Optimisation under Uncertainty
Session 3/4

I Workshop de Otimização sob Incerteza - UFSCar

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Outline of this lecture

What is robust optimisation

An alternative paradigm for taking uncertainty into account:

- Permeated by the notion of worst-case;
- Control of the degree of conservatism;
- Parallels with chance constraints and risk measures.

In robust optimisation, feasibility is the key concern

- Can be extended to objective function performance requirements;
- May or may not be scenario-based;
- Static v. adaptable: the presence of recourse decisions;
- Exception: distributionally robust optimisation.

Robust optimisation approaches

The key notion in robust optimisation is tie to that of an uncertainty set

- The “region” U within the uncertainty support Ξ within which parameter realisation does not turn the solution infeasible;
- Tractability is closely tied to the geometry of such uncertainty sets.

$$\begin{aligned}
\min_x.\ & c^\top x \\
\text{s.t.: } & Ax \leq b \\
& x \in X
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\min_x.\ & c^\top x \\
\text{s.t.: } & A(\eta)x \leq b, \ \forall \eta \in U \subseteq \Xi \\
& x \in X
\end{aligned}$$

$$\Downarrow$$

$$\begin{aligned}
\min_x.\ & c^\top x \\
\text{s.t.: } & \max_{\eta \in U \subseteq \Xi} A(\eta)x \leq b \\
& x \in X.
\end{aligned}$$

Robust counterparts

Let $\tilde{a}_{ij} \in J_i$ be the uncertain elements in the matrix $A_{m \times n}$

- Random variables $\tilde{a}_{ij}$ with "central value" a_{ij} and "maximum deviation" $\hat{a}_{ij}$;
- symmetric and with bounded support $\tilde{a}_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$.

Let $\eta_{ij} = \frac{(\tilde{a}_{ij} - a_{ij})}{\hat{a}_{ij}}$. Thus $\eta_{ij} \in [-1, 1]$ and follows the same distribution as $\tilde{a}_{ij}$, but centred in zero and scaled.

Our robust counterpart[1] is the following bilevel problem:

$$
\begin{aligned}
\min_x .\ & c^\top x \\
\text{s.t.: } & a_{ij}x_j + \max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j \right\} \leq b_i, \forall i \in [m] \\
& x_j \geq 0, \ \forall j \in [n].
\end{aligned}
\tag{RC}
$$

[1] Assuming, w.l.g., $a_{ij} \geq 0$, $\forall i \in [m], \forall j \in [n]$

Uncertainty set geometries

Box uncertainty set [Soyster, 1973]

- Maximum protection level;
- All parameters take their worst-possible value;
- Simple, but highly conservative.

The uncertainty set is

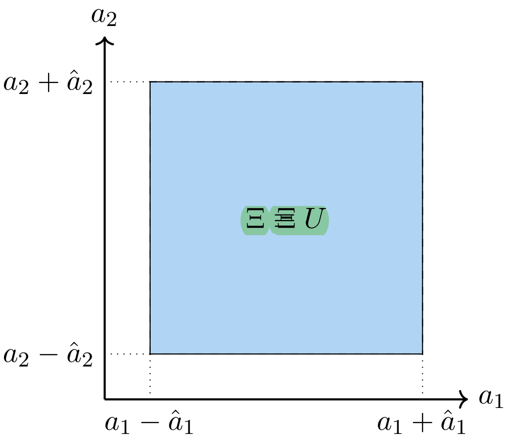
$$U_i = \{\eta_i : ||\eta_i||_1 \leq |J_i|\} \equiv \{\eta_{ij} : |\eta_{ij}| \leq 1, \forall j \in J_i\}$$

The lower-level problem becomes

$$\max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : |\eta_{ij}| \leq 1, \forall j \in J_i \right\} = \sum_{j \in J_i} \hat{a}_{ij} x_j.$$

Uncertainty set geometries

Box uncertainty set [Soyster, 1973]



Uncertainty set geometries

Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]

- Softens extreme-case protection;
- Parametrically controlled;
- Leads to smooth sets;
- (MI)SOCPs which are more computationally demanding.

The uncertainty set is

$$U_i = \{\eta_i : ||\eta_i||_2 \leq \Gamma_i\} \equiv \left\{ \eta_{i\cancel{j}} : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma_i^2, \forall j \in J_i \right\}$$

The lower-level problem becomes

$$\underset{\eta_i \in U_i}{\text{max.}} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma_i^2, \forall j \in J_i \right\}.$$

Uncertainty set geometries

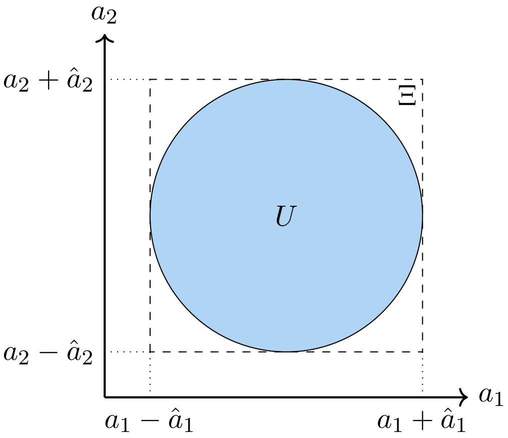
Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]

Again, this uncertainty set has a closed-form solution:

$$
\begin{aligned}
&\max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : \sum_{j \in J_i} \eta_{ij}^2 \le \Gamma_i^2, \forall j \in J_i \right\} \\
&= \max_{\eta_i \in U_i} \left\{ \sqrt{\left(\sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j \right)^2} : \sum_{j \in J_i} \eta_{ij}^2 \le \Gamma_i^2, \forall j \in J_i \right\} \\
&= \max_{\eta_i \in U_i} \left\{ \sqrt{\left(\sum_{j \in J_i} \eta_{ij} \right)^2 \left(\sum_{j \in J_i} \hat{a}_{ij} x_j \right)^2} : \sum_{j \in J_i} \eta_{ij}^2 \le \Gamma_i^2, \forall j \in J_i \right\} \\
&= \Gamma_i \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 x_j^2}
\end{aligned}
$$

Uncertainty set geometries

Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]



Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]

- Allows for controlling conservatism;
- Retains problem complexity;
- Budget of uncertainty lacks interpretability.

The uncertainty set is

$$U_i = \{\eta_i : ||\eta_i||_1 \leq \Gamma_i\} \equiv \left\{ \eta_{i\cancel{j}} : \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i, \forall j \in J_i \right\}.$$

The lower-level problem becomes

$$\max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i, \; 0 \leq \eta_{ij} \leq 1, \; \forall j \in J_i \right\}.$$

Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]

In this case, the lower-level problem does not admit a closed form. However, it is a linear program.

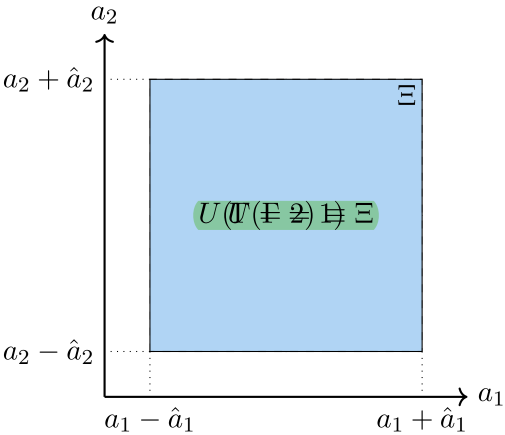
- Strong duality (primal-dual equivalence) is available;
- True for any convex[2] lower-level problem.

$$
\begin{aligned}
\max_{\eta_i \in U_i} \ & \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j \\
\text{s.t.: } & \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i \ (\pi_i) \\
& 0 \leq \eta_{ij} \leq 1, \ (p_{ij}) \ \forall j \in J_i
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
\min_{\pi_i, p_i} \ & \Gamma_i \pi_i + \sum_{j \in J_i} p_{ij} \\
\text{s.t.: } & \pi_i + p_{ij} \geq \hat{a}_{ij} x_j, \forall j \in J_i \\
& p_{ij} \geq 0, \ \forall j \in J_i \\
& \pi_i \geq 0.
\end{aligned}
$$

[2] Satisfying some constraint qualification.

Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]



Robust counterparts

Let us consider a knapsack problem of the form

$$
\begin{aligned}
\text{max. } & c^\top x \\
\text{s.t.: } & \sum_{j \in [n]} a_j x_j \leq b \\
& 0 \leq x_j \leq 1, \ \forall j \in [n].
\end{aligned}
$$

The robust counterparts for the previous uncertainty sets are:

Box

$$
\begin{aligned}
\text{max. } & c^\top x \\
\text{s.t.: } & \sum_{j \in [n]} (a_j + \hat{a}_{ij}) x_j \leq b \\
& 0 \leq x_j \leq 1, \ \forall j \in [n].
\end{aligned}
$$

Ellipsoid

$$
\begin{aligned}
\text{max. } & c^\top x \\
\text{s.t.: } & \sum_{j \in [n]} a_j x_j + \Gamma \sqrt{\sum_{j \in [n]} (\hat{a} x_j)^2} \leq b \\
& 0 \leq x_j \leq 1, \ \forall j \in [n].
\end{aligned}
$$

Robust counterparts

Let us consider a knapsack problem of the form

$$
\begin{aligned}
\text{max. } & c^\top x \\
\text{s.t.: } & \sum_{j \in [n]} a_j x_j \leq b \\
& 0 \leq x_j \leq 1, \; \forall j \in [n].
\end{aligned}
$$

The robust counterparts for the previous uncertainty sets are:

Polyhedral

$$
\begin{aligned}
\text{max. } & c^\top x \\
\text{s.t.: } & \sum_{j \in [n]} a_j x_j + \Gamma \pi + \sum_{j \in [n]} p_j \leq b \\
& \pi + p_j \geq \hat{a}_j x_j, \; \forall j \in [n] \\
& 0 \leq x_j \leq 1, p_j \geq 0, \; \forall j \in [n] \\
& \pi \geq 0.
\end{aligned}
$$

On constraint violation probabilities

Arguably, Bertsimas & Sim (2004) raised attention to robust optimisation with "the price of robustness".

- The price refers to the optimality traded off for feasibility guarantees;
- Quantifying these trade-offs can be done:
  1. Using theoretical bounds;
  2. Via simulating solution performance.
- In my own experience, theoretical bounds are often loose.

For example, [Bertsimas and Sim, 2004] show the probability of violation of constraint $i \in [m]$ to be

$$P^{\text{vio}} = P\left(a_{ij}x_j + \max_{\eta_i \in U_i}\left\{\sum_{j \in J_i} \eta_{ij}\hat{a}_{ij}x_j\right\} > b_i\right) \leq e^{\frac{-\Gamma^2}{2|J_i|}},$$

Multi-stage robust optimisation

We focus on 2-stage adjustable robust optimisation (ARO) problems:

$$\begin{aligned} \min\ & c^\top x + \max_{\xi \in U \subset \Xi} \min_y q^\top y(\xi) \\ \text{s.t.: } & Ax = b,\ x \geq 0 \\ & T(\xi)x + Wy(\xi) = h(\xi),\ \forall \xi \in U \subset \Xi \\ & y(\xi) \geq 0,\ \forall \xi \in \Xi. \end{aligned} \tag{ARO}$$

- Only RHS uncertainty: $T(\xi) = T$, $W(\xi) = W$, and $q(\xi) = q\ \forall \xi \in \Xi$;
- Assumption often necessary to eliminate quadratic dependence between ξ and decision variables;
- Not necessary if the uncertainty set is discrete and finite (scenarios)

A side note: min-max, minimum regret and related

If the uncertainty set is a finite and discrete set of scenarios, we have that

$$\begin{aligned}\min_{x}.\ & c^\top x + \max_{s \in U}.\ \min_{y}.\ q_s^\top y_s \\ \text{s.t.:}\ & T_s x + W_s y_s \leq h_s,\ \forall s \in U \\ & x \in X\end{aligned}$$

is a tractable ARO [Mulvey et al., 1995]. Variants include:

Min-max

$$\begin{aligned}\min_{x}.\ & c^\top x + \theta \\ & \theta \geq q_s^\top y_s,\ \forall s \in U \\ \text{s.t.:}\ & T_s x + W_s y_s \leq h_s,\ \forall s \in U \\ & x \in X\end{aligned}$$

Min-regret

$$\begin{aligned}\min_{x}.\ & c^\top x + \theta \\ & \theta \geq q_s^\top y_s - q_s^\top y_s^\star,\ \forall s \in U \\ \text{s.t.:}\ & T_s x + W_s y_s \leq h_s,\ \forall s \in U \\ & x \in X\end{aligned}$$

where $y_s^\star$ is optimal for $s \in U$.

Affinely adjustable robust optimisation [Ben-Tal et al., 2004]

One idea for modelling adjustability is using affine policies:

- Replace $y(\xi)$ with $\alpha + \beta\xi$;
- $h(\xi)$ is assumed affinely dependent on ξ, e.g.: $h(\xi) = h + \hat{h}\xi$.

Then ARO becomes:

$$\begin{aligned}
\min_{x,\alpha,\beta}\ & c^\top x + \theta \\
\text{s.t.: } & \theta \geq q^\top(\alpha + \beta\xi),\ \forall \xi \in U \\
& Ax = b,\ x \geq 0 \\
& Tx + W(\alpha + \beta\xi) \leq h(\xi),\ \forall \xi \in U \\
& y(\xi) \geq 0,\ \forall \xi \in U.
\end{aligned}$$

Similar to the static case, computational tractability can be achieved:

- Requires that U is a box or ellipsoidal set;
- For a practical example, see [Ben-Tal et al., 2005]

Adjustable robust optimisation

An alternative approach: looking closer at the inner problem as a bilevel optimisation problem.

Let us restate our ARO in a simplified notation. For that, let

- $X = \{x \in \mathbb{R}^{n_1} : Ax = b, x \geq 0\}$;
- $Y = \{y \in \mathbb{R}^{n_2} : y \geq 0\}$;
- Uncertainty in RHS only, with $h(\xi) = h - \hat{h}\xi$, and $\xi \in \left[\,\underline{\xi}, \overline{\xi}\,\right]$

Then we have that ARO is equivalent to

$$\min_{x \in X} c^\top x + \mathcal{Q}(x), \tag{ARO}$$

where

$$\mathcal{Q}(x) = \left\{ \underset{\xi \in U}{\text{max.}} \; \underset{y \in Y}{\text{min.}} \; q^\top y : Tx = (h - \hat{h}\xi) - Wy \right\}.$$

Adjustable robust optimisation & CCG

Let us assume that an oracle is available such that, for a given $\overline{x} \in X$ it evaluates $\mathcal{Q}(x)$ and returns associated $(\overline{\xi}, \overline{y})$, if they exist.

In addition, let us assume that the uncertainty set is finitely representable:

- Scenarios, but an intractable amount of them (e.g., samples, or data)
- Polyhedral set (finite extreme points and rays)

In that case, we can employ column-and-constraint generation (CCG) [Zeng and Zhao, 2013] to solve ARO:

Main problem M^k: $\overline{x}^{k+1}$

$$\begin{aligned}
\min_{x,y,\theta} \ & c^\top x + \theta \\
& \theta \geq q^\top y_l, \ l \in [k] \\
& x \in X \\
& Tx = h - \hat{h}\overline{\xi}_l - W y_l, \ l \in [k] \\
& y_l \in Y, \ l \in [k].
\end{aligned}$$

Oracle $\mathcal{Q}(\overline{x}^{k+1})$: $\overline{\xi}^{k+1}$

$$\begin{aligned}
\max_{\xi \in U} \ \min_{y \in Y} \ & q^\top y \\
\text{s.t.: } & T\overline{x}^{k+1} = (h - \hat{h}\xi) - Wy.
\end{aligned}$$

Adjustable robust optimisation & CCG

In summary, the CCG method can be stated as

1. **Initialisation.** $LB = -\infty$, $UB = \infty$, $k = 0$.
2. **Solve the main problem M^k.** Let $\underline{z}^k = c^\top x^k + \theta^k$, where $\operatorname{argmin} M^k = (x^k, \theta^k, (y_l^k)_{l=1}^k)$. Make $LB = \underline{z}^k$.
3. **Solve** $\mathcal{Q}(x^k)$. Let $\operatorname{argmin} \mathcal{Q}(x^k) = (\overline{\xi}^{k+1}, \overline{y}^{k+1})$, if it exists. Let $\overline{z}^k = c^\top x^k + \mathcal{Q}(x^k)$. Make $UB = \min\left\{UB, \overline{z}^k\right\}$. **If** $UB = LB < \epsilon$, **return** x^k.
4. **Add column and constraints to M^k.** If $\mathcal{Q}(x^k)$ is feasible, create columns y_{k+1} and, together with the constraints

$$\theta \geq q^\top y_{k+1} \tag{1}$$

$$Tx = h - \hat{h}\overline{\xi}_{k+1} - W y_{k+1}, \; y_{k+1} \in Y, \tag{2}$$

add them to M^k, forming M^{k+1}. Make $k = k + 1$ and return to Step 2. If $\mathcal{Q}(x^k)$ is not feasible, then only (2) is created.

Adjustable robust optimisation & CCG

Practical remarks

Essentially, CCG for ARO is a delayed-generation approach of the min-max formulation

- Can thus be useful when too many scenarios are available;
- Convergence relies on a finiteness argument on the uncertainty set.

CCG can be seen as a primal equivalent to Benders decomposition

- One can use the same column generation approach in the context of the L-shaped method [Van Slyke and Wets, 1969];
- This can help as a way to transmit "recourse information" to the main problem.

Adjustable robust optimisation & CCG

On solving $\mathcal{Q}(x)$

Recall that $\mathcal{Q}(x)$ is of the form

$$
\begin{aligned}
\mathcal{Q}(x) = \max_{u} \ & q^\top y \\
\text{s.t.: } & u \in \mathcal{U} \\
& y \in \operatorname*{argmin}_{y} q^\top y \\
& \qquad \text{s.t.: } Tx = h - \hat{h}\xi - Wy \\
& \qquad\qquad y \in Y.
\end{aligned}
$$

This is a bilevel model and can be solved using dedicated methods.

- Most techniques rely on posing optimality conditions of the lower-level problem to yield an equivalent single-level (tractable) problem;
- Thus, lower-level convexity (plus CQ) is often a requirement.

Adjustable robust optimisation & CCG

On solving $\mathcal{Q}(x)$

Example: assume that $Y = \mathbb{R}^{n_2}_{+}$. We can use strong duality to reformulate the lower-level problem, obtaining

$$\begin{aligned}\mathcal{Q}(x) = &\max_{\xi,\pi} (h - \hat{h}\xi - Tx)^\top \pi \\ \text{s.t.: } &\pi^\top W \leq q^\top \\ &\xi \in U.\end{aligned}$$

$\mathcal{Q}(x)$ is solvable, if:

1. ξ is integer or has a discrete domain, since $\xi^\top \pi$ can be reformulated exactly (e.g., [Rintamäki et al., 2023]);
2. if $-(\hat{h}\xi)^\top \pi + (h - Tx)^\top \pi$ is a concave bilinear function in π and ξ;
3. if applying a global solver (e.g., Gurobi's spatial branch-and-bound method) is feasible from a computational standpoint.

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