Homework Assignment 1.1

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Exersice a

The matrix A_σ is given by:

$$A_\sigma = \begin{bmatrix} \sigma + 1 & 3 \\ -2 & \sigma - 1 \end{bmatrix}.$$

The characteristic equation for A_σ is:

$$\det(A_\sigma - \lambda I) = 0,$$
$$\lambda^2 - 2\sigma\lambda + (\sigma^2 + 5) = 0.$$

The eigenvalues are solutions to this quadratic equation:

$$\lambda = \frac{2\sigma \pm \sqrt{(2\sigma)^2 - 4(\sigma^2 + 5)}}{2}.$$

$$(2\sigma)^2 - 4(\sigma^2 + 5) = 4\sigma^2 - 4\sigma^2 - 20 = -20,$$

$$\lambda = \frac{2\sigma \pm \sqrt{-20}}{2}.$$

Thus, the eigenvalues are the complex conjugate pair:

$$\lambda_{1,2} = \sigma \mp i\sqrt{5}.$$

Exersice b

I used the following code in Mathematica to get the anwser:

```
(* Define the system of equations *)
eqs = {
  x'[t] == (\[Sigma] + 1)x[t] + 3y[t],
  y'[t] == -2x[t] + (\[Sigma] - 1)y[t],
  x[0] == u,
  y[0] == v
};

(* Solve the system *)
sol = DSolve[eqs, {x[t], y[t]}, t]
```

Exersice c

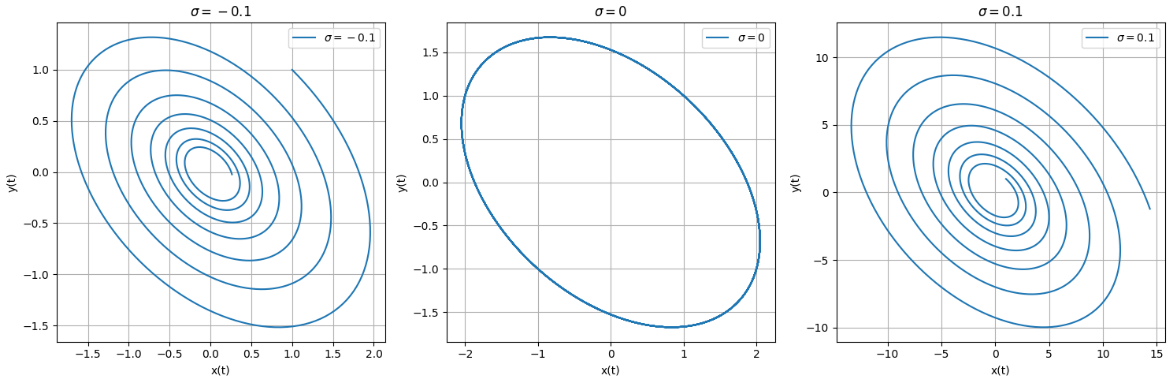

Figure 1: representative trajectories for different values of σ.

Exersice d

The period T of one closed orbit is given by:

$$T = \frac{2\pi}{\omega}$$

where $\omega = \sqrt{5}$. Substituting ω, the period is:

$$T = \frac{2\pi}{\sqrt{5}}$$

Exersice e

With $\sigma = 0$, the equations become:

$$x(t) = u\cos(\sqrt{5}t) + \frac{u+3v}{\sqrt{5}}\sin(\sqrt{5}t), \quad y(t) = v\cos(\sqrt{5}t) - \frac{2u+v}{\sqrt{5}}\sin(\sqrt{5}t).$$

The angular frequency is $\omega = \sqrt{5}$, giving the period:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{5}}.$$

The coefficient matrix A is:

$$A = \begin{bmatrix} u & \frac{u+3v}{\sqrt{5}} \\ v & -\frac{2u+v}{\sqrt{5}} \end{bmatrix}.$$

The matrix Q is:

$$Q = A^T A = \begin{bmatrix} \frac{21}{5} & -\frac{3}{\sqrt{5}} \\ -\frac{3}{\sqrt{5}} & \frac{14}{5} \end{bmatrix}.$$

The eigenvalues λ are found by solving:

$$\det(Q - \lambda I) = \lambda^2 - 7\lambda + \frac{49}{5} = 0 \implies \lambda = \frac{7 \pm 2\sqrt{5}}{2}.$$

Let $a^2 = \frac{7}{2\sqrt{5}}(\sqrt{5}+1)$ and $b^2 = \frac{7}{2\sqrt{5}}(\sqrt{5}-1)$. The ratio of a and b is:

$$\frac{a}{b} = \sqrt{\frac{a^2}{b^2}} = \sqrt{\frac{\sqrt{5}+1}{\sqrt{5}-1}}.$$

Exersice f

The eigenvector corresponding to $\lambda = \frac{7+2\sqrt{5}}{2}$ satisfies:

$$(Q - \lambda I)\begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

This simplifies to:

$$y = \frac{1-\sqrt{5}}{2}x.$$

Normalize the eigenvector:

$$\text{Norm factor} = \frac{1}{\sqrt{1 + \left(\frac{1-\sqrt{5}}{2}\right)^2}} = \frac{\sqrt{10-2\sqrt{5}}}{2}.$$

The final normalized eigenvector is:

$$\frac{2}{\sqrt{10-2\sqrt{5}}}\begin{bmatrix} 1 \\ \frac{1-\sqrt{5}}{2} \end{bmatrix}.$$

2.2\2.2.py

```python
#%% Import libraries
import numpy as np
import matplotlib.pyplot as plt
#%% Functions
# Define the analytical solution for x(t) and y(t)
def analytical_solution(t, sigma, u, v):
    omega = np.sqrt(5)  # Frequency
    exp_term = np.exp(sigma * t)

    x = exp_term * (u * np.cos(omega * t) + (u + 3 * v) / omega * np.sin(omega * t))
    y = exp_term * (v * np.cos(omega * t) - (2 * u + v) / omega * np.sin(omega * t))

    return x, y
#%% Main b)
# Time range for the plots
t = np.linspace(0, 20, 1000)

# Parameters for initial conditions
u, v = 1, 1

sigma_values = [-1/10, 0, 1/10]

fig, axes = plt.subplots(1, 3, figsize=(15, 5))

for i, sigma in enumerate(sigma_values):
    x, y = analytical_solution(t, sigma, u, v)

    axes[i].plot(x, y, label=f"$\\sigma = {sigma}$")
    axes[i].set_title(f"$\\sigma = {sigma}$")
    axes[i].set_xlabel("x(t)")
    axes[i].set_ylabel("y(t)")
    axes[i].legend()
    axes[i].grid()

plt.tight_layout()
plt.show()

```