The Gaussian effective potential and stochastic partial differential equations

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Abstract

We investigate arbitrary stochastic partial differential equations subject to translation invariant and temporally white noise correlations from a nonperturbative framework. The method that we expose first casts the stochastic equations into a functional integral form, then it makes use of the Gaussian effective potential approach, which is an useful tool for describing symmetry breaking. We apply this method to the Kardar Parizi Zhang equation and find that the system exhibits spontaneous symmetry breaking in (1+1), (2+1) and (3+1) Euclidean dimensions, providing insight into the evolution of the system configuration due to the presence of noise correlations. A simple and systematic approach to the renormalization, without explicit regularization, is employed.

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1 Introduction

Among some of the fundamental tools for modeling systems, where noise is an essential ingredient, one finds stochastic partial differential equations (SPDE's). They may be applied in a multitude of situations such as turbulence, pattern formation, interface growth or driven diffusion [1, 2]. In the last few years, a close relationship between SPDE's and quantum field theory has emerged providing a way to investigate phenomena involving classical physics by techniques coming from quantum physics [3, 4, 5, 6, 7, 8, 9]. In particular, a perturbative method based on the one-loop effective potential (1-LEP) associated to arbitrary stochastic partial differential equations was developed [10]. This method has given some insight into spontaneous symmetry breaking and related phase structures of some interesting stochastic models [11],[12].

In this paper, our aim is to explore a *variational* framework, within the quantum field theory approach, in order to investigate stochastic partial differential equations. We adress this matter through the use of the Gaussian effective potential (GEP), which is a simple,

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nonperturbative approach and a useful tool for describing symmetry breaking in scalar field theories [13, 14] as well as in scalar electrodynamics [15]. We also use the stochastic quantization program described in [7, 8, 10] to get a "classical action" related to a stochastic partial differential equation. Then, following closely the variational approach developed in [15], we calculate the GEP for arbitrary stochastic partial differential equations subject to translation-invariant and temporally white noise correlations.

We apply this method to a particular stochastic field theory with considerable physical interest, the Kardar-Parisi-Zhang (KPZ) equation [16]. This equation describes a non-linear continuous dynamical model for the growth of surfaces. We found that the KPZ equation exhibits a spontaneous symmetry breaking in (1+1), (2+1) and (3+1) Euclidean dimensions and we believe that, as our approach is nonperturbative we are able to observe spontaneous symmetry breaking in the (3+1) case, in contrast with the perturbative calculation in [11].

Our paper is organized as follows: In the next section we review the procedure to obtain the generating functional $Z[J]$ and identify in it the corresponding "classical action" in the form that will be useful for our calculations. In section 3 we develop the general procedure involved in the variational approach that we use, the Gaussian effective potential, and apply it, in section 4, to the Kardar-Parisi-Zhang equation where we discuss the symmetry breaking as well as the renormalization in a close analogy with the $\lambda\phi^4$ scalar field theory within the framework of the Gaussian effective potential [13]. In section 5 we summaryse our results and make some final remarks.

2 The Generating Functional

In order to obtain the effective potential associated to a stochastic partial differential equation, it is first necessary to define what is the corresponding "action" . In this section we will review some of the hypothesis and definitions used to implement the functional integral formalism leading to this "action"(see also refs. [8, 9, 10]). In the expressions below we will use the following condensed notations, $f(\mathbf{x}, t) \mapsto f(x)$, $d^dxdt \mapsto dx$.

Consider the general class of stochastic partial differential equations

$$D\phi(x) = F(\phi(x)) + \eta(x) \;, \tag{1}$$

where D is any linear differential operator, involving arbitrary time and space derivatives, which does *not* explicitly involve the field ϕ. The function $F(\phi)$ is any forcing term, generally nonlinear in the field ϕ. The function $\eta(x)$ denotes the source of noise. The nature (white, power-law, colored, pink, $1/\mathbf{f}$-noise, or shot noise) and probability distribution of the noise need not yet be specified.

The definition of averages in stochastic systems is more subtle than in systems which do not describe a random behavior, since we must consider the presence of noise and its respective probability distribution. Let us then consider that *given a particular configuration of the noise η, the differential equation* (1) *is assumed to have an unique solution $\phi_s(x|\eta)$*. Then, for any functional $Q[\phi]$, we can define the stochastic average (over the noise) as

$$\langle Q[\phi]\rangle \equiv \int (\mathcal{D}\eta)\; \mathcal{P}[\eta]\; Q[\phi_s(x|\eta)] \;, \tag{2}$$

where $\mathcal{P}[\eta]$ is the probability density functional of the noise which is arbitrary and normalized to 1.

It is convenient to do an algebraic manipulation in (2) to take off the explicit dependence on the unique solution ϕ_s, due to fact that, in general, the SPDEs do not have solutions that can be written as a closed analytical expression. Therefore, we make use of the following identity,

$$\phi_s(x|\eta) \equiv \int (\mathcal{D}\phi)\ \phi\ \delta[\phi - \phi_s(x|\eta)]\ . \tag{3}$$

Using the property [1]

$$\delta[G(\phi)] = \frac{\delta[\phi - \phi_s(x|\eta)]}{|det(\frac{\delta G(\phi)}{\delta\phi})|}\ , \tag{4}$$

where $G(\phi_s) \equiv D\phi_s - F(\phi_s) - \eta = 0$ and the Jacobian functional determinant is defined by

$$\mathcal{J} \equiv det\left(D - \frac{\delta F}{\delta\phi}\right)\ , \tag{5}$$

we can now insert (4) in (3) to obtain

$$\phi_s(x|\eta) \equiv \int (\mathcal{D}\phi)\ \phi\ \delta[D\phi - F(\phi) - \eta]\ |\mathcal{J}|\ . \tag{6}$$

It is easy to see that one also has the identity

$$Q[\phi_s(x|\eta)] \equiv \int (\mathcal{D}\phi)\ Q[\phi]\ \delta[D\phi - F(\phi) - \eta]\ |\mathcal{J}|\ , \tag{7}$$

which, inserted into (2), provides the following expression for the stochastic average over the noise

$$\begin{aligned} \langle Q[\phi]\rangle &= \int(\mathcal{D}\eta)\int(\mathcal{D}\phi)\ \mathcal{P}[\eta]\ Q[\phi]\ \delta[D\phi - F(\phi) - \eta]\ |\mathcal{J}| \\ &= \int(\mathcal{D}\phi)\ \mathcal{P}[D\phi - F(\phi)]\ Q[\phi]\ |\mathcal{J}|\ . \end{aligned} \tag{8}$$

With this, one eliminates the necessity of knowing the analytical formula of $\phi_s(x|\eta)$ and now, the conection with quantum field theory can be done through the analogy of the expression (8) with the generating functional $Z[J]$, when $Q[\phi]$ assumes the form

$$Q[\phi] = \exp\left(\int dx\ J(x)\ \phi(x)\right)\ . \tag{9}$$

Then,

$$\begin{aligned} Z[J] &\equiv \left\langle \exp\left(\int dx\ J\ \phi\right)\right\rangle \\ &= \int(\mathcal{D}\phi)\ \mathcal{P}[D\phi - F(\phi)]\ \exp\left(\int dx\ J\ \phi\right)\ |\mathcal{J}|\ . \end{aligned} \tag{10}$$

This key result will enable us to calculate the effective potential in a direct way. At this stage we will make some assumptions about the noise. We assume that the noise probability distribution is Gaussian with zero mean so that the only non-vanishing cumulant is the second order one. If the noise has a non-zero mean, one can always redefine the forcing term $F[\phi]$ to make the noise be of zero mean without loss of generality. Arbitrary Gaussian noise is enough since it allows us to write the noise probability distribution as

[1]note that this is the functional generalization of the delta-function property $\delta(f(x)) = \frac{\delta(x-x_0)}{|\frac{df(x)}{dx}|_{x=x_0}}$ where $f(x)$ is assumed to have only one zero at $x = x_0$.

$$\mathcal{P}[\eta] = \exp\left[-\frac{1}{2}\int\int dx\, dy\, \frac{\eta(x)\,\eta(y)}{\langle\eta(x)\eta(y)\rangle}\right] \ , \tag{11}$$

where

$$\begin{aligned} \langle\eta(x)\eta(y)\rangle &= \frac{1}{N}\int(\mathcal{D}\eta)\,\eta(x)\,\eta(y)\exp\left(-\frac{1}{\mathcal{A}}\int\int dx'dy'\,\eta\, G_\eta^{-1}\,\eta\right) \\ &= \mathcal{A}\, G_\eta(x,y)\,, \end{aligned} \tag{12}$$

is the two-point noise correlation function. The correlation amplitude $\mathcal{A}$ will tell us if one is in the regime of absence of noise correlations ($\mathcal{A}=0$) or not ($\mathcal{A}\neq 0$). As we are interested in the influence of the noise correlations in the stochastic partial differential equation 'vacua', it is convenient to set $\mathcal{A}=1$ in nonperturbative (GEP) approach.[2] The generating functional is thus

$$\begin{aligned} Z[J] = \quad & \int(\mathcal{D}\phi)\,\exp\left(\int dx\, J\,\phi\right)\,|\mathcal{J}|\,\times \\ & \times\exp\left[-\frac{1}{2\mathcal{A}}\int\int dx\,dy\,(D\phi - F(\phi))\, G_\eta^{-1}\,(D\phi - F(\phi))\right] . \end{aligned} \tag{13}$$

This expression contains all the physics of (1). The noise η has been completely eliminated and survives only through the explicit appeareance of its two-point function. Since the generating functional is now given as a functional integral over the physical field ϕ, all the standard machinery of statistical and quantum field theory can be brought to bear. So, modulo the determinant of the Jacobian, the physics of a stochastic partial differential equation can be extracted from a functional integral based on a "classical action" $S[\phi, J]$, the exponent in (13) , such that

$$S[\phi, J] = \frac{1}{2}\int\int dxdy\,(D\phi - F(\phi))\, G_\eta^{-1}\,(D\phi - F(\phi)) + \int dx\, J\,\phi\,. \tag{14}$$

3 The Variational approach

Once obtaining the expression for the action through the definition of the generating functional analogue in a stochastic partial differential equations context, we can now use quantum field theory techniques to analyse the physics of a particular SPDE. Our intent is to investigate the effective potential (which, in some sense, gives a picture of how the noise correlations modify the 'classical' potential) through a nonperturbative method, the Gaussian effective potential.

As discussed in [13] within a quantum mechanics and quantum field theory context, the Gaussian effective potential is more reliable in both quantitative and qualitative terms when the quantum fluctuations become more relevant and is exactly renormalization-group invariant because, as the Gaussian effective potential has a nonperturbative character, it is not necessary to renormalize it order by order conforming to a perturbative procedure. In our calculations we will follow closely the procedure described in [15] with the necessary adaptation to our particular problem.

[2] In the perturbative approach, $\mathcal{A}$ is a loop counting parameter, analogous to $\hbar$ in quantum field theory.

First we will make in the action the following shift on the field [3]

$$\phi(x) \rightarrow \hat{\phi}(x) + f(\phi_0). \tag{15}$$

This shift is done to explore the SPDE's symmetries and to introduce the constant field ϕ_0 which in the quantum field theory context can be identified as being the classical field. The function $f(\phi_0)$ and the interpretation of ϕ_0 will depend of the particular stochastic partial differential equation. In fact, the Gaussian effective potential will be a function of the possible values of ϕ_0.

After this field transformation, it is convenient to split the action S into two parts. The Gaussian part S_G defined as

$$S_G[\hat{\phi}] = \frac{1}{2}\int dx\, \hat{\phi}\, G^{-1}\hat{\phi}\,, \tag{16}$$

corresponds to the quadratic terms which will generate the propagator analogue of quantum field theory and the interaction part S_I, that will generate the vertex function analogue. The variational calculation is introduced through the maximization of the effective action with respect to the propagator function $\tilde{G}$ with the purpose of obtains the optimum propagator for the desired system. Thus, the entire action becomes

$$S[\hat{\phi}, J, \phi_0] = S_G[\hat{\phi}] + S_I[\hat{\phi}, J, \phi_0]. \tag{17}$$

The variational principle beneath our approach comes from the functional generalization of the Jensen inequality [8] $\langle \exp f \rangle \geq \exp \langle f \rangle$, i.e,

$$\frac{Z[J]}{N} \geq \exp[\frac{1}{N}\int \mathcal{D}\hat{\phi}\,(S_G - S)\, \exp(-S_G)]. \tag{18}$$

Since S_G is quadratic in the fields, the normalization factor is

$$N = \int \mathcal{D}\hat{\phi}\, \exp(-S_G) = (\det G^{-1})^{-\frac{1}{2}}. \tag{19}$$

The generating functional of connected Green's functions is given by $W[J] = -\ln Z[J]$. So,

$$W[J] \leq -\ln N - \frac{1}{N}\int \mathcal{D}\hat{\phi}\,(S_G - S)\, \exp(-S_G). \tag{20}$$

The term $-\ln N$ can be written in a more convenient form in the Fourier basis

$$-\ln N = \frac{1}{2}\ln\det \tilde{G}^{-1} = \frac{1}{2}Tr\ln \tilde{G}^{-1} = \frac{\mathcal{V}}{2}\int dk\, \ln \tilde{G}^{-1}\,, \tag{21}$$

where $\mathcal{V}$ is the space-time volume ($\int dx$). A more useful object, however, is the effective action which is defined as the functional Legendre transform of $W[J]$,

$$\Gamma[\phi_0] = \int dx\, J\phi_0 - W[J]. \tag{22}$$

Then, the inequality (20) becomes

$$\Gamma[\phi_0] \leq \ln N + \langle S_G - S\rangle_G\, /_{J=0}, \tag{23}$$

[3]Notice that the functional measure is unchanged, i.e, $\mathcal{D}\phi = \mathcal{D}\hat{\phi}$

where the symbol G outside the brackets indicates that the functional average is Gaussian. The right hand side of (23) can be calculated analytically and the variational approach will consist to minimize it with respect to the function $\tilde{G}$ and to obtain the best value of the effective action. Defining V_G as

$$V_G(\phi_0) \equiv -\frac{\Gamma[\phi_0]}{\mathcal{V}}\,, \tag{24}$$

we have the Gaussian effective action

$$\Gamma^{GEA}[\phi_0] = \min_{\tilde{G}} \left[-\frac{\mathcal{V}}{2} \int dk \, \ln \tilde{G}^{-1} + \langle S_G - S \rangle_G \,/_{J=0} \right] \equiv -\min_{\tilde{G}} [\mathcal{V}\, V_G(\phi_0)] \tag{25}$$

Now we can define the Gaussian effective potential as

$$\overline{V}_G(\phi_0) \equiv \min_{\tilde{G}} \left[\frac{1}{2} \int dk \, \ln \tilde{G}^{-1} + \frac{1}{\mathcal{V}} \langle S_I \rangle_G \,/_{J=0} \right], \tag{26}$$

where the bar above V_G means that $\tilde{G}$ was replaced for its optimized value $\overline{\tilde{G}}$.

4 The Gaussian effective potential associated to the KPZ equation

Now we are ready to write down the Gaussian effective potential for a particular stochastic partial differential equation. The Kardar-Parisi-Zhang equation

$$\left(\frac{\partial}{\partial t} - \nu \nabla^2 \right) \phi(x) = F_0 + \frac{\lambda}{2} (\nabla \phi)^2 + \eta(x)\,, \tag{27}$$

is a natural extension of the Edwards-Wilkinson model constructed taking to account the relevants symmetry principles to growth phenomena description [4] [17]. The field $\phi(x)$ is taken to be the height of the surface (typically defined over a plane). The nonlinear term increases the height of the interface by adding more 'material' to the parts of the surface where the local slope is larger, and the average height of the surface increases. This situation may be contrasted to the effect of the linear term, which reorganizes the surface height in such a way that the total mass remains unchanged. The result of this two combined factors is a good description of the surface growth phenomena. The inclusion of the term F_0 is necessary, as we will soon see, to complete the renormalization program. This constant term can be seen as a consequence of the transformation $\phi \to \phi + F_0 t$ which represents just a change in the average velocity of the surface growth with respect to the laboratory frame of reference.

An important symmetry of the KPZ equation which is relevant for our analysis is the invariance under the transformation

$$\begin{aligned} \mathbf{x} &\to \mathbf{x} - \lambda \vec{\phi_0} t, \\ t &\to t, \\ \phi(\mathbf{x}, t) &\to \phi(\mathbf{x}, t) + \vec{\phi_0} \cdot \mathbf{x}. \end{aligned} \tag{28}$$

[4] In this paper we will restrict our analysis to the surface growth interpretation, but in the fluid dynamical interpretation (i.e., the vorticity-free Burgers equation), the KPZ equation can be used as a model for turbulence, structure development in the early universe, driven diffusion and flame fronts.

Here, ϕ_0 is a constant field and this symmetry amounts to choosing a different coordinate system that is tilted at an angle θ with respect the vertical with

$$\tan\theta = \left|\vec{\phi_0}\right| , \tag{29}$$

and for this reason this transformation is often referred to as tilt invariance, but this transformation will remain a symmetry only if the two-point noise correlation function is translation-invariant and temporally white. A convenient choice is [5].

$$\langle \eta(x)\eta(y)\rangle = G_\eta(x,y) \equiv \delta(x-y) . \tag{30}$$

Using the white noise in the action (14) we have

$$S[\phi] = \frac{1}{2}\int dx\,(D\phi - F(\phi))^2 = \frac{1}{2}\int dx\,\left[\dot{\phi} - \nu\nabla^2\phi - F_0 - \frac{\lambda}{2}(\nabla\phi)^2\right]^2 . \tag{31}$$

To investigate symmetry breaking through the Gaussian effective potential one must perform a shift on the field with $\langle\phi\rangle_{vacuum} \neq 0$. In our case, a convenient choice is $\phi(x) \to \hat{\phi}(\mathbf{x}) + \vec{\phi_0}\cdot\mathbf{x}$, with $\vec{\phi_0}$ being a N-dimensional vector such that $\vec{\phi_0} \equiv (\phi_0, ..., \phi_0)$. So,

$$\begin{aligned} \dot{\phi} &= 0 , \\ \nabla^2\phi(x) &= \nabla^2\hat{\phi}(\mathbf{x}) , \\ \nabla\phi(x) &= \nabla\hat{\phi}(\mathbf{x}) + \vec{\phi_0} . \end{aligned} \tag{32}$$

Then, fixing the surface tension parameter ν to 1 in Eq (27), the action becomes

$$\begin{aligned} S[\hat{\phi}, J, \phi_0] = \frac{T}{2}\int d^dx\,\left[\nabla^2\hat{\phi}\,\nabla^2\hat{\phi} + F_0\,\lambda\,(\nabla\hat{\phi})^2 + F_0\,\lambda\,\phi_0^2\right] \\ + \left[\frac{\lambda^2}{4}(\nabla\hat{\phi})^4 + \frac{\lambda^2}{4}\phi_0^4 + \frac{3}{2}\lambda^2\phi_0^2(\nabla\hat{\phi})^2 + \,odd\,terms\right] + \int d^dx\,\hat{\phi}\,J , \end{aligned} \tag{33}$$

where $T = \int\, dt$ and the odd terms were put apart because they vanish when we perform the field and momentum odd integrations. The term with F_0^2 alone is not physically relevant, being a constant that can be subtracted. First we must to know the optimum propagator expression to define S_I. In the Fourier basis, the real function $\hat{\phi}(\mathbf{x})$ is

$$\hat{\phi}(\mathbf{x}) = \int \frac{d^dk}{(2\pi)^d}\,\tilde{\phi}(\mathbf{k})\,e^{-i\mathbf{k}\cdot\mathbf{x}}. \tag{34}$$

Then[6],in a Fourier basis S_G becomes

$$S_G[\tilde{\phi}] = \frac{T}{2}\int \frac{d^dk}{(2\pi)^d}\,\tilde{\phi}(\mathbf{k})\,\tilde{G}^{-1}(\mathbf{k})\,\tilde{\phi}^*(\mathbf{k}) . \tag{35}$$

Using (25),

[5] Remember that we set $\mathcal{A} = 1$

[6] Note that $\int d^dx\,e^{-i\mathbf{k}_1\cdot\mathbf{x}}\,e^{i\mathbf{k}_2\cdot\mathbf{x}} = \delta^d(\mathbf{k}_1 - \mathbf{k}_2)$ and $\hat{\phi}^2 = \hat{\phi}\,\hat{\phi}^*$

$$\frac{\delta\Gamma^{GEA}[\phi_0]}{\delta\tilde{G}(k')} = \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\mathbf{k}^4\delta^d(\mathbf{k}-\mathbf{k}^{'}) + \frac{3}{4}\lambda^2\int\frac{d^dk}{(2\pi)^d}\mathbf{k}^2\tilde{G}(\mathbf{k})\left(\int\frac{d^dk}{(2\pi)^d}\mathbf{k}^2\delta^d(\mathbf{k}-\mathbf{k}^{'})\right) + $$
$$+\frac{3}{4}\lambda^2\phi_0^2\int\frac{d^dk}{(2\pi)^d}\mathbf{k}^2\delta^d(\mathbf{k}-\mathbf{k}^{'}) - \frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\tilde{G}^{-1}(\mathbf{k})\delta^d(\mathbf{k}-\mathbf{k}^{'}) = 0\,. \quad (36)$$

Then,

$$\bar{\tilde{G}}^{-1}(\mathbf{k}) = \mathbf{k}^4 + \mathbf{k}^2\bar{\Omega}^2\,, \quad (37)$$

where the value of $\bar{\Omega}$ that provides a solution must satisfy the relation usually called $\bar{\Omega}$-equation,

$$\bar{\Omega}^2 = F_0\lambda + \frac{3}{2}\lambda^2\left[\left(\int\frac{d^dk}{(2\pi)^d}\frac{1}{(\mathbf{k}^2+\bar{\Omega}^2)}\right) + \phi_0^2\right]. \quad (38)$$

With the optimum propagator defined, we are ready to evaluate the interaction part of the action in a coordinate basis.

$$S_I[\hat{\phi},\phi_0] = S[\hat{\phi},\phi_0] - S_G[\hat{\phi},\phi_0] = T\int d^dx\,\left[F_0\frac{\lambda}{2}(\nabla\hat{\phi})^2 + F_0\frac{\lambda}{2}\phi_0^2 + \frac{\lambda^2}{8}(\nabla\hat{\phi})^4 + \frac{\lambda^2}{8}\phi_0^4\right]$$
$$+\left[\frac{3}{4}\lambda^2\phi_0^2(\nabla\hat{\phi})^2 - \frac{1}{2}(\nabla\hat{\phi})^2\,\bar{\Omega}^2 + odd\,terms\right]. \quad (39)$$

Now that we have the expression for S_I, using (26) we are able to calculate the Gaussian effective potential. To evaluate $\langle S_I\rangle$ we will perform the following two Gaussian functional integrations in a Fourier basis:

$$\frac{1}{N}\int\mathcal{D}\hat{\phi}\,(\nabla\hat{\phi})^2\,\exp(-S_G) = \frac{(-i)(i)}{N}\int\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\,\mathbf{k}_1\,\mathbf{k}_2\,e^{-i\mathbf{k}_1\cdot\mathbf{x}}\,e^{i\mathbf{k}_2\cdot\mathbf{x}}\,\times$$
$$\times\int(\prod_{\mathbf{k}}d\tilde{\phi}(\mathbf{k}))\,\tilde{\phi}(\mathbf{k}_1)\,\tilde{\phi}^*(\mathbf{k}_2)\,\exp(-S_G) = \int\frac{d^dk}{(2\pi)^d}\,\mathbf{k}^2\,\bar{\tilde{G}}\,(\mathbf{k})\,, \quad (40)$$

notice that we have just one combination which make this integral even in the fields: $\mathbf{k}_1 = \mathbf{k}_2$.

$$\frac{1}{N}\int\mathcal{D}\hat{\phi}\,(\nabla\hat{\phi})^4\,\exp(-S_G) =$$
$$= \frac{(-i)^2(i)^2}{N}\int\frac{d^dk_1}{(2\pi)^d}\frac{d^dk_2}{(2\pi)^d}\frac{d^dk_3}{(2\pi)^d}\frac{d^dk_4}{(2\pi)^d}\,\mathbf{k}_1\,\mathbf{k}_2\,\mathbf{k}_3\,\mathbf{k}_4\,e^{-i\mathbf{k}_1\cdot\mathbf{x}}\,e^{i\mathbf{k}_2\cdot\mathbf{x}}\,e^{-i\mathbf{k}_3\cdot\mathbf{x}}\,e^{i\mathbf{k}_4\cdot\mathbf{x}}\,\times$$
$$\times\int(\prod_{\mathbf{k}}d\tilde{\phi}(\mathbf{k}))\,\tilde{\phi}(\mathbf{k}_1)\,\tilde{\phi}^*(\mathbf{k}_2)\,\tilde{\phi}(\mathbf{k}_3)\,\tilde{\phi}^*(\mathbf{k}_4)\,\exp(-S_G)$$
$$= 3\left(\int\frac{d^dk}{(2\pi)^d}\,\mathbf{k}^2\,\bar{\tilde{G}}\,(\mathbf{k})\right)^2. \quad (41)$$

The factor 3 emerges because there are three combinations of $k's$ resulting in an even integral in the fields.

Then, defining [7]

$$I_1(\bar{\Omega}) \equiv \frac{1}{2}\int \frac{d^dk}{(2\pi)^d}\left[\ln(\mathbf{k}^2+\bar{\Omega}^2)-\ln(\mathbf{k}^2)\right] = \frac{1}{2}\int \frac{d^{d-1}k}{(2\pi)^{d-1}}(\mathbf{k}^2+\bar{\Omega}^2)^{1/2}\,, \tag{42}$$

$$I_0(\bar{\Omega}) \equiv \int \frac{d^dk}{(2\pi)^d}\frac{1}{(\mathbf{k}^2+\bar{\Omega}^2)} = \frac{1}{2}\int \frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{(\mathbf{k}^2+\bar{\Omega}^2)^{1/2}}\,, \tag{43}$$

we have the Gaussian effective potential in (d+1) dimensions

$$\overline{V}_G(\phi_0) = I_1(\bar{\Omega}) + C + \frac{\lambda^2}{8}\phi_0^4 + \frac{3}{8}\lambda^2\, I_0^2(\bar{\Omega}) + \\ +\frac{3}{4}\lambda^2\phi_0^2 I_0(\bar{\Omega}) + F_0\,\frac{\lambda}{2}\,I_0(\bar{\Omega}) + F_0\,\frac{\lambda}{2}\,\phi_0^2 - \frac{1}{2}\bar{\Omega}^2 I_0(\bar{\Omega})\,, \tag{44}$$

where C is an infinite constant coming from the integral $\int dk \ln k^2$.

However, there are two subtleties to beware of: first, (38) may have more than one solution, and one must take care to select the right one. In particular, the solution must be a minimum, not a maximum of $\overline{V}_G$. Second, the global minimum of $\overline{V}_G(\phi_0)$ may not be a solution of (38) at all, but may occur at one or another end point of the range $0 < \bar{\Omega} < \infty$. Another important remark is that through of equations (44) and (38), we have a close analogy between the Gaussian effective potential for the KPZ problem and the GEP of $\lambda\phi^4$ theory. Indeed, interpreting $F_0\lambda$ as being a bare mass term and rescaling the parameter λ, we have an expression identical to the one obtained in obtained in [13], apart from the constant term C and tha fact that that we will perform the integrals I_1 and I_0 over an Euclidean space. As, in the KPZ problem, the field $\phi(x)$ is taken to be the heigth of the surface, typically defined over a two dimensional plane, we will restrict our analysis to (2+1) and (3+1) Euclidean space dimensions.

4.1 Renormalization

To perform the renormalization program, we will define $\Delta_0^2 \equiv F_0\lambda$ and renormalize the bare parameter Δ_0 as

$$\Delta_0^2 = \Delta_R^2 - \frac{3}{2}\lambda^2 I_0(\bar{\Omega}_0)\,, \tag{45}$$

where $\bar{\Omega}_0$ is the solution to the $\bar{\Omega}$ equation at $\phi_0 = 0$. So, the $\bar{\Omega}$ equation associated to V_G becomes

$$\bar{\Omega}^2 = \Delta_R^2 + \frac{3}{2}\lambda^2[(I_0(\bar{\Omega}) - I_0(\bar{\Omega}_0)) + \phi_0^2]\,, \tag{46}$$

with

$$\bar{\Omega}_0^2 = \Delta_R^2\,. \tag{47}$$

[7]The RHS will be useful when we perform the renormalization program.

We will follow the same renormalization program as the one in [13] which discard an explicit regularization because the calculations will involve only differences $I_N(\bar{\Omega}) - I_N(\Delta_R)$. Substituting (45) and (46) into (44) we have

$$\overline{V}_G(\phi_0) = I_1(\bar{\Omega}) + C + \frac{\lambda^2}{8}\phi_0^4 - \frac{3}{8}\,\lambda^2\, I_0^2(\bar{\Omega})\,. \tag{48}$$

Defining a ϕ_0-independent divergent constant.

$$D \equiv \overline{V}_G(\phi_0 = 0) = I_1(\bar{\Omega}_0) + C - \frac{3}{8}\,\lambda^2\,[I_0(\bar{\Omega}_0)]^2\,, \tag{49}$$

we see that D represents the vacuum energy density of the $\phi_0 = 0$ vacuum. The presence of D has no physical consequences, since only energy differences, not absolute energies, are measurable. So, we can follow the usual practice of redefining the zero of the energy scale such that $\overline{V}_G(\phi_0 = 0) = 0$, subtracting D from $\overline{V}_G(\phi)$.

4.2 KPZ (2+1)

In (2+1) Euclidean dimensions the integrals I_1 and I_0 are divergent and we shall now see explicitly how the equation (45) leads to a manifestly finite result. By means of the formula for $I_N(\bar{\Omega}) - I_N(\Delta_R)$ from table **II** of [13] and setting the renormalized parameter $\Delta_R = 1$, we have

$$I_1(\bar{\Omega}) - I_1(1) = \frac{1}{2}(\bar{\Omega}^2 - 1)I_0(1) - \frac{L_2(\bar{\Omega}^2)}{8\pi} \tag{50}$$

$$I_0(\bar{\Omega}) - I_0(1) = -\frac{L_1(\bar{\Omega}^2)}{4\pi} \tag{51}$$

where

$$L_1(\bar{\Omega}^2) = \ln\bar{\Omega}^2 \tag{52}$$

$$L_2(\bar{\Omega}^2) = \bar{\Omega}^2\ln\bar{\Omega}^2 - (\bar{\Omega}^2 - 1) \tag{53}$$

Substituting (45) into (44) and using the identity $(I_0 - I_0(1))^2 = I_0^2 - 2I_0I_0(1) + I_0^2(1)$ leads to

$$\overline{V}_G(\phi_0) = D + [I_1 - I_1(1)] + \frac{1}{2}(1 - \bar{\Omega}^2)I_0 + \frac{\lambda^2}{8}\phi_0^4 + \frac{1}{2}\phi_0^2 + \\ + \frac{3}{4}\lambda^2\,[I_0 - I_0(1)]\,\phi_0^2 + \frac{3}{8}\lambda^2\,[I_0 - I_0(1)]^2\;. \tag{54}$$

Inserting (50) and (51) in (54), gives the renormalized Gaussian effective potential

$$\overline{V}_G(\phi_0) = D + \frac{\phi_0^2}{2} + \frac{\lambda^2}{8}\phi_0^4 - \frac{L_2(\bar{\Omega}^2)}{8\pi} - \frac{L_1(\bar{\Omega}^2)}{4\pi}\left[\frac{1}{2}(1 - \bar{\Omega}^2) + \frac{3}{4}\lambda^2\phi_0^2 - \frac{3}{8}\lambda^2\frac{L_1(\bar{\Omega}^2)}{4\pi}\right]\,. \tag{55}$$

Using (46) and (53) to simplify (55) by eliminating the explicit logarithms, leaving us with

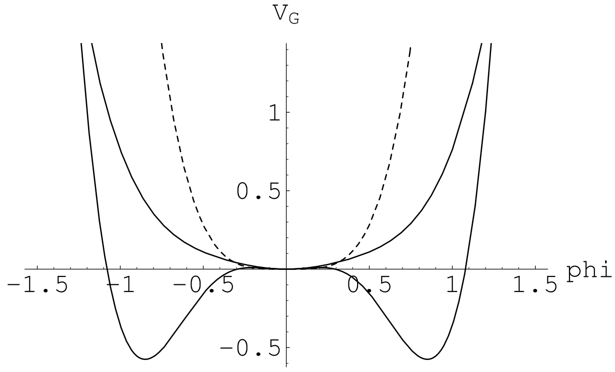

Figure 1: The Gaussian effective potential for KPZ in (2+1) Euclidean dimensions. The single-well continuous line shows the symmetric vacuum when $\lambda < \lambda_c$. The double-well indicates a phase transition that makes which the symmetry associated with the vacuum of the system be spontaneously broken and the dashed line shows the zero noise correlation ($\mathcal{A} = 0$) regime.

$$\overline{\mathcal{V}}_G(\phi_0) \equiv \overline{V}_G(\phi_0) - D = -\frac{\lambda^2}{4}\phi_0^4 + \frac{(\bar{\Omega}^2 - 1)}{3\lambda^2}\left[1 + \frac{3\lambda^2}{8\pi} + \frac{1}{2}(\bar{\Omega}^2 - 1)\right]. \tag{56}$$

In the regime where noise correlations are present ($\mathcal{A} \neq 0$), the influence of noise on the nonlinear term, which governs the addition of material on surface, is such that when λ reaches a critical value, a phase transition associated with the vacuum symmetry of the system occurs, i.e, the symmetry is spontaneously broken, and now, the vacuum has two degenerate minima. It is convenient to introduce the critical parameter λ_c, which is the value where the transition from the single-well to double-well behavior takes place. For $\lambda < \lambda_c$ the global minimum of the Gaussian effective potential is at $\phi_0 = 0$, while for $\lambda > \lambda_c$ the global minima is symmetric and occur at $\phi_0 = \pm\, c$ with $c \neq 0$ (see Figure 1). For $\lambda = \lambda_c$ the Gaussian effective potential has three exactly degenerate minima. We find

$$\lambda_c \cong 4.51903\,. \tag{57}$$

4.3 KPZ (3+1)

Because the algebra of the divergent integrals in (3+1) Euclidean dimensions is so similar to that in (2+1) Euclidean dimensions, we may simply take over the results in (55) except that now

$$L_1(\bar{\Omega}^2) = (\bar{\Omega} - 1)\,, \tag{58}$$

$$L_2(\bar{\Omega}^2) = \frac{1}{3}(\bar{\Omega} - 1)^2\,(2\,\bar{\Omega} + 1). \tag{59}$$

The $\bar{\Omega}$ equation (46) now becomes

$$\frac{1}{2}(\bar{\Omega}^2 - 1) = \frac{3}{4}\lambda^2[\phi_0^2 - \frac{(\bar{\Omega} - 1)}{4\pi}]\,. \tag{60}$$

If we use (60) to simplify (55) we can express the renormalized Gaussian effective potential in the convenient form

$$\overline{\mathcal{V}}_G(\phi_0) = \frac{1}{2}\phi_0^2 + \frac{\lambda^2}{8}\phi_0^4 - \frac{(\bar{\Omega}-1)^2}{24\pi}\left[1 + \frac{9\lambda^2}{16\pi} + 2\bar{\Omega}\right] . \tag{61}$$

By computing (61) with $\bar{\Omega}$ given by (60) we have obtained the results shown in Figure 2. Again, we see a transition from single-well to double-well behavior, with the critical λ being

$$\lambda_c \cong 4.96258\,, \tag{62}$$

which we believe to indicate a second-order phase transition, stemming from the analogy with a ϕ^4 scalar field.

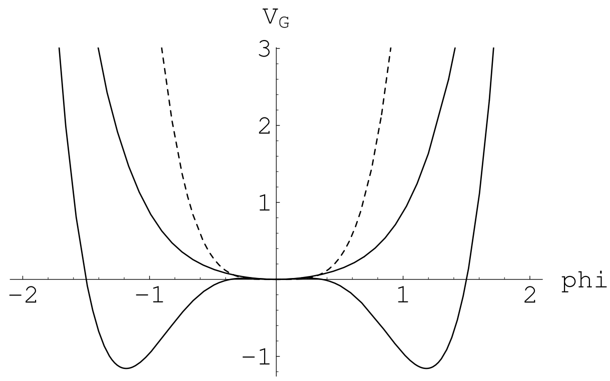

Figure 2: The Gaussian effective potential for KPZ in (3+1) Euclidean dimensions. The single-well continuous line shows the symmetric vacuum when $\lambda < \lambda_c$. The double-well indicates a phase transition that makes which the symmetry associated with the vacuum of the system be spontaneously broken and the dashed line shows the zero noise correlation ($\mathcal{A} = 0$) regime.

5 Summary

We have introduced in this paper a new approach to the investigation of arbitrary stochastic partial differential equations subject to white noise correlations. This approach relies partially on a nonperturbative calculation of the effective potential, the Gaussian effective potential. We made use of the variational approach developed in [15] and the stochastic quantization program [8],[7],[10] such that we obtain the Gaussian effective potential for arbitrary SPDEs.

The Gaussian effective potential, being nonperturbative includes not only the leading orders of the effective potential as the 1-loop effective potential but it also contains contributions coming from all orders. The advantage of this approach is that when correlations become large, the GEP is more reliable and it is renormalization group invariant[13], unlike perturbative calculations that are not prescription independent. So, in the same way as in the analysis of quantum fluctuactions we have shown, through this computation of the effective potential and investigation of the symmetry properties and patterns of symmetry breaking, how it is possible to apply the same general ideas, in a nonperturbative framework, for stochastic partial differential equations.

As a first example, we applied the method on the Kardar-Parisi-Zhang equation subject to a static field. The static field is convenient to obtain a close parallel with the $\lambda\phi^4$ field theory along the lines of [11] as well as to compare the perturbative and nonperturbative results, the choice made also provides $I_N(\Omega)$ integrals that are less divergent. Another outcome of the analogy with $\lambda\phi^4$ in our approach is the indication of ultraviolet renormalization to all orders. Notice that as in [11] we are not looking into the infrared region, an issue that has been considered in [5, 6]. Due to a noise correlation effect, the analogous of quantum fluctuations in the quantum field theory context, we have found in (1+1), (2+1) and (3+1) Euclidean dimensions a second-order phase transition. The Gaussian effective potential evolves from a single-well to a double-well shape as λ increases, characterizing a spontaneous symmetry breaking. In the surface growth interpretation, where the field is taken to be the height of the surface, the spontaneous symmetry breaking of the symmetric vacuum corresponds to a transition between a configuration where the surface starts with zero slope ($\lambda < \lambda_c$) and evolves to a configuration where distinct domains of the surface exhibit different slopes of same modulo described by the double-well behavior obtained with the Gaussian effective potential in $\lambda > \lambda_c$ regime. This behaviour changes the configuration (roughness) of the surface. We believe that as our approach is nonperturbative the spontaneous symmetry breaking in the (3+1) case is present, in contrast with the 1-loop calculation in [11].

The possibility of aplication of the Gaussian effective potential to other stochastic partial differential equations is very attractive and may furnish tools for the analysis of symmetries and phase structure. We hope that the present paper provides some stimulation for further research in this direction.

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