Stochastic programming and robust optimisation

Lecture 3/5

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Outline of this lecture

Introduction

Chance constraints

Risk measures

Outline of this lecture

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Chance constraints

Risk measures

Beyond expected values

Many settings require that a risk profile is imposed:

- Expected values assume a risk neutral stance;
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- Risk neutral means that the product probability $\times$ outcome is the single factor under consideration.

In many settings, the decision-maker may have other implicit priorities:

- Impose (probabilistic) guarantees on feasibility
  $\Rightarrow$ **chance constraints**;
- Avoid being exposed to the possibility of a high-loss
  $\Rightarrow$ **risk measures**.

Outline of this lecture

Feasibility guarantees

A solution may reveal itself infeasible once the uncertainty unveils

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- Appealing in settings in which safety and resilience are requirements.

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Let us first consider the case without recourse. Our problem is thus

$$\begin{aligned}
\text{min. } & c^\top x \\
\text{s.t.: } & Ax = b \\
& T(\xi)x = h(\xi), \ \forall \xi \in \Xi \\
& x \geq 0.
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Notice the static nature of the problem

- once a decision x is made, we observe the realisation of the uncertainty ξ;
- no correction decision is allowed.

Feasibility guarantees

There are two main paradigms on how to model feasibility requirements for the constraint $T(\xi)x = h(\xi), \ \forall \xi \in \Xi$:

1. Impose that $T(\xi)x = h(\xi)$ holds for a set of realisations $U \subseteq \Xi$ (robust optimisation)
2. Impose that the probability of $T(\xi)x = h(\xi)$ holding is at least a given threshold (chance constraints)

As one may suspect, the two approaches are interrelated.

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As one may suspect, the two approaches are interrelated.

Imposing a chance constraint means that a solution is deemed acceptable if, for a given confidence level α, we have that

$$\mathbb{P}(T(\xi)x = h(\xi)) \geq \alpha.$$

Types of chance constraints

Let $T(\xi)$ be a $m_2 \times n_1$ matrix, where $T(\xi)_i$ represents its i^{th}-row, and $h(\xi)$ a m_2 vector with components $h(\xi)_i$.

There are two types of chance constraints:

1. **Individual chance constraints** (ICC):

$$p_i(x) := \mathbb{P}((T(\xi)_i)^\top x = h(\xi)_i) \geq \alpha_i, \ \forall i \in [m_2],$$

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JCCs are typically more challenging from a tractability standpoint

- ICCs can be used to approximate a JCC;
- **Bonferroni inequality**: for a given x, if $p_i(x) > \alpha_i$, $\forall i \in [m_2]$, where $\alpha_i = 1 - \frac{(1-\alpha)}{m_2}$, then $p(x) \geq \alpha$.

Tractability of chance constraints

Tractability issues stem from the properties of the feasible sets generated by chance constraints. Let

$$C(\alpha_1, \dots, \alpha_{m_2}) := \bigcap_{i \in [m_2]} = C_i(\alpha_i), \text{ where}$$
$$C_i(\alpha_i) = \{x \in \mathbb{R}^n : p_i(x) \geq \alpha_i\}.$$

- No general result that guarantees the convexity of $C_i(\alpha_i)$;
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- Particular cases do exist for important distributions which lead to the convexity of $C(\alpha_1, \dots, \alpha_{m_2})$.

For example, assume that $T(\xi) = T$, $\forall \xi \in \Xi$, and that $h(\xi) = \xi$. For the univariate case, we have that

$$C(\alpha) = \left\{x \in \mathbb{R}^{n_1} : Tx \geq F^{-1}(\alpha)\right\}.$$

Tractability of chance constraints

One general result that is known and can be informative in the multivariate case is the following:

Theorem 1

Let $T(\xi) = T$, $\forall \xi \in \Xi$, and $h(\xi) = \xi$, where $\xi \in \mathbb{R}^{m_2}$ is a random vector with density function f. If $\log(f)$ is concave (assuming $\log(0) = -\infty$), then $C(\alpha)$ is closed and convex for all $\alpha \in [0, 1]$.

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- **Main case:** $\xi \sim$ Normal(μ, Σ) with vector mean μ and covariance matrix Σ;
- Uniform case also "trivial" to show that holds;
- For a list of "other distributions": [Nemirovski and Shapiro, 2007]

Tractability of chance constraints

Another important known case is this: $T(\xi)$ is a $1 \times n$ random vector and $h(\xi) = h$, $\forall \xi \in \Xi$.

Theorem 2

Assume that $T(\xi) = \xi = (\xi_i)_i = 1^{n_1}$ *is the only random parameter, where* $\xi \sim$ *Normal*(μ, Σ) *with* $\mu = (\mu_i)_{i=1}^{n_1}$ *a vector of means and* Σ *the covariance matrix. Then*

$$C(\alpha) = \left\{x \in \mathbb{R}^{n_1} : \mu^\top x \geq h + \Phi^{-1}(\alpha)\sqrt{x^\top \Sigma x}\right\}$$

Tractability of chance constraints

Proof.

The random variable $\xi^\top x$ is a multivariate normal with mean $\mu^\top x$ and variance $x^\top \Sigma x$. Letting Z follow a standard normal, we have that

$$\mathbb{P}(\xi^\top x \geq h) \geq \alpha \Leftrightarrow \mathbb{P}\left(\frac{\xi^\top x - \mu^\top x}{\sqrt{x^\top \Sigma x}} \geq \frac{h^\top x - \mu^\top x}{\sqrt{x^\top \Sigma x}}\right) \geq \alpha$$

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Notice that the constraint is convex if $\Phi^{-1}(\alpha) \geq 0$, i.e., $\alpha \in [1/2, 1]$.

Discretisation of chance constraints

An alternative way of handling chance constraints is using scenarios:

- Allows for general (discrete) distributions;
- Convex problems by construction (for an originally convex problem);
- Allows for recourse decisions;
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Let us consider again our scenario-based deterministic equivalent 2SSP:

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\text{min. } & c^\top x + \sum_{s \in S} P_s q_s^\top y_s \\
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Let $v_s \in \{0, 1\}$, $u_s \in \mathbb{R}$, $\forall s \in S$, and M be a sufficiently large (big-M) value.

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Let $v_s \in \{0, 1\}$, $u_s \in \mathbb{R}$, $\forall s \in S$, and M be a sufficiently large (big-M) value. Then, we can reformulate our chance-constrained problem as

$$\begin{aligned}
\text{min. } & c^\top x + \sum_{s \in S} P_s q_s^\top y_s && \text{(1a)} \\
\text{s.t.: } & Ax = b && \text{(1b)} \\
& T_s x + W_s y_s = h_s + u_s, \ \forall s \in S && \text{(1c)} \\
& |u_s| \le M v_s, \ \forall s \in S && \text{(1d)} \\
& \sum_{s \in S} P_s v_s \le 1 - \alpha && \text{(1e)} \\
& x \ge 0 && \text{(1f)} \\
& y_s \ge 0, u_s \in \mathbb{R}, v_s \in \{0, 1\} \ \forall s \in S, && \text{(1g)}
\end{aligned}$$

where α be our feasibility likelihood.

Discretisation of chance constraints

Some final remarks:

- $|u_s| \leq Mv_s, \ \forall s \in S$, can be easily linearised, and is not necessary when the chance-constraints are one-sided (i.e., $\leq$ or $\geq$);

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- Notice that one binary variable per scenario is required;
- The above can be circumvented using integrated chance constraints.
  - Alternative, one can impose limits on the expected infeasibility (variables $u_s, \forall s \in S$)
  - This is achieved by replacing (1d) and (1e) with

$$\sum_{s \in S} P_s u_s \leq \beta$$

where β is a limit on the expected amount of infeasibility;

Chance constraints

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Risk measures

Measuring risk

Recall that we seek to find a solution x that optimises

$$\min_x \mathbb{E}_\xi \left[F(x,\xi)\right],$$

where:

- $F(x,\xi) = \left\{c^\top x + Q(x,\xi) : x \in X\right\}$;
- $Q(x,\xi) = \text{min.}_{\;y} \left\{q(\xi)^\top y : W(\xi)y = h(\xi) - T(\xi)x, y \geq 0\right\}$;
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That is, we choose $x^\star = \operatorname{argmin}_x \mathbb{E}_\xi \left[F(x,\xi)\right]$.

- Each $x' \in X$ has an associated a probability distribution $f_x(\xi)$ which maps a cost $F(x',\xi)$ to the probability of scenario ξ;
- Thus, x' is preferred over x'' if $\mathbb{E}_\xi \left[F(x',\xi)\right] < \mathbb{E}_\xi \left[F(x'',\xi)\right]$.

Measuring risk

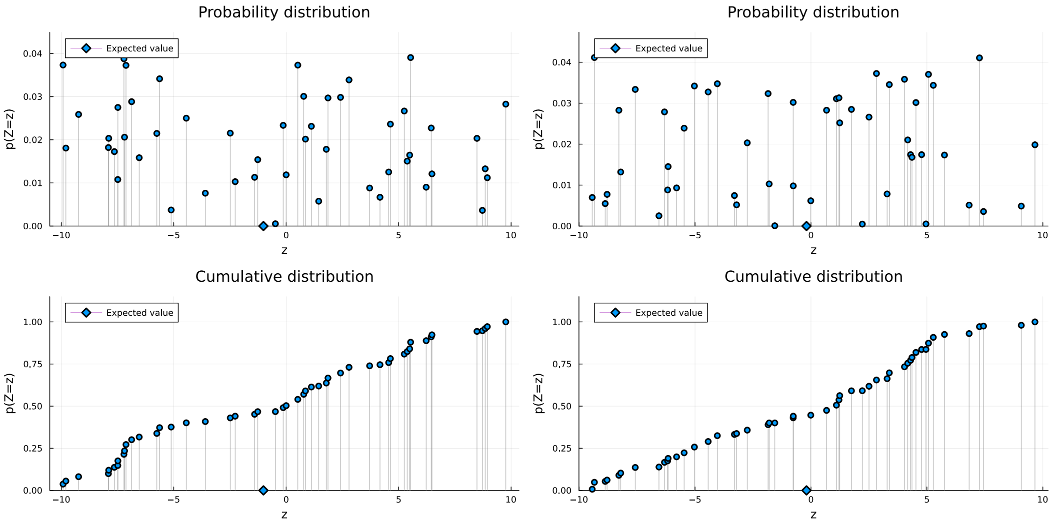

Figure: Comparing two solutions: the solution generating the cost distribution on the left is preferred, as it has a lower expected value

Measuring risk

However, choosing between distributions using their expected values neglects information about the dispersion:

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- Tails of the cost distribution are often relevant from a decision-making standpoint.

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However, choosing between distributions using their expected values neglects information about the dispersion:

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- Tails of the cost distribution are often relevant from a decision-making standpoint.

To capture more information about such tails we can define risk measures $r : X \to \mathbb{R}$ such that

- r associates the random variable $F(x, \xi)$ generated by the solution x with a real-valued risk $r_\xi(x)$
- Analogously, x' can be chosen over x'' if $r_\xi\left[F(x', \xi)\right] < r_\xi\left[F(x'', \xi)\right]$.

Trading off risk and expected return

Being two conflicting objectives, risk and return are typically considered under a bi-objective standpoint, e.g., using

1. Weighted terms in the objective function:

$$\min_x \mathbb{E}_\xi \left[F(x,\xi)\right] + \beta r_\xi \left[F(x,\xi)\right],$$

   where $\beta = 0$ represents a risk-neutral stance and risk aversion increases as $\beta \to \infty$;

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2. A risk exposition budget δ:

$$\begin{aligned} \min_x \ & \mathbb{E}_\xi \left[F(x,\xi)\right] \\ \text{s.t.: } & r_\xi \left[F(x,\xi)\right] \leq \delta. \end{aligned}$$

Coherent risk measures

[Artzner et al., 1999] provides axiomatic definitions for coherent risk measures:

1. **Translation invariance:** $r_\xi\left[F(x,\xi)+a\right] = r_\xi\left[F(x,\xi)\right] + a$ for $a \in \mathbb{R}$.
2. **Subadditivity:** $r_\xi\left[F(x',\xi)+F(x'',\xi)\right] \leq r_\xi\left[F(x',\xi)\right] + r_\xi\left[F(x'',\xi)\right]$
3. **Positive homogeneity:** $r_\xi\left[F(x,\xi)\times a\right] = r_\xi\left[F(x,\xi)\right] \times a$ for $a \in \mathbb{R}$.
4. **Monotonicity:** if for every ξ, we have that $F(x',\xi) \leq F(x'',\xi)$, then $r_\xi\left[F(x',\xi)\right] \leq r_\xi\left[F(x'',\xi)\right]$

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This has been further developed by [Rockafellar, 2007], who establishes the role of coherence in optimisation problems. A coherent risk measure:

- Preserves convexity;
- Preserves certainty;
- Is insensitive to scaling.

Conditional value-at-risk

The most widespread risk measure in the context of optimisation is the Conditional Value-at-Risk (CVaR)

- A coherent risk measure widely used in other areas;
- Empirical results on its efficacy in production planning: [Alem et al., 2020]

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Let X be a random variable and F_X its cumulative distribution function. Then, for a confidence level α, the Value-at-Risk (VaR_α) is defined as

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The conditional VaR_α represents the expectation of X in the conditional distribution of its α-upper tail, i.e.,

$$CVaR_\alpha(X) = \min_{\eta \in \mathbb{R}} \left\{ \eta + \frac{\mathbb{E}[X - \eta]^+}{(1-\alpha)} \right\},$$

where $[\,\cdot\,]^+ = \max\{0, \cdot\,\}$.

Conditional value-at-risk

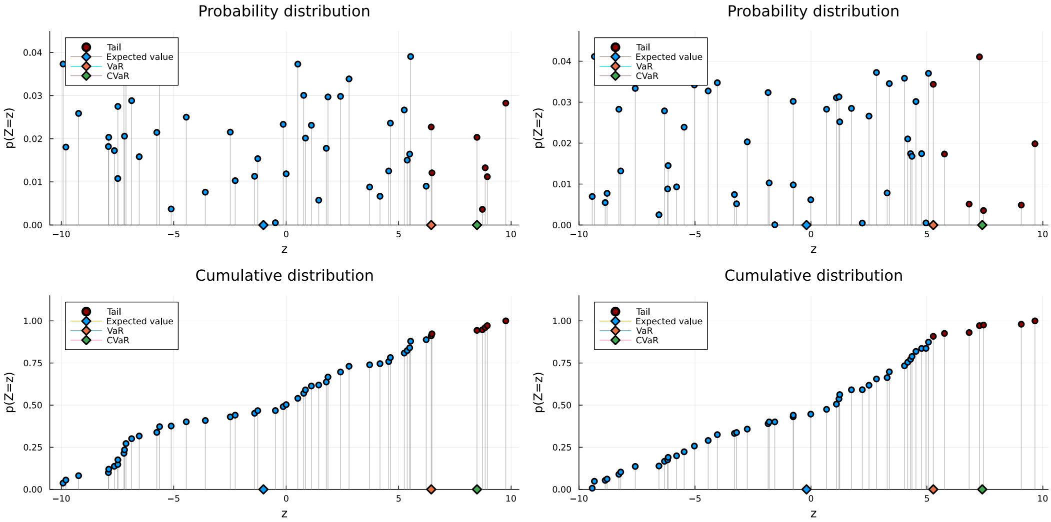

Figure: Comparing two solutions: the solution on the right has better (smaller) $\text{CVaR}_{90\%}$

Conditional value-at-risk in SP models

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- Requires discretisation to handle the expected value;
- In the context of optimisation, this means that no additional binary variables are needed;
- This contrasts with VaR (or chance constraints), which need such variables.

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& y_s \geq 0, \ \forall s \in S.
\end{aligned}$$

Conditional value-at-risk in SP models

Let us define the following auxiliary terms:

- $\beta \in [0, 1]$: weight for the risk term,
- α: confidence level;
- $\eta \geq 0$: represent the value at risk (VaR);
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Then, the risk-averse scenario-based deterministic equivalent 2SSP is

$$
\begin{aligned}
\text{min. } & (1-\beta)\left[c^\top x + \sum_{s \in S} P_s q_s^\top y_s\right] + \beta\left[\eta + \frac{\sum_{s \in S} P_s \pi_s}{1-\alpha}\right] \\
\text{s.t.: } & Ax = b, x \geq 0 \\
& T_s x + W_s y_s = h_s, \ \forall s \in S \\
& \pi_s \geq c^\top x + q_s^\top y_s - \eta, \ \forall s \in S \\
& y_s \geq 0, \pi_s \geq 0, \ \forall s \in S \\
& \eta \geq 0.
\end{aligned}
$$

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Some final practical remarks:

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  - Serve as proxy to other risk-aversion paradigms (worst-case minimisation or distributional robustness)

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