Efficiently Solving Stochastic Mixed-Integer Problems combining Gauss-Siedel and Penalty-Based methods

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Introduction

We are interested in solving problems of the form

$$\begin{aligned}\zeta^{SIP} := \ & \min_{x,y} c^\top x + \sum_{s \in S} p_s(q_s^\top y_s) \\ \text{s.t.: } & x \in X \\ & y_s \in Y_s(x), \ \forall s \in S,\end{aligned}$$

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Sets $X \subset \mathbb{R}^{n_x}$ and $Y_s(x) \subset \mathbb{R}^{n_y \times |S|}$ define linear constraints and integrality restrictions on x and y.

An alternative equivalent formulation allow us to exploit the block-angular structure that this problem naturally presents.

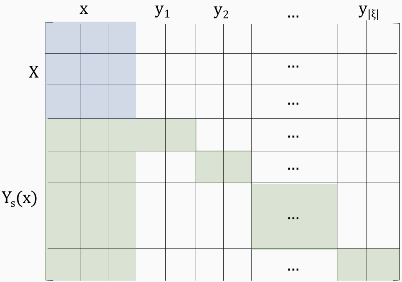
Exploiting in this case means: *find means to solve small bits in parallel rather than massive MIPs*.

$$\zeta^{SIP} := \min_{x,y,z} \sum_{s \in S} p_s(c^\top x_s + q_s^\top y_s)$$

$$\text{s.t.: } x_s - z = 0, \ \forall s \in S \text{ (NAC)}$$

$$x_s \in X, \ \forall s \in S$$

$$y_s \in Y_s(x_s), \ \forall s \in S.$$



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Lagrangian Relaxation

The most straightforward approach is to relax NAC. First, let

$$\begin{aligned}
\zeta^{LR}(\omega) := \; & \min_{x,y,z} \sum_{s \in S} p_s L_s(x_s, y_s, z, \omega) \\
& \text{s.t.: } x_s \in X, \; \forall s \in S \\
& \qquad y_s \in Y_s(x_s), \; \forall s \in S,
\end{aligned}$$

where $\omega := (\omega_s)_{s \in S} \in \Omega := \{\omega \mid \sum_{s \in S} p_s^\top \omega_s = 0\}$ and

$$\begin{aligned}
L_s(x_s, y_s, z, \omega_s) &:= c^\top x_s + q_s^\top y_s + \omega_s^\top (x_s - z), \; \forall s \in S \\
&:= (c + \omega_s)^\top x_s + q_s^\top y_s, \; \forall s \in S.
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Then, we focus on solving the Lagrangian Dual

$$\zeta^{LD} := \max_{\omega \in \Omega} \zeta^{LR}(\omega).$$

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Augmented Lagrangian Relaxation

Alternatively, we can define the (augmented) Lagrangian as

$$\begin{aligned}\zeta_\rho^{LR+}(\omega) := \ & \min_{x,y,z} \sum_{s \in S} p_s L^s(x_s, y_s, z, \omega) + \psi_\rho^s(x_s - z) \\ & \text{s.t.: } x_s \in X, \ \forall s \in S \\ & \qquad y_s \in Y_s(x_s), \ \forall s \in S,\end{aligned}$$

where $\omega = (\omega_s)_{s \in S} \in \Omega$ and $\psi_\rho^s : \mathbb{R}^{n_x} \mapsto \mathbb{R}$ is an appropriate penalty function that depends on the penalty parameter ρ.

Augmented Lagrangian Relaxation

Most common choice: $\psi_{\rho}^{s}(u_s) := \frac{\rho}{2}||u_s||_2^2$ for each $s \in S$, which provides smoothness to the original Lagrangian dual function.

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Main motivations

[Feizollahi et al., 2016, Boland and Eberhard, 2015] have shown that the augmented Lagrangian dual is capable of asymptotically achieving zero duality gap only if the weight ρ is allowed to go to infinity.

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Main motivations

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However, it is possible to circumvent this drawback if the augmentation of the Lagrangian dual is made *using a norm as the penalty function*. In this case, the theory suggests that it is possible to attain strong duality for a finite value of ρ.

The Framework: Combining Penalty-based and Gauss-Siedel Approaches

Combining Penalty-based and Gauss-Siedel Approaches

What leads our definition of a good penalty function is this result:

Theorem 1 (based on [Feizollahi et al., 2016, Thm. 5])

Consider a feasible MIP problem given whose problem data is formed from rational entries and with its optimal value bounded. If $\psi : \prod_{s \in S} \mathbb{R}^{n_x} \mapsto \mathbb{R}$ is a summed augmenting function $\psi(u) := \sum_{s \in S} \psi^s_\rho(u_s)$ for problem $\zeta^{LR+}_\rho(\omega)$ such that

1. $\psi(0) = 0$
2. $\psi(u) \geq \delta > 0, \forall u \notin V$
3. $\psi(u) \geq \gamma ||u||_\infty, \forall u \in V$

for some open neighbourhood V of 0, and positive scalars $\delta, \gamma > 0$, then there exists a finite ρ such that $\zeta^{LD+}_\rho = \zeta^{LR+}_\rho(\omega) = \zeta^{SIP}$, for any ω.

Combining Penalty-based and Gauss-Siedel Approaches

We propose a class of augmenting functions based on the use of positive basis [Davis, 1954].

Given discrepancy a vector $u := (u_s)_{s \in S} \in \prod_{s \in S} \mathbb{R}^{n_x}$, we define for each scenario s the penalty function

$$\psi_\rho^s(u_s) := \underline{\rho}_s^\top [u_s]^- + \overline{\rho}_s^\top [-u_s]^-,$$

where $\rho = (\underline{\rho}_s, \overline{\rho}_s)_{s \in S} \in \mathbb{R}_{>0}^{2n_x|S|}$ and $[v]^- := -\min\{0, v\}$ (performed component wise), where in this case $v \in \mathbb{R}^{n_x}$. Then we define

$$\psi_\rho(u) := \sum_{s \in S} \psi_\rho^s(u_s) = \left(\sum_{s \in S} \underline{\rho}_s^\top [u_s]^- + \sum_{s \in S} \overline{\rho}_s^\top [-u_s]^- \right).$$

Combining Penalty-based and Gauss-Siedel Approaches

The reason for using positive basis is because it encompasses a set of properties that allows us to arrive at the following result:

Corollary 8 (highlights)

Assume that ζ^{SIP} is feasible, its optimal value is finite, and the data which defines it is rational. Then the optimal value of the augmented Lagrangian dual problem ζ_ρ^{LD+} using an augmenting function of the form of

$$\psi_1^N(u) := \sum_{i=1}^{l} \max\{\mathbf{n}_i^\top u, 0\}$$

where any positive basis $N := \{\mathbf{n}_1, \ldots, \mathbf{n}_l\}$ is equal to the optimal value of ζ^{SIP} for some finite ρ; that is,

$$\zeta_\rho^{LD+} = \zeta_\rho^{LR+}(\omega) = \zeta^{SIP}$$

Combining Penalty-based and Gauss-Siedel Approaches

Which gives us hope because it is easy to show that, picking a positive basis to be $N_\rho = \{\overline{\rho}_{s,i} e_{i+(s-1)n_x} \mid s \in S, i \in \{1, \dots, n_x\}\} \cup \{-\underline{\rho}_{s,i} e_{i+(s-1)n_x} \mid s \in S, i \in \{1, \dots, n_x\}\}$, we have

$$\begin{aligned}
\psi_1^{N_\rho}(u) &= \sum_{s \in S} \sum_{i=1,\dots,n_x} \overline{\rho}_{s,i} \max\{0, u_{s,i}\} \\
&\quad + \sum_{s \in S} \sum_{i=1,\dots,n_x} \underline{\rho}_{s,i} \max\{0, -u_{s,i}\} \\
&= \left(\sum_{s \in S} \underline{\rho}_s^\top [u_s]^- + \sum_{s \in S} \overline{\rho}_s^\top [-u_s]^- \right) \\
&= \sum_{s \in S} \psi_\rho^s(u) = \psi_\rho(u)
\end{aligned}$$

Combining Penalty-based and Gauss-Siedel Approaches

Building upon what we have so far, we end-up with the following augmented Lagrangian problem

$$\begin{aligned}\zeta_{\rho}^{LR+}(\omega): \ \min_{x,y,z} & \sum_{s\in S} p_s([c+\omega]^\top x_s + q_s^\top y_s) \\ & + \sum_{s\in S} \underline{\rho}_s^\top [x_s - z]^- + \sum_{s\in S} \overline{\rho}_s^\top [z - x_s]^- \\ \text{s.t.: } & x_s \in X, \ \forall s \in S \\ & y_s \in Y_s(x_s), \ \forall s \in S,\end{aligned}$$

which, with the help of Thm 1 and Cor 8, can be written as

$$\begin{aligned}\zeta_{\rho}^{LR+}(0): \ \min_{x,y,z} & \sum_{s\in S} p_s(c^\top x_s + q_s^\top y_s) + \sum_{s\in S} \underline{\rho}_s^\top [x_s - z]^- + \sum_{s\in S} \overline{\rho}_s^\top [z - x_s]^- \\ \text{s.t.: } & x_s \in X, \ \forall s \in S \\ & y_s \in Y_s(x_s), \ \forall s \in S.\end{aligned}$$

Combining Penalty-based and Gauss-Siedel Approaches

Nonlinear Block Gauss-Siedel Method

In principle, one can only solve ζ_{ρ}^{LR+} individually scenario wise if the variable z has a known (or fixed) value. This motivates the application of Nonlinear Block Gauss-Siedel (GS) approach.

Suppose we want to solve:

$$\begin{aligned} \zeta := \min_{x,z} \ & f(x,z) \\ \text{s.t.: } & x \in X, z \in Z, \end{aligned}$$

with f is convex, but not necessarily differentiable and sets X and Z closed, but not necessarily convex.

Combining Penalty-based and Gauss-Siedel Approaches

The idea is to use the following algorithmic setting.

Algorithm 1 A block GS method

1: **initialise** $(x^0, z^0) \in X \times Z$
2: **for** $k = 1, \dots, k_{\max}$ **do**
3: $\quad x^k \leftarrow \operatorname{argmin}_x \left\{ f(x, z^{k-1}) : x \in X \right\}$
4: $\quad z^k \leftarrow \operatorname{argmin}_z \left\{ f(x^k, z) : z \in Z \right\}$
5: $\quad k \leftarrow k + 1$
6: **end for**
7: **return** $(x^{k_{\max}}, z^{k_{\max}})$

Combining Penalty-based and Gauss-Siedel Approaches

Proposition 11

For problem ζ, let f be continuous and bounded from below, and let X and Z be compact. Then the limit points (x^*, z^*) of the sequence $\{(x^k, z^k)\}$ generated by iterations of Algorithm 1 are partial minima, i.e.

$$f(x^*, z^*) \leq f(x, z^*), \ \forall x \in X,$$
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$$f(x^*, z^*) \leq f(x, z^*),\ \forall x \in X,$$
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The tricky bit: In the more general setting where f is non-differentiable and/or X and Z are non-convex, a partial minimum is not necessarily a global, or even a local, minimum.

The algorithm

Implementation aspects

Let $\phi^\rho(x, y, z, \rho) := \sum_{s \in S} p_s \phi^\rho_s(x_s, y_s, z, \mu_s)$, where

$$\phi^\rho_s(x_s, y_s, z, \mu_s) := \left\{ c^\top x_s + q_s^\top y_s + \underline{\mu}_s^\top [x_s - z]^- + \overline{\mu}_s^\top [z - x_s]^- \right\}$$

and $(\underline{\mu}_s, \overline{\mu}_s) := (\frac{1}{p_s}\underline{\rho}_s, \frac{1}{p_s}\overline{\rho}_s)$ for each $s \in S$.

For a given $\rho^k_s = (\underline{\rho}^k_s, \overline{\rho}^k_s)_{s \in S}$ and an initial $z^{0,0}$, we iterate between subproblems $l = 0, 1, \ldots, l_{\max}$:

$$\begin{aligned}
(x^{k,l+1}, y^{k,l+1})_{s \in S} \leftarrow \operatorname*{argmin}_{x,y} \ & \phi^\rho(x, y, z^{k,l}, \rho^k) \\
\text{s.t.: } & x_s \in X, \ \forall s \in S \\
& y_s \in Y_s(x_s), \ \forall s \in S; \\
z^{k,l+1} \leftarrow \operatorname*{argmin}_{z} \ & \phi^\rho(x^{k,l+1}, y^{k,l+1}, z, \rho^k),
\end{aligned}$$

followed by $l = l + 1$ and successive repetition until *partial convergence* is approximately achieved.

Proposed Algorithm

Algorithm 2 Alternating direction method for SMIP

initialise $\rho^0 = (\underline{\rho}^0, \overline{\rho}^0)$, $\hat{z}^0$, ϵ, γ, β, $l_{\max}$, $k_{\max}$
for $k = 1, \ldots, k_{\max}$ **do**
  $z^{k,0} \leftarrow \hat{z}^{k-1}$
  for $l = 1, \ldots, l_{\max}$ **do**
    for $s \in S$ **do**
      $(x_s^{k,l}, y_s^{k,l}) \leftarrow \text{argmin}_{x,y} \left\{ \phi^{\rho,k}(x_s, y_s, z^{k,l-1}, \rho^k) : x_s \in X, y_s \in Y_s(x_s) \right\}$
    end for
    $z^{k,l} \leftarrow \text{argmin}_z \; \phi^{\rho,k}(x^{k,l}, y^{k,l}, z, \rho^k)$
    $\Gamma \leftarrow \phi^{\rho,k}(x^{k,l-1}, y^{k,l-1}, z^{k,l-1}, \rho^k) - \phi^{\rho,k}(x^{k,l}, y^{k,l}, z^{k,l}, \rho^k)$
    if $\Gamma \leq \epsilon$ **or** $l = l_{\max}$ **then**
      $(\hat{x}_s^k, \hat{y}_s^k) \leftarrow (x_s^{k,l}, y_s^{k,l})$ for all $s \in S$
      $\hat{z}^k \leftarrow z^{k,l}$
      break
    end if
    $l \leftarrow l + 1$
  end for
  if $||\hat{x}^k - \hat{z}^k||_2^2 \leq \epsilon$ **or** $k = k_{\max}$ **then**
    return $((\hat{x}_s^k, \hat{y}_s^k)_{s \in S}, \hat{z}^k)$
  else
    $\underline{\rho}_s^k = \underline{\rho}_s^{k-1} + \gamma[\hat{x}_s^k - \hat{z}^k]^-$ for all $s \in S$
    $\overline{\rho}_s^k = \overline{\rho}_s^{k-1} + \gamma[\hat{z}^k - \hat{x}_s^k]^-$ for all $s \in S$
end if
$k \leftarrow k + 1$
end for

Computational Experiments and Numerical Results

Experimental setting

- Three classes of problems from literature: SSLP, CAP, and DCAP;
- 50 random instances for 2 problems in each class: a total of 300 instances;
- The performance was compared Progressive Hedging (PH) method (which combines GS and $\psi_\rho^s(u_s) := \frac{\rho}{2}||u_s||_2^2$);
- instances were solved with three parameter choices for PH (different choices of ρ) and 12 combinations of parameter choices for PBGS (different choices of ρ^0, β and γ);
- Hardware: Intel i7 CPU with 3.40GHz and 8GB of RAM; software: AIMMS 3.14 using CPLEX 12.6.3.

Experimental Results

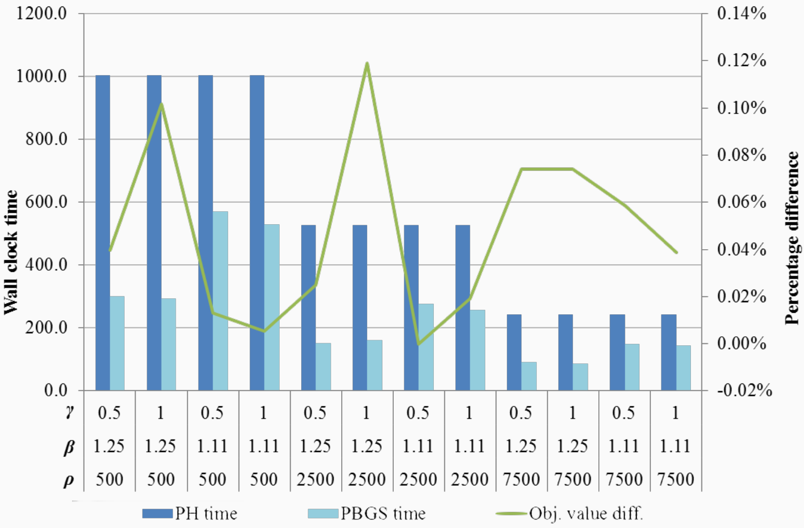

Figure 1: CAP111

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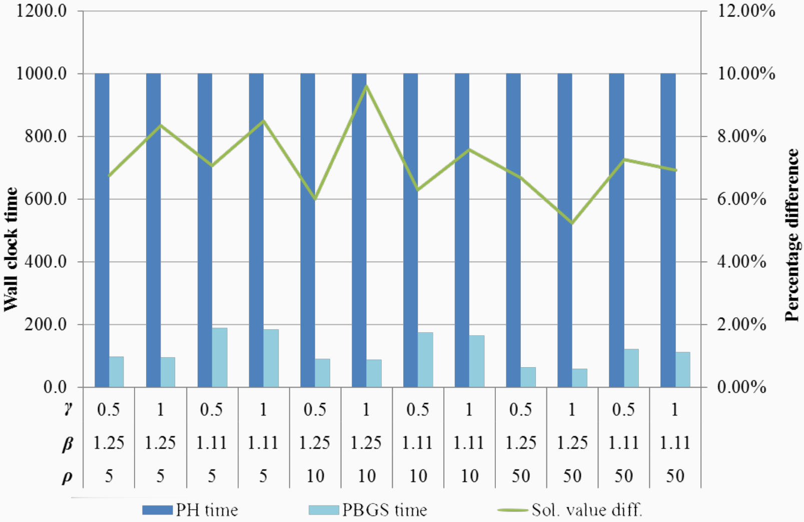

Figure 2: DCAP342

Experimental Results

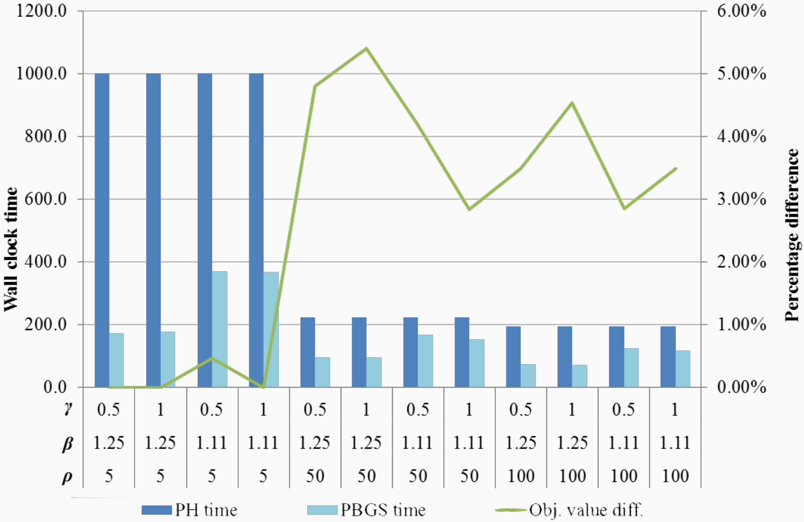

Figure 3: SSLP10-50

Take-aways: Conclusions and Future Directions

Conclusions

Key points

- A competitive approach in terms of computational efficient;
- Readily amenable to parallelisation, which is a key point for dealing with large-scale SMIPs;
- Theoretical results are encouraging regarding alternative frameworks for defining penalty functions.

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Future directions

- Extensions of the block Gauss-Seidel approach into non-smooth non-separable problems.
- Better understanding of how updates of the penalty coefficients can improve the block Gauss-Seidel behaviour.
- Evaluate the proposed approach in contexts other than SMIPs and considering its extension to the multi-stage case.

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