J_u}$. Finally, we call $[Du]_c$ the Cantor part.

Definition 2.20 (SBV)**.** We define the space of special functions of bounded variation by

$$\mathrm{SBV}(\Omega) := \{u \in \mathrm{BV}(\Omega) \;:\; [Du]_\mathrm{c} = 0\}\ .$$

Now, using the space SBV, the Mumford–Shah problem can be rewritten in the following sense.

Problem 2.21: Segmentation with Mumford–Shah in SBV

Given $u^{\text{input}} \in L^2(\Omega)$ and $\mu, \eta > 0$. Then minimize

$$E^{\mathrm{MS}}(u) := \frac{1}{2}\int_\Omega (u - u^{\text{input}})^2 \,\mathrm{d}x + \frac{\mu}{2}\int_{\Omega\setminus J_u} |Du|^2 \,\mathrm{d}x + \eta\mathscr{H}^{d-1}(J_u)$$

over all $u \in \mathrm{SBV}(\Omega, [0,1])$.

Convex Lifting in SBV.

Theorem 2.22 (Lifting to the Epigraph)**.** *For $v \in \mathrm{BV}(\Omega, [0,1])$ with $v(x,0) = 0$ and $v(x,1) = 1$, we define*

$$\tilde{E}^{lift}(v) = \sup_{\phi \in K} \int_{\Omega \times [0,1]} \phi(x,t) dD\chi_{\mathrm{epi}\, u}(x,t)\,,$$

where the set K is given by

$$\begin{aligned} K := \Big\{ &\phi = (\phi^x, \phi^t) \in C(\Omega \times \mathbb{R}, \mathbb{R}^{d+1}) \; : \\ &\phi^t(x,t) \geqslant \frac{1}{4}|\phi^x(x,t)|^2 - \frac{1}{2}(t - u^{input}(x))^2\,, \\ &\left| \int_{t^-}^{t^+} \phi^x(x,t)\, \mathrm{d}t \right| \leqslant \eta \text{ for all } x \in \Omega \text{ and all } 0 \leqslant t^- < t^+ \leqslant 1 \Big\} \end{aligned}$$

Then, for all $u \in \mathrm{SBV}(\Omega, [0,1])$, we have $E^{MS}(u) = \tilde{E}^{lift}(\chi_{\mathrm{epi}\, u})$.

Proof.
Step 1: $\tilde{E}^{\mathrm{lift}}(\chi_{\mathrm{epi}\, u}) \leqslant E^{\mathrm{MS}}(u)$
We consider the interface

$$\Gamma_u = \{(x,t) \in \Omega \times [0,1] \; : \; u(x) = t\}\;.$$

Now, since $u \in \mathrm{SBV}$, we can define a normal on this interface

$$\nu_{\Gamma_u}(x,t) = \begin{cases} \dfrac{(Du, -1)}{\sqrt{|Du|^2 + 1}} & \text{if } x \in \Omega \backslash J_u\,, \\ (\nu_u, 0) & \text{if } x \in J_u\,, \end{cases}$$

where ν_u is the outer unit normal on J_u. Then, for $\phi \in C(\Omega \times \mathbb{R}, \mathbb{R}^{d+1})$, we obtain

$$\begin{aligned} &\int_{\Omega \times [0,1]} \phi \; \mathrm{d}D\chi_{\mathrm{epi}\, u}(x,t) = \int_{\Gamma_u} \phi \nu_{\Gamma_u} \; \mathrm{d}\mathscr{H}^d(x,t) \\ &= \int_{\Omega \backslash J_u} \phi^x(x, u(x)) Du(x) - \phi^t(x, u(x)) \; \mathrm{d}x + \int_{J_u} \left(\int_{u^-(x)}^{u^+(x)} \phi^x(x,t) \; \mathrm{d}t \right) \nu_u(x) \; \mathrm{d}\mathscr{H}^{d-1}(x) \end{aligned}$$

Moreover, for $\phi \in K$, we have

$$\int_{J_u} \left(\int_{u^-(x)}^{u^+(x)} \phi^x(x,t) \; \mathrm{d}t \right) \nu_u(x) \; \mathrm{d}\mathscr{H}^{d-1}(x) \leqslant \eta \mathscr{H}^{d-1}(J_u)$$

and

$$\begin{aligned} &\sup_{\phi = (\phi^x, \phi^t) \in K} \phi^x(x, u(x)) \cdot Du(x) - \phi^t(x, u(x)) \\ \leqslant &\sup_{\phi^x \in C(\Omega \times \mathbb{R}, \mathbb{R}^d)} \phi^x(x, u(x)) \cdot Du(x) - \frac{1}{4}\phi^x(x, u(x))^2 + \frac{1}{2}(u(x) - u^{\mathrm{input}}(x))^2 \\ = &|Du(x)|^2 + \frac{1}{2}(u(x) - u^{\mathrm{input}}(x))^2 \end{aligned}$$

Therefore,

$$\sup_{\phi\in K}\int_{\Omega\times[0,1]}\phi(x,t)\,\mathrm{d}D\chi_{\mathrm{epi}\,u}(x,t)\leqslant E^{\mathrm{MS}}(u)\,.$$

Step 2: can choose $\phi\in K$ s.t. $\tilde{E}^{\mathrm{lift}}(\chi_{\mathrm{epi}\,u})=E^{\mathrm{MS}}(u)$
In analogy to the proof of Theorem 2.4, we can construct such a ϕ for smooth data. In general, we have to approximate ϕ. □

Thus, we consider the following relaxed convex optimization problem.

Problem 2.23: Segmentation with Mumford–Shah in SBV

Let $u^{\text{input}}\in L^2(\Omega,[0,1])$. Then minimize

$$\tilde{E}^{\mathrm{lift}}(v)=\sup_{\phi\in K}\int_\Omega\int_{\mathbb{R}}\phi(x,t)dDv(x,t)$$

over all $v\in\mathrm{BV}(\Omega\times\mathbb{R},[0,1])$ with $v(x,0)=0$ and $v(x,1)=1$. Here, the convex set K is given by

$$\begin{aligned}K:=\Big\{&\phi=(\phi^x,\phi^t)\in C(\Omega\times\mathbb{R},\mathbb{R}^{d+1})\ :\\&\phi^t(x,t)\geqslant\frac{1}{4}|\phi^x(x,t)|^2-\frac{1}{2}(t-u^{\text{input}}(x))^2\,,\\&\left|\int_{t^-}^{t^+}\phi^x(x,t)\,\mathrm{d}t\right|\leqslant\eta\text{ for all }x\in\Omega\text{ and all }0\leqslant t^-<t^+\leqslant1\Big\}\end{aligned}$$

Remark 2.24. So far, it is unclear, if for a minimizer v of $\tilde{E}^{\mathrm{lift}}$, the set $\{(x,t)\in\Omega\times[0,1]\ :\ v(x)>t\}$ is the epigraph of a minimizer of E^{MS}. However, the lifted problem has been used in [Poc+09] and leads to reasonable numerical results.

Application of Proximal Splitting Algorithm. As for the depth map from stereo camera, we divide the unit cube $[0,1]^3$ into a set of cubic cells $\mathscr{C}_h=(C_\alpha)_\alpha$ with side length h. First, we define discrete sets

$$\begin{aligned}K_h=\{&\phi_h=(\phi_h^x,\phi_h^t)\in\mathbb{R}^{|\mathscr{C}_h|^2}\times\mathbb{R}^{|\mathscr{C}_h|}\ :\\&\phi_\alpha^t\geqslant\frac{1}{4}|\phi_\alpha^x|^2-\frac{1}{2}(t_\alpha-u_\alpha^{\text{input}})^2\text{ for all }C_\alpha\in\mathscr{C}_h\,,\\&h\left|\sum_{C_\alpha\,:\,x_\alpha=x_h,\,t_h^-\leqslant t_\alpha\leqslant t_h^+}(\phi_\alpha^{x_1},\phi_\alpha^{x_2})\right|\leqslant\eta\ \text{ for all }0\leqslant t_h^-<t_h^+\leqslant1\text{ and all }x_h\}\,.\end{aligned}$$

and

$$C_h=\{v_h\in C^0([0,1]^3,[0,1])\ :\ v_h\text{ multilinear on each cell},v_h(x,0)=0\,,\ v_h(x,1)=1\}\,.$$

Moreover, we define $\nabla_h\colon C_h \to \mathbb{R}^{|\mathscr{C}_h|^2} \times \mathbb{R}^{|\mathscr{C}_h|}$ by

$$\nabla_h(v_h)_\alpha = \sum_{q\in Q} \omega(q) D v_h(q)\,,$$

where e.g. Q is a tensor-product Simpson quadrature.

Thus, we arrive at the saddle point problem

$$\inf_{v_h} \sup_{\phi_h} \langle \phi_h, \nabla_h v_h\rangle + \mathcal{I}_{C_h} - \mathcal{I}_{K_h}\,,$$

which is equivalent to the primal problem $\inf_{v_h} F_h(L_h v_h) + G_h(v_h)$ and equivalent to the dual problem $\sup_{\phi_h} -F_h^*(\phi_h) - G_h^*(-L_h^*\phi_h)$ with $F_h^* = \mathcal{I}_{K_h}$, $G_h = \mathcal{I}_{C_h}$, and $L_h = \nabla_h$.

As for the depth map from stereo camera, the projection onto C_h can computed pointwise, i.e. it decouples for every cell C_α. However, the projection onto the set K_h is now coupled in the variable t because of the constraint $h\left|\sum_{C_\alpha\,:\,x_\alpha=x_h,\,t_h^-\leqslant t_\alpha\leqslant t_h^+}(\phi_\alpha^{x_1},\phi_\alpha^{x_2})\right| \leqslant \eta$. Thus, for every discrete point in the two dimensional space x_h, the projection onto K_h requires to solve a coupled nonlinear optimization problem.

Instead, we will use an auxiliary variable to obtain a problem, where each projection can be performed pointwise. We observe for the second constraint in the set K_h, the map

$$M_h(\phi_h) := \sum_{C_\alpha\,:\,x_\alpha=x_h,\,t_h^-\leqslant t_\alpha\leqslant t_h^+} (\phi_\alpha^{x_1},\phi_\alpha^{x_2})$$

defines a linear map into the space of triples x_h, t_h^-, t_h^+. Furthermore, for the first constraint in K_h, we define for each cell C_α a set

$$K_\alpha = \left\{(\phi^t,\phi^{x_1},\phi^{x_2})\in\mathbb{R}^3 \;:\; \phi^t \geqslant \frac{1}{4}(\phi^{x_1})^2 + \frac{1}{4}(\phi^{x_2})^2 - \frac{1}{2}(t_\alpha - u_\alpha^{\text{input}})^2\right\}\,.$$

Thus, we can rewrite the dual problem as

$$\begin{aligned}
&= \sup_{\phi_h}\; -\mathcal{I}_{K_h}(\phi_h) - \mathcal{I}^*_{C_h}(-L_h^*\phi_h)\\
&= \sup_{\phi_h}\; -\sum_{C_\alpha}\mathcal{I}_{K_\alpha}(\phi_h) - \sum_{(x_h,t_h^-,t_h^+)} \mathcal{I}_{|\cdot|\leqslant\frac{\eta}{h}}(-M_h\phi_h) - \mathcal{I}^*_{C_h}(-L_h^*\phi_h)\\
&= \sup_{\phi_h,\psi_h}\; \mathcal{I}_=(\phi_h,\psi_h) - \sum_{C_\alpha}\mathcal{I}_{K_\alpha}(\phi_h) - \sum_{(x_h,t_h^-,t_h^+)} \mathcal{I}_{|\cdot|\leqslant\frac{\eta}{h}}(-M_h\psi_h) - \mathcal{I}^*_{C_h}(-L_h^*\phi_h)\,.
\end{aligned}$$

Note that this corresponds to a primal problem $\inf_{v_h,w_h} \tilde{F}_h(\tilde{L}_h(v_h,w_h)) + \tilde{G}_h(v_h,w_h)$ with

$$\begin{aligned}
&\tilde{L}_h(v_h,w_h) = (L_1 v_h, L_2 w_h)\,, \quad \text{where } L_1 v_h = \nabla_h v_h \text{ and } L_2^*\psi_h = M_h\psi_h\\
&\tilde{F}_h^*(\phi_h,\psi_h) = \mathcal{I}_=(\phi_h,\psi_h) + \sum_{C_\alpha}\mathcal{I}_{K_\alpha}(\phi_h)\\
&\tilde{G}_h(v_h,w_h) = \mathcal{I}_{C_h}(v_h) + G_2(w_h)\,, \quad \text{where } G_2^*(w_h') = \sum_{(x_h,t_h^-,t_h^+)}\mathcal{I}_{|\cdot|\leqslant\frac{\eta}{h}}(w_h')
\end{aligned}$$

In fact, we have enlarged the space for ψ_h by an auxiliary dual varible ψ_h, which is actually equal to ψ_h. It turned out that this corresponds to enlarging the space for v_h by an auxiliary primal variable w_h in the space of all triples (x_h, t_h^-, t_h^+).

Finally, we want to discuss the projection onto K_α in detail.

Lemma 2.25 (Projection onto K_α)**.** *The projection onto K_α is given by*

$$\mathrm{proj}_{K_\alpha}(\phi) = \begin{cases} \phi & \text{if } \phi \in K \\ \phi_{pr} & \text{if } \phi \notin K \end{cases}$$

where ϕ_{pr} is given by

$$\begin{aligned} \phi^t_{pr} &= \phi^t + \lambda \\ \phi^{x_1}_{pr} &= \frac{\phi^{x_1}}{1+\frac{\lambda}{2}} \\ \phi^{x_2}_{pr} &= \frac{\phi^{x_2}}{1+\frac{\lambda}{2}} \end{aligned} \tag{2.6}$$

and λ is a solution of

$$\lambda(1+\frac{\lambda}{2})^2 + (\frac{1}{2}(t_\alpha - u_\alpha^{input})^2 + \phi^t)(1+\frac{\lambda}{2})^2 - \frac{(\phi^{x_1})^2 + (\phi^{x_2})^2}{4} = 0\,. \tag{2.7}$$

Proof. For simplicity, we set $c = \frac{1}{2}(t_\alpha - u_\alpha^{\text{input}})^2$.
We define the Lagrangian

$$\begin{aligned} L(\phi_{\text{pr}}, \lambda) =& \frac{1}{2}(\phi^t_{\text{pr}} - \phi^t)^2 + \frac{1}{2}(\phi^{x_1}_{\text{pr}} - \phi^{x_1})^2 + \frac{1}{2}(\phi^{x_2}_{\text{pr}} - \phi^{x_2})^2 \\ & - \lambda\left(\phi^t_{\text{pr}} + \frac{1}{4}(\phi^{x_1}_{\text{pr}})^2 + \frac{1}{4}(\phi^{x_2}_{\text{pr}})^2 - c\right)\,. \end{aligned}$$

For the projection ϕ_{pr}, we have to solve the optimality condition, i.e. $(\phi_{\text{pr}}, \lambda)$ is a saddle point of L and thus, solves

$$\begin{aligned} 0 &= \partial_{\phi^t_{\text{pr}}} L(\phi_{\text{pr}}, \lambda) = \phi^t_{\text{pr}} - \phi^t - \lambda \\ 0 &= \partial_{\phi^{x_1}_{\text{pr}}} L(\phi_{\text{pr}}, \lambda) = \phi^{x_1}_{\text{pr}} - \phi^{x_1} + \frac{1}{2}\lambda\phi^{x_1}_{\text{pr}} \\ 0 &= \partial_{\phi^{x_2}_{\text{pr}}} L(\phi_{\text{pr}}, \lambda) = \phi^{x_2}_{\text{pr}} - \phi^{x_2} + \frac{1}{2}\lambda\phi^{x_2}_{\text{pr}} \\ 0 &= \partial_{\lambda} L(\phi_{\text{pr}}, \lambda) = -\phi^t_{\text{pr}} + \frac{1}{4}(\phi^{x_1}_{\text{pr}})^2 + \frac{1}{4}(\phi^{x_2}_{\text{pr}})^2 - c \end{aligned}$$

The first three equations lead to

$$\begin{aligned} \phi^t_{\text{pr}} &= \phi^t + \lambda \\ \phi^{x_1}_{\text{pr}}(1 + \frac{1}{2}\lambda\phi^{x_1}_{\text{pr}}) &= \phi^{x_1} \\ \phi^{x_2}_{\text{pr}}(1 + \frac{1}{2}\lambda\phi^{x_2}_{\text{pr}}) &= \phi^{x_2} \end{aligned}$$

which already shows (2.6). Now, inserting in the fourth equation gives

$$
\begin{aligned}
0 &= -\phi^t_{\mathrm{pr}} + \frac{1}{4}(\phi^{x_1}_{\mathrm{pr}})^2 + \frac{1}{4}(\phi^{x_2}_{\mathrm{pr}})^2 - c \\
&= -\phi^t - \lambda + \frac{1}{4}\frac{(\phi^{x_1})^2}{(1+\frac{\lambda}{2})^2} + \frac{1}{4}\frac{(\phi^{x_2})^2}{(1+\frac{\lambda}{2})^2} - c\,.
\end{aligned}
$$

From that we get (2.7) □

[Lecture 06 – 05/25/2020]

[Lecture 07 – 06/08/2020]

3. Transport Problems

3.1. Optical Flow

In the following, we consider two images u^A, u^B taken from the same camera at times t^A, t^B (see Figure 3.1).

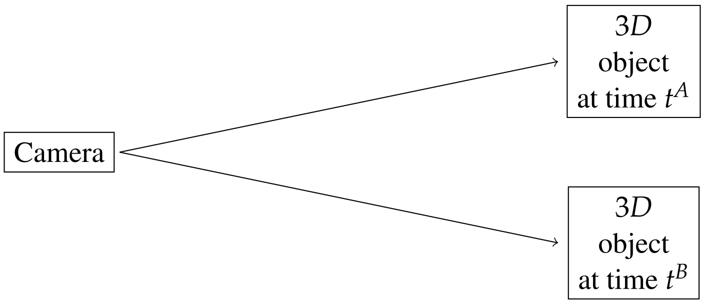

Figure 3.1: A camera takes two images u^A and u^B at time steps t^A and t^B.

W.l.o.g. we may assume $t^A = 0$ and $t^B = T$. Our goal is to estimate the motion of the $3D$ object during the time interval $[0, T]$. Here, we will discuss a static and a dynamic perspective and then use a linearized model for a computational method.

The Static Optical Flow Problem. In the static model, we want to find a disparity map $w\colon \Omega \to \mathbb{R}^d$ s.t.

$$u^A(x) = u^B(x + w(x))$$

for all $x \in \Omega$. As for the depth map from stereo camera, we can only expect that this relation is approximately satisfied. Furthermore, we assume that $w \in \mathrm{BV}(\Omega, \mathbb{R}^d)$. Therefore, we consider the following problem.

Problem 3.1: Static Optical Flow

Given $u^A, u^B \in \mathrm{BV}(\Omega)$. Then minimize

$$E^{\mathrm{sOF}}(w) = \frac{T}{2} \int_\Omega |u^B(x + w(x)) - u^A(x)|^2 \,\mathrm{d}x + \eta |Dw|_{TV}(\Omega)$$

over all $w \in \mathrm{BV}(\Omega, \mathbb{R}^d)$.

Note that this is a nonconvex and nonsmooth optimization problem in w. Different to the problems in Chapter 1, a convex lifting onto functions on $\Omega \times \mathbb{R}$ is not possible, since the lifting argument must be performed for each component of the disparity map w. Thus, a lifting could only be possible to functions on $\Omega \times \mathbb{R}^d$, which even for $d = 2$ becomes computationally quite expensive.

The Dynamic Optical Flow Problem. In the dynamic model, we introduce a time interval $[t^A, t^B] = [0, T]$. More precisely, we want to compute a motion/velocity field $v\colon [0,T] \times \Omega \to \mathbb{R}^d$. Therefore, we consider the transport of a point x^A in the image u^A. In fact, such a point is transported to a point x^B in the image u^B. Now, we consider a transport path $x\colon [0,T] \to \Omega$ with the boundary conditions $x(0) = x^A$ and $x(T) = x^B$. Then the motion field v satisfies $x'(t) = v(t, x(t))$. Next, we introduce a path of images $u\colon [0,T] \times \Omega \to [0,1]$. Since the pixel value is constant along the transport, we observe

$$u(t, x(t)) = \text{const} = u^A(x^A) = u^B(x^B)\,.$$

Hence, we obtain that the so-called material derivative vanishes:

$$\begin{aligned} 0 = \frac{d}{dt} u(t, x(t)) &= \partial_t u(t, x(t)) + \partial_x u(t, x(t)) \cdot x'(t) \\ &= \partial_t u(t, x(t)) + \partial_x u(t, x(t)) \cdot v(t, x(t))\,. \end{aligned}$$

Due to usual artefacts in real images, we can only expect that this transport equation is approximately satisfied for each pixel value. Moreover, we assume that both the images and the motion field are in the space BV. Thus, we arrive at the following problem.

Problem 3.2: Dynamic Optical Flow

Given $u^A, u^B \in \mathrm{BV}(\Omega)$. Then minimize

$$E^{\mathrm{dOF}}(u, v) = \frac{1}{2} \int_{[0,T]\times\Omega} |\partial_t u + \partial_x u \cdot v|^2 \, \mathrm{d}(t,x) + \eta \left(|Du|_{TV} + |Dv|_{TV}\right)([0,T] \times \Omega)$$

over all pairs $(u, v)\colon [0,T] \times \Omega \to [0,1] \times \mathbb{R}^d$ with $u(0) = u^A$ and $u(T) = u^B$.

The functional E^{dOF} is convex in each argument, i.e. in u and v, but it is not jointly convex in (u, v).

The Linearized Optical Flow Problem. Thus, for computational reasons, we will study a linearized model. Therefore, we use the linearization

$$u^B(x + w(x)) \approx u^B(x) + Du^B(x) \cdot w(x)\,.$$

Then, the static optical flow problem becomes to

Problem 3.3: Linearized Optical Flow

Given $u^A, u^B \in \mathrm{BV}(\Omega)$. Then minimize

$$E^{\mathrm{lOF}}(w) = \frac{T}{2} \int_{\Omega} |u^B(x) + Du^B(x) \cdot w(x) - u^A(x)|^2 \, \mathrm{d}x + \eta |Dw|_{TV}(\Omega)$$

over all $w \in \mathrm{BV}(\Omega, \mathbb{R}^d)$.

This is a convex problem and can be solved by a proximal splitting algorithm. Note that the linearization $u^B(x) + Du^B(x) \cdot w(x) - u^A(x)$ of the static model is also obtained by a time discretization of the transport equation:

$$\partial_t u(t, x) + \partial_x u(t, x) \cdot v(t, x) \approx \frac{u^B(x) - u^A(x)}{T} + Du^B(x) \cdot v(t^B, x)\,.$$

Application of Proximal Splitting Algorithm. For a discretization, we take into account discrete images u_h^A, u_h^B and a discrete disparity map w_h, which are all given by their values on cells C_α. Then, to define a discrete version of E^{lOF}, we have to define the product $Du_h^B(x) \cdot w_h(x)$. Therefore, for a discrete image u_h, we use the average value

$$(Du_h)_\alpha := \frac{\sum_{\beta\,:\,I_{\alpha,\beta}\in\mathscr{I}_h}(u_\beta - u_\alpha)}{\sum_{\beta\,:\,I_{\alpha,\beta}\in\mathscr{I}_h} 1}$$

Recall that $\mathscr{I}_h$ is the set of interfaces. Thus, we can define

$$G_h(w_h) = T\frac{h^d}{2}\sum_{C_\alpha}|u_\alpha^B - u_\alpha^A + (Du_h^B)_\alpha \cdot w_\alpha|^2\,.$$

The computation of the proximal map decouples for each cell:

$$\begin{aligned} p_\alpha &= \left(\mathrm{prox}_{\tau G_h}(w_h)\right)_\alpha \\ &= \arg\min_{\widehat{w}_\alpha} \frac{h^d}{2}\sum_{i=1,\dots,d}(\widehat{w}_\alpha^i - w_\alpha^i)^2 + \tau T\frac{h^d}{2}(u_\alpha^B - u_\alpha^A + \sum_{i=1,\dots,d}(Du_h^B)_\alpha \cdot \widehat{w}_\alpha^i)^2\,. \end{aligned}$$

Thus, it can be computed from the optimality condtion

$$0 = p_\alpha - w_\alpha + \tau T(u_\alpha^B - u_\alpha^A + Du_\alpha^B \cdot p_\alpha)Du_\alpha^B$$

and is given by the solution of the linear system

$$(\mathbb{1} + \tau T Du_\alpha^B (Du_\alpha^B)^T)p_\alpha = w_\alpha - \tau T(u^B - u^A)Du_\alpha^B\,.$$

As for e.g. the denoising model, we choose

$$F_h(q_h) = \eta \sum_{I_{\alpha,\beta}\in\mathscr{I}_h} h^{d-1}|q_{\alpha,\beta}|$$

and $L_h = \nabla_h$. We recall that we have already computed $\mathrm{prox}_{\sigma F_h^*}(q_h) = \mathrm{proj}_{[-\eta,\eta]}(q_{\alpha,\beta})$. Thus, we arrive at the following algorithm:

Algorithm 3.6 Proximal splitting algorithm for linearized optical flow model.

function LINEARIZEDOPTICALFLOW($u_h^A, u_h^B, w_h^0, q_h^0, z_h^0 = w_h^0$)
  for $n = 1, \dots, N$ **do**

$$\begin{aligned} q_h^{n+1} &= \mathrm{proj}_{[-\eta,\eta]}\left(q_h^n + \sigma\nabla_h z_h^n\right) \\ w_\alpha^{n+1} &= (\mathbb{1} + \tau T Du_\alpha^B(Du_\alpha^B)^T)^{-1}\left(w_\alpha^n - \tau(\mathrm{div}_h\, q_h^{n+1})_\alpha + \tau T(u_\alpha^B - u_\alpha^A)Du_\alpha^B\right) \\ z_h^{n+1} &= w_h^{n+1} + (w_h^{n+1} - w_h^n) \end{aligned}$$

  end for
end function

3.2. Image Registration

More generally, instead of two images of the same $3D$ object at different times, we will consider matchings between two arbitrary images u^A and u^B. E.g. we can compare two different MRI images of a human brain. In fact, this leads to the same problem as the static optical flow model 3.1. However, we will study the image registration problem for images in L^∞ instead of BV. Then, to rigorously proof existence of optimal matchings, we need a higher Sobolev regularity. This approach is motivated by interpreting the matching as an elastic deformation. Therefore, we define the set of admissible deformations by

$$\mathcal{A} = \left\{ \phi \in W^{m,2}(\Omega, \Omega) \;:\; \det D\phi > 0 \text{ a.e. in } \Omega \,,\; \phi = \mathrm{id} \text{ on } \partial\Omega \right\} .$$

Here, we assume that $m > 1 + \frac{d}{2}$. In fact, this allows an embedding $W^{m,2} \to C^{1,\alpha}$.

Repetition 3.4 (Sobolev embedding). Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary.
Let $m \in \mathbb{N}_+$, $k \in \mathbb{N}$, $p \in [1, \infty)$, and $\alpha \in [0, 1]$ s.t. $m - \frac{d}{p} > k + \alpha$.
Then the embedding $\mathrm{id}\colon W^{m,p}(\Omega) \to C^{k,\alpha}(\Omega)$ is continuous and compact.

Moreover, we consider an energy density function $W\colon \mathbb{R}^d \to \mathbb{R}$, which satisfies the following assumptions.

(W1) W is non-negative, (sufficiently) smooth, and polyconvex, i.e. there exists a convex function $\tilde{W}$ s.t. $W(B) = \tilde{W}(M(B))$ for all $B \in \mathbb{R}^{d,d}$, where $M(B)$ denotes the minors of B.

(W2) There are constants $\beta_0, \beta_1, s > 0$ s.t.

$$W(A) \geqslant \beta_0 (\det A)^{-s} - \beta_1$$

for every invertible matrix A with $\det A > 0$. Moreover, we set $W(A) = \infty$ for $\det A \leqslant 0$.

(W3) We have that $W(\mathbb{1}) = 0$, $DW(\mathbb{1}) = 0$ and

$$\frac{1}{2} D^2 W(\mathbb{1})(B, B) = \frac{\lambda}{2} (\operatorname{tr} B)^2 + \mu \operatorname{tr}\left(\left(\frac{B + B^T}{2} \right)^2 \right) \qquad \forall B \in \mathbb{R}^{d,d} .$$

Remark 3.5 (On the assumptions (W1-3)).

(1) Note that the assumption (W3) means that the linearization of W around id correponds to the quadratic form of linearized elasticity. The values λ, μ are called Lamé–Navier parameters. Here, we actually only need the assumptions (W1-2). However, (W3) gives W a physical interpretation as an elastic energy density function. Moreover, we will make use of (W3) by studying the mage interpolation problem.

(2) In general, convexity of W is a to strong requirement on W. Thus, we only assume polyconvexity in (W1). Note that for $d = 2$ the vector $M(B)$ of minors is given by $M(B) = (B, \det B) \in \mathbb{R}^5$ and for $d = 3$ it is given by $M(B) = (B, \det B, \mathrm{cof} B) \in \mathbb{R}^{19}$, where $\mathrm{cof} B = (\det B) B^{-T}$ if B is invertible.

(3) The assumption (W2) will allow to control that a sequence of diffemorphisms converges to a diffemorphism.

Example 3.6 (Ogden Material)**.** A suitable choice for an isotropic and rigid body motion invariant energy density W in the case $d = 2$, which fulfills the assumptions (W1-3), is given by

$$W(D\phi) = a_1 \left(\operatorname{tr}(D\phi^T D\phi)\right)^q + a_2(\det D\phi)^r + a_3(\det D\phi)^{-s} + a_4$$

with appropriate coefficients $a_1, a_2, a_3, a_4 > 0$ depending on the Lamé–Navier parameters λ and μ. This is a special case of a so-called Ogden material.

Then we consider the following problem.

Problem 3.7: Image Registration

Given $u^A, u^B \in L^\infty(\Omega, [0,1]$. Let $\delta, \gamma > 0$ and $m > 1 + \frac{d}{2}$. Then minimize

$$E^{\text{reg}}(\phi) = \int_\Omega \frac{1}{\delta}|u^B \circ \phi - u^A|^2 + W(D\phi) + \gamma|D^m\phi|^2 \,\mathrm{d}x$$

over all $\phi \in \mathcal{A}$.

Proposition 3.8 (Existence of Optimal Deformations)**.** *Let $u^A, u^B \in L^\infty(\Omega, [0,1])$. Under the above assumptions (W1-2), there exists a minimizer of E^{reg} over all deformations $\phi \in \mathcal{A}$. Moreover, ϕ is a diffeomorphism and $\phi^{-1} \in C^{1,\alpha}(\Omega)$ for $\alpha \in (0, m - 1 - \frac{d}{2})$.*

Proof.
Step 1: Convergence of minimizing sequence
Since W is nonnegative by (W1), we have that $0 \leqslant \inf_{\phi\in\mathcal{A}} E^{\text{reg}}(\phi)$. Moreover, $\mathrm{id} \in \mathcal{A}$ and

$$E^{\text{reg}}(\mathrm{id}) = \int_\Omega \frac{1}{\delta}|u^B(x) - u^A(x)|^2 + W(\mathbb{1}) + \gamma|0|^2 \,\mathrm{d}x < \infty\,.$$

Thus, there exists a minimizing sequence $(\phi^j)_{j\in\mathbb{N}} \subset \mathcal{A}$ with monotonously decreasing energy $E^{\text{reg}}(\phi^j) < \infty$ s.t. $E^{\text{reg}}(\phi^j) \to \inf_{\phi\in\mathcal{A}}$ for $j \to \infty$. In particular, $\overline{E} = E^{\text{reg}}(\phi^1) < \infty$ is an upper bound. Now, because of the higher order term $|D^m\phi|^2$, we have that $(\phi^j)_j$ is bounded in $W^{m,2}$. Thus, there is a weakly convergent subsequence $\phi^j \rightharpoonup \phi$ in $W^{m,2}$. By the Sobolev embedding theorem, we have that $\phi^j \to \phi$ in $C^{1,\alpha}$ for $\alpha \in (0, m - 1 - \frac{d}{2} > 0)$.

Step 2: The limit deformation ϕ belongs to $\mathcal{A}$.
We will control the measure of the set

$$S_\varepsilon := \left\{x \in \Omega \;:\; \det D\phi \leqslant \varepsilon\right\}$$

for sufficiently small $\varepsilon > 0$. Indeed, by using (W2) and Fatou's lemma, we obtain

$$\begin{aligned}\beta_0\varepsilon^{-s}|S_\varepsilon| \leqslant \beta_0 \int_{S_\varepsilon} (\det D\phi)^{-s} \,\mathrm{d}x &\leqslant \int_{S_\varepsilon} W(D\phi) \,\mathrm{d}x + \beta_1|\Omega| \\ &\leqslant \liminf_{j\to\infty} \int_{S_\varepsilon} W(D\phi^j) \,\mathrm{d}x + \beta_1|\Omega| \leqslant \overline{E} + \beta_1|\Omega|\,.\end{aligned}$$

Thus,

$$|S_\varepsilon| \leqslant \frac{(\overline{E} + \beta_1|\Omega|)\varepsilon^s}{\beta_0} \quad \to \quad 0 \quad \text{for } \varepsilon \to 0\,.$$

Hence, $\det D\phi > 0$ a.e. on Ω.

Step 3 : ϕ is a diffeomorphism (without proof)

Step 4: Convergence of the matching energy functional.
Now, using that ϕ^j and ϕ are diffeomorphisms, we can estimate

$$\begin{aligned}
&\left|\int_\Omega |u^B \circ \phi^j - u^A|^2 - |u^B \circ \phi - u^A|^2 \,\mathrm{d}x\right| \\
&\leqslant \int_\Omega (|u^B \circ \phi^j - u^A| + |u^B \circ \phi - u^A|)|u^B \circ \phi^j - u^B \circ \phi| \,\mathrm{d}x \\
&\leqslant \left(\|u^B \circ \phi^j\|_{L^2} + \|u^B \circ \phi\|_{L^2} + 2\|u^A\|_{L^2}\right) \|u^B \circ \phi^j - u^B \circ \phi\|_{L^2} \\
&\leqslant C\left(\|u^B\|_{L^\infty(\Omega)} + \|u^A\|_{L^\infty(\Omega)}\right) \|u^B \circ \phi^j - u^B \circ \phi\|_{L^2} .
\end{aligned}$$

It remains to show that $\|u^B \circ \phi^j - u^B \circ \phi\|_{L^2} \to 0$ for $j \to \infty$.
Therefore, we approximate u^B in $L^2(\Omega, [0,1])$ by a sequence $(u_i^B)_{i\in\mathbb{N}} \subset C^\infty(\Omega, [0,1])$. Then we can estimate

$$\|u^B \circ \phi^j - u^B \circ \phi\|_{L^2} \leqslant \|u^B \circ \phi^j - u_i^B \circ \phi^j\|_{L^2} + \|u_i^B \circ \phi^j - u_i^B \circ \phi\|_{L^2} + \|u_i^B \circ \phi - u^B \circ \phi\|_{L^2} .$$

First, for every i, since u_i^B is continuous, we have that $\|u_i^B \circ \phi^j - u_i^B \circ \phi\|_{L^2} \to 0$ for $j \to \infty$.
Furthermore, we have that

$$\begin{aligned}
\|u^B \circ \phi^j - u_i^B \circ \phi^j\|_{L^2}^2 &= \int_\Omega (u^B - u_i^B)^2 \det(D(\phi^j)^{-1}) \,\mathrm{d}x \\
&\leqslant \|\det(D(\phi^j)^{-1})\|_{L^\infty} \|u^B - u_i^B\|_{L^2}^2 \\
&\leqslant C\|u^B - u_i^B\|_{L^2}^2 \quad \to 0 \quad \text{for } i \to \infty .
\end{aligned}$$

Step 5: ϕ is a minimizer of E^{reg}
We consider the minimizing sequence $(\phi^j)_{j\in\mathbb{N}}$.
First, for any $\varepsilon > 0$, there exists $j(\varepsilon) \in \mathbb{N}$ s.t.

$$E^{\mathrm{reg}}(\phi^j) \leqslant E^{\mathrm{reg}}(\phi^{j(\varepsilon)}) \leqslant \inf_{\phi\in\mathcal{A}} +\varepsilon$$

for all $j \geqslant j(\varepsilon)$.
Furthermore, by Step 4, we can assume that

$$\left|\int_\Omega |u^B \circ \phi^j - u^A|^2 - |u^B \circ \phi - u^A|^2 \,\mathrm{d}x\right| \leqslant \varepsilon$$

for all $j \geqslant j(\varepsilon)$.
Now, since W is polyconvex by assumption (W1), the functional $\int_\Omega W(D\phi) + \gamma|D^m\phi|^2 \,\mathrm{d}x$ is lower semicontinuous. Thus, we obtain

$$\begin{aligned}
E^{\mathrm{reg}}(\phi) &= \int_\Omega W(D\phi) + \gamma|D^m\phi|^2 + \frac{1}{\delta}|u^B \circ \phi - u^A|^2 \,\mathrm{d}x \\
&\leqslant \liminf_{j\to\infty} \int_\Omega W(D\phi^j) + \gamma|D^m\phi^j|^2 + \frac{1}{\delta}|u^B \circ \phi^j - u^A|^2 \,\mathrm{d}x + \frac{\varepsilon}{\delta} \\
&\leqslant \inf_{\phi\in\mathcal{A}} +\varepsilon + \frac{\varepsilon}{\delta} .
\end{aligned}$$

Hence, ϕ is a minimizer of E^{reg}. $\square$

Numerical Solution with Gradient Descent. Recall, that we have only assumed that $u^A, u^B \in L^\infty(\Omega)$. However, by computing the first variation

$$DE^{\text{reg}}(\phi)(\widehat{\phi}) = \int_\Omega DW(D\phi) : D\widehat{\phi} + 2\gamma D^m\phi : D^m\widehat{\phi} + \frac{2}{\delta}(u^B \circ \phi - u^A) Du^B \circ \phi \cdot \widehat{\phi} \,\mathrm{d}x\,,$$

we observe that the derivative Du^B appears. Thus, for a numerical solution, on the domain $\Omega = [0,1]^d$, we take into account cubic cells C_α and use the space

$$U_h = \{u_h \in C^0(\Omega, [0,1]) \;:\; u_h \text{ multilinear on each cell}\}$$

to discretize $u_h^A, u_h^B \in U_h$. Also for the deformation, we choose

$$\Phi_h = \{\phi_h \in C^0(\Omega, \mathbb{R}^d) \;:\; \phi_h \text{ multilinear on each cell, } \phi_h = \mathrm{id} \text{ on } \partial\Omega\}\,.$$

Then, discrete versions of $\int_\Omega W(D\phi)\,\mathrm{d}x$ and $\int_\Omega \frac{1}{\delta}|u^B \circ \phi - u^A|^2\,\mathrm{d}x$ are obtained by a Simpson quadrature. But, since ϕ_h is only in $W^{1,2}$, the space Φ_h does not allow a conforming approximation of the higher order term $\int_\Omega \gamma |D^m\phi|^2\,\mathrm{d}x$. In fact, for the numerical simulation, suitable results can be achieved without this term. However, by defining the mass matrix M_h with entries $M_{ij} = \int_\Omega \phi_i \cdot \phi_j\,\mathrm{d}x$ and the stiffness matrix S_h with entries $S_{ij} = \int_\Omega D\phi_i : D\phi_j\,\mathrm{d}x$, a nonconforming approximation of the higher order term is given by $M_h(M_h^{-1}S_h)^{\frac{m}{2}}\phi_h \cdot (M_h^{-1}S_h)^{\frac{m}{2}}\phi_h$. For computational reasons, M_h^{-1} is usually approximated with the inverse of the lumped mass matrix.

Remark 3.9 (On the elastic energy density function W). In practice, the Ogden material as in Example 3.6 is not necessary. E.g. the density $W(D\phi) = |D\phi - \mathbb{1}|^2$ already produces good results.

[Lecture 07 – 06/08/2020]

[Lecture 08 – 06/15/2020]

3.3. Image Interpolation

Again, we consider two images $u^A, u^B \in L^\infty(\Omega, [0,1])$. As for the dynamic optical flow model, we are interested in finding an interpolation path of images $u : [0,T] \times \Omega \to [0,1]$ between u^A and u^B. Here, we will study a time discrete version as proposed in [BER15]. More precisely, we consider a vector of images of length $K+1$:

$$U = (u^0 = u^A, u^1, \dots, u^{K-1}, u^K = u^B) \in L^\infty(\Omega, [0,1])^{K+1},$$

which we call a discrete path. Now, as for the image registration problem, for each pair (u^{k-1}, u^k) with $k = 1, \dots, K$, we consider a deformation $\phi^k \in \mathcal{A}$. This leads to a vector of deformations of length K:

$$\Phi = (\phi^1, \dots, \phi^K) \in \mathcal{A}^K .$$

For each pair of images, we take into account the registration energy and define

$$E^{\text{reg}}(u^{k-1}, u^k, \phi^k) = \int_\Omega \frac{1}{\delta}|u^k \circ \phi^k - u^{k-1}|^2 + W(D\phi^k) + \gamma |D^m \phi^k|^2 \,\mathrm{d}x .$$

Then we consider the following optimization problem.

Problem 3.10: Time Discrete Metamorphosis for Image Interpolation

Given $u^A, u^B \in L^\infty(\Omega, [0,1]$. Let $\delta, \gamma > 0$ and $m > 1 + \frac{d}{2}$. Then minimize

$$E_K^{\text{meta}}(U) = K \sum_{k=1}^K \inf_{\phi^k \in \mathcal{A}} E^{\text{reg}}(u^{k-1}, u^k, \phi^k)$$

over all $U \in L^\infty(\Omega, [0,1])^{K+1}$ with $u^0 = u^A$ and $u^K = u^B$.

Furthermore, we define the energy explicitly depending on U and Φ as

$$E_K^{\text{meta}}(U, \Phi) = K \sum_{k=1}^K E^{\text{reg}}(u^{k-1}, u^k, \phi^k) .$$

Proposition 3.11 (Consistency of the time discrete path energy)**.** *Let Ω be convex. We consider a sufficiently smooth path of images $u\colon [0,1] \times \Omega \to [0,1]$ and a sufficiently smooth time dependent motion field $v\colon [0,1] \times \Omega \to \mathbb{R}^d$. Then, for a step size $\tau = \frac{1}{K}$, we define interpolated images $u_K^k = u(\frac{k}{K})$ and motion fields $v_K^k = v(\frac{k}{K})$ for $k = 0, \dots, K$. Furthermore, we define deformations $\phi_K^k = \frac{1}{K} v_K^k + \mathrm{id}$. Then the time discete path energy $E_K^{meta}(u_K^0, \dots, u_K^K, \phi_K^1, \dots, \phi_K^K)$ converges for $K \to \infty$ to the corresponding continuous path energy*

$$E^{meta}(u, v) := \int_0^1 \int_\Omega \frac{1}{\delta} |\partial_t u + \partial_x u \cdot v|^2 + \frac{\lambda}{2} (\operatorname{tr} \varepsilon(v))^2 + \mu \operatorname{tr}(\varepsilon(v)^2) + \gamma |D^m v|^2 \,\mathrm{d}x\,\mathrm{d}t ,$$

where $\varepsilon(v) = \frac{Dv + (Dv)^T}{2}$ is the symmetrized gradient.

Proof. First, for the matching term, we use the Taylor expansion around id:

$$\begin{aligned} u_K^k \circ \phi_K^k &= u_K^k + \partial_x u_K^k \cdot (\phi_K^k - \mathrm{id}) + \text{higher order terms} \\ &= u_K^k + \tau \partial_x u_K^k \cdot v_K^k + O(\tau^2) \end{aligned}$$

and obtain

$$\begin{aligned} & K \sum_{k=1}^{K} \int_\Omega |u_K^k \circ \phi_K^k - u_K^{k-1}|^2 \,\mathrm{d}x \\ &= \frac{1}{\tau} \sum_{k=1}^{K} \int_\Omega |u_K^k - u_K^{k-1} + \tau \partial_x u_K^k \cdot v_K^k + O(\tau^2)|^2 \,\mathrm{d}x \\ &= \frac{\tau^2}{\tau} \sum_{k=1}^{K} \int_\Omega \left| \frac{u_K^k - u_K^{k-1}}{\tau} + \partial_x u_K^k \cdot v_K^k + O(\tau) \right|^2 \,\mathrm{d}x \\ &\xrightarrow{K \to \infty} \int_0^1 \int_\Omega |\partial_t u + \partial_x u \cdot v|^2 \,\mathrm{d}x \,\mathrm{d}t \,. \end{aligned}$$

Also for the energy density function W and the higher order term $|D^m \phi|^2$, we use a Taylor expansion around $\mathbb{1}$ and make use of the assumption (W3):

$$\begin{aligned} & K \sum_{k=1}^{K} \int_\Omega W(D\phi_K^k) + \gamma |D^m \phi_K^k|^2 \,\mathrm{d}x \\ &= \frac{1}{\tau} \sum_{k=1}^{K} \int_\Omega W(\mathbb{1}) + \tau DW(\mathbb{1})(Dv_K^k) + \frac{\tau^2}{2} D^2 W(\mathbb{1})(Dv_K^k, Dv_K^k) + O(\tau^3) + \gamma \tau^2 |D^m v_K^k|^2 \,\mathrm{d}x \\ &= \tau \sum_{k=1}^{K} \int_\Omega \left(\frac{\lambda}{2} \left(\operatorname{tr} Dv_K^k \right)^2 + \mu \operatorname{tr} \left(\frac{Dv_K^k + (Dv_K^k)^T}{2} \right)^2 \right) + O(\tau) + \gamma |D^m v_K^k|^2 \,\mathrm{d}x \\ &\xrightarrow{K \to \infty} \int_0^1 \int_\Omega \frac{\lambda}{2} (\operatorname{tr} \varepsilon(v))^2 + \mu \operatorname{tr}(\varepsilon(v)^2) + \gamma |D^m v|^2 \,\mathrm{d}x \,\mathrm{d}t \,. \end{aligned}$$

□

Now, we will study optimizers of this functional for fixed deformations.

Proposition 3.12 (Existence of optimal images for fixed deformations)**.** *Let $u^A, u^B \in L^\infty(\Omega, [0,1])$. Moreover, let $K \geqslant 2$ and $\Phi \in \mathcal{A}^K$. Then, there exists a unique minimizer of $E_K^{meta}(\cdot, \Phi)$ over all $U \in (L^\infty(\Omega, [0,1]))^{K+1}$ with $u_0 = u^A$, $u_K = u^B$.*

Proof.
Step 1: Convergence of Minimizing Sequence
First, we consider a vector in $L^\infty(\Omega, [0,1])^{K+1}$ defined by

$$(u^A, u^A, \dots, u^A, u^B) \,.$$

We observe that

$$\begin{aligned} E_K^{\mathrm{meta}}((u^A, u^A, \dots, u^A, u^B), \Phi) &\leqslant \frac{C}{\delta} (\|u^A\|_\infty + \|u^B\|_\infty) + K \sum_{k=1}^{K} \int_\Omega W(D\phi^k) + \gamma |D^m \phi^k|^2 \,\mathrm{d}x \\ &\leqslant \bar{C} < \infty \,. \end{aligned}$$

Thus, there exists a minimizing sequence

$$U_j = (u_j^0 = u^A, u_j^1, \dots, u_j^{K-1}, u_j^K = u^B) \subset L^\infty(\Omega, [0,1])^{K+1}$$

s.t. $E_K^{\text{meta}}(U_j, \Phi) \leqslant C$ for all $j \in \mathbb{N}$. Then we can estimate

$$\|u_j^k\|_2 \leqslant \|u_j^{k+1} \circ \phi^{k+1} - u_j^k\|_2 + \|u_j^{k+1} \circ \phi^{k+1}\|_2 \leqslant \left(\frac{\delta \bar{C}}{K}\right)^{\frac{1}{2}} + \|u_j^{k+1} \circ \phi^{k+1}\|_2 . \tag{3.1}$$

Now, starting from $k = K-1$, we observe that $\|u_j^{K-1}\|_2 \leqslant \left(\frac{\delta \bar{C}}{K}\right)^{\frac{1}{2}} + C\|u^B\|_\infty$. Thus, by induction, we obtain that $\|u_j^k\|_2$ is uniformly bounded for all k. Hence, there exists a weakly convergent subsequence $U_j \rightharpoonup U$ in $L^2(\Omega, [0,1])^{K+1}$.

Step 2: Uniqueness of Minimizer
We want to compute the Euler-Lagrange equation $\partial_{u^k} E_K^{\text{meta}}(U, \Phi) = 0$ for a single image u^k. Note that u^k only appears two times in the functional E_K^{meta}. Now, we use the transformation rule

$$\begin{aligned}
&\int_\Omega (u^k \circ \phi^k - u^{k-1})^2 + (u_{k+1} \circ \phi^{k+1} - u^k)^2 \, \mathrm{d}x \\
&= \int_\Omega (u^k - u^{k-1} \circ (\phi^k)^{-1})^2 (\det D\phi^k)^{-1} \circ (\phi_k)^{-1} + (u_{k+1} \circ \phi^{k+1} - u_k)^2 \, \mathrm{d}x .
\end{aligned}$$

Thus, the Euler-Lagrange equation is given by the pointwise condition

$$\left((u^k - u^{k-1} \circ (\phi^k)^{-1}) \left((\det D\phi^k)^{-1} \circ (\phi_k)^{-1} \right) + (u^k - u_{k+1} \circ \phi^{k+1}) \right)(x) = 0$$

for a.e. $x \in \Omega$. Hence, we have that

$$u^k(x) = \frac{u^{k+1} \circ \phi^{k+1}(x) + (u^{k-1} \circ (\phi^k)^{-1}(x))((\det D\phi^k)^{-1} \circ (\phi^k)^{-1}(x))}{1 + (\det D\phi^k)^{-1} \circ (\phi^k)^{-1}(x)}$$

for a.e. $x \in \Omega$.
This leads to a linear system of equations for $(u^1, \dots, u^{K-1})$, where evaluations at deformed positions are combined with evaluations at non-deformed positions, which we can consider as a block tridiagonal operator equation.
Therefore, for each $x \in \Omega$, we define the discrete transport path

$$X(x) = (X^0(x), X^1(x), X^2(x), \dots, X^K(x))^T \in \mathbb{R}^{K+1}$$

with $X^0(x) = x$ and $X^k(x) = \phi^k(X^{k-1}(x))$ for $k \in \{1, \dots, K\}$. Furthermore, we define the vector of inner images (i.e. without u^A and u^B) as

$$U^{\text{inner}}(\Phi)(x) := (u^1(X^1(x)), u^2(X^2(x)), \dots, u^{K-1}(X^{K-1}(x)))^T \in \mathbb{R}^{K-1} .$$

Then we obtain for $K \geqslant 3$ and a.e. $x \in \Omega$ a linear system of equations

$$A(\Phi)(x) U^{\text{inner}}(\Phi)(x) = R(\Phi)(x) \tag{3.2}$$

on $\mathbb{R}^{K-1}$. In this case, $A(\Phi)(x) \in \mathbb{R}^{K-1,K-1}$ is a tridiagonal matrix with

$$
\begin{aligned}
(A(\Phi)(x))_{k,k+1} &= -\frac{1}{1+(\det D\phi^k)^{-1} \circ (\phi^k)^{-1}(X^k(x))} \\
&= -\frac{1}{1+(\det D\phi^k)^{-1}(X^{k-1}(x))}, \\
(A(\Phi)(x))_{k,k} &= +1, \\
(A(\Phi)(x))_{k,k-1} &= -\frac{(\det D\phi^k)^{-1} \circ (\phi^k)^{-1}(X^k(x))}{1+(\det D\phi^k)^{-1} \circ (\phi^k)^{-1}(X^k(x))} \\
&= -\frac{(\det D\phi^k)^{-1}(X^{k-1}(x))}{1+(\det D\phi^k)^{-1}(X^{k-1}(x))}.
\end{aligned}
$$

Furthermore, the RHS $R(\Phi)(x) \in \mathbb{R}^{K-1}$ is given by

$$
R(\Phi)(x) = \left(\frac{u^A(x)(\det D\phi^1)^{-1}(x)}{1+(\det D\phi^1)^{-1}(x)}, 0, \ldots\ 0, \frac{u^B(X^K(x))}{1+(\det D\phi^{K-1})^{-1}(X^{K-2}(x))} \right)^T .
$$

Now, for a.e. $x \in \Omega$, the tridiagonal matrix $A(\Phi)(x)$ is diagonally dominant, i.e.

$$
\begin{aligned}
&1 = |A_{k,k}| \geqslant \sum_{l \neq k} |A_{k,l}| = 1 \quad \text{for all } k = 2, \ldots, K-2 \\
&1 = |A_{1,1}| > \sum_{l \neq 1} |A_{1,l}| = |A_{1,2}| = \frac{1}{1+(\det D\phi^1)^{-1}(X^0(x))} \\
&1 = |A_{K-1,K-1}| > \sum_{l \neq K-1} |A_{K-1,l}| = |A_{K-1,K-2}| = \frac{(\det D\phi^{K-1})^{-1}(X^{K-2}(x))}{1+(\det D\phi^{K-2})^{-1}(X^{K-2}(x))}.
\end{aligned}
$$

Thus, $A(\Phi)(x)$ is invertible. Hence, for all $x \in \Omega$, there exists a unique solution $U(\Phi)(x)$ solving the linear system (3.2). ◻

Now, we are in the position to prove the existence of discrete geodesics making use of the existence of a minimizing family of deformations for the energy E_K^{meta} and a given discrete image path as a consequence of Proposition 3.8 and the existence of an optimal discrete image path for a given family of deformations as stated in Proposition 3.12.

Theorem 3.13 (Existence of Time Discrete Image Interpolation)**.** *Let $u^A, u^B \in L^\infty(\Omega, [0,1])$ and $K \geqslant 2$. Then there exists a minimizer of E_K^{meta} over all $U \in L^\infty(\Omega, [0,1])^{K+1}$ with $u^0 = u^A$ and $u^K = u^B$.*

Proof. Let $(U_j = (u_j^0 = u^A, u_j^1, \ldots, u_j^{K-1}, u_j^K = u^B))_{j \in \mathbb{N}} \subset L^\infty(\Omega, [0,1])^{K-1}$ be a minimizing sequence. Note that $E_K^{\mathrm{meta}}(U_j)$ can be assumed to be uniformly bounded, since the vector $(u^A, \ldots, u^A, u^B, \mathrm{id}, \ldots, \mathrm{id})$ has finite energy. By Proposition 3.8, for every U_j, there exists a vector of optimal deformations $\Phi_j = (\phi_j^1, \ldots, \phi_j^K) \in \mathcal{A}^K$ with $E_K^{\mathrm{meta}}(U_j, \Phi_j) \leqslant E_K^{\mathrm{meta}}(U_j, \widehat{\Phi})$ for all $\widehat{\Phi} \in \mathcal{A}^K$. Furthermore, we can assume hat U_j already minimizes the energy $E_K^{\mathrm{meta}}(\widehat{U}, \Phi_j)$ over all $\widehat{U} \in L^\infty(\Omega, [0,1])^{K+1})$ with $\widehat{u}^0 = u^A$ and $\widehat{u}^K = u^B$, since otherwise we would further reduce the energy.

Now, because of the higher order term, we have that the deformations (ϕ_j^k) are uniformly bounded in $W^{m,2}(\Omega,\mathbb{R}^d)$ for $k=1,\dots,K$. Thus, by the Sobolev embedding, there exists a subsequence s.t. $\Phi^j \to \Phi = (\phi^1,\dots,\phi^K)$ in $C^{1,\alpha}(\Omega,\mathbb{R}^d)$ for $0<\alpha<m-1-\frac{d}{2}$. Following the same line of arguments as in Step 2 of the proof of Proposition 3.8, we observe that $\det D\phi^k > 0$ for a.e. $x\in\Omega$ and thus $\Phi\in\mathcal{A}^K$.
Using the estimate (3.1), we observe that the images u_j^k are uniformly bounded in $L^2(\Omega,[0,1])$. Hence, a subsequence of $(u_j^k)_{j\in\mathbb{N}}$ converges weakly in L^2 to some u^k. Now, by a uniform bound of $(\phi_j^k)^{-1}$ in $C^{1,\alpha}$, we obtain that for any $\xi\in C^\infty$

$$\begin{aligned}\int_\Omega u_j^k\circ\phi_j^k\xi\,\mathrm{d}x &= \int_\Omega \det D(\phi_j^k)^{-1}u_j^k\xi\circ(\phi_j^k)^{-1}\,\mathrm{d}x\\ &\to \int_\Omega \det D(\phi^k)^{-1}u^k\xi\circ(\phi^k)^{-1}\,\mathrm{d}x = \int_\Omega u^k\circ\phi^k\xi\,\mathrm{d}x\,,\end{aligned}$$

i.e. $u_j^k\circ\phi_j^k$ converges distributionally to $u^k\circ\phi^k$ for $k\to\infty$. In analogy to (3.1), we also observe that $\|u_j^k\circ\phi_j^k\|_2$ is uniformly bounded and thus, there is a weakly converging subsequence. Since we already know the distrubutional limit, we must have that $u_j^k\circ\phi_j^k \rightharpoonup u^k\circ\phi^k$ in L^2. Finally, by the lower semicontinuity of the L^2 norm and of the functional $\phi\mapsto\int_\Omega W(D\phi)+\gamma|D^m\phi|^2\,\mathrm{d}x$, it follows that $U=(u^0=u^A,u^1,\dots,u^{K-1},u^K=u^B)$ is a minimizer of E_K^{meta}. □

Numerical Method for Image Interpolation. The proof of Theorem 3.13 also allows to derive a numerical solution scheme by alternatingly minimizing the functional E_K^{meta} in the vector of image for fixed deformations, and vice-versa to optimize E_K^{meta} in the vector of deformations for fixed images. For the optimization in the deformations for a fixed vector U, we observe that minimizing $E_K^{\mathrm{meta}}(U,\Phi)$ over all Φ decouples in K registration problems by minimizing $E^{\mathrm{reg}}(u^{k-1},u^k,\phi^k)$ for all $k=1,\dots,K$. Furthermore, for fixed Φ, in analogy to Proposition 3.12, we observe that minimizing a discrete version of $E_K^{\mathrm{meta}}(U,\Phi)$ over all U leads to linear system. Therefore, we define a deformed mass matrix $M_h(\phi_h,\psi_h)$ depending on discete deformations ϕ_h,ψ_h by

$$M_h(\phi_h,\psi_h)_{ij} = \sum_{C_\alpha}\sum_{q=1,\dots,9}\omega(q)\xi_i(\phi_h(x(C_\alpha,q)))\xi_j(\psi_h(x(C_\alpha,q)))\,,$$

where (ξ_i) denotes the nodal-valued finite element basis, ω is a quadrature weight corresponding to a Simpson quadrature, and $x(C_\alpha,q)$ is the coordinate of the quadrature point q on the cell C_α. Thus, the discrete matiching energy functional is given by

$$\begin{aligned}&K\sum_{k=1}^K\frac{1}{\delta}\sum_{C_\alpha}\sum_{q=1,\dots,8}\omega(q)|u_h^k\circ\phi_h^k(x(C_\alpha,q))-u_h^{k-1}(x(C_\alpha,q))|^2\\ &=\frac{K}{\delta}\sum_{k=1}^K M_h(\phi_h^k,\phi_h^k)u_h^k\cdot u_h^k - 2M_h(\phi_k,\mathrm{id})u_h^k\cdot u_h^{k-1} + M_h(\mathrm{id},\mathrm{id})u_h^{k-1}\cdot u_h^{k-1}\,.\end{aligned}$$

Now, for discrete images u_h^k with $k=1,\dots,K-1$, we obtain the Euler-Lagrange equation

$$0=\frac{2K}{\delta}\left((M_h(\phi_h^k,\phi_h^k)+M_h(\mathrm{id},\mathrm{id}))u_h^k - M_h(\phi^k,\mathrm{id})u_h^{k-1} - M_h(\phi_h^{k+1},\mathrm{id})^T u_h^{k+1}\right)\,.$$

Algorithm 3.7 Alternating gradient descent algorithm for image interpolation with time discrete metamorphosis model.

function IMAGEINTERPOLATION(u_h^A, u_h^B, U_h^0, $\Phi_h^0 = \mathrm{id}^K$)
  for $n = 1, \dots, N$ **do**
    for $k = 1, \dots, K$ **do**
      compute $(\Phi_h^n)_k = \arg\min_{\widehat{\phi}_h} E^{\mathrm{reg}}((U_h^{n-1})_{k-1}, (U_h^{n-1})_k, \widehat{\phi}_h)$ via gradient descent
    end for
    solve linear system $A_h(\Phi_h^n)U_h^n = R_h(u_h^A, u_h^B, \Phi_h^n)$
  end for
end function

Hence, the minimizer $U_h = (u_h^1, \dots, u_h^{K-1})$ of the fully discrete functional $E_{K,h}^{\mathrm{meta}}(u_h^A, \cdot, u_h^B, \Phi_h)$ is given by the solution of a block-tridiagonal linear system $A_h(\Phi_h)U_h = R_h(u_h^A, u_h^B, \Phi_h)$.

Remark 3.14 (On the image interpolation algorithm). To initialize the vector U_h^0, we can use just a linear interpolation, i.e. $(U_h^0)_k = \frac{k}{K}u_h^B + (1 - \frac{k}{K})u_h^A$. In practice, if the images u_h^A and u_h^B are quite different, it is usefull to apply a multilevel scheme both in space and time. More precisely, we start with coarser space resultion and a smaller K. Then, the solution on this coarser level can be prolongated to a next finer level until reaching the original resolution.

Remarks On Time Discrete Geodesic Calculus The approach above, by defining a time discrete path energy, is following a general concept proposed in [RW15] to discretize geodesic curves on Banach manifolds.
Using the same concept, a discrete version of a geodesic exponential map can be defined. Therefore, we consider two images u^0 and u^1. Then, we want to find a minimizer of the energy

$$E(u^2, \phi^1, \phi^2) = 2\left(E^{\mathrm{reg}}(u^0, u^1, \phi^1) + E^{\mathrm{reg}}(u^1, u^2, \phi^2)\right) .$$

Note that by computing the Euler-Lagrange equation, ϕ^1 can directly be computed by a registration problem between u^0 and u^1. It turns out that the remaining system in (ϕ^2, u^2) can be solved by first computing a fixed point iteration for ϕ^2 and then use a simple update for u^2.
Furthermore, applying De-Casteljau's algorithm by using miminizers of the discrete path energy instead of straight lines, we can compute Bezier curves between images.

[Lecture 08 – 06/15/2020]
[Lecture 09 – 06/22/2020]

3.4. Optimal Transport

In the following, we will study different formulations of the optimal transport problem. We will see similarities to the metamorphosis model for image interpolation, where we aimed to approximately preserve intensity values (i.e. pointwise pixel values), whereas for the optimal transport model we globally want to preserve mass (i.e. the integral over the pixel values). For a general introduction, we refer to [San15], and for computational methods, we refer to [PC17].

Monge formulation. A first version of the optimal transport problem was already formulated in 1781 by Monge [Mon81], who asked for the minimal cost to transport a pile of sand into a hole of the same volume. For a mathematical model, source and sink are described by Borel probability measures $\mu^A, \mu^B \in \mathscr{P}(\Omega)$. Note that the set of all Borel probability measures on Ω is defined as a subset of positive Radon measures

$$\mathscr{P}(\Omega) = \{\mu \in \mathscr{M}^+(\Omega) \ : \ \mu(\Omega) = 1\}\,.$$

In our context, we understand μ^A and μ^B as images. More precisely, for images $u \in L^\infty(\Omega, [0,1])$, we have $\mu = u\mathscr{L}$. Here we cover the more general case of possibly singular measures.
Now, to define a transport of the mass represented by the measure μ^A, we take into account a transport map $T\colon \Omega \to \Omega$. Then, to guarantee that the mass of μ^A is transported by T to the measure μ^B, a matching condition is required.

Definition 3.15 (Pushforward)**.** Let $\mu \in \mathscr{P}(\Omega)$ and $T\colon \Omega \to \Omega$ Borel measurable. We define the pushforward $T_\# \mu$ of μ through T as

$$T_\#\mu(E) \coloneqq \mu(T^{-1}(E)) \quad \text{for all } E \in \mathscr{B}(\Omega)\,.$$

Thus, we want to have that $T_\#\mu^A = \mu^B$. By duality of Radon measures, this is equivalent to

$$\int_\Omega f(T(x)) \,\mathrm{d}\mu^A(x) = \int_\Omega f(x)\,\mathrm{d}\mu^B(x) \quad \forall f \in C(\Omega)\,. \tag{3.3}$$

Note that for $f = 1$, we have $\int_\Omega \mathrm{d}\mu^A(x) = \int_\Omega \mathrm{d}\mu^B(x)$, i.e. (3.3) is a mass-preserving condition.

Moreover, a transport cost function $c\colon \Omega \times \Omega \to [0,\infty]$ describes the cost to move a particle from a position $x \in \Omega$ to a position $y \in \Omega$. In the following, we will assume Ω to be convex and restrict the transport cost function to the Euclidean distance $c(x,y) = \frac{1}{2}|x-y|^2$. Then Monge's problem in its general formulation is to find a transport map T having minimal transport cost.

Problem 3.16: Monge Formulation of Optimal Transport

Let $\mu^A, \mu^B \in \mathscr{P}(\Omega)$. Then minimize

$$E^{\text{Monge}}(T) = \int_\Omega c(x, T(x))\,\mathrm{d}\mu^A(x) = \frac{1}{2}\int_\Omega |T(x)-x|^2\,\mathrm{d}\mu^A(x)\,. \tag{3.4}$$

over all transport maps $T\colon \Omega \to \Omega$ Borel measurable s.t. $T_\#\mu^A = \mu^B$.

Unfortunately, Monge's problem (3.4), in general, does not admit existence nor uniqueness.

Example 3.17 (Nonexistence and Nonuniqueness for Monge's Problem)**.**

(1) Let $\Omega = [-1,1]$, $\mu^A = \delta_0$ and $\mu^B = \frac{1}{2}(\delta_{-1} + \delta_1)$, where δ_p denotes the Dirac measure at the point $p \in \Omega$. Then there does not exist a transport map T between μ^A and μ^B, since otherwise

$$f(T(0)) = \int_\Omega f(x)\,\mathrm{d}T_\# \mu^A(x) = \int_\Omega f(x)\,\mathrm{d}\mu^B(x) = \frac{1}{2}(f(-1) + f(1))$$

for all $f \in C(\Omega)$. In other words, we cannot split a single point.

(2) Let $\Omega = [0,1]^2$, $\mu^A = \frac{1}{2}(\delta_{(0,0)} + \delta_{(1,1)})$ and $\mu^B = \frac{1}{2}(\delta_{(1,0)} + \delta_{(0,1)})$. Then an optimal transport map could map $(0,0)$ to $(0,1)$ and $(1,1)$ to $(1,0)$, but also the opposite way is optimal.

Connection to the Metamorphosis model. Compared to the time discrete metamorphosis model, we have no regularity assumption on the transport map T. However, under the assumptions that $\mu^A = u^A\mathscr{L}$, $\mu^B = u^B\mathscr{L}$, and $T = \phi \in \mathcal{A}$, we can use the transformation rule for the matching condition (3.3) to obtain

$$\int_\Omega f(\phi(x))u^A(x)\,\mathrm{d}x = \int_\Omega f(y)u^B(y)\,\mathrm{d}y = \int_\Omega f(\phi(x))\det D\phi(x)u^B(\phi(x))\,\mathrm{d}x \quad \forall f \in C(\Omega)\,.$$

Hence, we have that

$$\det D\phi(x)u^B(\phi(x)) = u^A(x) \tag{3.5}$$

for a.e. $x \in \Omega$. By taking into account an L^2 penalization instead of the pointwise constraint (3.5), we define a corresponding registration energy

$$E^{\mathrm{reg}}(u^A, u^B, \phi) := \int_\Omega \frac{1}{\delta}|\det D\phi(x)u^B(\phi(x)) - u^A(x)|^2 + \frac{1}{2}|\phi - \mathrm{id}|^2 u^A(x)\,\mathrm{d}x\,.$$

Then for a discrete path of images $U = (u^0, \dots, u^K) \in L^\infty(\Omega, [0,1])^{K+1}$ and a vector of deformations $\Phi \in \mathcal{A}^K$, in analogy to the time discrete metamorphosis model, we define an energy

$$E_K^{\mathrm{OT},\delta}(U, \Phi) = K \sum_{k=1}^K E^{\mathrm{reg}}(u^{k-1}, u^k, \phi^k)\,.$$

We observe the following functional in the limit for $K \to \infty$.

Proposition 3.18 (Consistency of the time discrete path energy)**.** *Let Ω be convex. We consider a sufficiently smooth path of images $u\colon [0,1] \times \Omega \to [0,1]$ and a sufficiently smooth time dependent motion field $v\colon [0,1] \times \Omega \to \mathbb{R}^d$. Then, for a step size $\tau = \frac{1}{K}$, we define interpolated images $u_K^k = u(\frac{k}{K})$ and motion fields $v_K^k = v(\frac{k}{K})$ for $k = 0, \dots, K$. Furthermore, we define deformations $\phi_K^k = \frac{1}{K}v_K^k + \mathrm{id}$. Then the time discrete path energy $E_K^{OT,\delta}(u_K^0, \dots, u_K^K, \phi_K^1, \dots, \phi_K^K)$ converges for $K \to \infty$ to the corresponding continuous path energy*

$$E^{OT,\delta}(u, v) := \int_{[0,1]} \int_\Omega \frac{1}{\delta}|\partial_t u + \mathrm{div}(uv)|^2 + \frac{1}{2}u|v|^2\,\mathrm{d}x\,\mathrm{d}t\,, \tag{3.6}$$

Proof. First, for the transport cost we easily get

$$K\sum_{k=1}^{K}\int_{\Omega}|\phi_K^k-\mathrm{id}|^2 u_K^{k-1}\,\mathrm{d}x=\sum_{k=1}^{K}\tau\int_{\Omega}|v_k^K|^2u_K^{k-1}\,\mathrm{d}x\xrightarrow{K\to\infty}\int_{[0,1]}\int_{\Omega}|v|^2u\,\mathrm{d}x\,\mathrm{d}t\,.$$

For the pushforward term, we use the Taylor expansions

$$\begin{aligned}\det D\phi_K^k&=\mathbb{1}+\tau\,\mathrm{tr}\left(\frac{\phi_K^k-\mathrm{id}}{\tau}\right)+O(\tau^2)=\mathbb{1}+\tau\,\mathrm{div}(v_K^k)+O(\tau^2)\\ u_K^k\circ\phi_K^k&=u_K^k+\tau\nabla u_K^k\cdot v_K^k+O(\tau^2)\end{aligned}$$

and obtain

$$\begin{aligned}&K\sum_{k=1}^{K}\int_{\Omega}|\det(D\phi_K)u_K^k\circ\phi_K^k-u_K^{k-1}|^2\,\mathrm{d}x\\ &=\sum_{k=1}^{K}\tau\int_{\Omega}\left|\frac{u_K^k-u_K^{k-1}}{\tau}+\mathrm{div}(v_K^k)u_K^k+\nabla u_K^k\cdot v_K^k+O(\tau)\right|^2\mathrm{d}x\\ &\xrightarrow{K\to\infty}\int_{[0,1]}\int_{\Omega}|\partial_t u+\mathrm{div}(uv)|^2\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

□

By interpreting u as mass, we have that $\int_{[0,1]}\int_{\Omega}\frac{1}{2}u|v|^2\,\mathrm{d}x\,\mathrm{d}t$ is the kinetic energy functional.

Benamou–Brenier Formulation of Optimal Transport. Using the exact constraint in (3.6) instead of an L^2 penalization, we arrive at the Benamou–Brenier formulation

Problem 3.19: Benamou–Brenier formulation for optimal transport

Given $\mu^A=u^A\mathscr{L}$ and $\mu^B=u^B\mathscr{L}$, minimize

$$E^{\mathrm{BB}}(u,v)=\int_{[0,1]}\int_{\Omega}|v(t,x)|^2u(t,x)\,\mathrm{d}x\,\mathrm{d}t$$

over all pairs (u,v) in the set of solutions to the continuity equation

$$\begin{aligned}\mathcal{CE}(u^A,u^B):=\Big\{(u,v)\colon[0,1]\times\Omega\to[0,1]\times\mathbb{R}^d\ :\ &\partial_t u+\mathrm{div}(uv)=0\,,\\ &u(0)=u^A,u(1)=u^B,\\ &v\cdot n=0\text{ on }\partial\Omega\Big\}\,.\end{aligned}$$

This dynamic formulation of the optimal transport problem goes back to Benamou and Brenier [BB00]. As the push-forward condition in the static case, the continuity equation leads to a mass-preservation, since for any $t \in [0,1]$, we have that

$$0 = \int_0^t \int_\Omega \partial_t u(s,x) + \operatorname{div}(u(s,x)v(s,x)) \,\mathrm{d}x \,\mathrm{d}s = \int_\Omega u(t,x) - u(0,x) \,\mathrm{d}x + \int_0^t \int_{\partial\Omega} uv \cdot n \,\mathrm{d}x \,\mathrm{d}t$$

$$= \int_\Omega u(t,x) \,\mathrm{d}x - \int_\Omega u^A(x) \,\mathrm{d}x \,.$$

Concerning a numerical optimization scheme, we observe that the minimization problem 3.19 is nonconvex in the pair (u,v). However, we can apply a change of variables by considering the pair $(u, m = uv)$ of mass and momentum. More precisely, we define a density for the kinetic energy by

$$\theta(u,m)(t,x) = \begin{cases} \frac{|m(t,x)|^2}{2u(t,x)} & \text{if } u(t,x) > 0 \\ 0 & \text{if } u(t,x) = 0 \text{ and } m(t,x) = 0 \\ \infty & \text{otherwise} \end{cases} \tag{3.7}$$

Lemma 3.20. *The function θ as defined in (3.7) is convex, lsc, and 1-homogeneous.*

Proof. easy to check □

Then, we consider the following problem.

Problem 3.21: Benamou–Brenier formulation in mass and momentum variable

Given $\mu^A = u^A \mathscr{L}$ and $\mu^B = u^B \mathscr{L}$, minimize

$$E^{\mathrm{BB}}(u,m) = \int_{[0,1]} \int_\Omega \theta(u(t,x), m(t,x)) \,\mathrm{d}x \,\mathrm{d}t$$

over all pairs (u,m) in the set of solutions to the continuity equation

$$\begin{aligned} \mathcal{CE}(u^A,u^B) := \Big\{ (u,m) \colon [0,1] \times \Omega \to [0,1] \times \mathbb{R}^d \ : \ & \partial_t u + \operatorname{div}(m) = 0\,, \\ & u(0) = u^A, u(1) = u^B, \\ & m \cdot n = 0 \text{ on } \partial\Omega \Big\}\,. \end{aligned}$$

Note that $\mathcal{CE}(u^A,u^B)$ is a convex set in the pair of variables (u,m).

Therefore, we can apply a proximal splitting algorithm to a discrete version of the functional $E(u,m) = F(u,m) + G(u,m)$, where $F(u,m) = \int_0^1 \int_\Omega \theta(u,m) \,\mathrm{d}x \,\mathrm{d}t$ and $G(u,m) = \mathcal{I}_{\mathcal{CE}(u^A,u^B)}(u,m)$. Note that $L = \mathrm{id}$.

First, we compute the proximal map of F^*.

Proposition 3.22 (Convex Conjugate of Kinetic Energy)**.** *For the function θ as defined in (3.7) we have that $\theta^* = \mathcal{I}_K$ is an indicator function of the convex set*

$$K = \left\{ (u,m) \in \mathbb{R} \times \mathbb{R}^d \ : \ u + \frac{|m|^2}{2} \leqslant 0 \right\}.$$

Proof. The convex conjugate is defined by

$$\theta^*(u', m') = \sup_{u,m} u'u + m' \cdot m - \theta(u, m) .$$

First, we consider the case $u > 0$. Then

$$\begin{aligned}\sup_{u,m} u'u + m' \cdot m - \theta(u, m) &= \sup_{u,m} u'u + m' \cdot m - \frac{|m|^2}{2u} \\ &= \sup_{u,m} \frac{u}{2}\left(2u' + \frac{2m' \cdot m}{u} - \frac{m \cdot m}{u^2}\right) \\ &= \sup_{u,m} -\frac{u}{2}\left(m' - \frac{m}{u}\right)^2 + u\left(u' + \frac{|m'|^2}{2}\right) \\ &= \sup_{u,m} u\left(-\frac{1}{2}\left(m' - \frac{m}{u}\right)^2 + \left(u' + \frac{|m'|^2}{2}\right)\right) .\end{aligned}$$

In the case that $u' + \frac{|m'|^2}{2} \leqslant 0$ (i.e. if $(u', m') \in K$)), we observe that the supremum is 0, since $\left(m' - \frac{m}{u}\right)^2 \geqslant 0$ and by assumption $u > 0$. Otherwise, if $u' + \frac{|m'|^2}{2} > 0$, we can choose $m' = \frac{m}{u}$, s.t. $\left(m' - \frac{m}{u}\right)^2 = 0$, and we obtain that the supremum is ∞.
Finally, we can check the other cases for (u, m). □

Thus, the proximal map of F^* is given by pointwise projections onto the set K, which we specify in the following.

Lemma 3.23 (Projection onto K)**.** *The projection of a point $(u, m) \in \mathbb{R} \times \mathbb{R}^d$ onto the set K is given by*

$$\operatorname{proj}_K(u, m) = (u^{pr}, m^{pr}) = \begin{cases} (u, m) & \text{if } (u, m) \in K , \\ \left(u + 1 - \dfrac{1}{\sigma}, \sigma m\right) & \text{if } (u, m) \notin K , \end{cases}$$

where $\sigma \in \mathbb{R}$ is defined as the solution of the equation $\sigma^3|m|^2 + 2(1 + u)\sigma - 2 = 0$.

Proof. In the case that $(u, m) \notin K$, the projection lies on the boundary ∂K, which can be parametrized by a map $\gamma\colon \mathbb{R}^d \to \partial K$ defined as $\gamma(b) = \left(-\frac{|b|^2}{2}, b\right)$. Hence, the vector $(1, b) \in \mathbb{R}^{d+1}$ spans the normal space at a point $(a, b) \in \partial K$. Now, for $(u, m) \in \mathbb{R} \times \mathbb{R}^d$, we search for the orthogonal projected point $(u^{\mathrm{pr}}, m^{\mathrm{pr}}) \in \partial K$, which satisfies the relation $(u^{\mathrm{pr}}, m^{\mathrm{pr}}) + \tau(1, m^{\mathrm{pr}}) = (u, m)$ for some $\tau \in \mathbb{R}$. We set $\sigma = (1 + \tau)^{-1}$, which leads to $(u^{\mathrm{pr}}, m^{\mathrm{pr}}) = (u + 1 - \frac{1}{\sigma}, \sigma m)$. Since $(u^{\mathrm{pr}}, m^{\mathrm{pr}}) \in \partial K$, we obtain $\sigma^3|m|^2 + 2(1 + u)\sigma - 2 = 0$. □

Note that this polynomial equation of order three can be solved by a simple Newton method. Furthermore, it is important to choose a discretization, s.t. the function θ can indeed be updated pointwise.

Next, we compute the proximal map of G. Since $G = \mathcal{I}_{\mathcal{CE}(u^A, u^B)}$, we directly have to solve a projection problem.

Proposition 3.24 (Projection onto Solutions to the Continuity Equation)**.** *Given $p = (u, m)$, the solution $p^{pr} = (u^{pr}, m^{pr})$ to the projection problem*

$$p^{pr} = \operatorname{proj}_{\mathcal{CE}(u^A,u^B)}(p) = \underset{\widehat{p}\in\mathcal{CE}(u^A,u^B)}{\arg\min} \left\| p - \widehat{p} \right\|^2 . \tag{3.8}$$

is given by

$$p^{pr} = p + \frac{1}{2} D_{t,x}\phi^{pr} , \tag{3.9}$$

where $\phi^{pr} \colon [0,1] \times \Omega \to \mathbb{R}$ is defined by solving

$$\frac{1}{2} \int_0^1 \int_\Omega D_{t,x}\phi^{pr} \cdot D_{t,x}\widehat{\phi} \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \widehat{\phi}(1)u^B - \widehat{\phi}(0)u^A \,\mathrm{d}x - \int_0^1 \int_\Omega p \cdot D_{t,x}\widehat{\phi} \,\mathrm{d}x\,\mathrm{d}t$$

for all $\widehat{\phi}$.

Proof. The associated Lagrangian to the minimization problem (3.8) is given by

$$\begin{aligned} L\left(p^{\mathrm{pr}}, \phi^{\mathrm{pr}}\right) &= \int_0^1 \int_\Omega |p^{\mathrm{pr}} - p|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^1 \int_\Omega \phi^{\mathrm{pr}}(\partial_t u^{\mathrm{pr}} + \operatorname{div} m^{\mathrm{pr}}) \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^1 \int_\Omega |p^{\mathrm{pr}} - p|^2 \,\mathrm{d}x\,\mathrm{d}t - \int_0^1 \int_\Omega p^{\mathrm{pr}} \cdot D_{t,x}\phi^{\mathrm{pr}} \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \phi^{\mathrm{pr}}(1)u^B - \phi^{\mathrm{pr}}(0)u^A \,\mathrm{d}x \end{aligned}$$

with a Lagrange multiplier $\phi^{\mathrm{pr}} \colon [0,1] \times \Omega \to \mathbb{R}$. The corresponding Euler–Lagrange equations for a saddle point $(p^{\mathrm{pr}}, \phi^{\mathrm{pr}})$ are given by

$$\int_0^1 \int_\Omega p^{\mathrm{pr}} \cdot D_{t,x}\widehat{\phi} \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \widehat{\phi}(1)\, u^B - \widehat{\phi}(0)\, u^A \,\mathrm{d}x \quad \forall \widehat{\phi} , \tag{3.10}$$

$$\int_0^1 \int_\Omega \widehat{p} \cdot D_{t,x}\phi^{\mathrm{pr}} \,\mathrm{d}x\,\mathrm{d}t = \int_0^1 \int_\Omega 2(p^{\mathrm{pr}} - p)\, \widehat{p} \,\mathrm{d}x\,\mathrm{d}t \quad \forall \widehat{p} . \tag{3.11}$$

Testing (3.11) with $\widehat{p} = D_{t,x}\widehat{\phi}$ and then using (3.10) gives

$$\begin{aligned} \int_0^1 \int_\Omega \frac{1}{2} D_{t,x}\phi^{\mathrm{pr}} \cdot D_{t,x}\widehat{\phi} \,\mathrm{d}x\,\mathrm{d}t &= \int_0^1 \int_\Omega (p^{\mathrm{pr}} - p) \cdot D_{t,x}\widehat{\phi} \,\mathrm{d}x\,\mathrm{d}t \\ &= \int_\Omega \widehat{\phi}(1)u^B - \widehat{\phi}(0)u^A \,\mathrm{d}x - \int_0^1 \int_\Omega p \cdot D_{t,x}\widehat{\phi} \,\mathrm{d}x\,\mathrm{d}t . \end{aligned}$$

This is a weak form of a Laplace equation and can be uniquely solved in ϕ^{pr} up to a mean value. Finally, by testing (3.11) with $\widehat{p} = 1$, we observe the update $p^{\mathrm{pr}} = p + \frac{1}{2} D_{t,x}\phi^{\mathrm{pr}}$. □

Thus, we have to solve a discrete version of the time-space Laplace equation $-\Delta_{t,x}\phi^{\mathrm{pr}} = \operatorname{div}_{t,x}(p)$. Therefore, for the time-space domain $[0,1] \times [0,1]^2$, we choose a simplicial mesh and ϕ_h in the finite element space $\{\phi_h \in C([0,1]^3) \ :\ \phi_h \text{ linear on each simplex}\}$. Because of the update formula (3.9), we must choose u_h and m_h in the space $\{(u_h, m_h) \colon [0,1]^3 \to \mathbb{R} \times \mathbb{R}^2 \ :\ (u_h, m_h) \text{ constant on each simplex}\}$. This also allows to perform the projection onto K pointwise (i.e. one projection for each simplex).

[Lecture 09 – 06/22/2020]
[Lecture 10 – 07/06/2020]

Kantorovich formulation. Now, we will study the Kantorovich [Kan42; Kan48] formulation of the optimal transport problem, which can be interpreted as a convex lifting of the Monge's formulation. However, it will be a true relaxation, which is useful, since we have seen that Monge's formulation does not always allow existence of an optimal transport map. Thus, we embed the transport map $T\colon \Omega \to \Omega$ between μ^A and μ^B into the product space $\Omega \times \Omega$ by considering a so-called transport plan $\pi = (\mathrm{id} \times T)_\# \mu^A$, i.e.

$$\int_{\Omega\times\Omega} f(x,y)\,\mathrm{d}\pi(x,y) = \int_\Omega f(x,T(x))\,\mathrm{d}\mu^A(x) \quad \forall f \in C(\Omega\times\Omega)\,.$$

Since T fulfills the pushforward matching condition (3.3), the transport plan π satisfies the marginal constraints

$$(\mathrm{proj}_1)_\#\pi = \mu^A \quad \text{and} \quad (\mathrm{proj}_2)_\#\pi = \mu^B\,,$$

where proj_i for $i = 1,2$ denotes the projection on the i-th component. More generally, we define the set of all Borel probability measures on the product space with the above marginal constraints by

$$\Pi(\mu^A,\mu^B) = \left\{\pi \in \mathscr{P}(\Omega\times\Omega) \;:\; (\mathrm{proj}_1)_\#\pi = \mu^A\,,\; (\mathrm{proj}_2)_\#\pi = \mu^B\right\}\,.$$

Then we consider the following problem.

Problem 3.25: Kantorovich formulation of optimal transport

Let $\mu^A, \mu^B \in \mathscr{P}(\Omega)$. Then minimize

$$E^{\mathrm{Kan}}(\pi) = \int_{\Omega\times\Omega} c(x,y)\,\mathrm{d}\pi(x,y) = \int_{\Omega\times\Omega} |x-y|^2\,\mathrm{d}\pi(x,y)$$

over all $\pi \in \Pi(\mu^A,\mu^B)$.

Then, the following existence result holds.

Theorem 3.26 (Existence of Solutions)**.** *Suppose that $c\colon \Omega\times\Omega \to [0,\infty]$ is lsc and bounded from below. Then Kantorovich's problem 3.25 admits a solution.*

Proof. We consider a minimizing sequence $(\pi_n)_{n\in\mathbb{N}} \subset \Pi(\mu^A,\mu^B)$. Since by definition of $\Pi(\mu^A,\mu^B)$, all π_n are probability measures, there is a subsequence $\pi_n \overset{*}{\rightharpoonup} \pi$ in $\mathscr{M}^+(\Omega\times\Omega)$. We can easily check that $\pi \in \Pi(\mu^A,\mu^B)$. In the case that c is continuous, we directly observe the lsc of E^{Kan} w.r.t. to the weak-* convergence. For a lsc c, we can approximate c by an increasing sequence of continuous functions c_n. □

Therefore, we also must have a solution for the Example 3.17.

Example 3.27 (Splitting of Mass)**.** Let $\Omega = [-1,1]$, $\mu^A = \delta_0$ and $\mu^B = \frac{1}{2}(\delta_{-1} + \delta_1)$. Then $\pi = \frac{1}{2}(\delta_{(0,-1)} + \delta_{(0,-1)})$ is the optimal transport plan.

Even more, in the case $c(x,y) = |x-y|^2$, the Kantorovich problem allows to define a metric on the space of probability measures on Ω, the so-called Wasserstein distance

$$\mathcal{W}(\mu^A,\mu^B) = \inf_{\pi\in\Pi(\mu^A,\mu^B)} E^{\mathrm{Kan}}(\pi)\,. \tag{3.12}$$

Remark 3.28 (Connection to Monge and Benamou–Brenier formulation). Under the condition that the initial measure μ^A is absolutely continuous w.r.t. the Lebesgue measure on Ω, uniqueness of the optimal transport plan can be established by applying Brenier's polar factorization result [Bre91], which allows decomposing a density function into a gradient of a convex function up to a concatenation with a measure-preserving map. In this case, the solution to Monge's and Kantorovich's problem coincide. More precisely, if $\mu^A = u^A \mathscr{L}$, there exists a unique optimal transport map T solving Monge's problem, and $T = D\psi$ is the μ^A-a.e. unique gradient of a convex function ψ. Moreover, the unique optimal transport plan solving Kantorovich's problem is given by $\pi = (\mathrm{id} \times D\psi)_\# \mu^A$. Furthermore, the solution of the Benamou–Brenier problem is given by $\mu(t) = ((1-t)\mathrm{id} + tT)_\# \mu^A$.

For a computational scheme we consider empirical measures

$$\mu^A = \sum_{i=1}^{I} a_i \delta_{x_i}, \quad \mu^B = \sum_{j=1}^{J} b_j \delta_{x_j},$$

where $x_i, x_j \in [0,1]^d$, $a_i, b_j \in \mathbb{R}_+$, and $\sum_{i=1}^{I} a_i = 1 = \sum_{j=1}^{J} b_j$. Note that in the case of imaging applications, we can choose $I = J = |\mathscr{C}_h|$ and identifty x_α with a midpoint of a cell C_α. We identify a coupling $\pi \in \Pi(\mu^A, \mu^B)$ by a matrix $P \in \mathbb{R}^{I \times J}$. More precisely, the marginal constraints are given by

$$\begin{aligned} P\mathbf{1}_J &= (\sum_{j=1}^{J} P_{ij})_{i=1,\dots,I} = (a_i)_{i=1,\dots,I} = \mathbf{a}\,, \\ P^T\mathbf{1}_I &= (\sum_{i=1}^{I} P_{ji})_{j=1,\dots,J} = (b_j)_{j=1,\dots,J} = \mathbf{b}\ , \end{aligned} \tag{3.13}$$

where $\mathbf{1}_N \in \mathbb{R}^N$ denotes the vector with all entries equal 1. Now, the cost function c defines for each pair (x_i, x_j) a corresponding cost $C_{ij} = c(x_i, x_j)$, s.t. in the discrete setup, we can identifty the cost function with a cost matrix $C \in \mathbb{R}^{I \times J}$. Finally, the Kantorovich problem in the discrete setup is given by minimizing $\langle C, P \rangle$ over all $P \in \mathbb{R}^{I \times J}$ s.t. the discrete marginal constraints (3.13) are satisfied and P represents a probability measure, i.e. $P_{ij} \geqslant 0$ for all (i,j). Note that by the discrete marginal constraints, it automatically follows that $\sum_{i,j} P_{ij} = \sum_i (\sum_j P_{ij}) = \sum_i a_i = 1$. Thus, the Kantorovich problem in this discrete setting is a linear optimization problem, which is typically solved with the simplex algorithm.

Entropic regularization of Kantorovich Formulation. First, for a discrete coupling matrix $P \in \mathbb{R}^{I \times J}$, we define the entropy functional

$$E^{\mathrm{ent}}(P) = -\sum_{i=1}^{I} \sum_{j=1}^{J} P_{ij}(\log(P_{ij}) - 1)\,.$$

Then, for a regularization parameter $\varepsilon > 0$, we consider the following regularized Kantorovich formulation. *i.e.*, the optimization problem

Problem 3.29: Entropic regularized Kantorovich problem

Given empirical probability measures μ^A, μ^B. Then minimize

$$\langle P, C\rangle - \varepsilon E^{\mathrm{ent}}(P)$$

over all discrete couplings $\pi \in \Pi(\mu^A, \mu^B)$ represented by a coupling matrix P.

This problem was proposed in [Ben+15]. As for the Wasserstein metric in (3.12), we define

$$\mathcal{W}_\varepsilon(\mu^A, \mu^B) := \inf_{\pi \in \Pi(\mu^A, \mu^B)} E^{\mathrm{Kan}}(\pi) - \varepsilon E^{\mathrm{ent}}(P)\,.$$

However, $\mathcal{W}_\varepsilon(\mu^A, \mu^B)$ does not define a metric, but we can prove convergence for $\varepsilon \to 0$.

Lemma 3.30 (Convergence of the regularized Wasserstein distance)**.**

(1) For every $\varepsilon > 0$, there exists a unique minimizer P_ε of Problem 3.29.

(2) We have convergence

$$P_\varepsilon \to \arg\min \{-E^{ent}(P) \;:\; P \textit{ solution of Kantorovich problem }\}$$

for $\varepsilon \to 0$. I.e., in the limit we obtain the solution of the Kantorovich problem with maximal entropy.

Proof.

(1) Follows by strict convexity of $-E^{\mathrm{ent}}$.

(2) Since $P_\varepsilon \in \mathbb{R}^{I\times J}$ represent probability measures, there is a subsequence $P_\varepsilon \to P$ for $\varepsilon \to 0$. From this convergence, we also obtain that P is a coupling matrix between μ^A and μ^B. Now, we compare P with a solution $\tilde{P}$ of the Kantorovich problem. First, for any ε, we have that $\langle C, \tilde{P}\rangle \leqslant \langle C, P_\varepsilon\rangle$. By optimality of P_ε for the regularized problem, we have that $\langle C, P_\varepsilon\rangle - \varepsilon E^{\mathrm{ent}}(P_\varepsilon) \leqslant \langle C, \tilde{P}\rangle - \varepsilon E^{\mathrm{ent}}(\tilde{P})$. Together, we obtain

$$0 \leqslant \langle C, P_\varepsilon\rangle - \langle C, \tilde{P}\rangle \leqslant \varepsilon\left(E^{\mathrm{ent}}(P_\varepsilon) - E^{\mathrm{ent}}(\tilde{P})\right)\,.$$

Since E^{ent} is continous, by passing to the limit $\varepsilon \to 0$, we observe that $\langle C, P\rangle = \langle C, \tilde{P}\rangle$. Hence, P is a minimizer of the Kantorovich problem. Furthermore, we get $E^{\mathrm{ent}}(\tilde{P}) \leqslant E^{\mathrm{ent}}(P)$.

□

Lemma 3.31 (Represenation of regularized solution)**.** *Let P be the unique solution of Problem 3.29. Then there exist $u \in \mathbb{R}^I_+$ and $v \in \mathbb{R}^J_+$ s.t.*

$$P_{ij} = u_i G^\varepsilon_{ij} v_j\,.$$

Here, G^ε is given by the Gibbs kernel $G^\varepsilon_{ij} = e^{-\frac{C_{ij}}{\varepsilon}}$

Proof. We can define the Lagrangian

$$\begin{aligned} L(P,\lambda^A,\lambda^B) &= \langle C,P\rangle - \varepsilon E^{\text{ent}}(P) - \langle \lambda^A, P\mathbf{1}-\mathbf{a}\rangle - \langle \lambda^B, P^T\mathbf{1}-\mathbf{b}\rangle \\ &= \sum_{i,j} C_{ij}P_{ij} + \varepsilon P_{ij}(\log(P_{ij})-1) - \lambda_i^A P_{ij} - \lambda_j^B P_{ij} + \sum_i \lambda_i^A a_i + \sum_j \lambda_j^B b_j\,. \end{aligned}$$

The optimality condition in P_{ij} is given by

$$0 = \partial_{P_{ij}} L = C_{ij} + \varepsilon\log(P_{ij}) - \lambda_i^A - \lambda_j^B\,.$$

Hence, we obtain that

$$P_{ij} = e^{\frac{\lambda_i^A}{\varepsilon}} e^{-\frac{C_{ij}}{\varepsilon}} e^{\frac{\lambda_j^B}{\varepsilon}}\,.$$

Thus, by defining $u_i = e^{\frac{\lambda_i^A}{\varepsilon}} > 0$ and $v_j = e^{\frac{\lambda_j^B}{\varepsilon}} > 0$, we arrive at the representation of P. □

Algorithm 3.8 Sinkhorn algorithm.

function SINKHORN(v^0, ε)
    for $n = 1,\dots,N$ **do**

$$u_i^{n+1} = \frac{a_i}{(G^\varepsilon v^n)_i}$$
$$v_j^{n+1} = \frac{b_j}{(G^\varepsilon u^{n+1})_j}$$

    end for
end function

Theorem 3.32 (Convergence of Sinkhorn)**.** *Let $v^0 \in \mathbb{R}_+^J$ be an initial value and $\varepsilon > 0$. Assume that $I = J$. For the iterates of the Sinkhorn algorithm 3.8, we have convergence $(u^n, v^n) \to (u^*, v^*)$ for $n \to \infty$. Then P^* defined by $P^*_{ij} = u_i^* G_{ij}^\varepsilon v_j^*$ is the unique solution of the regularized Kantorovich problem 3.29.*

Proof (sketch).
Step 1: Define a logarithmic metric on $\mathbb{R}_+^I$
We consider

$$\begin{aligned} d_{log,TV}(u,v) &:= |\log(u)-\log(v)|_{TV} \\ &= \max_i\,(\log(u_i)-\log(v_i)) + \max_j\,\big(\log(v_j)-\log(u_j)\big) \\ &= \max_i\,(\log(u_i)-\log(v_i)) - \min_j\,\big(\log(u_j)-\log(v_j)\big) \\ &= \max_i \log(u_i/v_i) - \min_j \log(u_j/v_j)\,. \end{aligned}$$

For $d_{log,TV}$ we make the following observations:

(a) $d_{log,TV}$ defines a distance on $\mathbb{R}_+^I/\sim$ (i.e. $u \sim v$ if $u = cv$). (easy to check)

(b) $d_{log,TV}(u,v) = d_{log,TV}(\mathbf{1}/u, \mathbf{1}/v) = d_{log,TV}(u/v, \mathbf{1})$

(c) $d_{log,TV}(G^\varepsilon u, G^\varepsilon v) \leqslant C(G^\varepsilon) d_{log,TV}(u,v)$ with $C(G^\varepsilon) < 1$. (very technical)

Step 2: We apply the properties of $d_{log,TV}$ to the iteration
First, we estimate

$$\begin{aligned}
d_{log,TV}(u^{n+1}, u^*) &= d_{log,TV}(\mathbf{a}/G^\varepsilon v^n, \mathbf{a}/G^\varepsilon v^*) && \text{by definition of } u \\
&= d_{log,TV}(G^\varepsilon v^*/G^\varepsilon v^n, \mathbf{1}) && \text{by property (b)} \\
&= d_{log,TV}(G^\varepsilon v^*, G^\varepsilon v^n) && \text{by property (b)} \\
&\leqslant C(G^\varepsilon) d_{log,TV}(v^*, v^n) && \text{by property (c).}
\end{aligned}$$

In analogy, $d_{log,TV}(v^{n+1}, v^*) \leqslant C(G^\varepsilon) d_{log,TV}(u^*, u^{n+1})$. Hence, we obtain that

$$\begin{aligned}
d_{log,TV}(u^{n+1}, u^*) &\leqslant C(G^\varepsilon)^{2n+1} d_{log,TV}(v^0, u^*) &&\to 0 \quad \text{for } n \to \infty \\
d_{log,TV}(v^{n+1}, v^*) &\leqslant C(G^\varepsilon)^{2n+2} d_{log,TV}(v^0, v^*) &&\to 0 \quad \text{for } n \to \infty\,.
\end{aligned}$$

Step 3: Convergence of P
We set $P^0_{ij} = u^0_i G^\varepsilon_{ij} v^0_j$ and define the iteration

$$\begin{aligned}
P^{2n+1}_{ij} &= u^{n+1}_i G^\varepsilon_{ij} v^n_j\,, \\
P^{2n+2}_{ij} &= u^{n+1}_i G^\varepsilon_{ij} v^{n+1}_j\,.
\end{aligned} \tag{3.14}$$

Now, we can express the iterates in terms of the limit

$$P^{2n}_{ij} = u^n_i G^\varepsilon_{ij} v^n_j = \frac{u^n_i}{u^*_i} u^*_i G^\varepsilon_{ij} v^*_j \frac{v^n_j}{v^*_j} = \frac{u^n_i}{u^*_i} P^*_{ij} \frac{v^n_j}{v^*_j}\,.$$

Then, we can estimate

$$\begin{aligned}
\|\log(P^{2n}) - \log(P^*)\|_\infty &= \max_{ij} \left| \log \left(\frac{\frac{u^n_i}{u^*_i} P^*_{ij} \frac{v^n_j}{v^*_j}}{P^*_{ij}} \right) \right| \\
&= \max_{ij} |\log(\frac{u^n_i}{u^*_i}) + \log(\frac{v^n_j}{v^*_j})| \\
&\leqslant d_{log,TV}(u^n, u^*) + d_{log,TV}(v^n, v^*) \ \to 0 \text{ for } n \to \infty\,.
\end{aligned}$$

In analogy for P^{2n+1}. □

Remark 3.33 (On the Sinkhorn Iteration). The iteration (3.14) can be interpreted as

$$\begin{aligned}
P^{n+1} &= \text{proj}^{KL}_{P \in \mathbb{R}^{IJ}\,:\, P\mathbf{1} = \mathbf{a}}(P^n)\,, \\
P^{n+2} &= \text{proj}^{KL}_{P \in \mathbb{R}^{IJ}\,:\, P^T\mathbf{1} = \mathbf{b}}(P^{n+1})\,.
\end{aligned}$$

Here, the Kullback–Leibler divergence is defined as

$$KL(P, G^\varepsilon) = \sum_{ij} P_{ij} \log(\frac{P_{ij}}{G^\varepsilon_{ij}})$$

Thus, we alternatingly compute projections w.r.t. the Kullback–Leibler divergence.

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