Linear Algebra and its Applications 436 (2012) 1425–1435

Contents lists available at SciVerse ScienceDirect

Linear Algebra and its Applications

journal homepage: www.elsevier.com/locate/laa

Spectral distances of graphs

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ARTICLE INFO

Article history:
Received 11 June 2011
Accepted 9 August 2011
Available online 3 September 2011

Submitted by R.A. Brualdi

AMS classification:
05C50

Keywords:
Spectral distance
Adjacency matrix
Graph energy
Cospectrality measure
Spectral diameter

ABSTRACT

Let $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$ be the adjacency spectrum of a graph G on n vertices. The spectral distance $\sigma(G_1, G_2)$ between n vertex graphs G_1 and G_2 is defined by

$$\sigma(G_1, G_2) = \sum_{i=1}^{n} |\lambda_i(G_1) - \lambda_i(G_2)|.$$

Here we provide some initial results regarding this quantity. First, we give some general results concerning the spectral distances between arbitrary graphs, and compute these distances in some particular cases. Certain relation with the theory of graph energy is identified. The spectral distances bounded by a given constant are also considered. Next, we introduce the cospectrality measure and the spectral diameter, and obtain specific results indicating their relevance for the theory of cospectral graphs. Finally, we give and discuss some computational results and conclude the paper by a list of conjectures.

1. Introduction

We consider only simple graphs, i.e. finite undirected graphs without loops or multiple edges. If G is such a graph with the vertex set $V_G = \{v_1, v_2, \ldots, v_n\}$, the *adjacency matrix* of G is the $n \times n$ matrix $A_G = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between the vertices i and j, and $a_{ij} = 0$ otherwise. The *eigenvalues* (resp. *spectrum*) of G, denoted by

$$\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G), \tag{1}$$

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doi:10.1016/j.laa.2011.08.019

are just the eigenvalues (resp. spectrum) of A_G. At some points we shall suppress the graph name from our notation.

We assume that the reader is familiar with the standard terminology and notation on graphs and their spectra (compare [1]). For example, P_n (resp. C_n) denotes a path (resp. cycle) on n vertices, etc.

Let G_1 and G_2 be the graphs with n vertices. The following sum

$$\sigma(G_1, G_2) = \sum_{i=1}^{n} |\lambda_i(G_1) - \lambda_i(G_2)| \tag{2}$$

will be referred to as the *spectral distance* between the graphs G_1 and G_2. (Note that the concept of the spectral distance appears in the literature; it is defined in many different ways and it mainly regards to the applications of the spectral graph theory. These references and the corresponding results will not be listed here.) Some problems concerning the distance between the spectra of graphs has been posed recently by R.A. Brualdi (see [3]) – besides (2), another definition based on the Euclidean norm was suggested. We have chosen this one mostly because of its close connection with the well studied graph energy in some particular cases – see the next section. Here we give some initial results that refer to this topic.

The paper is organized as follows. In Section 2 we give certain general results on spectral distances. In Section 3, in order to describe nature of spectral distances, we compute them for particular pairs of graphs. Certain interesting results are obtained (these results can be considered as the initial examples, as well). In Section 4, we introduce the cospectrality measure, spectral eccentricity and spectral diameter along with an illustrative example showing their importance for the theory of graph spectra. Finally, some computational results, their consequences and arising conjectures are given and discussed in Section 5.

2. Some general results

Through this and the following sections we shall deal with the graph energy defined as

$$E(G) = \sum_{i=1}^{n} |\lambda_i(G)|.$$

Now we prove some general statements. Clearly, we have $E(G) = \sigma(G, nK_1)$. Moreover, we have the following lemma.

Lemma 2.1. *Let $n^-(G_1)$ and $n^-(G_2)$ be the numbers of (strictly) negative eigenvalues of G_1 and G_2, respectively. Denote $k = \max\{n^-(G_1), n^-(G_2)\}$ and $l = |n^-(G_1) - n^-(G_2)|$, and assume that $|\lambda_i(G_1)| \geqslant |\lambda_i(G_2)|$ holds for $i \leqslant n-k$ and $i \geqslant n-k+l$. Then*

$$\sigma(G_1, G_2) = E(G_1) - E(G_2) + 2 \sum_{n-k<i<n-k+l} |\lambda_i(G_2)|.$$

Proof. For $i \leqslant n-k$ and $i \geqslant n-k+l$ we have $|\lambda_i(G_1) - \lambda_i(G_2)| = |\lambda_i(G_1)| - |\lambda_i(G_2)|$, while for the remaining values of i we have $|\lambda_i(G_1) - \lambda_i(G_2)| = |\lambda_i(G_1)| + |\lambda_i(G_2)|$. Thus,

$$\begin{aligned}\sigma(G_1, G_2) &= \sum_{i=1}^{n-k}(|\lambda_i(G_1)| - |\lambda_i(G_2)|) + \sum_{n-k<i<n-k+l}(|\lambda_i(G_1)| + |\lambda_i(G_2)|)\\ &\quad + \sum_{i=n-k+l}^{n}(|\lambda_i(G_1)| - |\lambda_i(G_2)|) = E(G_1) - E(G_2) + 2\sum_{n-k<i<n-k+l}|\lambda_i(G_2)|. \qquad \square\end{aligned}$$

The following result is an immediate consequence of the previous lemma.

Corollary 2.1. *If G_1 and G_2 are bipartite and $\lambda_i(G_1) \geqslant \lambda_i(G_2)$, $i = 1, \ldots, \lfloor n/2 \rfloor$, then*

$$\sigma(G_1, G_2) = E(G_1) - E(G_2).$$

Lemma 2.2. *Given a graph G, let G_1 and G_2 be its proper induced subgraphs on m vertices. Then $\sigma(G_1, G_2) \leqslant \sum_{i=1}^{\min\{m,n-m\}} (\lambda_i(G) - \lambda_{n-\min\{m,n-m\}+i}(G))$. In addition, if each of G_1 and G_2 is obtained by deletion of a single vertex from G, we have $\sigma(G_1, G_2) \leqslant \lambda_1(G) - \lambda_n(G)$.*

Proof. Using the Interlacing Theorem (see [1], Theorem 0.10) we get $\lambda_i(G) \geqslant \lambda_i(G_1), \lambda_i(G_2) \geqslant \lambda_{n-m+i}(G),\ i = 1, \ldots, m$. Thus, we have

$$\sum_{i=1}^{m} |\lambda_i(G_1) - \lambda_i(G_2)| \leqslant \sum_{i=1}^{m} (\lambda_i(G) - \lambda_{n-m+i}(G)) = \sum_{i=1}^{\min\{m,n-m\}} (\lambda_i(G) - \lambda_{n-\min\{m,n-m\}+i}(G)).$$

If $m = n - 1$, the last sum reduces to $\lambda_1(G) - \lambda_n(G)$, and the proof follows. $\square$

We shall prove the theorem that gives a bound for the spectral distance between two arbitrary graphs.

Theorem 2.1. *Let*

$$\min\{\lambda_i(G_1), \lambda_i(G_2)\} < \max\{\lambda_{i+1}(G_1), \lambda_{i+1}(G_2)\} \tag{3}$$

holds for exactly t different values of i $(1 \leqslant i \leqslant n-1)$ denoted by $k_1, \ldots, k_t$, then

$$\begin{aligned} \sigma(G_1, G_2) \leqslant\ & \max\{\lambda_1(G_1), \lambda_1(G_2)\} - \min\{\lambda_{k_1}(G_1), \lambda_{k_1}(G_2)\} \\ & + \max\{\lambda_{k_1+1}(G_1), \lambda_{k_1+1}(G_2)\} - \min\{\lambda_{k_2}(G_1), \lambda_{k_2}(G_2)\} \\ & + \cdots + \max\{\lambda_{k_t+1}(G_1), \lambda_{k_t+1}(G_2)\} - \min\{\lambda_n(G_1), \lambda_n(G_2)\}. \end{aligned} \tag{4}$$

If additionally, $\max\{\lambda_1(G_1), \lambda_1(G_2)\} - \min\{\lambda_{k_1}(G_1), \lambda_{k_1}(G_2)\}$, $\max\{\lambda_{k_1+1}(G_1), \lambda_{k_1+1}(G_2)\} - \min\{\lambda_{k_2}(G_1), \lambda_{k_2}(G_2)\}$, …, $\max\{\lambda_{k_t+1}(G_1), \lambda_{k_t+1}(G_2)\} - \min\{\lambda_n(G_1), \lambda_n(G_2)\} \leqslant s$*, then*

$$\sigma(G_1, G_2) \leqslant s(t+1).$$

Proof. If $\min\{\lambda_i(G_1), \lambda_i(G_2)\} \geqslant \max\{\lambda_{i+1}(G_1), \lambda_{i+1}(G_2)\}$ holds for $i = 1, \ldots, k_1 - 1$, then we have

$$\begin{aligned} \sum_{i=1}^{k_1} |\lambda_i(G_1) - \lambda_i(G_2)| &= \max\{\lambda_1(G_1), \lambda_1(G_2)\} - \min\{\lambda_1(G_1), \lambda_1(G_2)\} \\ &\quad + \max\{\lambda_2(G_1), \lambda_2(G_2)\} - \min\{\lambda_2(G_1), \lambda_2(G_2)\} \\ &\quad + \cdots + \max\{\lambda_{k_1}(G_1), \lambda_{k_1}(G_2)\} - \min\{\lambda_{k_1}(G_1), \lambda_{k_1}(G_2)\} \\ &\leqslant \max\{\lambda_1(G_1), \lambda_1(G_2)\} - \min\{\lambda_1(G_1), \lambda_1(G_2)\} \\ &\quad + \min\{\lambda_1(G_1), \lambda_1(G_2)\} - \min\{\lambda_2(G_1), \lambda_2(G_2)\} \\ &\quad + \cdots + \min\{\lambda_{k_1-1}(G_1), \lambda_{k_1-1}(G_2)\} - \min\{\lambda_{k_1}(G_1), \lambda_{k_1}(G_2)\} \\ &= \max\{\lambda_1(G_1), \lambda_1(G_2)\} - \min\{\lambda_{k_1}(G_1), \lambda_{k_1}(G_2)\}, \end{aligned}$$

and similarly for $k_i + 1 \leqslant i \leqslant k_{i+1},\ i = 1, \ldots, t-1$ and $k_t + 1 \leqslant i \leqslant n$. Therefore, we have

$$\begin{aligned}\sigma(G_1, G_2) = &\sum_{i=1}^{k_1} |\lambda_i(G_1) - \lambda_i(G_2)| + \sum_{i=k_1+1}^{k_2} |\lambda_i(G_1) - \lambda_i(G_2)| + \ldots + \sum_{i=k_t+1}^{n} |\lambda_i(G_1) - \lambda_i(G_2)| \\ &\leqslant \max\{\lambda_1(G_1), \lambda_1(G_2)\} - \min\{\lambda_{k_1}(G_1), \lambda_{k_1}(G_2)\} \\ &\quad + \max\{\lambda_{k_1+1}(G_1), \lambda_{k_1+1}(G_2)\} - \min\{\lambda_{k_2}(G_1), \lambda_{k_2}(G_2)\} \\ &\quad + \cdots + \max\{\lambda_{k_t+1}(G_1), \lambda_{k_t+1}(G_2)\} - \min\{\lambda_n(G_1), \lambda_n(G_2)\} \\ &\leqslant s(t+1). \quad \square\end{aligned}$$

We have the following simple consequence.

Corollary 2.2. *Using the notation of the previous theorem, if s and t are fixed constants (they do not depend on n), then $\sigma(G_1, G_2)$ does not grow with n.*

The following examples demonstrate the results of the previous theorem and corollary (mainly, the spectra of the corresponding graphs are not listed, but one can determine them by using Table 1 – see the next section).

Example 1 (*the bound (4) attained*)**.** It is a matter of routine to prove that the bound (4) is attained whenever $G_2 = nK_1$. Moreover, let us consider the well–known Petersen graph (denoted by P; having the eigenvalues[2]: 3, $[1]^5$ and $[-2]^4$) and $5K_2$. Using the notation of the previous theorem, we get $t = 3$ and $k_1 = 7$, $k_2 = 8$, $k_3 = 9$. So, the following holds

$$8 = \sigma(P, 5K_2) \leqslant \lambda_1(P) - \lambda_7(P) + \sum_{i=8}^{10}(\lambda_i(5K_2) - \lambda_i(P)) = 8.$$

Example 2 ($\sigma(G_1, G_2)$ *does not grow with n*)**.** Consider the graphs P_n and $P_{n-1} + K_1$ ('+' stands for the union of graphs). Since the eigenvalues of P_{n-1} interlace the eigenvalues of P_n, while the only eigenvalue of K_1 is equal to 0, we get $t = 0$, and, for example, we can take $s = 4$. So, we have

$$\sigma(P_n, P_{n-1} + K_1) < 4, \quad \text{for any } n \geqslant 2.$$

(Note that P_n and $P_{n-1} + K_1$ make an example of graphs satisfying the conditions of Corollary 2.1, as well.)

Example 3 ($\sigma(G_1, G_2)$ *depends on n*)**.** Consider the graphs P_n and nK_1. Here, if n is even, we have $t = n - 2$ (the inequality (3) holds whenever $i \neq n/2$), while if n is odd, we have $t = n - 3$ (the same inequality holds whenever $\lfloor n/2 \rfloor \neq i \neq \lceil n/2 \rceil$). If we take $s = 2$, we obtain the following inequalities

$$\sigma(P_n, nK_1) \leqslant 2(n-1) \ \ (\text{if } n \text{ is even}) \ \ \text{i.e. } \sigma(P_n, nK_1) \leqslant 2(n-2) \ \ (\text{if } n \text{ is odd}).$$

And indeed, $\sigma(P_n, nK_1) = E(P_n) \to \infty$ when $n \to \infty$ (compare [2]).

Consider now K_{2n} and $2K_n$. We have $t = 0$ and $s = 2n$ with attaining the bound (4), i.e. $\sigma(K_{2n}, 2K_n) = 2n \to \infty$ when $n \to \infty$.

Example 4 (*the bound (4) is too rough in some cases*)**.** Consider the graphs $K_{1,n}$ and $K_{1,n-1} + K_1$. Using Theorem 2.1, similarly as in the previous examples, we get $t = 0$ and $s = 2\sqrt{n}$. Thus, $\sigma(K_{1,n}, K_{1,n-1} + K_1) \leqslant 2\sqrt{n}$. But, by direct computation we get $\sigma(K_{1,n}, K_{1,n-1} + K_1) = 2(\sqrt{n} - \sqrt{n-1}) \to 0$ when $n \to \infty$.

In the next section we shall give more results regarding some particular graphs.

[2] In the exponential notation, the exponents stand for the multiplicities of the eigenvalues.

Table 1
Some specific graphs and their spectra.

Graph	Eigenvalues
P_n	$2\cos\frac{i\pi}{n+1},\ i=1,\ldots,n$
C_n	$2\cos\frac{2i\pi}{n},\ i=1,\ldots,n$
Z_n	0 and $2\cos\frac{(2i-1)\pi}{2n-2},\ i=1,\ldots,n-1$
W_n	$2,\ [0]^2,\ -2$ and $2\cos\frac{i\pi}{n-3},\ i=1,\ldots,n-4$
K_n	$n-1$ and $[-1]^{n-1}$
K_{n_1,n_2}	$\sqrt{n_1n_2},\ [0]^{n_1+n_2-2}$ and $-\sqrt{n_1n_2}$
CP_n	$2n-2,\ [0]^n$ and $[-2]^{n-1}$

3. Examples of spectral distances

Since this paper provides initial results concerning spectral distances, we compute them for particular pairs of graphs. It turns out that some interesting results arise. We use the corresponding spectra listed in Table 1 (the graphs Z_n and W_n are depicted in Fig. 1 – their spectra can be found in [1], p. 77; the cocktail party graph CP_n has $2n$ vertices and it is a complement of nK_2).

We start with the following result.

Theorem 3.1. $\sigma(P_n, C_n) = 2,\ n \geqslant 3.$

Proof. Assume first that n is odd. Concerning the spectra of both graphs (see Table 1), we get

$$\lambda_1(C_n) > \lambda_i(C_n) = \lambda_{i+1}(C_n) > \lambda_{i+2}(C_n) = \lambda_{i+3}(C_n), \quad i = 2, 4, 6, \ldots, n-3$$

and

$$\lambda_i(C_n) > \lambda_i(P_n) > \lambda_{i+1}(P_n) > \lambda_{i+1}(C_n), \quad i = 1, 3, 5, \ldots, n-2,\ \lambda_n(C_n) > \lambda_n(P_n).$$

Using the above (in)equalities, we obtain

$$\begin{aligned}
\sigma(P_n, C_n) &= \lambda_1(C_n) - \lambda_1(P_n) + \sum_{i=2}^{n} |\lambda_i(P_n) - \lambda_i(C_n)| \\
&= \lambda_1(C_n) - \lambda_1(P_n) + \sum_{i \text{ even},\ i=2}^{n-1} (\lambda_i(P_n) - \lambda_i(C_n) + \lambda_i(C_n) - \lambda_{i+1}(P_n)) \\
&= 2 + \sum_{i=1}^{n} (-1)^n \lambda_i(P_n) = 2,
\end{aligned}$$

where the last equality follows from the facts that P_n is bipartite and n is odd.

Assume now that n is even. We have

$$\lambda_1(C_n) > \lambda_i(C_n) = \lambda_{i+1}(C_n) > \lambda_{i+2}(C_n) = \lambda_{i+3}(C_n) > \lambda_n(C_n), \quad i = 2, 4, 6, \ldots, n-4,$$

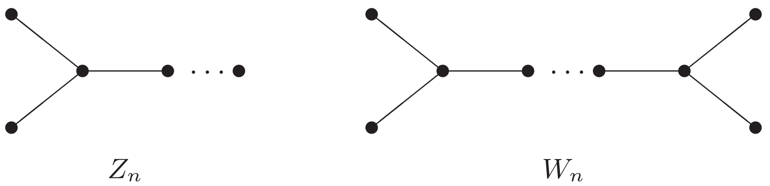

Fig. 1. Two graphs from Table 1.

and $\lambda_i(C_n) > \lambda_i(P_n) > \lambda_{i+1}(P_n) > \lambda_{i+1}(C_n)$, for $i = 1, 3, \ldots, n-1$. Thus, we have

$$\sigma(P_n, C_n) = 2\lambda_1(C_n) + \sum_{i=1}^{n}(-1)^i\lambda_i(P_n) = 4 - 2\sum_{i=1}^{n}(-1)^i \cos\frac{i\pi}{n+1} = 2,$$

where the last equality follows from the bipartiteness of P_n and the known trigonometrical identity $\sum_{i=1}^{n/2}(-1)^i \cos\frac{i\pi}{n+1} = 1/2$. □

Similarly, we can prove the next result.

Theorem 3.2. $\sigma(Z_{2n-1}, C_{2n-1}) = 2, \ n \geqslant 2$.

Proof. The eigenvalues of these graphs satisfy the same (in)equalities as the eigenvalues of P_n and C_n when n is odd (see the first part of the proof of Theorem 3.1). Thus, we have

$$\sigma(Z_{2n-1}, C_{2n-1}) = 2 + \sum_{i=1}^{n}(-1)^n\lambda_i(Z_{2n-1}) = 2. \quad \square$$

We shall now prove the following theorems.

Theorem 3.3. *Considering the distances among the graphs P_n, C_n, Z_n and W_n we get the following results:*

(i) $\sigma(P_n, Z_n) < 1$ whenever n is odd; $\sigma(P_n, Z_n) < 1$ whenever n is even and $n \geqslant 30$.
(ii) Let $n \geqslant 5$. $\sigma(P_n, W_n) < 2$ whenever n is odd and $n \geqslant 11$; $\sigma(P_n, W_n) < 2$ whenever n is even.
(iii) $\sigma(C_n, Z_n) < 3$ whenever n is even and $n \geqslant 4$.
(iv) Let $n \geqslant 5$. $\sigma(C_n, W_n) < 3$ whenever n is odd and $n \geqslant 7$; $\sigma(C_n, W_n) < 3$ whenever n is even.
(v) Let $n \geqslant 5$. $\sigma(Z_n, W_n) < 1$ whenever n is odd and $n \geqslant 31$; $\sigma(Z_n, W_n) < 1$ whenever n is even.

Sketch Proof. Concerning the distances $|\lambda_i(P_n) - \lambda_i(Z_n)|, \ i = 1, \ldots, n$, it can be checked that $\sigma(P_{2k}, Z_{2k}), \sigma(P_{2k-1}, Z_{2k-1}) < \sigma(P_{2k-2}, Z_{2k-2})$ holds for any $k \geqslant 2$. In addition, by direct computation we get $\sigma(P_{30}, Z_{30}) < 1$ and $\sigma(P_{2k-1}, Z_{2k-1}) < 1, \ k \leqslant 15$. Therefore, we get (i).

The remaining cases are considered in the similar way. □

Theorem 3.4. *Let G be an arbitrary graph, and let $n^*(G)$ denote the number of its eigenvalues which are greater than or equal to -1. Then the following holds:*

(i) $\sigma(K_n, G) = 2(n^* - 1 + \sum_{i=2}^{n^*}\lambda_i(G))$.

$$\text{(ii)}\ \sigma(K_{n_1,n_2}, G) = \begin{cases} E(G) - 2\sqrt{n_1n_2}, & \sqrt{n_1n_2} < -\lambda_n(G) \\ E(G) + 2\lambda_n(G), & \sqrt{n_1n_2} \in [-\lambda_n(G), \lambda_1(G)) \\ E(G) + 2(\sqrt{n_1n_2} - \lambda_1(G) + \lambda_n(G)), & \sqrt{n_1n_2} \geqslant \lambda_1(G). \end{cases}$$

Proof. (i) We get

$$\begin{aligned}
\sigma(K_n, G) &= n - 1 - \lambda_1(G) + \sum_{i=2}^{n}|\lambda_i(G) + 1| \\
&= n - 1 - \lambda_1(G) + \sum_{i=2}^{n^*}(\lambda_i(G) + 1) - \sum_{i=n^*+1}^{n}(\lambda_i(G) + 1) \\
&= n - 1 - \lambda_1(G) + n^* - 1 + \sum_{i=2}^{n^*}\lambda_i(G) - n + n^* - \sum_{i=n^*+1}^{n}\lambda_i(G) \\
&= 2(n^* - 1) - \lambda_1(G) + \sum_{i=2}^{n^*}\lambda_i(G) - \sum_{i=n^*+1}^{n}\lambda_i(G) = 2(n^* - 1 + \sum_{i=2}^{n^*}\lambda_i(G)),
\end{aligned}$$

Table 2
Spectral distances among some particular graphs.

	K_n	K_{n_1,n_2}	$CP_{n/2}$
P_n	$2\left(\left\lfloor \frac{2(n+1)}{3} \right\rfloor - 1 + \sum_{i=2}^{n-\left\lfloor \frac{2(n+1)}{3} \right\rfloor} \lambda_i(P_n)\right)$	$2(\sqrt{n_1n_2} - 2\lambda_1(P_n)) + E(P_n)$	$2(n - \lambda_1(P_n) + \lambda_{n/2}(P_n) - 2)$
C_n	$2\left(2\left\lfloor \frac{n}{3} \right\rfloor + \sum_{i=2}^{2\left\lfloor \frac{n}{3} \right\rfloor+1} \lambda_i(C_n)\right)$	$2(\sqrt{n_1n_2} - 2 + \lambda_n(C_n)) + E(C_n)$	$2(n + \lambda_{n/2}(C_n) - 4)$
Z_n	$2\left(\left\lfloor \frac{2(n+1)}{3} \right\rfloor - 1 + \sum_{i=2}^{n-\left\lfloor \frac{2(n+1)}{3} \right\rfloor} \lambda_i(Z_n)\right)$	$2(\sqrt{n_1n_2} - 2\lambda_1(Z_n)) + E(Z_n)$	$2(n - \lambda_1(Z_n) - 2)$
W_n	$2\left(\left\lfloor \frac{2n}{3} \right\rfloor + \sum_{i=2}^{n-\left\lfloor \frac{2n}{3} \right\rfloor-1} \lambda_i(W_n)\right)$	$2(\sqrt{n_1n_2} - 4) + E(W_n)$	$2(n-4)$
K_n	0	$2(n-2)$	n
K_{n_1,n_2}	$2(n-2)$	0	$2(n-4)$
$CP_{n/2}$	n	$2(n-4)$	0
Remarks	for $P_n, Z_n : n \geqslant 6$ for $C_n : n \geqslant 4$ for $W_n : n \geqslant 7$ for $K_{n_1,n_2} : 2 \leqslant n = n_1 + n_2$ for $CP_{n/2}$: n even	$n_1 + n_2 = n$ for $P_n, C_n, Z_n, W_n : n \geqslant 5$ for $K_n : n \geqslant 2$ for $CP_{n/2}$: n even and $n \geqslant 6$	n even for $P_n, C_n, Z_n, W_n : n \geqslant 6$ for $K_{n_1,n_2} : 6 \leqslant n = n_1 + n_2$

where the last equality follows from $\sum_{i=1}^{n} \lambda_i(G) = 0$.
(ii) Here we have

$$\sigma(K_{n_1,n_2}, G) = E(G) - \lambda_1(G) + \lambda_n(G) + |\lambda_1(G) - \sqrt{n_1n_2}| + |\lambda_n(G) + \sqrt{n_1n_2}|.$$

Discussing the possible values of $\sqrt{n_1n_2}$ we get the above equalities. □

The next theorem concerns the remaining spectral distances between the graphs of Table 1.

Theorem 3.5. *The spectral distances between the graphs of Table 2 are given in the corresponding columns. Some restrictions (eliminating certain impossible cases or distances that are already computed earlier) are listed at the bottom of each column.*

Proof. Each distance is computed directly using Table 1. For computing spectral distances for graphs K_n and K_{n_1,n_2}, respectively, Theorem 3.4 is exploited, while the property of bipartiteness simplified some expressions. □

By considering Table 2, one can conclude that certain spectral distances have very simple expressions. Note that $\sigma(CP_{n/2}, W_n) = \sigma(CP_{n/2}, K_{n_1,n_2})$. Additionally, if $n \equiv_4 0$ we get $\lambda_{n/2}(C_n) = 0$, which means that the spectral distance between $CP_{n/2}$ and C_n is also equal to the previous two. Some similar conclusions can be derived by analyzing the results obtained.

In addition, we shall consider relatively small spectral distances. Namely, the problem posed in [3] consists of evaluation of the non–negative bound ϵ such that $\sigma(G_1, G_2) \leqslant \epsilon$. In that case we can say that G_1 and G_2 are ϵ–cospectral. Clearly, G_1 and G_2 are 0–cospectral if and only if they have common spectrum, while G_1 and G_2 are *almost cospectral* if the bound ϵ is sufficiently small. Some of candidates for such a bound follow.

Concerning the results from the previous sections, we immediately obtain some examples. Namely, from Lemma 2.2 we get that $\sigma(G_1, G_2) \leqslant \lambda_1(G) - \lambda_n(G)$ whenever G_1 and G_2 are vertex–deleted subgraphs of G. In other words, if $\lambda_1(G) \leqslant \epsilon/2$, we have $\sigma(G_1, G_2) \leqslant \epsilon$. Additionally, if G has a small number of distinct eigenvalues, say k, then $\sigma(G_1, G_2)$ reduces to the sum of distances between at most $k+2$ corresponding eigenvalues. It seems that, in these cases, $\sigma(G_1, G_2)$ can be much smaller than

$\lambda_1(G) - \lambda_n(G)$. Small spectral distances can also be obtained by applying Theorem 2.1 to the specially chosen graphs.

Some infinite families of graphs whose spectral distance is less than or equal to 1, 2 or 3 are given in Theorems 3.1–3.5, while the following lemma refer to the non–cospectral graphs having the arbitrary small distances.

Lemma 3.1. *$\sigma(K_{n_1,n_2}, K_{n_1-l,n_2} + lK_1) \leqslant \epsilon$ whenever $n_1 \geqslant \frac{(\epsilon^2+4ln_2)^2}{16\epsilon^2 n_2}$.*

Proof. We have

$$\sigma(K_{n_1,n_2}, K_{n_1-l,n_2} + lK_1) = 2\left(\sqrt{n_1 n_2} - \sqrt{(n_1 - l)n_2}\right) \leqslant \epsilon.$$

Solving the previous inequality we obtain the result. □

So, there are graphs having very small distances. Moreover, if (in the previous lemma) l and n_2 are fixed, we have $\sigma(K_{n_1,n_2}, K_{n_1-l,n_2} + lK_1) \to 0$ when $n_1 \to \infty$. Consequently, the following question arises: what is the good choice for ϵ so that we can say that the corresponding graphs are almost cospectral? For example, if we take $\epsilon = 1$, we get that the graphs (with n vertices) are almost cospectral if the average distance between the corresponding eigenvalues does not exceed $1/n$. The future research could provide some other bound or improve this one. Regarding Table 3 (Section 5), $1/2$ is one of candidates for ϵ, as well.

4. Cospectrality measure, spectral eccentricity and spectral diameter

Let X denotes an arbitrary subset of a set of graphs on n vertices. Then, the *cospectrality* of $G \in X$ is defined by (cf. [3])

$$\mathrm{cs}_X(G) = \min\{\sigma(G,H) : H \in X,\ H \neq G\},$$

followed by an additional function (*cospectrality measure*)

$$\mathrm{cs}(X) = \max\{\mathrm{cs}_X(G) : G \in X\},$$

measuring how far apart the spectrum of a graph in X can be from the spectrum of any other graph belonging to the same set. Both functions can be considered in the light of the results concerning the small spectral distances obtained in the previous section. Here we define the *spectral eccentricity* and the *spectral diameter* of $G \in X$ by

$$\mathrm{secc}_X(G) = \max\{\sigma(G,H) : H \in X\}$$

and

$$\mathrm{sdiam}(X) = \max\{\mathrm{secc}_X(G) : G \in X\},$$

respectively. Clearly, the spectral diameter of a set of graphs measures how large can be the spectral distance between two graphs belonging to it.

We shall consider the above functions in the following example.

Example 5. Let $X = \{G_0, \ldots G_n\}$, where $G_0 = CP_n$, while G_i is obtained by inserting exactly i edges into G_0. Thus, $G_n = K_{2n}$.

The graph G_i ($i = 1, \ldots, n$) is in fact equal to $K_{2i} \nabla CP_{n-i}$. Using the formula for the complete product of graphs (cf. [1], Theorem 2.7), we get

$$\begin{aligned} P_{G_i}(x) &= (x-2i+1)(x+1)^{2i-1}\left(x(x+2)\right)^{n-i} + (x-2(n-i)+2)x^{n-i}(x+2)^{n-i-1}(x+1)^{2i} \\ &\quad - (x+1)^{2i}\left(x(x+2)\right)^{n-i} \\ &= (x+2)^{n-i-1}(x+1)^{2i-1}x^{n-i}\left(x^2+(3-2n)x+2(1-n-i)\right). \end{aligned}$$

The last factor gives the remaining two eigenvalues of G_i ($0 \leqslant i < n$): λ_1 and λ_{n+i+1}. Let $0 \leqslant i < j \leqslant n$. Taking into account that $\lambda_1(G_i) + \lambda_{n+i+1}(G_i) = \lambda_1(G_j) + \lambda_{n+j+1}(G_j) = 2n-3$, whenever $j < n$ (and using Theorem 3.4 (i) if $j = n$), we get

$$\sigma(G_i, G_j) = 2(j-i).$$

Thus, we compute

$$\mathrm{cs}_X(G_i) = 2, \text{ for any } i = 1, \ldots, n, \text{ and therefore } \mathrm{cs}(X) = 2.$$

And similarly,

$$\mathrm{secc}_X(G_i) = \max\{2i, 2(n-i)\} \text{ for } i = 1, \ldots, n, \text{ and therefore } \mathrm{sdiam}(X) = 2n.$$

In other words, the cospectrality measure is constant for any n, while the spectral diameter is equal to the spectral distance between CP_n and K_{2n}.

Consideration of the above functions for special classes of graphs and obtaining some bounds for cospectrality measure and spectral diameter should be interesting topics for future researches.

5. Computational results and some conjectures

Here we give some computational results, corresponding comments and some arising conjectures. We shall consider:

(a) the class of all graphs,
(b) the class of connected regular graphs,
(c) the class of connected bipartite graphs, and
(d) the class of trees

on n vertices (where n is relatively small). We include only those orders n for which the corresponding class contains at least 3 graphs. We have calculated all the distances between these graphs and the results are presented in Table 3: first column contains the least spectral distance (different from zero), second column contains the largest spectral distance, the third column involves the number of pairs of (nonisomorphic) cospectral graphs, while the remaining four columns contain the number of spectral distances belonging to the specific numerical ranges.

A graph having the energy greater than $2n-2$ is called hyperenergetic. There are no such graphs for $n \leqslant 7$. On the other hand, there are exactly 20 hyperenergetic graphs of order 8. In the last column of Table 3, 20 of 29 spectral distances among the graphs of order 8 from the class (a) are distances between one of graphs from this class and $8K_1$. The remaining 9 spectral distances are equal to 14, and only 2 of them do not include $8K_1$. If we continue analysis of the graphs whose spectral distance is greater than $2n-2$, we shall observe that such pairs of graphs exist in the class (b). There are 32 (resp. 186) such pairs of graphs of order 10 (resp. 11), and the complete graph belongs to each of these pairs.

The remaining observations correspond to some of the following conjectures.

Conjectures (all posed by the second author – Z.S.[3]):

[3] Conjecture 1 is assumed by R.A. Brualdi, as well (private communication).

Table 3
The data on spectral distances concerning some special classes of graphs.

n	$\min\sigma \neq 0$	$\max\sigma$	$\sigma = 0$	$0 < \sigma \leqslant 0.5$	$0 < \sigma \leqslant 1$	$0 < \sigma \leqslant 2$	$2n-2 \leqslant \sigma$
				Graphs			
3	0.83	4	0	0	1	4	1
4	0.54	6	0	0	5	29	1
5	0.47	8	1	2	33	211	1
6	0.30	10	5	26	395	3214	1
7	0.22	12	58	345	9283	112937	2
8	0.12	14.33	963	13862	544525	12244923	29
				Connected regular graphs			
6	2	8	0	0	0	1	0
7	4	11.21	0	0	0	0	0
8	2	13.65	0	0	0	1	0
9	0.93	16	0	0	2	16	2
10	0.90	20	4	0	14	514	34
11	0.52	22.45	28	0	204	8636	186
				Connected bipartite graphs			
4	0.54	2	0	0	1	3	0
5	0.70	3.43	0	0	3	7	0
6	0.30	6	0	7	21	78	0
7	0.23	7.59	0	25	134	448	0
8	0.13	10	8	244	1449	6155	0
9	0.10	11.96	102	2136	16071	85262	0
10	0.06	14.94	690	29396	294681	2075391	0
11	0.05	16.42	6416	505931	6985367	$> 10^8$	0
				Trees			
5	0.70	2.54	0	0	1	2	0
6	0.48	4.25	0	1	3	10	0
7	0.42	5.56	0	1	12	32	0
8	0.32	7.29	1	10	48	139	0
9	0.25	8.68	5	37	184	539	0
10	0.20	10.38	4	119	750	2577	0
11	0.17	11.79	33	418	2995	11786	0

1. *The spectral distance between any two graphs of order n does not exceed* $E_n^{\max} = \max\{E(G), G$ has n vertices$\}$.

The computational results confirm this assumption for $n \leqslant 8$. Clearly, we have $\sigma(nK_1, G) \leqslant E_n^{\max}$, for any graph G on n vertices. Is it true that $\sigma(G, \widehat{G}) \leqslant E_n^{\max}$, where $E(\widehat{G}) = E_n^{\max}$? Even if it is true, the conjecture remains open.

2. *Let R_1 and R_2 be the graphs having the maximum spectral distance among the connected regular graphs of order n. Then one of them is K_n.*

The computational results confirm this assumption for $n \leqslant 11$. The same results show that there is no some clear rule for the other graph. In some cases, the maximum spectral distance is attained when the other graph is a cycle, while for $n = 10$ it is the Petersen graph. Also, in some cases there is more than one such graph. So, there is another intriguing problem: if one graph is K_n, describe the other.

3. *Let B_1 and B_2 be the graphs having the maximum spectral distance among the connected bipartite graphs of order n. Then one of them is $K_{\lceil n/2\rceil, \lfloor n/2\rfloor}$.*

The computational results confirm this assumption for $n \leqslant 11$. Like in the previous case, there is no clear rule for the other graph.

4. *The spectral distance between any two trees of order n does not exceed $\sigma(P_n, K_{n-1,1})$.*

The computational results confirm this assumption even for $n \leqslant 16$. We can provide the following lemma as well.

Lemma 5.1. *If T_n is an arbitrary tree of order n, then $\sigma(T_n, K_{n-1,n}) \leqslant \sigma(P_n, K_{n-1,1})$.*

Proof. Having in mind that $\lambda_1(P_n) \leqslant \lambda_1(T_n) \leqslant \lambda_1(K_{n-1,1})$ and $E(P_n) \geqslant E(T_n)$ (see [2]) hold for any n, we get

$$\begin{aligned}\sigma(T_n, K_{n-1,1}) &= 2(\sqrt{n-1} - 2\lambda_1(T_n)) + E(T_n) \leqslant 2(\sqrt{n-1} - 2\lambda_1(P_n)) + E(P_n)\\ &= \sigma(P_n, K_{n-1,1}). \quad \square\end{aligned}$$

5. $0.945 \approx \lim_{n\to\infty} \sigma(P_n, Z_n) = \lim_{n\to\infty} \sigma(W_n, Z_n) = \frac{1}{2}\lim_{n\to\infty} \sigma(P_n, W_n)$; $\lim_{n\to\infty} \sigma(C_{2n}, Z_{2n}) = 2\ (= \sigma(C_{2n-1}, Z_{2n-1}))$.

This conjecture can be considered as an extension of the results given in Theorem 3.5. The corresponding limit points were not important for our research, but the computational results suggest that they are correct.

6. *For any $\epsilon > 0$, there are graphs G_1 and G_2 having no common eigenvalues such that $\sigma(G_1, G_2) < \epsilon$ holds.*

Lemma 3.1 gives the graphs having an arbitrary small spectral distance. The same can be proven for, say, $K_{n+1,n}$ and $K_{n,n-1}$, but all these graphs have 0 as an eigenvalue of the large multiplicity. On the other hand, the spectra of P_{2n-1} and C_{2n-1} have no any common eigenvalue and, due to Theorem 3.1, $\sigma(P_{2n-1}, C_{2n-1}) = 2$. So, their distance is small, but not arbitrarily small. Finally, regarding Table 3, the least distance (different from zero) decreases in n, and so this conjecture could be true.

If $\mathcal{G}_n$ (resp. $\mathcal{R}_n$, $\mathcal{B}_n$ and $\mathcal{T}_n$) denotes the set of all graphs (resp. all connected regular graphs, all connected bipartite graphs and all trees) with n vertices, then Conjectures 1–4 can be stated in the following form.

Is it true that

1'. $\mathrm{sdiam}(\mathcal{G}_n) = E_n^{\max}$?
2'. $\mathrm{sdiam}(\mathcal{R}_n) = \mathrm{secc}_{\mathcal{R}_n}(K_n)$?
3'. $\mathrm{sdiam}(\mathcal{B}_n) = \mathrm{secc}_{\mathcal{B}_n}(K_{\lceil n/2\rceil,\lfloor n/2\rfloor})$?
4'. $\mathrm{sdiam}(\mathcal{T}_n) = \sigma(P_n, K_{n-1,1})$?

Acknowledgement

The work is supported by Serbian Ministry of Education and Science, Projects 174012 and 174033.

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