Penalty-based Alternating Direction Method for Solving Large-Scale Mixed-Integer Stochastic Problems

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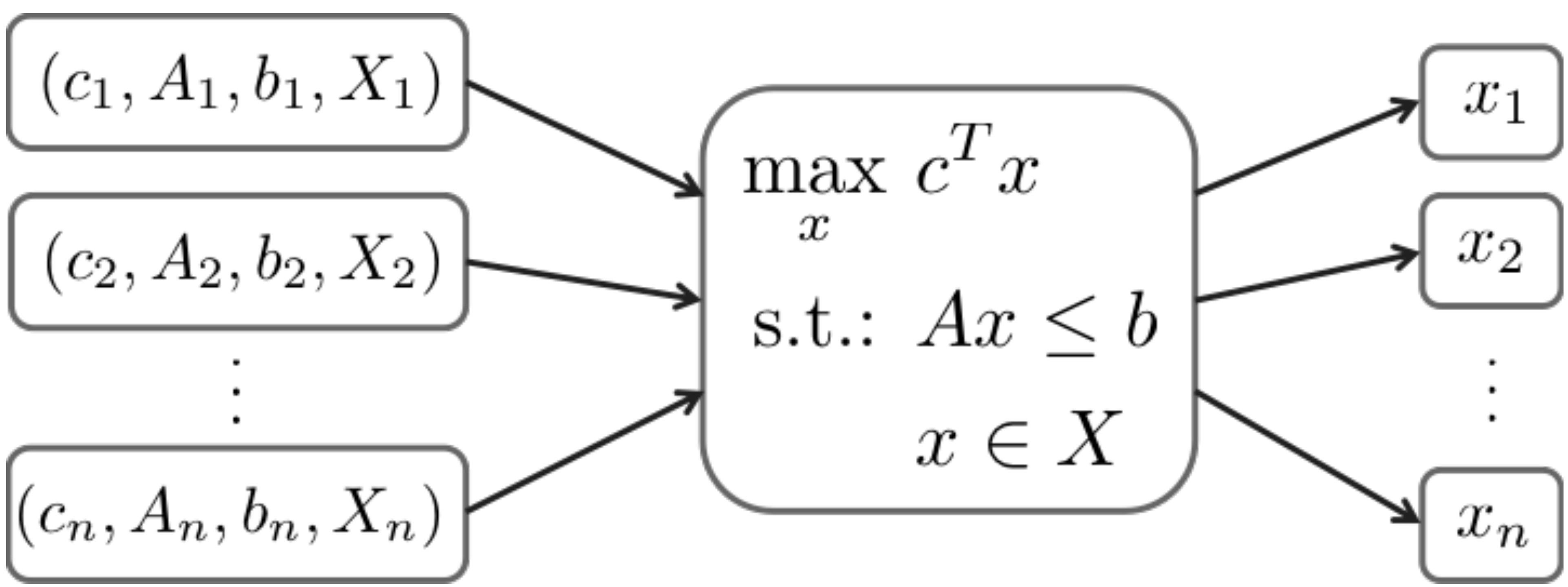
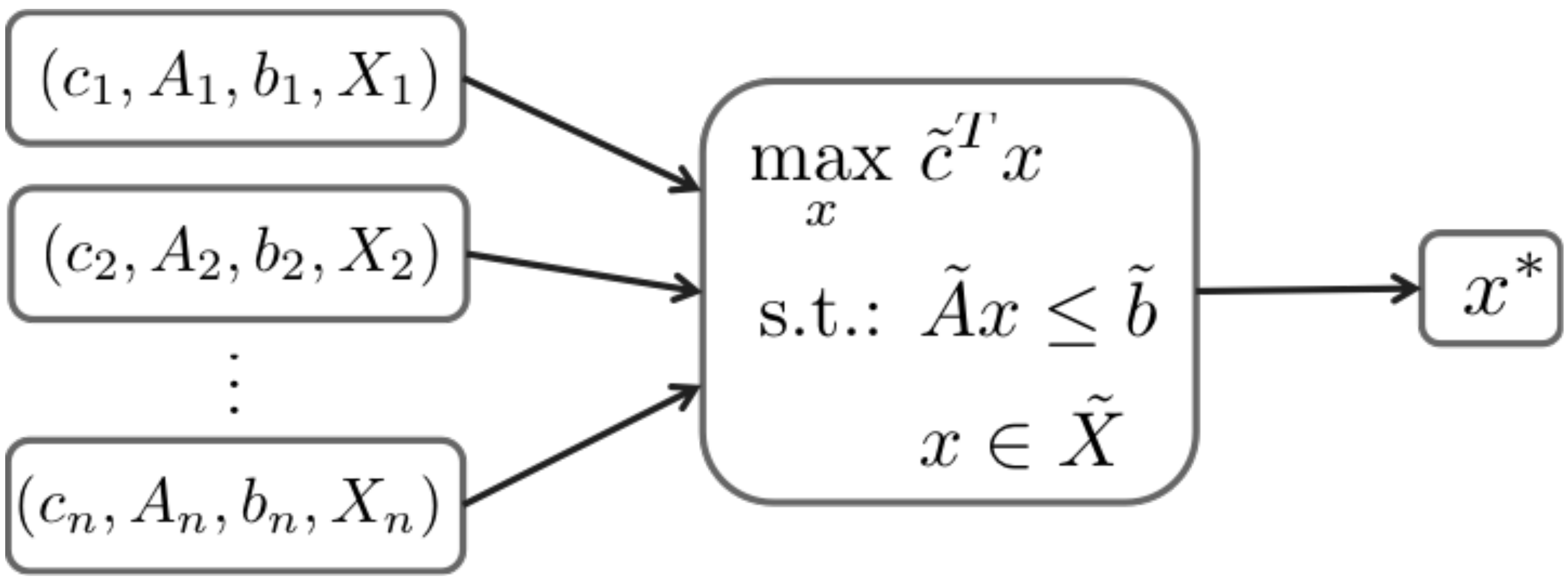
Optimisation under Uncertainty

Mathematical programming-based methods are **not able to explicitly consider uncertainty** in their evaluations



Optimisation under Uncertainty

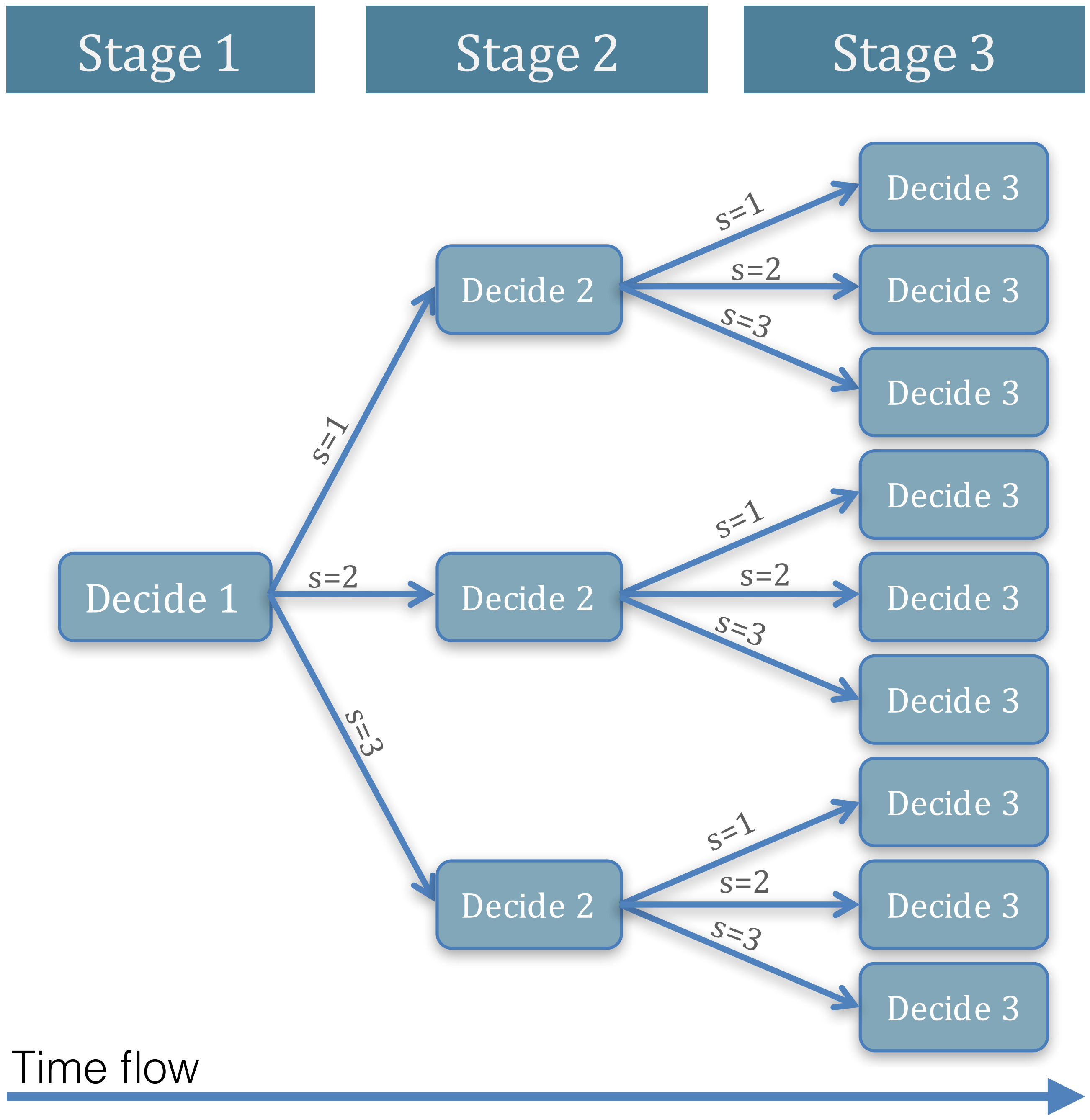
To be useful, a **unique** strategy must be defined beforehand for **meaningful decision making**



K-stage stochastic problems

Basic steps:

1. Break time flow into points of interest
2. Gather decisions that must be made up to each point
3. Explicitly represent scenarios



2-stage stochastic problems

Putting it all together, we have the following:

$$\begin{aligned}\max_x \; & c^\top x + \mathbb{E}_\Omega\left[Q(x,\omega)\right]\\ \text{s.t.: } & Ax \leq b\\ & x \in X\end{aligned}$$

where

$$\begin{aligned}Q(x,\omega) = \max \; & q_\omega^\top y\\ \text{s.t.: } & T_\omega x + W_\omega y = h_\omega\\ & y \in Y\end{aligned}$$

2-stage stochastic problems

Which in turn is (deterministically) equivalent to

$$
\begin{aligned}
\max_{x,y} \; & c^\top x + \sum_{\omega \in \Omega} p_\omega \left(q_\omega^\top y_\omega \right) \\
\text{s.t.:} \; & Ax \leq b \\
& x \in X \\
& T_\omega x + W_\omega y_\omega = h_\omega, \forall \omega \in \Omega \\
& y_\omega \in Y_\omega, \forall \omega \in \Omega
\end{aligned}
$$

Progressive Hedging

(Rockafellar and Wets, 1991)

Using the "Augmented Lagrangian" framework to obtain a relaxation for the original problem…

$$\max_{x,y,z} \sum_{\omega\in\Omega} p_\omega \left(c^\top x_\omega + q_\omega^\top y_\omega\right)$$
$$x_\omega \in X, \forall \omega \in \Omega$$
$$y_\omega \in Y_\omega, \forall \omega \in \Omega$$
$$x_\omega = z, \forall \omega \in \Omega$$

$$\max_{x,y,z} \sum_{\omega\in\Omega} p_\omega \phi_\omega(\mu, x_\omega, y_\omega, z)$$
$$x_\omega \in X, \forall \omega \in \Omega$$
$$y_\omega \in Y_\omega, \forall \omega \in \Omega$$

where

$$\phi_\omega(\mu, x_\omega, y_\omega, z) = c^\top x_\omega + q_\omega^\top y_\omega + \mu_\omega^\top (x_\omega - z) + \frac{\rho}{2}||x_\omega - z||_2^2$$

Progressive Hedging

… we can concentrate our efforts in obtaining optimal bounds solving this relaxation.

Augmented Lagrangian
Dual Problem

$$\min_{\mu} \left\{ \max_{x,y,z} \sum_{\omega \in \Omega} p_\omega \phi_\omega(\mu, x_\omega, y_\omega, z) \right\}$$

$$x_\omega \in X, \forall \omega \in \Omega$$

$$y_\omega \in Y_\omega, \forall \omega \in \Omega$$

Primal Problem

$$\max_{x,y,z} \sum_{\omega \in \Omega} p_\omega \phi_\omega(\mu, x_\omega, y_\omega, z)$$

$$x_\omega \in X, \forall \omega \in \Omega$$

$$y_\omega \in Y_\omega, \forall \omega \in \Omega$$

(x, y, z) $\downarrow$ $\uparrow$ μ

$$\mu_\omega^{k+1} = \mu_\omega^k + \rho(x_\omega - z), \ \forall \omega \in \Omega$$

Dual Problem

Progressive Hedging

To obtain separability, Alternate Direction Method can be used to solve the Primal Problem

Primal Problem

$$\max_{x,y} \sum_{\omega\in\Omega} p_\omega \phi_\omega(\mu, x_\omega, y_\omega, \bar{z})$$

$$x_\omega \in X, \forall \omega \in \Omega$$

$$y_\omega \in Y_\omega, \forall \omega \in \Omega$$

μ

$$\mu_\omega^{k+1} = \mu_\omega^k + \rho(x_\omega - z), \ \forall \omega \in \Omega$$

Dual Problem

(x, y)

z

$$\max_z \sum_{\omega\in\Omega} p_\omega \phi_\omega(\mu, \bar{x}_\omega, \bar{y}_\omega, z)$$

$$= \sum_{\omega\in\Omega} p_\omega \bar{x}_\omega$$

Penalty-based ADM

Develop an alternative relaxation based a polyhedral penalty function

$$\max_{x,y,z} \sum_{\omega \in \Omega} p_\omega \left(c^\top x_\omega + q_\omega^\top y_\omega \right)$$
$$x_\omega \in X, \forall \omega \in \Omega$$
$$y_\omega \in Y_\omega, \forall \omega \in \Omega$$
$$x_\omega \leq z, \forall \omega \in \Omega$$
$$x_\omega \geq z, \forall \omega \in \Omega$$

$$\max_{x,y,z} \sum_{\omega \in \Omega} p_\omega \phi_\omega^\rho(\mu)$$
$$x_\omega \in X, \forall \omega \in \Omega$$
$$y_\omega \in Y_\omega, \forall \omega \in \Omega$$

where

$$\phi_\omega^\rho(\mu, x_\omega, y_\omega, z) = c^\top x_\omega + q_\omega^\top y_\omega + \mu_\omega^\top (x_\omega - z) + \rho |x_\omega - z|$$

Penalty-based ADM

Combining results from Eberhard & Boland (2015) and Feizollahi et al. (2016), we have that:

*for any norm (**but not necessarily the squared norm**) and any Lagrangian multiplier, there exists a finite penalty such that*

$z_{LD} = z_{IP}.$

Hence, we can set $(\mu)_{\omega\in\Omega} = 0$, which leads to

$$\phi_\omega(\mu, x_\omega, y_\omega, z) = c^\top x_\omega + q_\omega^\top y_\omega + \rho|x_\omega - z|$$

and concentrate on finding **suitable penalties**.

Penalty-based ADM

"The Good Penalty Hunting"

$$\begin{aligned}\phi_\omega(\mu, x_\omega, y_\omega, z) &= c^\top x_\omega + q_\omega^\top y_\omega + \rho|x_\omega - z| \\ &= c^\top x_\omega + q_\omega^\top y_\omega + \rho([x_\omega - z]^- + [z - x_\omega]^-)\end{aligned}$$

where $[\ \cdot\ ]^- = -\min\{0,\ \cdot\ \}$

In fact, we adapt this penalty function to:

$$\phi_\omega^\rho(x_\omega, y_\omega, z) = c^\top x_\omega + q_\omega^\top y_\omega + \underline{\rho}_\omega^\top [x_\omega - z]^- + \overline{\rho}_\omega^\top [z - x_\omega]^-$$

Penalty-based ADM

Adapting the framework, we end up with a familiar setting

Primal Problem

$$\max_{x,y} \sum_{\omega\in\Omega} p_\omega \phi_\omega^\rho(x_\omega, y_\omega, \bar{z})$$

$$x_\omega \in X, \forall \omega \in \Omega$$

$$y_\omega \in Y_\omega, \forall \omega \in \Omega$$

$$\rho = (\underline{\rho}, \overline{\rho})_{\omega\in\Omega}$$

$$\underline{\rho}_\omega^{k+1} = \underline{\rho}_\omega + \gamma[x_\omega - z]^-$$

$$\overline{\rho}_\omega^{k+1} = \overline{\rho}_\omega + \gamma[z - x_\omega]^-$$

Penalty update

(x, y)

z

$$\max_{z} \sum_{\omega\in\Omega} p_\omega \phi_\omega^\rho(\bar{x}_\omega, \bar{y}_\omega, z)$$

Computational Results

Preliminary experiments have been conducted with two distinct test-sets:

- 100 instances of the Stochastic Server Location Problem (SSLP) from Ntamio and Sen (2008)
  - SSLP 10-50-100 and SSLP 15-30-100
- 100 instances of the Capacited Facility Location Problem (CAP) from Bodur et. al (2014)
  - CAP101-250 and CAP111-250

Computational Results

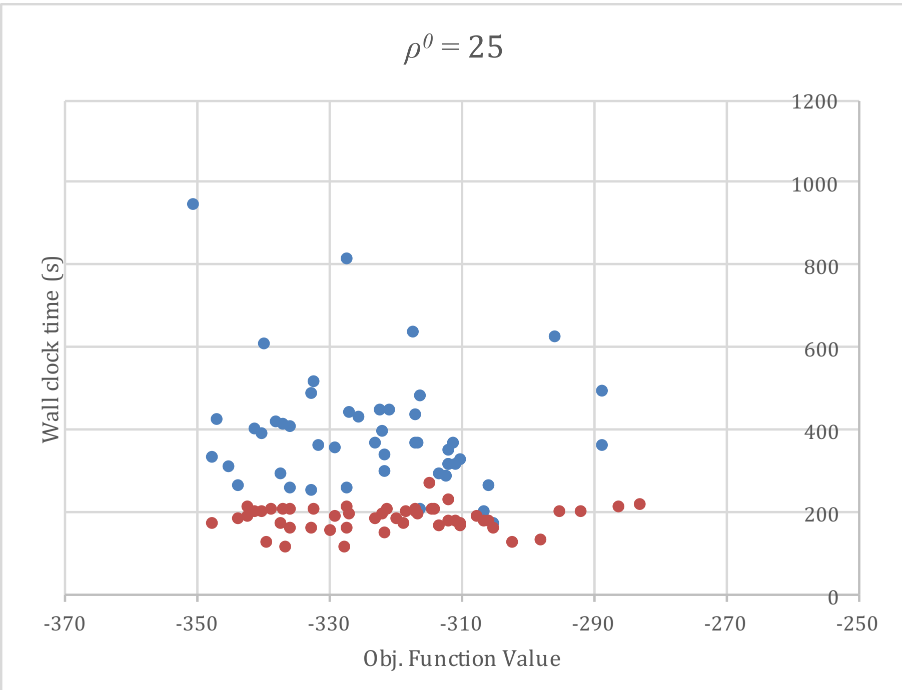
Results Summary

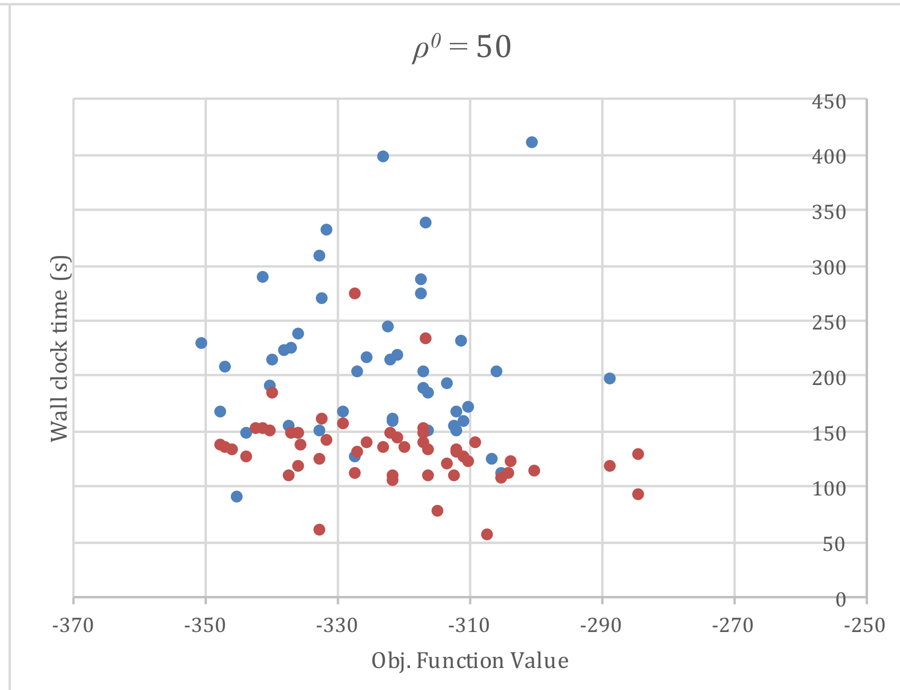
Problem	Penalty	Avg. obj. dif. (%)	Avg. speed up	PH feas. rate
SSLP 10-50-100	25	0.604%	2.232	96%
	50	0.191%	1.574	94%
	100	0.360%	1.321	82%
SSLP 15-30-100	25	1.634%	1.318	76%
	50	-1.447%	1.042	84%
	100	-1.452%	1.908	74%
CAP 101-250	5000	0.232%	2.284	92%
	7500	0.105%	2.247	98%
	10000	0.120%	2.517	100%
CAP 111-250	5000	0.020%	2.560	100%
	7500	0.059%	2.357	100%
	10000	0.087%	2.644	98%

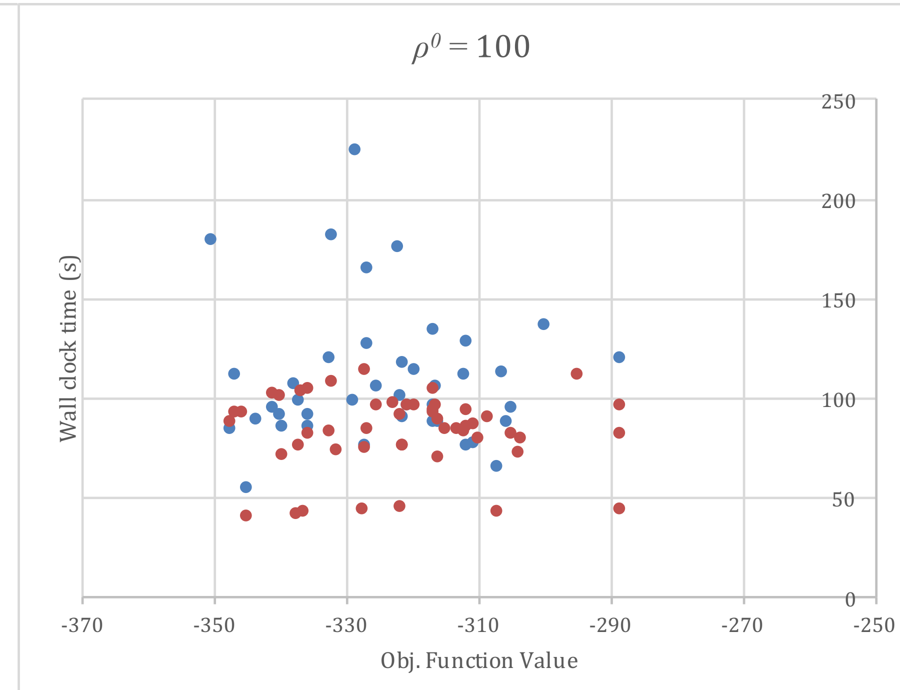
Computational Results

PH
ADM

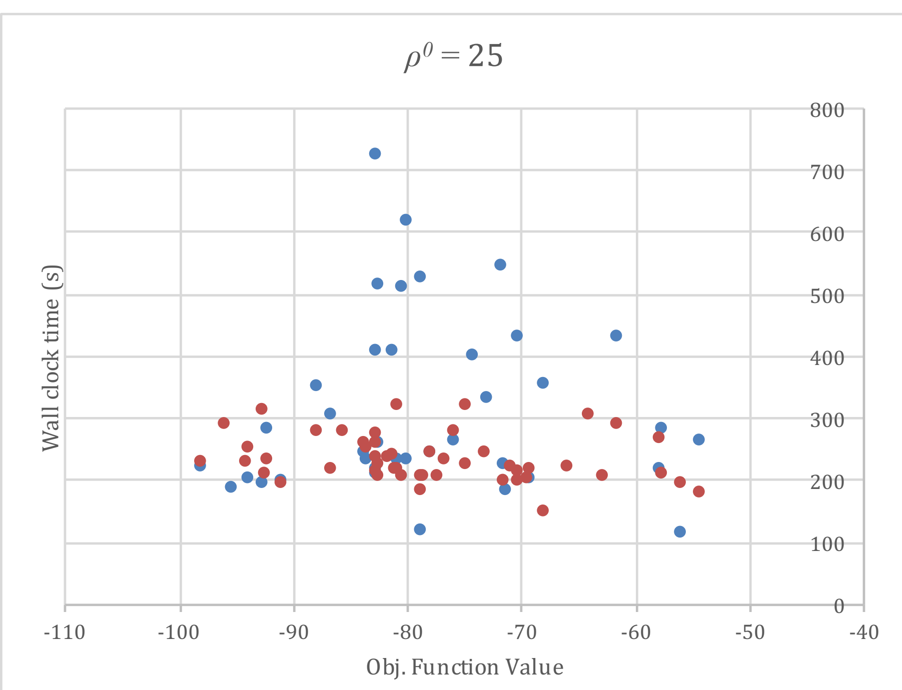
SSLP 10-50-100

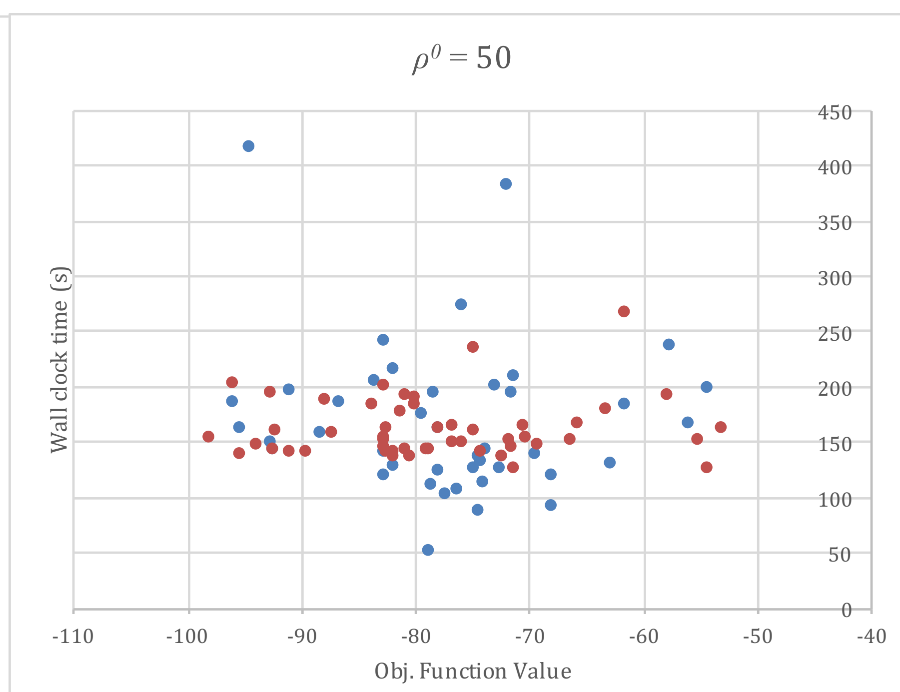


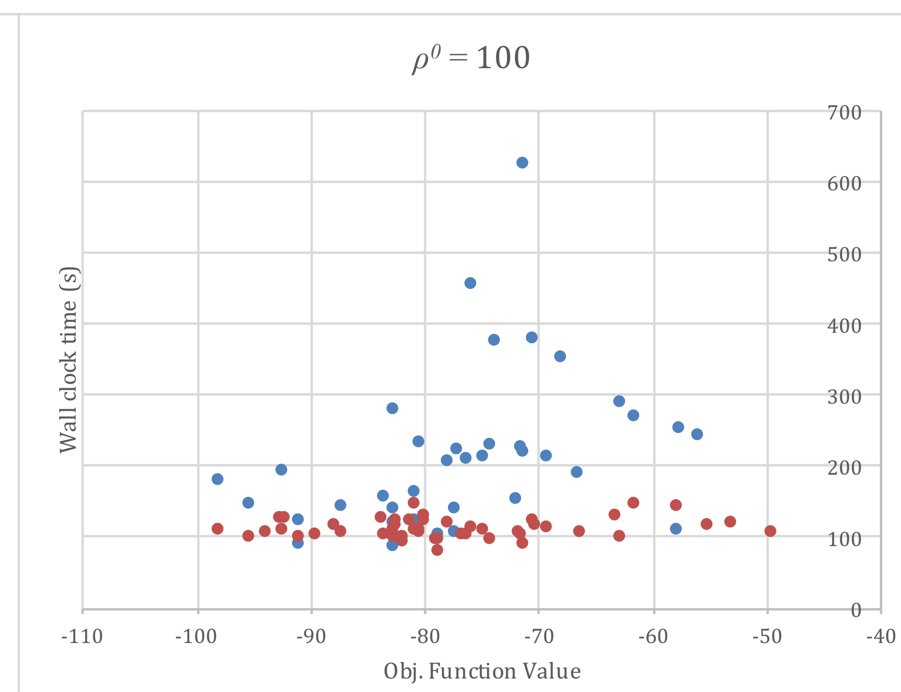




SSLP 15-30-100



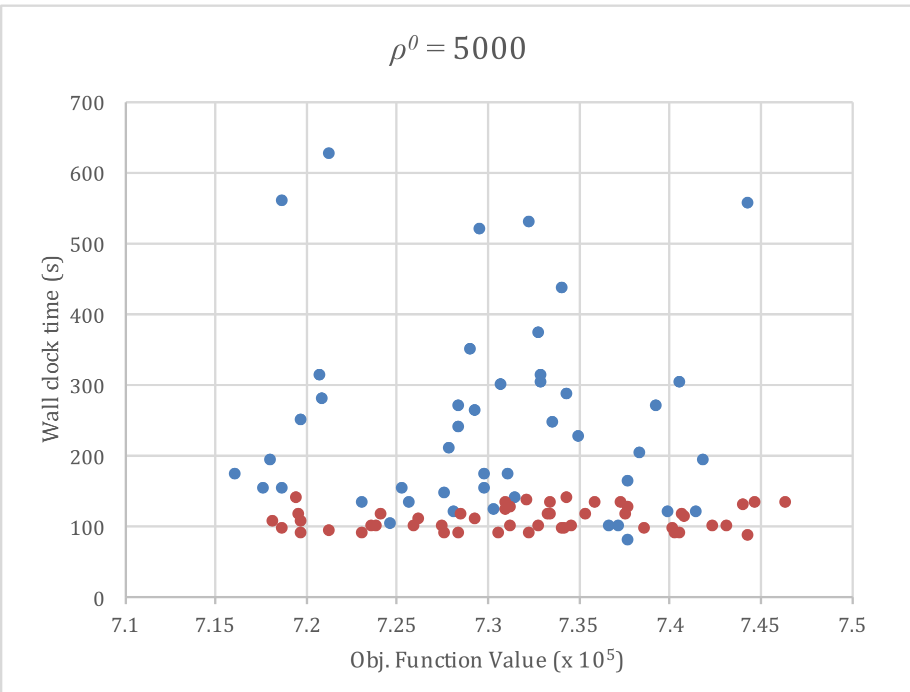


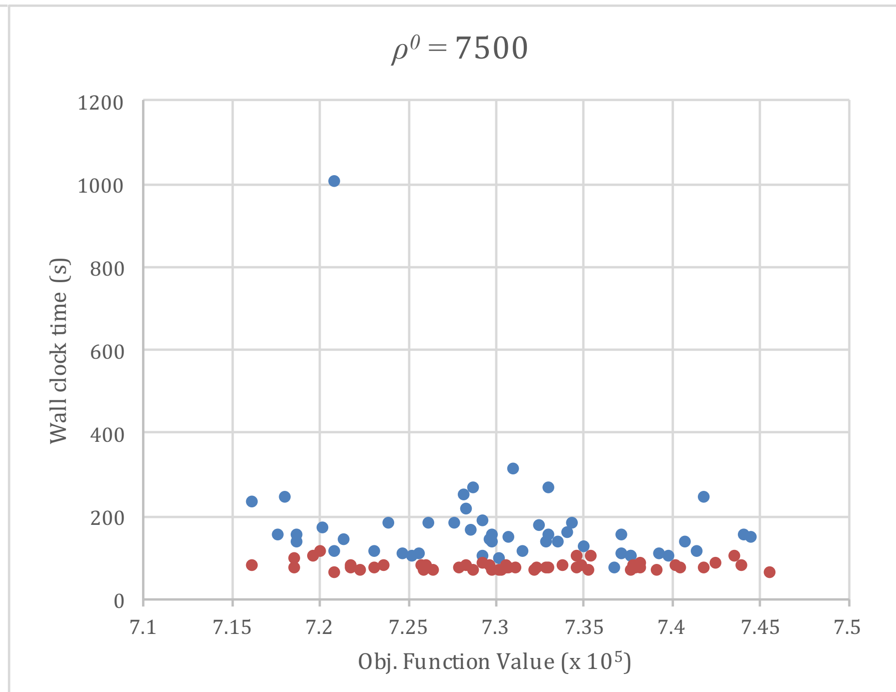


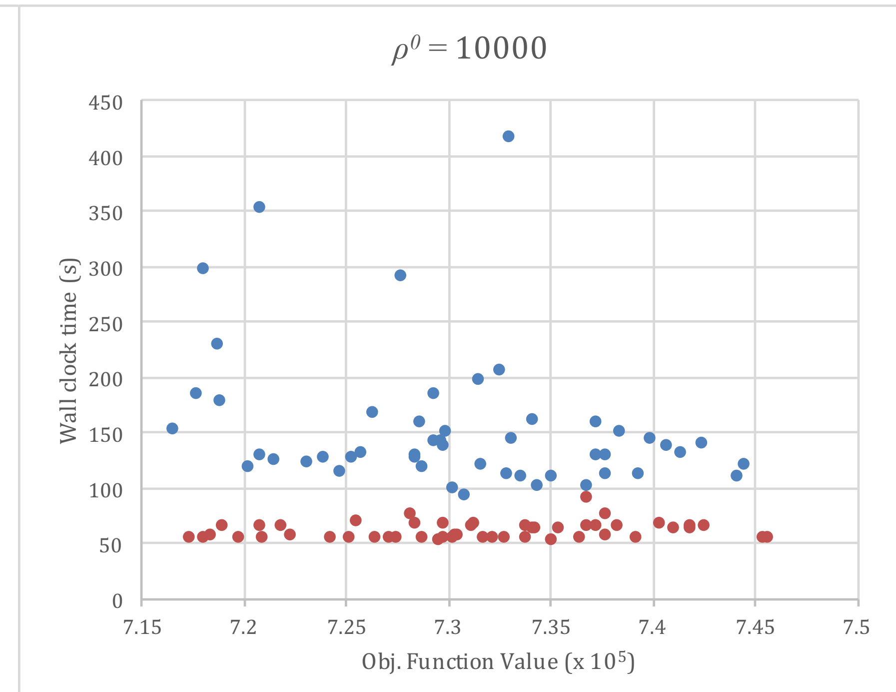
Computational Results

PH
ADM

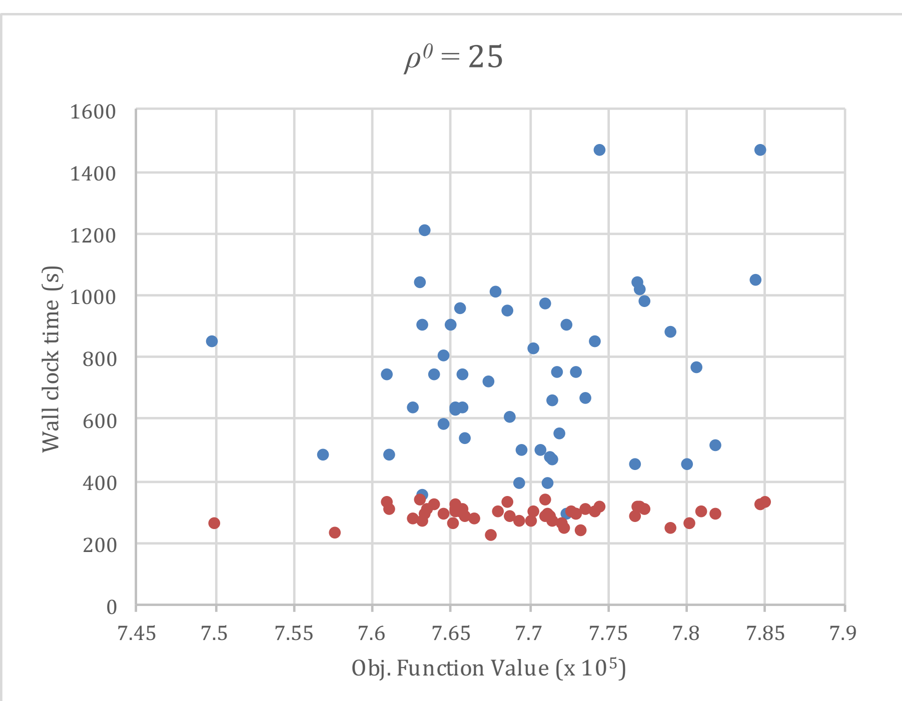
CAP101-250

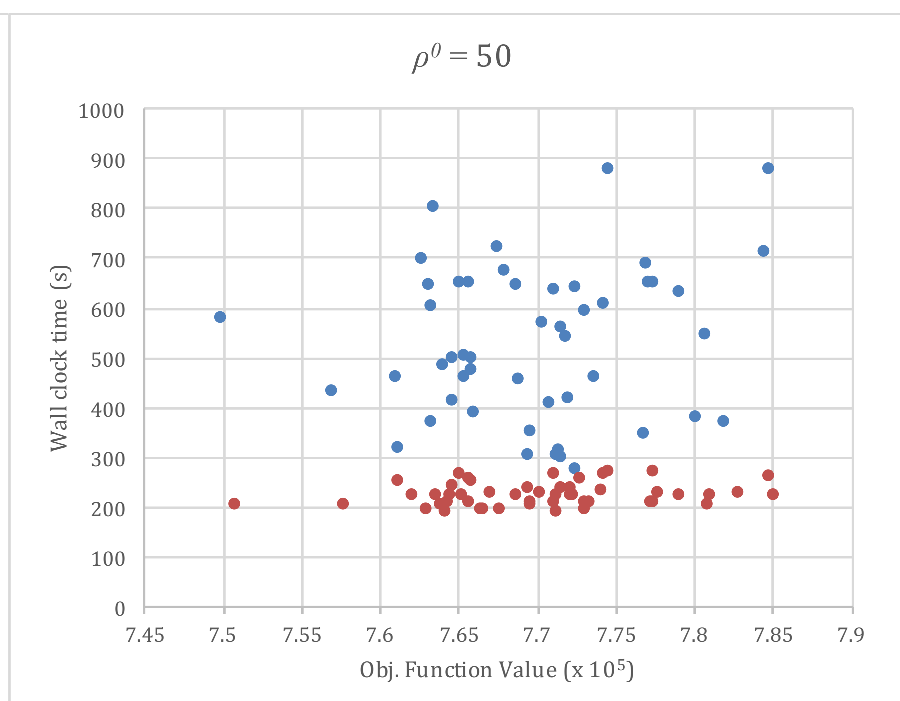


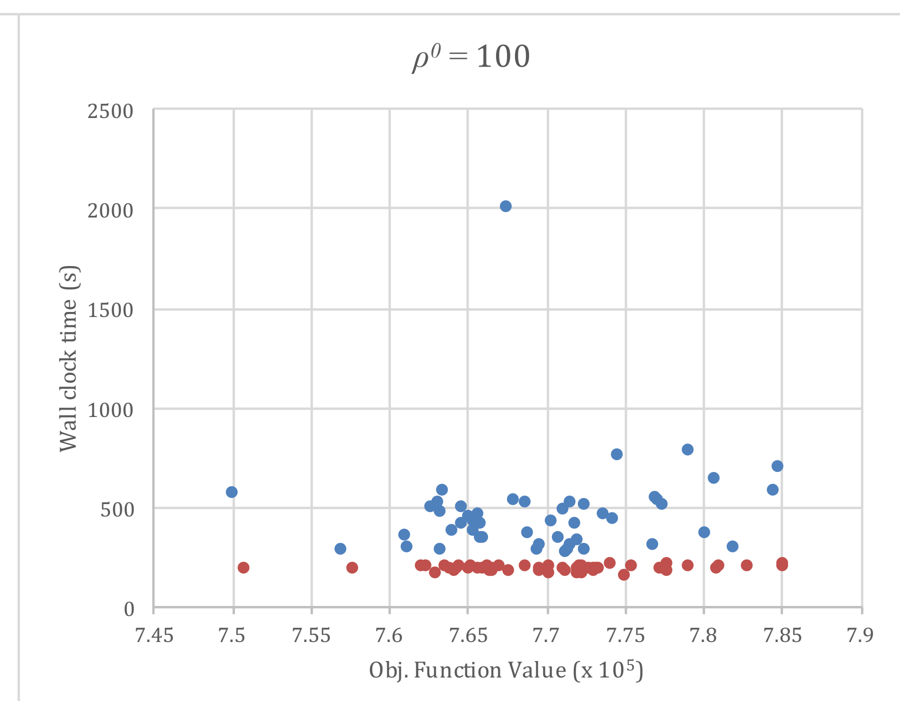




CAP111-250







Conclusions

Take-aways:

1. As the numerical results suggests, there is a trade-off between solution quality and solution time.

2. The use of a polyhedral penalty term seems to cause premature convergence, possibly due to non-smoothness of the augmented objective function.

3. For problems with complete recourse, the method provides feasible solutions during the iterative process and does not present cycling behaviour.

Next steps and further research directions:

- Develop optimal convergence proof (if possible).

- Computations: try other instances and multi-stage problems.

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Q/A

Thank you!

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