

MS-EV0017 - Stochastic programming and robust optimisation

Lecture 4/5

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Outline of this lecture

Introduction

Robust optimisation

Adjustable robust optimisation

What is robust optimisation

An alternative paradigm for taking uncertainty into account:

- ▶ Permeated by the notion of **worst-case**;
- ▶ Control of the degree of **conservatism**;
- ▶ Parallels with **chance constraints** and **risk measures**.

In robust optimisation, **feasibility** is the key concern:

- ▶ Can be extended to objective function performance requirements;
- ▶ May or may not be **scenario-based**;
- ▶ Static v. adaptable: the presence of **recourse decisions**;
- ▶ Exception: distributionally robust optimisation.

Robust optimisation approaches

The key notion in robust optimisation is tied to that of an **uncertainty set**

- ▶ Feasibility is considered **constraint-wise**, i.e., for each row A_i of A (static case)
- ▶ The “region” U within the uncertainty support Ξ within which **parameter realisation** does not turn the solution infeasible;
- ▶ Tractability is closely tied to the **geometry** of such uncertainty sets.

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.:} \quad & A_i^\top x \leq b \\ & x \in X \end{aligned}$$

\Downarrow

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.:} \quad & A_i(\eta)^\top x \leq b, \quad \forall \eta \in U \subseteq \Xi \\ & x \in X \end{aligned}$$

\Downarrow

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.:} \quad & \max_{\eta \in U \subseteq \Xi} A_i(\eta)^\top x \leq b \\ & x \in X. \end{aligned}$$

Robust counterparts

Let $\tilde{a}_{ij} \in J_i$ be the **uncertain elements** in the matrix $A_{m \times n}$

- ▶ Random variables \tilde{a}_{ij} with “central value” a_{ij} and “maximum deviation” \hat{a}_{ij} ;
- ▶ symmetric and with bounded support $\tilde{a}_{ij} \in [a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]$.

Let $\eta_{ij} = \frac{(\tilde{a}_{ij} - a_{ij})}{\hat{a}_{ij}}$. Thus $\eta_{ij} \in [-1, 1]$ and follows the **same distribution** as \tilde{a}_{ij} , but centred in zero and scaled.

Our robust counterpart¹ is the following **bilevel** problem:

$$\begin{aligned} \min_x \quad & c^\top x \\ \text{s.t.} \quad & a_{ij}x_j + \max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j \right\} \leq b_i, \forall i \in [m] \\ & x_j \geq 0, \forall j \in [n]. \end{aligned} \tag{RC}$$

¹Assuming, w.l.g., $a_{ij} \geq 0, \forall i \in [m], \forall j \in [n]$

Uncertainty set geometries

Box uncertainty set [Soyster, 1973]

- ▶ **Maximum** protection level;
- ▶ All parameters take their worst-possible value;
- ▶ Simple, but highly conservative.

The **uncertainty set** is

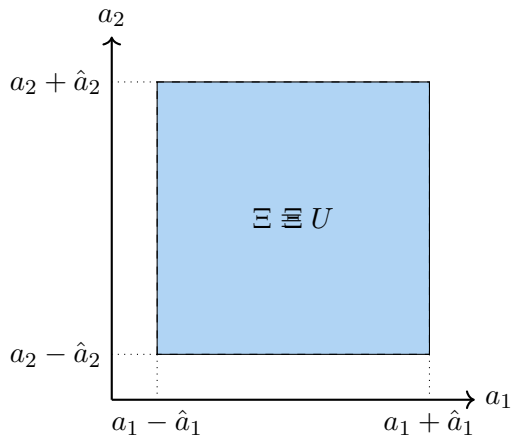
$$U_i = \{\eta_i : \|\eta_i\|_1 \leq |J_i|\} \equiv \{\eta_{ij} : |\eta_{ij}| \leq 1, \forall j \in J_i\}$$

The **lower-level problem** becomes

$$\max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : |\eta_{ij}| \leq 1, \forall j \in J_i \right\} = \sum_{j \in J_i} \hat{a}_{ij} x_j.$$

Uncertainty set geometries

Box uncertainty set [Soyster, 1973]



Uncertainty set geometries

Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]

- ▶ Softens extreme-case protection;
- ▶ Parametrically controlled;
- ▶ Leads to smooth sets;
- ▶ (MI)SOCPs which are more computationally demanding.

The uncertainty set is

$$U_i = \{\eta_i : \|\eta_i\|_2 \leq \Gamma_i\} \equiv \left\{ \eta_{ij} : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma_i^2, \forall j \in J_i \right\}$$

The lower-level problem becomes

$$\max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma_i^2, \forall j \in J_i \right\}.$$

Uncertainty set geometries

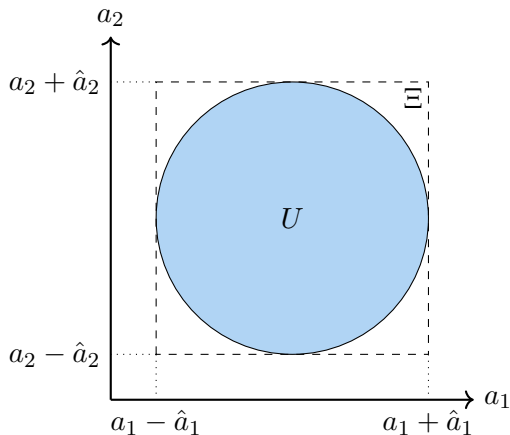
Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]

Again, this uncertainty set has a **closed-form** solution:

$$\begin{aligned} & \max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma_i^2, \forall j \in J_i \right\} \\ &= \max_{\eta_i \in U_i} \left\{ \sqrt{\left(\sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j \right)^2} : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma_i^2, \forall j \in J_i \right\} \\ &= \max_{\eta_i \in U_i} \left\{ \sqrt{\left(\sum_{j \in J_i} \eta_{ij} \right)^2 \left(\sum_{j \in J_i} \hat{a}_{ij} x_j \right)^2} : \sum_{j \in J_i} \eta_{ij}^2 \leq \Gamma_i^2, \forall j \in J_i \right\} \\ &= \Gamma_i \sqrt{\sum_{j \in J_i} \hat{a}_{ij}^2 x_j^2} \end{aligned}$$

Uncertainty set geometries

Ellipsoidal uncertainty set [Ben-Tal and Nemirovski, 1999]



Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]

- ▶ Allows for controlling conservatism;
- ▶ Retains problem complexity;
- ▶ **Budget of uncertainty** lacks interpretability.

The **uncertainty set** is

$$U_i = \{\eta_i : \|\eta_i\|_1 \leq \Gamma_i\} \equiv \left\{ \eta_{ij} : \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i, \forall j \in J_i \right\}.$$

The **lower-level problem** becomes

$$\max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j : \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i, 0 \leq \eta_{ij} \leq 1, \forall j \in J_i \right\}.$$

Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]

In this case, the lower-level problem does not admit a closed form.
However, it is a **linear program**.

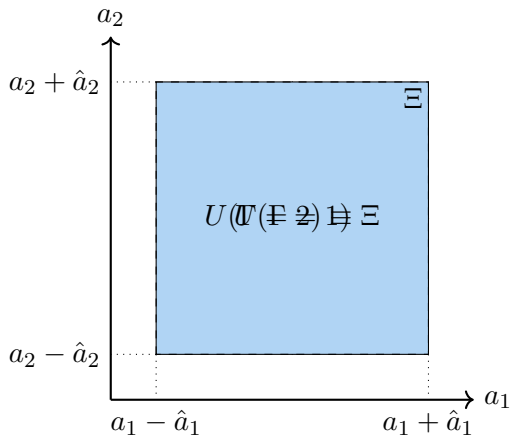
- ▶ Strong duality (primal-dual equivalence) is available;
- ▶ True for any **convex**² lower-level problem.

$$\begin{array}{ll} \max_{\eta_i \in U_i} \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j & \min_{\pi_i, p_i} \Gamma_i \pi_i + \sum_{j \in J_i} p_{ij} \\ \text{s.t.: } \sum_{j \in J_i} \eta_{ij} \leq \Gamma_i (\pi_i) & \Rightarrow \text{s.t.: } \pi_i + p_{ij} \geq \hat{a}_{ij} x_j, \forall j \in J_i \\ & p_{ij} \geq 0, \forall j \in J_i \\ 0 \leq \eta_{ij} \leq 1, (p_{ij}) \forall j \in J_i & \pi_i \geq 0. \end{array}$$

²Satisfying some constraint qualification.

Uncertainty set geometries

Polyhedral uncertainty set [Bertsimas and Sim, 2004]



Robust counterparts

Let us consider a knapsack problem of the form

$$\begin{aligned} \max. \quad & c^\top x \\ \text{s.t.:} \quad & \sum_{j \in [n]} a_j x_j \leq b \\ & 0 \leq x_j \leq 1, \quad \forall j \in [n]. \end{aligned}$$

The **robust counterparts** for the previous uncertainty sets are:

Box

$$\begin{aligned} \max. \quad & c^\top x \\ \text{s.t.:} \quad & \sum_{j \in [n]} (a_j + \hat{a}_j) x_j \leq b \\ & 0 \leq x_j \leq 1, \quad \forall j \in [n]. \end{aligned}$$

Ellipsoid

$$\begin{aligned} \max. \quad & c^\top x \\ \text{s.t.:} \quad & \sum_{j \in [n]} a_j x_j + \Gamma \sqrt{\sum_{j \in [n]} (\hat{a}_j x_j)^2} \leq b \\ & 0 \leq x_j \leq 1, \quad \forall j \in [n]. \end{aligned}$$

Robust counterparts

Let us consider a knapsack problem of the form

$$\begin{aligned} \max. \quad & c^\top x \\ \text{s.t.:} \quad & \sum_{j \in [n]} a_j x_j \leq b \\ & 0 \leq x_j \leq 1, \quad \forall j \in [n]. \end{aligned}$$

The **robust counterparts** for the previous uncertainty sets are:

Polyhedral

$$\begin{aligned} \max. \quad & c^\top x \\ \text{s.t.:} \quad & \sum_{j \in [n]} a_j x_j + \Gamma \pi + \sum_{j \in [n]} p_j \leq b \\ & \pi + p_j \geq \hat{a}_j x_j, \quad \forall j \in [n] \\ & 0 \leq x_j \leq 1, p_j \geq 0, \quad \forall j \in [n] \\ & \pi \geq 0. \end{aligned}$$

On constraint violation probabilities

Arguably, [Bertsimas & Sim \(2004\)](#) raised attention to robust optimisation with “the price of robustness”.

- ▶ The [price](#) refers to the optimality traded off for feasibility guarantees;
- ▶ Quantifying these trade-offs can be done:
 1. Using [theoretical](#) bounds;
 2. Via [simulating](#) solution performance.
- ▶ In my own experience, theoretical bounds are often loose.

For example, [Bertsimas and Sim, 2004] show the probability of violation of constraint $i \in [m]$ to be

$$P^{\text{vio}} = P \left(a_{ij}x_j + \max_{\eta_i \in U_i} \left\{ \sum_{j \in J_i} \eta_{ij} \hat{a}_{ij} x_j \right\} > b_i \right) \leq e^{\frac{-\Gamma^2}{2|J_i|}},$$

Multi-stage robust optimisation

We focus on 2-stage adjustable robust optimisation (ARO) problems:

$$\begin{aligned} \min \quad & c^\top x + \max_{\xi \in U \subset \Xi} \min_y q(\xi)^\top y(\xi) \\ \text{s.t.:} \quad & Ax = b, \quad x \geq 0 \\ & T(\xi)x + Wy(\xi) = h(\xi), \quad \forall \xi \in U \subset \Xi \\ & y(\xi) \geq 0, \quad \forall \xi \in \Xi. \end{aligned} \tag{ARO}$$

- ▶ Only **RHS** uncertainty: $T(\xi) = T$, $W(\xi) = W$, and $q(\xi) = q \quad \forall \xi \in \Xi$;
- ▶ Assumption often necessary to eliminate **quadratic** dependence between ξ and decision variables;
- ▶ Not necessary if the uncertainty set is **discrete** and **finite** (scenarios)

A side note: min-max, minimum regret and related

If the uncertainty set is a finite and discrete set of scenarios, we have that

$$\begin{aligned} \min_x. \quad & c^\top x + \max_{s \in U} \min_y. \quad q(\xi)_s^\top y_s \\ \text{s.t.:} \quad & T_s x + W_s y_s \leq h_s, \quad \forall s \in U \\ & x \in X \end{aligned}$$

is a **tractable** ARO [Mulvey et al., 1995]. Variants include:

Min-max

$$\begin{aligned} \min_{x,y,\theta}. \quad & c^\top x + \theta \\ \text{s.t.:} \quad & \theta \geq q_s^\top y_s, \quad \forall s \in U \\ & T_s x + W_s y_s \leq h_s, \quad \forall s \in U \\ & x \in X \end{aligned}$$

Min-regret

$$\begin{aligned} \min_{x,y,\theta}. \quad & c^\top x + \theta \\ \text{s.t.:} \quad & \theta \geq q_s^\top y_s - q_s^\top y_s^*, \quad \forall s \in U \\ & T_s x + W_s y_s \leq h_s, \quad \forall s \in U \\ & x \in X \end{aligned}$$

where y_s^* is optimal for $s \in U$.

Affinely adjustable robust optimisation [Ben-Tal et al., 2004]

One approach for modelling adjustability is using **affine policies**:

- ▶ Replace $y(\xi)$ with $\alpha + \beta\xi$;
- ▶ $h(\xi)$ is assumed **affinely dependent** on ξ , e.g.: $h(\xi) = h - \hat{h}\xi$.

Then ARO becomes:

$$\begin{aligned} \min_{x, \alpha, \beta} \quad & c^\top x + \theta \\ \text{s.t.:} \quad & \theta \geq q^\top (\alpha + \beta\xi), \quad \forall \xi \in U \\ & Ax = b, \quad x \geq 0 \\ & Tx + W(\alpha + \beta\xi) \leq h(\xi), \quad \forall \xi \in U \\ & y(\xi) \geq 0, \quad \forall \xi \in U. \end{aligned}$$

Similar to the static case, computational **tractability** can be achieved:

- ▶ Requires that U is a box or ellipsoidal set;
- ▶ For a practical example, see [Ben-Tal et al., 2005]

Adjustable robust optimisation

An **alternative** approach: looking closer at the inner problem as a **bilevel** optimisation problem.

Let us restate our ARO in a simplified notation. For that, let

- ▶ $X = \{x \in \mathbb{R}^{n_1} : Ax = b, x \geq 0\};$
- ▶ $Y = \{y \in \mathbb{R}^{n_2} : y \geq 0\};$
- ▶ Uncertainty in **RHS only**, with $h(\xi) = h - \hat{h}\xi$, and $\xi \in [\underline{\xi}, \bar{\xi}]$

Then we have that ARO is equivalent to

$$\min_{x \in X} c^\top x + \mathcal{Q}(x), \quad (\text{ARO})$$

where

$$\mathcal{Q}(x) = \left\{ \max_{\xi \in U} \min_{y \in Y} q^\top y : Tx = (h - \hat{h}\xi) - Wy \right\}.$$

Adjustable robust optimisation & CCG

Let us assume that an **oracle** is available such that, for a given $\bar{x} \in X$ it evaluates $\mathcal{Q}(x)$ and returns associated $(\bar{\xi}, \bar{y})$, if they exist.

In addition, let us assume that the uncertainty set is **finitely** representable:

- Scenarios, but an **intractable amount** of them (e.g., samples, or data)
- **Polyhedral** set (finite extreme points and rays)

In that case, we can employ **column-and-constraint generation** (CCG)

[Zeng and Zhao, 2013] to solve ARO:

Main problem M^k : \bar{x}^{k+1}

$$\min_{x, y, \theta} c^\top x + \theta$$

$$\theta \geq q^\top y_l, \quad l \in [k]$$

$$x \in X$$

$$Tx = h - \hat{h}\bar{\xi}_l - Wy_l, \quad l \in [k]$$

$$y_l \in Y, \quad l \in [k].$$

Oracle $\mathcal{Q}(\bar{x}^{k+1})$: $\bar{\xi}^{k+1}$

$$\max_{\xi \in U} \min_{y \in Y} q^\top y$$

$$\text{s.t.: } T\bar{x}^{k+1} = (h - \hat{h}\xi) - Wy.$$

Adjustable robust optimisation & CCG

In summary, the CCG method can be stated as

1. **Initialisation.** $LB = -\infty$, $UB = \infty$, $k = 0$.
2. **Solve the main problem M^k .** Let $\underline{z}^k = c^\top x^k + \theta^k$, where $\operatorname{argmin} M^k = (x^k, \theta^k, \{y_l^k\}_{l=1}^k)$. Make $LB = \underline{z}^k$.
3. **Solve $Q(x^k)$.** Let $\operatorname{argmax} Q(x^k) = (\bar{\xi}^{k+1}, \bar{y}^{k+1})$, if it exists. Let $\bar{z}^k = c^\top x^k + Q(x^k)$. Make $UB = \min \{UB, \bar{z}^k\}$. **If $UB = LB < \epsilon$, return x^k .**
4. **Add column and constraints to M^k .** If $Q(x^k)$ is feasible, create columns y_{k+1} and, together with the constraints

$$\theta \geq q^\top y_{k+1} \tag{1}$$

$$Tx = h - \hat{h}\bar{\xi}_{k+1} - Wy_{k+1}, \quad y_{k+1} \in Y, \tag{2}$$

add them to M^k , forming M^{k+1} . Make $k = k + 1$ and return to [Step 2](#). If $Q(x^k)$ is not feasible, then only (2) is created.

Adjustable robust optimisation & CCG

Practical remarks

Essentially, CCG for ARO is a **delayed-generation** approach of the min-max formulation

- ▶ Can thus be useful when **too many scenarios** are available;
- ▶ Convergence relies on a **finiteness argument** on the uncertainty set.

CCG can be seen as a **primal equivalent** to Benders decomposition

- ▶ One can use the same column generation approach in the context of the L-shaped method [Van Slyke and Wets, 1969];
- ▶ This can help as a way to **transmit** “recourse information” to the main problem.

Adjustable robust optimisation & CCG

On solving $\mathcal{Q}(x)$

Recall that $\mathcal{Q}(x)$ is of the form

$$\begin{aligned}\mathcal{Q}(x) = & \max_u q^\top y \\ & \text{s.t.: } \xi \in U \\ & y \in \operatorname{argmin}_y q^\top y \\ & \text{s.t.: } Tx = h - \hat{h}\xi - Wy \\ & y \in Y.\end{aligned}$$

This is a **bilevel model** and can be solved using dedicated methods.

- ▶ Most techniques rely on **posing optimality conditions** of the lower-level problem to yield an **equivalent single-level** (tractable) problem;
- ▶ Thus, **lower-level convexity** (plus CQ) is often a requirement.

Adjustable robust optimisation & CCG

On solving $\mathcal{Q}(x)$





Example: assume that $Y = \mathbb{R}_+^{n_2}$. We can use strong duality to reformulate the lower-level problem, obtaining

$$\begin{aligned}\mathcal{Q}(x) = & \max_{\xi, \pi} (h - \hat{h}\xi - Tx)^\top \pi \\ \text{s.t.: } & \pi^\top W \leq q^\top \\ & \xi \in U.\end{aligned}$$

$\mathcal{Q}(x)$ is solvable, if:

1. ξ is integer or has a discrete domain, since $\xi^\top \pi$ can be reformulated exactly (e.g., [Rintamäki et al., 2023]);
2. if $-(\hat{h}\xi)^\top \pi + (h - Tx)^\top \pi$ is a **concave bilinear** function in π and ξ ;
3. if applying a **global solver** (e.g., Gurobi's spatial branch-and-bound method) is feasible from a computational standpoint.

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