Nonperturbative Physical Analysis of Stochastic Equations

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Abstract

Arbitrary stochastic partial differential equations from a nonperturbative framework is investigated by a variational method that goes beyond the renormalization group analysis. The method first casts the stochastic equations into a functional integral form, then it makes use of the effective action to get a closed dimensional analytical expression of the field-field correlation function and the Gaussian effective potential to obtain, respectively, quantitative and qualitative physical information of a SPDE. As a first and comparative example, we apply this method to the Kardar-Parisi-Zhang equation and find the correlation function of this nonlinear and out of equilibrium system. When the field is assumed being self-affine, the scale invariance symmetry can be used to get the universal critical behavior of the system.

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1 Introduction

The purpose of this paper is to propose an advance on description of nonlinear and nonequilibrium systems with many degrees of freedom represented by stochastic partials differentials equations (SPDE's). Systems of this kind are important due the proximity of the complexity from the natural phenomena like turbulence[1, 2], pattern formation, driven diffusion, surface growth, polymers in a random media[4], bacterial colony growth[3], tumors, the formation of clounds in the upper atmosphere, some models for universe expansion and many other systems that can be modeled through SPDE's [5]. Note the diversity of knowledge that includes physics, biology, chemistry and technological applications.

In general there is not an analytical solution for a SPDE's, then we have to analyse it from other points of view like exploring the symmetries of the system. The method, that will be explained in the next sections, give us both qualitative and quantitative physical information and, in some cases, when the scalar field representing some natural variable

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(like a surface heigth, a velocity field in a fluid flow) is a self-affine function, the method can give us universal statistical properties of the system like the dynamical exponent z and the roughening exponent α or χ (both symbols are in the literature) in any space-time Euclidean dimensions through a scaling invariance, that means that the field fluctuations look statiscally the same when viewed at different length scales. This information plus the possible phase transitions(or spontaneus symmetry breaking SSB) that can be detected from the Gaussian effective potential (GEP)[6], can give us a reliable and complete system description of the SPDE's. The method uses a simple and nonperturbative variational approach in the effective action [7] of a particular system to find the field-field correlation function and the GEP to try to obtain a quantitative and qualitative information, respectively.

So, the correlation function(or propagator) gives the space-time influence between two points of the field and, when it is self-affine, we can use the scaling relation

$$C(r,t) = r^{2\alpha} f(\frac{t}{r^z}) \,, \tag{1}$$

to get the critical exponents. It is worth emphasizing the enormous simplification that occurs in the scaling description. There are only a very few quantities that define the asymptotic behavior of the system.

The variational approach has presents many advantages when compared with the renormalization-group(RG) approach. The main limitation of RG approach is due to the fact that the perturbative expansion parameter is, in general, the coupling constant of the nonlinear term. So, the RG is restrict to weak coupling regime, not giving a clear qualitative behavior of the system in other regimes. Then, the acquisition of physical information becomes more difficult, speculative and, in many cases, numerical methods will become the only option available. In the variational approach, besides the fact that do not restrict only the weak coupling regime, the qualitative and quantitative information is, in general, beyond the first order of the perturbative approach without the necessity of *exhaustive calculations* which can generate publishing mistakes as, for example, Kay Wiese pointed in [8] about the 2-loop calculations of the KPZ equation.

A first aplication of the variational method was employed in this paper on the KPZ equation[9], and the correlation function found is

$$G(r,t) = \mathcal{A}\, K_d\, r^{\frac{2-d}{2}} \int_0^\infty dk\, k^{\frac{d}{2}} J_{\frac{d}{2}-1}(2kr) \frac{e^{-2t(k^2\bar{\Omega}^2+k^4\nu^2)^{\frac{1}{2}}}}{(k^2\bar{\Omega}^2+k^4\nu^2)^{\frac{1}{2}}} \,, \tag{2}$$

where $\mathcal{A}$ is the noise correlation amplitude, K_d is a dimensional constant, $J_{\frac{d}{2}-1}(2kr)$ is a dimensional Bessel function and $\bar{\Omega}$ is a variational parameter. The nonlinear coupling constant λ is inside the $\bar{\Omega}$ expression. So, when we put $\bar{\Omega} = 0$ we fall in the linear regime represented for the Edwards-Wilkinson (EW) equation[10]. The linear propagator regime is exactly equal to Nattermann[11]. We can't find such generalized expression through RG approach and for $d > 1$ the strong λ regime is out of reach because of the absence of any RG fixed point.

The next sections will show that the union of quantum field theory(QFT) techniques, variational calculus and self-affine transformations is a powerfull tool to analyse nonlinear and nonequilibrium stochastic systems.

This paper is organized as follows: In the next section will be show the general procedure to obtain the effective action $\Gamma[\phi]$ of a SPDE through a nonperturbative (variational)

approach. So, to obtain the GEP and the correlation function will be a consequence. The section 3 shows how we can get the universality class of self-affine systems from the correlation function. An aplication of the method is in section 4, where the KPZ system is investigated. A discussion of the most important points of the paper and many others aplications of the method will be in the last section.

2 The Method

In order to obtain the effective potential associated to a stochastic partial differential equation, it is first necessary to define what is the corresponding "action" . In this section we will review some of the definitions used to implement the functional integral formalism leading to this "action"(see also [12, 13]). In the expressions below we will use the following condensed notations, $f(\mathbf{x},t) \mapsto f(x)$, $d^d x dt \mapsto dx$.

Consider the general class of stochastic partial differential equations

$$D\phi(x) = F(\phi(x), \partial_i \phi(x)) + \eta(x) \,, \tag{3}$$

where D is any linear differential operator, involving first time derivatives and arbitrary space derivatives, which does *not* explicitly involve the field ϕ. The function $F(\phi, \partial_i \phi)$ is generally nonlinear in the field ϕ and in the space derivatives of it. The function $\eta(x)$ denotes the source of noise. The nature (white, power-law, colored, pink, $1/\mathbf{f}$-noise, or shot noise) and the probability distribution of the noise need not yet be specified. Let us then consider that given a particular configuration of the noise η, the differential equation (3) is assumed to have an unique solution $\phi_s(x|\eta)$. Then with the functional probability distribution[1]

$$\mathcal{P}[\phi] = \langle \delta[\phi(x) - \phi_s(x|\eta)] \rangle_\eta = \int (\mathcal{D}\eta)\; \delta[\phi(x) - \phi_s(x|\eta)]\; \mathcal{P}[\eta] \,, \tag{4}$$

we can make averages of functionals using

$$\langle \mathcal{O}[\phi, J] \rangle_\phi = \int (\mathcal{D}\phi)\, \mathcal{O}[\phi, J]\, \mathcal{P}[\phi] \,. \tag{5}$$

It is convenient to do an algebraic manipulation in (4) to take off the explicit dependence on the unique solution ϕ_s, due to fact that, in general, the SPDEs do not have solutions that can be written as a closed analytical form. Therefore, we make use of the following functional delta identity

$$\delta[Q(\phi)] = \frac{\delta[\phi - \phi_s(x|\eta)]}{|det(\frac{\delta Q(\phi)}{\delta \phi})|} \,, \tag{6}$$

where $G(\phi) \equiv D\phi - F(\phi, \partial_i \phi) - \eta = 0$ and the Jacobian functional determinant is defined by

$$\mathcal{J} \equiv det(\frac{\delta Q(\phi)}{\delta \phi}) \,. \tag{7}$$

With this, one eliminates the necessity of knowing the analytical formula of $\phi_s(x|\eta)$ and now, the conection with quantum field theory can be done through the analogy of the expression (5) with the generating functional $Z[J]$ when $\mathcal{O}[\phi, J]$ assumes the form

$$\mathcal{O}[\phi, J] = \exp\left(\int dx\; J(x)\; \phi(x) \right) \,. \tag{8}$$

[1]See chapter 4 of Zinn-Justin() for details about the conservation of probability $\int (\mathcal{D}\phi) \dot{\mathcal{P}}[\phi] = 0$

Then, making the delta integral functional over the noise,

$$\begin{aligned} Z[J] &\equiv \left\langle \exp\left(\int dx\ J\ \phi\right)\right\rangle_{\phi} \\ &= \int (\mathcal{D}\phi)\ \mathcal{P}[D\phi - F(\phi,\partial_i\phi)]\ \exp\left(\int dx\ J\ \phi\right)\ |\mathcal{J}|\ . \end{aligned} \tag{9}$$

This key result will enable us to calculate the correlation function and the effective potential in a direct way. At this stage we will make some assumptions about the noise. We assume that the noise probability distribution is Gaussian with zero mean so that the only non-vanishing cumulant is the second order one. If the noise has a non-zero mean, one can always redefine the forcing term $F[\phi,\partial_i\phi]$ to make the noise be of zero mean without loss of generality. Arbitrary Gaussian noise is enough since it allows us to write the noise probability distribution as

$$\mathcal{P}[\eta] = \exp\left[-\frac{1}{2}\int\int dx\ dy\ \frac{\eta(x)\ \eta(y)}{\langle\eta(x)\eta(y)\rangle}\right]\ , \tag{10}$$

where

$$\begin{aligned} \langle\eta(x)\eta(y)\rangle &= \frac{1}{N}\int(\mathcal{D}\eta)\,\eta(x)\,\eta(y)\exp\left(-\frac{1}{\mathcal{A}}\int\int dx'dy'\,\eta(x')\,g^{-1}(x',y')\,\eta(y')\right) \\ &= \mathcal{A}\,g(x,y)\,, \end{aligned} \tag{11}$$

is the two-point noise correlation function. The correlation amplitude $\mathcal{A}$ will tell us if one is in the regime of absence of noise correlations ($\mathcal{A}=0$) or not ($\mathcal{A}\neq 0$).[2]

In general, just the presence of an amplitude of noise correlation is enough to get qualitative information though the GEP. So the white noise ($g(x,y)=\delta(x-y)$) is the best choice due the algebraic simplification. However, when we pursue quantitative information like the statistical behavior, the choice of the probability distribution $\mathcal{P}[\eta]$ and $g(x,y)$ function can alter the universal behavior of the critical exponents. Then, the proximity between theoretical and experimental data, in general, will depend of this choice. In [14], Zhang proposed that the discrepancy between the experiments and theory is due to the fact that in nature there is no Gaussian noise, as it is usually assumed in theoretical models. As an example, Zhang shows in a biological growth problem (using computational analysis) that microscopic details can indeed influence large scale behavior in a substantional way, thus violating the naive universality concept. As the functional integrals have a very limited class of solutions, we must keep the Gaussian probability distribution to perform the variational approach, but the function $g(x,y)$ is arbitraly.

So, the generating functional is

$$\begin{aligned} Z[J] = \quad & \int(\mathcal{D}\phi)\,\exp\left(\int dx\,J\,\phi\right)\,|\mathcal{J}|\,\times \\ & \times\exp\left[-\frac{1}{2\mathcal{A}}\int\int dx\,dy\,(D\phi - F(\phi,\partial_i\phi))\,g^{-1}(x,y)\,(D\phi - F(\phi,\partial_i\phi))\right]\ . \end{aligned} \tag{12}$$

This expression contains all the physics of (3). The noise η has been completely eliminated and survives only through the explicit appeareance of its two-point function. Since the

[2]In the perturbative approach, $\mathcal{A}$ is a loop counting parameter, analogous to $\hbar$ in quantum field theory. So the 1-loop analysis is restrict to weak amplitudes of the correlation function.

generating functional is now given as a functional integral over the physical field ϕ, all the standard machinery of statistical and quantum field theory can be brought to bear. So, modulo the determinant of the Jacobian, the physics of a stochastic partial differential equation can be extracted from a functional integral based on a "classical action" $S[\phi, J]$, the exponent in (12) , such that

$$S[\phi, J] = \frac{1}{2\mathcal{A}} \int\int dx\, dy\, (D\phi - F(\phi, \partial_i\phi))\, g^{-1}(x, y)\, (D\phi - F(\phi, \partial_i\phi)) + \int dx\, J\, \phi\,. \quad (13)$$

Once obtaining the expression for the action, we can use quantum field theory techniques to analyse the physics of a particular SPDE. First we will make in the action the following shift on the field

$$\phi(x) \to \hat{\phi}(x) + f(\phi_0). \quad (14)$$

This shift is done to explore the SPDE's symmetries, which will enable more physical information of the system, and to introduce the constant field ϕ_0 which in the quantum field theory context can be identified as being the classical field. The function $f(\phi_0)$ and the interpretation of ϕ_0 will depend of a particular mapping of a stochastic partial differential equation.

After this field transformation, it is convenient to split the action S into two parts. The Gaussian part S_G defined as

$$S_G[\hat{\phi}] = \frac{1}{2} \int\int dx\, dy\, \hat{\phi}(x)\, G^{-1}(x, y)\, \hat{\phi}(y)\,, \quad (15)$$

where $G(x, y)$ is the field-field correlation function, and the "interaction" part S_I. The variational calculus is introduced through the maximization of the effective action with respect to the correlation function in the Fourier space $\tilde{G}(k)$ with the purpose of obtains the optimum expression of the correlation function for the desired system. Thus, the entire action becomes

$$S[\hat{\phi}, J, \phi_0] = S_G[\hat{\phi}] + S_I[\hat{\phi}, J, \phi_0]\,. \quad (16)$$

The variational principle beneath our approach comes from the functional generalization of the Jensen inequality[15] $\langle \exp f \rangle \geq \exp \langle f \rangle$, i.e,

$$\frac{Z[J]}{N} \geq \exp[\frac{1}{N} \int \mathcal{D}\hat{\phi}\, (S_G - S)\, \exp(-S_G)]\,, \quad (17)$$

where the normalization factor is

$$N = \int \mathcal{D}\hat{\phi}\, \exp(-S_G)\,. \quad (18)$$

The generating functional of connected Green's functions is given by $W[J] = -\ln Z[J]$. So,

$$W[J] \leq -\ln N - \frac{1}{N} \int \mathcal{D}\hat{\phi}\, (S_G - S)\, \exp(-S_G). \quad (19)$$

The term $-\ln N$ can be written in a more convenient form in the Fourier basis

$$-\ln N = \frac{1}{2} \ln \det \tilde{G}^{-1}(k) = \frac{1}{2} Tr \ln \tilde{G}^{-1}(k) = \frac{\mathcal{V}}{2} \int dk\, \ln \tilde{G}^{-1}(k)\,, \quad (20)$$

where $\mathcal{V}$ is the space-time volume ($\int dx$). A more useful object, however, is the effective action which is defined as the functional Legendre transform of $W[J]$,

$$\Gamma[\phi_0] = \int dx\, J\phi_0 - W[J]\,. \tag{21}$$

Then, the inequality (19) becomes

$$\Gamma[\phi_0] \leq \ln N + \langle S_G - S\rangle_\phi \,/_{J=0}\,. \tag{22}$$

The right hand side of (22) can be calculated analytically and the variational approach will consist to minimize it with respect to the function $\tilde{G}(k)$ and to obtain the best value of the effective action[7], called of Gaussian effective action(GEA)

$$\overline{\Gamma}_G[\phi_0] \equiv \min_{\tilde{G}} \left[-\frac{\mathcal{V}}{2} \int dk\, \ln \tilde{G}^{-1} + \langle S_G - S\rangle_\phi \,/_{J=0} \right]. \tag{23}$$

The Gaussian effective potential is

$$\overline{V}_G(\phi_0) \equiv -\frac{\overline{\Gamma}_G[\phi_0]}{\mathcal{V}} = \min_{\tilde{G}} \left[\frac{1}{2} \int dk\, \ln \tilde{G}^{-1} + \frac{1}{\mathcal{V}} \langle S_I\rangle_\phi \,/_{J=0} \right]. \tag{24}$$

Through the equations (23) and (24), we are able to get physical information of any system modeled for a SPDE of the type $D\phi(x) = F(\phi(x), \partial_i\phi(x)) + \eta(x)$. To obtain qualitative information, we must explore the symmetries of a particular problem through the GEP. In general, the system will show a broken of stable or metastable configurations, giving phase transitions through the change of the coupling parameters. This type of analysis is possible because the systems described for SPDE's are always in evolution, i.e, we have the presence of nonequilibrium space pattern. So, it is natural that continuous symmetries will be broken, and effective potentials are a powerfull tool to analyse spontaneus symmetry breaking. As an example, in [16] we found a SSB in the (1+1), (2+1), and (3+1) KPZ system using the GEP. This SSB corresponds to a morphological transition on the surface after the deposition of the particles to increase the surface tension. In the next section, we will show how to get quantitative information from a stochastic equation.

3 Self-Affine Systems

The most important information of this paper is the two point field correlation function of a SPDE, and it is done with a great simplicity. In fact, using another methods, it is not possible, in general, to obtain a closed analytical expression of the CF from nonlinear and far from equilibrium systems at any space-time Euclidean dimensions. Here, we have just to minimize the Gaussian effective action (23) with respect to the function $\tilde{G}(k')$

$$\frac{\delta\Gamma_G[\phi_0]}{\delta\tilde{G}(k')} = 0\,, \tag{25}$$

and to isolate the optimum value of the correlation function. A lot of physical information can be extract from it and the most significative will be when the system is self-affine, because we can explore another simmetry called invariance under an anisotropic transformation, i.e, we have different scaling factors in different spatial directions, so $b\cdot\phi = (b_1x_1, b_2x_2, b_3x_3, ...)$ and invariance means that the rescaled function $b\cdot\phi$ is statisticaly identical with a part of the original function $\phi(x_1, x_2, x_3, ...)$. So there exists a generalized scaling

$$G(x,t) = b^{-2\alpha} G(b\,x, b^z t)\,, \tag{26}$$

where b is a scale factor, and α and z are known as the roughening and dynamic exponents which are also universal. As a direct consequence of (26), a scaling form for $G(x,t)$ can be obtained by choosing $b = \frac{1}{x}$

$$G(x,t) = x^{2\alpha}\, G(\frac{t}{x^z})\,, \tag{27}$$

a form that also explains the origin of the name "dynamic exponent" for z.

Then, all that we must do is to use (25) to get the correlation function and make an analogy with (27) to obtain the critical exponents at any Euclidean dimensions, which means universal statistical physical information. As we will show in the next section, we must transform the CF back to the coordinate space and this can make the analogy not so easy, but this quantitative information plus the qualitative one (GEP), go beyond of any other theoretical methods (like RG) to extract physical information from nonlinear SPDE's.

4 Aplication: The KPZ Equation

The Kardar-Parisi-Zhang equation

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\phi(x) = F_0 + \frac{\lambda}{2}(\nabla\phi)^2 + \eta(x)\,, \tag{28}$$

is a natural extension of the Edwards-Wilkinson model constructed taking to account the relevants symmetry principles to growth phenomena description[5]. In this paper we will restrict our analysis to the surface growth interpretation, but the KPZ equation can be mapped as a model for turbulence, structure development in the early universe, polymers in a random media, driven diffusion, flame fronts and others. The self-affine field $\phi(x)$ is taken to be the height of the surface (typically defined over a plane). The nonlinear term increases the height of the interface by adding more 'material' to the parts of the surface where the local slope is larger, and the average height of the surface increases. This situation may be contrasted to the effect of the linear term, which reorganizes the surface height in such a way that the total mass remains unchanged. The constant term F_0 can be seen as a consequence of the transformation $\phi \to \phi + F_0 t$ which represents just a change in the average velocity of the surface growth with respect to the laboratory frame of reference.

The KPZ equation is usually described in the literature with a white noise for symmetry reasons, so

$$\langle\eta(x)\eta(y)\rangle \equiv \mathcal{A}\,\delta(x-y)\,. \tag{29}$$

The action (13) becomes

$$S[\phi] = \frac{1}{2\mathcal{A}}\int dx\,(D\phi - F(\phi,\partial_i\phi))^2 = \frac{1}{2\mathcal{A}}\int dx\,\left[\dot{\phi} - \nu\nabla^2\phi - F_0 - \frac{\lambda}{2}(\nabla\phi)^2\right]^2\,, \tag{30}$$

and the shift on the field (14) is $\phi(x) \to \hat{\phi}(x) + \vec{\phi_0}\cdot\mathbf{x}$, with $\vec{\phi_0}$ being a N-dimensional vector such that $\vec{\phi_0} \equiv (\phi_0, ..., \phi_0)$. So,

$$\begin{aligned} \dot{\phi}(x) &= \dot{\hat{\phi}}(x)\,, \\ \nabla^2\phi(x) &= \nabla^2\hat{\phi}(x)\,, \\ \nabla\phi(x) &= \nabla\hat{\phi}(x) + \vec{\phi_0}\,. \end{aligned} \tag{31}$$

To perform the functional integral in the Gaussian effective action (23), we must go to the Fourier space such as

$$\begin{aligned} S[\tilde{\phi},J] &= \frac{1}{2\mathcal{A}}\left[\int \frac{dk}{(2\pi)^{d+1}}(-\omega^2)\tilde{\phi}^2+\nu^2\int \frac{dk}{(2\pi)^{d+1}}\mathbf{k}^4\tilde{\phi}^2\right] \\ &+ \left[F_0\lambda\int \frac{dk}{(2\pi)^{d+1}}\mathbf{k}^2\tilde{\phi}^2+\frac{\lambda^2}{4}(\int \frac{dk}{(2\pi)^{d+1}}\mathbf{k}^2\tilde{\phi}^2)^2+\frac{3}{2}\lambda^2\phi_0^2\int \frac{dk}{(2\pi)^{d+1}}\mathbf{k}^2\tilde{\phi}^2\right] \quad (32) \\ &+ \left[\mathcal{V}\,F_0\,\lambda\,\phi_0^2+\mathcal{V}\,\frac{\lambda^2}{4}\phi_0^4+\,odd\,terms\right]+\int dx\,\hat{\phi}\,J\,, \end{aligned}$$

and

$$S_G[\tilde{\phi}]=\frac{1}{2}\int \frac{dk}{(2\pi)^{d+1}}\,\tilde{\phi}(k)\,\tilde{G}^{-1}(k)\,\tilde{\phi}^*(k)\,, \quad (33)$$

where the odd terms were put apart because they vanish when we perform the field and momentum odd integrations. After performing functional integrals of the type

$$\frac{1}{N}\int \mathcal{D}\tilde{\phi}\,\tilde{\phi}^2\,\exp(-S_G)=\tilde{G}(k)\,, \quad (34)$$

we can write the Gaussian effective action (23) as

$$\begin{aligned} 2\mathcal{A}\,\Gamma_G[\phi_0] &= -\mathcal{A}\mathcal{V}\int dk\,\ln\tilde{G}^{-1}+\int \frac{dk}{(2\pi)^{d+1}}\,\omega^2\tilde{G}\,-\nu^2\int \frac{dk}{(2\pi)^{d+1}}\,\mathbf{k}^4\tilde{G} \\ &- F_0\lambda\int \frac{dk}{(2\pi)^{d+1}}\,\mathbf{k}^2\tilde{G}\,-\frac{3}{4}\lambda^2(\int \frac{dk}{(2\pi)^{d+1}}\,\mathbf{k}^2\tilde{G})^2 \quad (35) \\ &- \frac{3}{2}\lambda^2\phi_0^2\int \frac{dk}{(2\pi)^{d+1}}\,\mathbf{k}^2\tilde{G}\,+constant\,. \end{aligned}$$

Now, using (25), we have the optimum correlation function

$$\tilde{G}(k)=\frac{\mathcal{A}(2\pi)^{d+1}\,\delta^{d+1}(0)}{(\nu\mathbf{k}^2-i\omega)(\nu\mathbf{k}^2+i\omega)+\mathbf{k}^2\,\bar{\Omega}^2}\,, \quad (36)$$

or

$$\tilde{G}(k,k')=\frac{\mathcal{A}(2\pi)^{d+1}\,\delta^d(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega')}{(\nu\mathbf{k}^2-i\omega)\,(\nu(\mathbf{k}')^2-i\omega')+\mathbf{k}^2\,\bar{\Omega}^2}\,, \quad (37)$$

where was used $\mathcal{V}=(2\pi)^{d+1}\,\delta^{d+1}(0)$ and the variational parameter $\bar{\Omega}$ is

$$\bar{\Omega}^2=F_0\lambda+\frac{3}{2}\lambda^2\left[\left(\int \frac{dk}{(2\pi)^{d+1}}\,\mathbf{k}^2\tilde{G}(k)\right)+\phi_0^2\right]. \quad (38)$$

The $\bar{\Omega}$ -equation have a divergent integral and it will be renormalized in the appendix using F_0 as a bare parameter. The renormalized expression is

$$\bar{\Omega}^2=\frac{3}{2}\lambda^2\left(\frac{\mathcal{A}}{\nu^2}C_d\,\bar{\Omega}^d+\phi_0^2\right)\,, \quad (39)$$

where C_d is a dimensional constant. It is important to note that the correlation function has a term of λ^2, i.e, go beyond the first order in a nonperturbative expression, and in the linear limit $\lambda\to 0$, we recover the exactly CF of the Edwards-Wilkinson equation [5, 11]

$$\tilde{G}(k,k^{'}) = \frac{\mathcal{A}(2\pi)^{d+1}\,\delta^{d}(\mathbf{k}+\mathbf{k}^{'})\,\delta(\omega+\omega^{'})}{(\nu\mathbf{k}^{2}-i\omega)\,(\nu(\mathbf{k}^{'})^{2}-i\omega^{'})}\,. \tag{40}$$

Now we must to transform the CF (37) back to the coordinate space. Performing the ω integrals and the $\mathbf{k}^{'}$ integral, we have

$$G(r,t) = \frac{\mathcal{A}}{2}(2\pi)^{d+1}\int_{-\infty}^{\infty}\frac{d^{d}\mathbf{k}}{(2\pi)^{d}}\,\frac{e^{-2t(\mathbf{k}^{2}\bar{\Omega}^{2}+\mathbf{k}^{4}\nu^{2})^{\frac{1}{2}}}}{(\mathbf{k}^{2}\bar{\Omega}^{2}+\mathbf{k}^{4}\nu^{2})^{\frac{1}{2}}}\,e^{2i\mathbf{k}\cdot\mathbf{r}}\,. \tag{41}$$

This expression can be put in an elegant dimensional form performing an integration over a spherical shell with $\mathbf{k}\cdot\mathbf{r} = k\,r\,cos\theta$. So, carrying out the angular integrations yields

$$G(r,t) = \mathcal{A}\,K_{d}\,r^{\frac{2-d}{2}}\int_{0}^{\infty}dk\,k^{\frac{d}{2}}J_{\frac{d}{2}-1}(2kr)\frac{e^{-2t(k^{2}\bar{\Omega}^{2}+k^{4}\nu^{2})^{\frac{1}{2}}}}{(k^{2}\bar{\Omega}^{2}+k^{4}\nu^{2})^{\frac{1}{2}}}\,. \tag{42}$$

where K_d is a dimensional constant and $J_{\frac{d}{2}-1}(2kr)$ is a dimensional Bessel function. Although the growth process is local, through the lateral growth the information about the height of each of the neighbors spreads globally. When the entire interface becomes correlated, the interface width sature and the relaxation time τ depends on $k = |\mathbf{k}|$ as

$$\tau \propto \frac{1}{(k^{2}\bar{\Omega}^{2}+k^{4}\nu^{2})^{\frac{1}{2}}}\,. \tag{43}$$

So we can see that despite the leads of linear term, the nonlinear is significant and proportional to λ^2, which means that this nonperturbative method go beyond the RG analysis without the necessity of *exhaustive calculations*.

To extract the critical exponents from the linear regime $\lambda \to 0$, we have just to make the transformation $k \to \frac{k}{r}$ in (42) to get

$$G(r,t) = r^{2-d}f(\frac{t}{r^{2}})\,. \tag{44}$$

Comparing with (27), we have $\alpha = \frac{2-d}{2}$ and $z = 2$, that is the exactly critical exponents of the E-W equation. To do the same with the KPZ equation is not so easy...

5 Discussion

We have introduced in this paper a new nonperturbative method to get quantitative and qualitative physical information from nonlinear and out of equilibrium systems with many degrees of freedom represented for stochastic partial differential equations. We implement the functional integral formalism leading to a SPDE "action"(see also [12, 13]) and made use of the variational approach developed in [7] such that we obtain the Gaussian effective action for arbitrary SPDE's. So, we are able to obtain an analytical closed expression for the field-field correlation function and the Gaussian effective potential. If a particular system described for a stochastic equation have a self-affine function, the correlation function also can be used to analyse the statistical behavior through a scaling relation. As the systems described for SPDE's are always in evolution, i.e, we have the presence of nonequilibrium space pattern, it is natural that continuous symmetries will be broken and the Gaussian effective potential is a powerfull tool to analyse spontaneus symmetry breaking.

As a first example, we applied the method on the Kardar-Parisi-Zhang equation subject to a white noise. The most important result is the correlation function expression (in any space-time dimension) that in the linear limit $\lambda \to 0$ coincide exactly with of the Edwards-Wilkinson equation. A substantional nonlinear influence on the relaxation time was found.....

It is important to note that this nonperturbative method goes beyond the RG analysis and the possibility of further papers in this direction can have a diversity of knowledge that may includes physics, biology and chemistry.

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Appendix: The $\bar{\Omega}$-equation renormalization

Now, we will use the constant term F_0 as a bare parameter to make the $\bar{\Omega}$-equation renormalization. The divergent integral to be renormalized in (38) is

$$I_d(\bar{\Omega}) = \int \frac{dk}{(2\pi)^{d+1}}\, \mathbf{k}^2 \tilde{G}(k) = \int \frac{dk}{(2\pi)^{d+1}} \mathbf{k}^2 \frac{\mathcal{A}(2\pi)^{d+1}}{(\nu\mathbf{k}^2 - i\omega)(\nu\mathbf{k}^2 + i\omega) + \mathbf{k}^2\,\bar{\Omega}^2}\,. \tag{45}$$

Performing the integral in ω and after an integral over a spherical shell, we have

$$I_d(\bar{\Omega}) = \frac{\mathcal{A}}{\nu^2} C_d \int_0^\infty dk\, \frac{k^d}{(k^2 + \bar{\Omega}^2)^{\frac{1}{2}}}\,, \tag{46}$$

where $k = |\mathbf{k}|$ and C_d is a dimensional constant. Now we can use the "differentiate and integrate" trick process [17] to regularize the above expression with the conditions $\frac{d\, I_d(0)}{d\,\bar{\Omega}} = 0$ and $\frac{d^2\, I_d(0)}{d^2\,\bar{\Omega}} = infinity\ constant$. So, the $\bar{\Omega}$-equation stay

$$\bar{\Omega}^2 = F_0\lambda + \frac{3}{2}\mathcal{A}\frac{\lambda^2}{\nu^2} C_d(\bar{\Omega}^d + \Lambda_d) + \frac{3}{2}\lambda^2\phi_0^2\,, \tag{47}$$

where Λ_d is a dimensional infinity constant. Renormalizing the bare parameter F_0 as

$$F_0 = F_{renormalized} - \frac{3}{2}\mathcal{A}\frac{\lambda}{\nu^2} C_d \Lambda_d \tag{48}$$

and assuming $F_{renormalized} = 0$, we have the renormalized $\bar{\Omega}$-equation

$$\bar{\Omega}^2 = \frac{3}{2}\lambda^2 \left(\frac{\mathcal{A}}{\nu^2} C_d\, \bar{\Omega}^d + \phi_0^2\right)\,. \tag{49}$$

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