Property Demonstration on the Greedy Algorithm regarding Load Balancing Problem with Various Examples

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Recent Research Topic: Generative Model

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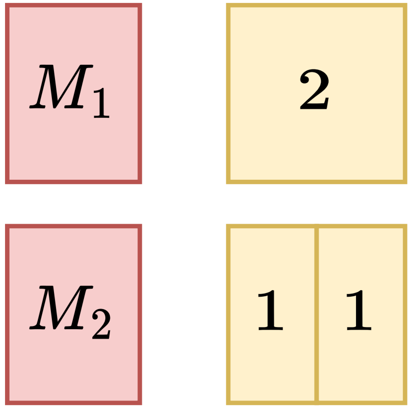
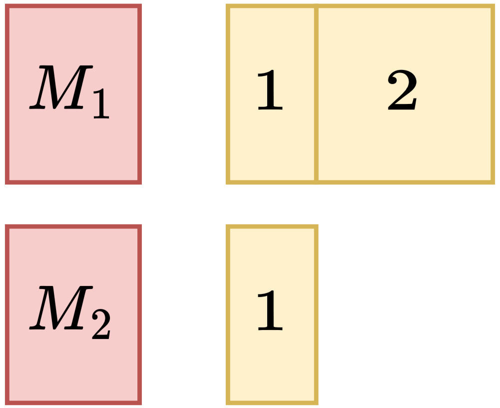
Task 4-1

2 identical machines $\{M_1, M_2\}$ and 3 jobs $\{J_1, J_2, J_3\}$ with processing time $t_1 = t_2 = 1, t_3 = 2$.

Consider the job sequence represented with time.
The following figure on the left is an instance of 211,121, where the makespan $T = 2 = T^*$. $\#\text{case} = 2 \cdot 2! = 4$.
The following figure on the right is an instance of 112, where the makespan $T = 3$. $\#\text{case} = 2! = 2$.





$$T_{\text{Avg}} = \frac{4 \cdot 2 + 2 \cdot 3}{4 + 2} = \frac{7}{3}$$

$$\frac{T_{\text{Avg}}}{T^*} = \frac{\frac{7}{3}}{2} = \frac{7}{6}$$

In the next page, the following formulation is obtained.

$$\frac{T_{\text{Avg}}}{T^*} = 1 + \frac{n^3 - 3n^2 + 4n - 2}{2n(n^2 - n + 1)}$$

Simply substitute n with 2,

$$\left.\frac{T_{\text{Avg}}}{T^*}\right|_{n=2} = 1 + \frac{2^3 - 3 \cdot 2^2 + 4 \cdot 2 - 2}{2 \cdot 2 \cdot (2^2 - 2 + 1)} = \frac{7}{6}$$

, which partially confirms the correctness of the formulation.

Task 4-1

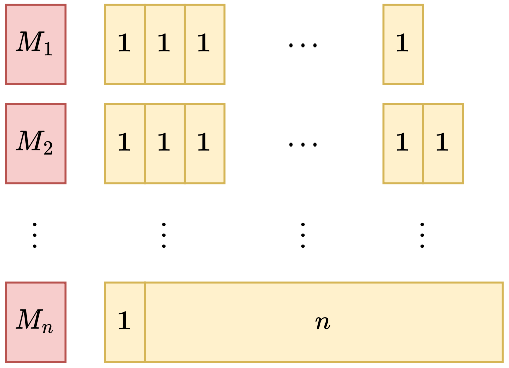
n identical machines $\{M_i\}, i \in [1, n]$ and $(n(n-1)+1)$ jobs $\{J_i\}, i \in [1, n^2-n+1]$ with processing time $t_1 = t_2 = \cdots = t_{n(n-1)} = 1, t_{n^2-n+1} = n$.

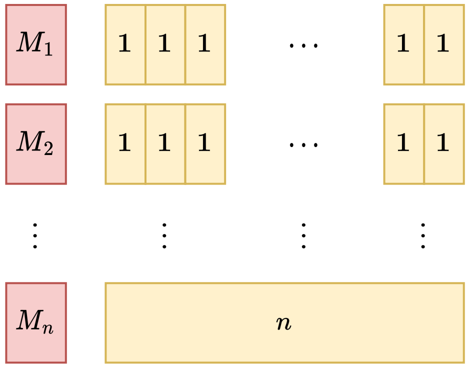
Consider the job sequence represented with time

$$1 \cdots 1 \color{red}{n} \color{black}{1} \cdots 1$$

, and the job J_{n^2-n+1} is at index i in the color red.
Obviously, the makespan $T(i) = \left\lfloor \frac{i-1}{n} \right\rfloor + n$.

The following figure on the left is an instance of $i = n+1$, where the makespan $T(n+1) = n+1$.
The following figure on the right is an instance of $i = 1$, where the makespan $T(1) = n = T^*$.





$$
\begin{aligned}
T_{\text{Avg}} &= \frac{\sum_{i=1}^{n(n-1)+1} [n(n-1)]! \cdot \left(\left\lfloor \frac{i-1}{n} \right\rfloor + n\right)}{[n(n-1)+1]!} \\
&= \frac{\sum_{i=1}^{n(n-1)+1} \left(\left\lfloor \frac{i-1}{n} \right\rfloor + n\right)}{n(n-1)+1} \\
&= \frac{n\left(n^2-n+1\right) + n - 1 + n\sum_{i=0}^{n-2} i}{n^2-n+1} \\
&= n + \frac{n - 1 + n \cdot \frac{n^2-3n+2}{2}}{n^2-n+1} \\
&= n + \frac{n^3-3n^2+4n-2}{2(n^2-n+1)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{T_{\text{Avg}}}{T^*} &= \frac{n + \frac{n^3-3n^2+4n-2}{2(n^2-n+1)}}{n} \\
&= 1 + \frac{n^3-3n^2+4n-2}{2n(n^2-n+1)}
\end{aligned}
$$

$$\lim_{n \to +\infty} \frac{T_{\text{Avg}}}{T^*} = 1 + \frac{1}{2} = \frac{3}{2}$$

Task 4-2

Consider the first example of Task 4-1.

The sorted job sequence represented with time is 211.

$$T_{\text{Sorted}} = 2 = T^*$$

$$\frac{T_{\text{Avg}}}{T_{\text{Sorted}}} = \frac{T_{\text{Avg}}}{T^*} = \frac{7}{6}$$

Consider the second example of Task 4-1.

The sorted job sequence represented with time is $n1 \cdots 1$.

$$T_{\text{Sorted}} = n = T^*$$

$$\frac{T_{\text{Avg}}}{T_{\text{Sorted}}} = \frac{T_{\text{Avg}}}{T^*} = 1 + \frac{n^3 - 3n^2 + 4n - 2}{2n(n^2 - n + 1)}$$

$$\lim_{n \to +\infty} \frac{T_{\text{Avg}}}{T_{\text{Sorted}}} = \frac{3}{2}$$

Task 4-3

Algorithm 1 A Solution to General Settings of Load Balancing

Require: machines $\mathcal{M} = \{M_i\}, i \in [1, m]$, jobs $\mathcal{J} = \{J_i\}, i \in [1, n]$, processing times $t : (\mathcal{M} \times \mathcal{J}) \to \mathbb{N}^+$, $t(M, J)$ represents the processing time of job J on machine M;

Ensure: $A : \mathcal{M} \to 2^{\mathcal{J}}$ with $(\bigcup_{M \in \mathcal{M}} A(M) = \mathcal{J}) \wedge (\bigcap_{M \in \mathcal{M}} A(M) = \varnothing)$, makespan T;

Sort jobs $\mathcal{J} = \{J_i\}, i \in [1, n]$ by $\max_{M \in \mathcal{M}} t(M, J_i)$ and obtain $(J'_1, \cdots, J'_n)$;
for $i \leftarrow 1, \cdots, n$ **do**
  $M' \leftarrow \arg\min_{M \in \mathcal{M}} \sum_{J \in A(M)} t(M, J)$
  $A(M') \leftarrow A(M') \cup J'_i$
end for
$T \leftarrow \max_{M \in \mathcal{M}} \sum_{J \in A(M)} t(M, J)$

	J1	J2	J3	J4	J5	J6	J7	J8	J9	J10	J11	J12	J13	J14	J15
M1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
M2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
M3	3	6	9	12	15	18	21	24	27	30	15	15	15	15	15

Sort the jobs by $\max t(M, J_i)$, obtain the job index sequence

[15,10,14,9,13,12,8,11,7,6,5,4,3,2,1]

Apply the greedy algorithm,

```
Job 15 is assigned to machine 1.
Job 10 is assigned to machine 2.
Job 14 is assigned to machine 3.
Job 9 is assigned to machine 1.
Job 13 is assigned to machine 3.
Job 12 is assigned to machine 2.
Job 8 is assigned to machine 1.
Job 11 is assigned to machine 3.
Job 7 is assigned to machine 1.
Job 6 is assigned to machine 1.
Job 5 is assigned to machine 2.
Job 4 is assigned to machine 1.
Job 3 is assigned to machine 3.
Job 2 is assigned to machine 1.
Job 1 is assigned to machine 1.

Machine 1: [15, 9, 8, 7, 6, 4, 2, 1]
T = 52
Machine 2: [10, 12, 5]
T = 54
Machine 3: [14, 13, 11, 3]
T = 54

makespan: 54
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The obtained makespan is 54.