Math H185 Homework 2

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(1) Let $f : U \to \mathbb{C}$ be a holomorphic function and γ a simple closed curve that is null-homotopic in U. Let $z_0, ..., z_n$ be a collection of $n+1$ distinct points in the region enclosed by γ (the set of points U_1 whose index is 1). Let

$$l_j(z) = \frac{\prod_{k\neq j}(z - z_k)}{\prod_{k\neq j}(z_j - z_k)}, \quad \omega(z) = \prod_{k=0}^{n}(z - z_j)$$

(a) Verify that $p(z) = \sum_{j=0}^{n} f(z_j)l_j(z)$ is the unique polynomial of degree at most n satisfying $p(z_j) = f(z_j)$ for each $j = 0, ..., n$.

Solution: Note that $l_j(z_i) = 0$ if $i \neq j$ and $l_j(z_j) = 1$. Thus $p(z_j) = f(z_j)$. Now suppose there were two polynomials p and q such that $q(z_j) = f(z_j) = p(z_j)$ for all $z_0, ..., z_n$. Then we have $p - q$ has $n + 1$ zeroes, despite it being at degree n. Thus $p - q$ is identically zero (we can verify by induction, clearly true for $n = 0$ or 1, then $p - q = (x - w)r(x)$, and $r(x)$ is degree $n - 1$ and has n zeros, thus is identically zero by induction hypothesis). Thus we have that $p(x) = q(x)$, verifying uniqueness.

(b) Prove the Hermite remainder formula

$$f(z) - p(z) = \frac{1}{2\pi i}\int_\gamma \frac{\omega(z)}{\omega(t)}\frac{f(t)}{t - z}dt$$

for all $z \in U_1$

Solution: We can use the residue theorem to evaluate

$$\frac{1}{2\pi i}\int_\gamma \frac{\omega(z)}{\omega(t)}\frac{f(t)}{t - z}dt$$

There are a few cases to consider. First suppose that $z \in \{z_0, ..., z_n\}$, then we have that $\omega(z) = 0$, and so the integral is zero, and $f(z) =$

$p(z)$ and z_i so we are done. Now suppose that $z \notin \{z_0, ..., z_n\}$ and $\text{Ind}_\gamma(z) = 1$. We ignore $\omega(z)$ for now since it is a constant and try and evaluate

$$\frac{1}{2\pi i}\int_\gamma \frac{f(t)}{\omega(t)(t-z)}dt$$

Note that for $t \notin \{z, z_0, ..., z_n\}$, the integrand is differentiable, so these points form poles. For now consider the pole z_j. It is a pole of order 1 (unless $f(z_j) = 0$ which won't be a problem) because

$$\lim_{t \to z_j}(z - z_j)\frac{f(t)}{\omega(t)(t-z)} \neq 0$$

because $f(z_j) \neq 0$ is holomorphic, thus continuous and $\frac{(t-z_j)}{\omega(t)}$ has a well-defined limit at $t = z_j$. So we have

$$\text{Res}_{z_j}\frac{f(t)}{\omega(t)(t-z)} = \lim_{t \to z_j}\frac{f(t)(t-z_j)}{\omega(t)(t-z)} = \frac{f(z_j)}{\prod_{k \neq j}(z_j - z_k) * (z_j - z)}$$

which is still true even if $f(z_j) = 0$, and thus not causing an issue in the case where it is not a pole of order 1. Then $t = z$ is also a pole of order 1 (again barring $f(z) = 0$) by similar reason to above and we have

$$\text{Res}_z\frac{f(t)}{\omega(t)(t-z)} = \lim_{t \to z}\frac{f(t)(t-z)}{\omega(t)(t-z)} = \frac{f(z)}{\omega(z)}$$

Thus we have that

$$\frac{1}{2\pi i}\int_\gamma \frac{1}{\omega(t)}\frac{f(t)}{t-z}dt = \sum_{j=0}^{n}\frac{f(z_j)}{\prod_{k \neq j}(z_j - z_k) * (z_j - z)} + \frac{f(z)}{\omega(z)}$$

Multiplying by $\omega(z)$ we get

$$\frac{1}{2\pi i}\int_\gamma \frac{\omega(z)}{\omega(t)}\frac{f(t)}{t-z}dt = \sum_{j=0}^{n}(-f(z_j))l_j(z) + f(z) = f(z) - p(z)$$

as desired.

(2) (a) Suppose P and Q are polynomials, and the degree of Q exceeds that of P by at least 2, and Q has no roots on the real axis. Prove that $\int_{-\infty}^{\infty}\frac{P(z)}{Q(z)}dz$ is $2\pi i$ times the sum of the residues of P/Q in the upper half plane.

Solution: Let $\gamma' : [0, \pi] \to \mathbb{C}$ where $\gamma(t) = Re^{it}$. Since Q has finitely many zeroes (it is a polynomial that is nontrivial), consider

$$R > \max\{|z| : Q(z) = 0, \mathrm{Im}(z) > 0\}$$

We have that the only possible candidates for residues are the points at which Q is zero. Define $\gamma = \gamma' \oplus [-R, R]$. We have that this is a simple closed curve, so $\mathrm{Ind}_\gamma(z) = 1$ for all z inside the semi-circle. Note by the residue theorem, we have

$$\int_\gamma \frac{P(z)}{Q(z)} dz = 2\pi i \sum_k \mathrm{Res}_{z_k} \frac{P(z)}{Q(z)}$$

where z_k are the poles inside the closed curve (and with how we defined R, it is all the residues in the upper half plane). Now note that

$$\int_\gamma \frac{P(z)}{Q(z)} dz = \int_{[-R,R]} \frac{P(z)}{Q(z)} dz + \int_{\gamma'} \frac{P(z)}{Q(z)} dz$$

Suppose $P(z)$ is degree n, and $Q(z)$ is degree at least $n+2$. Then we have there exists $\mu_Q > 0$ and $r_Q \geq 1$ such that for all $|z| \geq r_Q$, $|Q(z)| \geq \mu_Q |z|^{n+2}$. Thus we have $|Q(Re^{it})| \geq \mu_Q R^{n+2}$ for $R \geq r_Q$. Now.

$$|P(z)| = |a_n z^n + ... + a_1 z + a_0| \leq |a_n||z^n| + ... + |a_0| \leq (|a_n| + ... + |a_0|)|z|^n$$

if $|z| \geq 1$. Thus we have that $P(z) \leq \mu_P |z|^n$, or $|P(Re^{it})| \leq \mu_P R^n$ with $\mu_P > 0$. Therefore we have that for sufficiently large R,

$$\left| \frac{P(z)}{Q(z)} \right| \leq \frac{\mu_P R^n}{\mu_Q R^{n+2}} = \frac{\mu_P}{\mu_Q R^2}$$

Thus we have that

$$\lim_{R \to \infty} \sup\{|f(Re^{it})| : t \in [0.\pi]\} = 0$$

and so by Jordan's lemma,

$$\lim_{R \to \infty} \int_{\gamma'} \frac{P(z)}{Q(z)} dz = 0$$

(with a mild abuse of notation, the R changes for the defined γ'). Therefore we have that

$$\lim_{R \to \infty} \int_\gamma \frac{P(z)}{Q(z)} dz = \lim_{R \to \infty} \left(\int_{\gamma'} \frac{P(z)}{Q(z)} dz + \int_{[-R,R]} \frac{P(z)}{Q(z)} dz \right)$$

$$= \int_{-\infty}^{\infty} \frac{P(z)}{Q(z)} dz$$

and

$$\int_\gamma \frac{P(z)}{Q(z)} dz = 2\pi i \sum_k \text{Res}_{z_k} \frac{P(z)}{Q(z)}$$

for all R, so we are done.

(b) Use this method to compute $\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx$

Solution: Note

$$\frac{1}{(1+x^2)^2} = \frac{1}{(i+x)^2(i-x)^2}$$

Thus we can easily see that it has poles of order 2 at $x = i$ and $x = -i$, but only $x = i$ is in the upper-half. Thus we have

$$\text{Res}_i f = \lim_{z \to 1} \frac{\partial}{\partial x} \left(\frac{(i-x)^2}{(i+x)^2(i-x)^2} \right)$$

We get that the derivative of $\frac{1}{(i+x)^2}$ is $\frac{-2}{(i+x)^3}$, and so

$$\lim_{z \to i} - \frac{2}{(i+x)^3} = -\frac{2}{(2i)^3} = -\frac{i}{4}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^2} dx = 2\pi i \left(-\frac{i}{4}\right) = \frac{\pi}{2}$$

(3) Its is required to expand the function $\frac{1}{1+z^2} + \frac{1}{2+z}$ is a Laurent series about the origin. How many such expansions are there? In which region is each of them valid? Find the coefficients c_n explicitly for each of these expansions.

Solution: The reason for multiple different Laurent series is the location of the poles. The Laurent series for $\frac{1}{1+z^2}$ is different inside $B_1(0)$ and $\mathbb{C} \setminus B_1(0)$ since the poles are $i, -i$ which are a distance of 1 from the origin. The Laurent series for $\frac{1}{2+z}$ is different inside $B_2(0)$ and $\mathbb{C} \setminus B_2(0)$ since its pole is at $z = -2$.

First we consider $\frac{1}{1+z^2}$. We have that the coefficient of the Laurent series of this function when $k > 0$ is

$$\alpha_k = \frac{1}{2\pi i} \int_\gamma \frac{1}{z^{k+1}(1+z^2)} dz$$

We compute the residues. Note that $z = i$ and $z = -i$ are poles of order 1, so

$$\text{Res}_i \frac{1}{z^{k+1}(1+z^2)} = \lim_{z\to i} \frac{1}{z^{k+1}(z+i)} = \frac{1}{2i^{k+2}} = -\frac{1}{2i^k}$$

$$\text{Res}_i \frac{1}{z^{k+1}(1+z^2)} \lim_{z\to -i} \frac{1}{z^{k+1}(z-i)} = \frac{1}{2(-i)^{k+2}} = \frac{(-1)^{k+1}}{2i^k}$$

We also have that $z = 0$ is residue of order $k+1$, thus

$$\text{Res}_0 \frac{1}{z^{k+1}(1+z^2)} = \frac{1}{k!} \lim_{z\to 0} g^{(k)}(z)$$

where $g(z) = \frac{1}{(1+z^2)}$. Rather than trying to evaluate the and then want to die, we can realize that $\frac{1}{1+z^2}$ is holomorphic inside $B_1(0)$, so we can consider the power series by viewing it as a geometric series (since $|z^2| < 1$). Thus we have

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - ...$$

Thus we have by using the properties of power series

$$g^{(k)}(z) = \begin{cases} 0 & k \text{ odd} \\ (-1)^{\frac{k}{2}} k! & \text{otherwise} \end{cases}$$

and so we have

$$\text{Res}_0 \frac{1}{z^{k+1}(1+z^2)} = \begin{cases} 0 & k \text{ odd} \\ (-1)^{\frac{k}{2}} & \text{otherwise} \end{cases}$$

Then we have that

$$\alpha_k = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}}(0)} \frac{1}{z^{k+1}(1+z^2)} dz = \begin{cases} 0 & k \text{ odd} \\ (-1)^{\frac{k}{2}} & \text{otherwise} \end{cases}$$

(since the residues $z = -i, i$ aren't inside the curve), and

$$\alpha_k = \frac{1}{2\pi i} \int_{\Gamma_2(0)} \frac{1}{z^{k+1}(1+z^2)} dz = \begin{cases} 0 & k \text{ odd} \\ (-1)^{\frac{k}{2}} - \frac{1}{i^k} & \text{otherwise} \end{cases}$$

Now if $k > 0$ and we consider

$$\frac{1}{2\pi i} \int_{\gamma} \frac{z^{k-1}}{1+z^2} dz$$

We have that the integrand only has poles at $z = i$ and $z = -i$ of order 1, and after doing a bit of work

$$\mathrm{Res}_i \frac{z^{k-1}}{1+z^2} = \frac{i^{k-1}}{2i} = -\frac{i^k}{2}$$

and

$$\mathrm{Res}_{-i} \frac{z^{k-1}}{1+z^2} = \frac{(-i)^{k-1}}{-2i} = \frac{(-1)^k i^{k-1}}{2i} = \frac{(-1)^{k+1} i^k}{2}$$

Thus we have that

$$\alpha_{-k} = \frac{1}{2\pi i} \int_{\Gamma_{\frac{1}{2}}(0)} \frac{z^{k-1}}{1+z^2} dz = 0$$

$$\alpha_{-k} = \frac{1}{2\pi i} \int_{\Gamma_2(0)} \frac{z^{k-1}}{1+z^2} dz = \begin{cases} 0 & k \text{ odd} \\ -i^k & \text{otherwise} \end{cases}$$

Now we consider the other part, the $\frac{1}{2+z}$. We have that

$$\beta_k = \frac{1}{2\pi i} \int_\gamma \frac{1}{z^{k+1}(2+z)} dz$$

Note that this has a pole of order $k+1$ at $z = 0$ and a simple pole at $z = -2$. We have

$$\mathrm{Res}_0 \frac{1}{z^{k+1}(2+z)} dz = \frac{1}{k!} \lim_{z \to 0} g^{(k)}(z)$$

where $g(z) = \frac{1}{2+z}$, and so $g^{(k)}(z) = (-1)^k \frac{k!}{(2+z)^{k+1}}$. Thus the residue is $\frac{(-1)^k}{2^{k+1}}$. We also have that

$$\mathrm{Res}_{-2} \frac{1}{(2+z)z^{k+1}} = \lim_{z \to -2} \frac{1}{z^{k+1}} = \frac{(-1)^{k+1}}{2^{k+1}}$$

Therefore by the Residue Theorem we have that

$$\beta_k = \frac{1}{2\pi i} \int_{\Gamma_1(0)} \frac{1}{(2+z)z^{k+1}} dz = \frac{(-1)^k}{2^{k+1}}$$

$$\beta_k = \frac{1}{2\pi i} \int_{\Gamma_3(0)} \frac{1}{(2+z)z^{k+1}} dz = 0$$

If $k > 0$, now we consider

$$\beta_{-k} = \frac{1}{2\pi i}\int_\gamma \frac{z^{k-1}}{2+z}dz$$

This only has a simple pole at $z = -2$. Thus

$$\mathrm{Res}_{-2}\frac{z^{k-1}}{2+z} = \lim_{z\to -2} z^{k-1} = (-1)^{k+1}2^{k-1}$$

and thus

$$\beta_{-k} = \frac{1}{2\pi i}\int_{\Gamma_1(0)} \frac{z^{k-1}}{2+z}dz = 0$$

$$\beta_{-k} = \frac{1}{2\pi i}\int_{\Gamma_3(0)} \frac{z^{k-1}}{2+z}dz = (-1)^{k+1}2^{k-1}$$

Thus for the Laurent Series, inside $B_1(0)$ our coefficients are

$$c_k = \begin{cases} -\frac{1}{2^{k+1}} & k \text{ odd} \\ (-1)^{k/2} + \frac{1}{2^{k+1}} & \text{otherwise} \end{cases}$$

$$c_{-k} = 0$$

For the Laurent Series in $B_2(0)\setminus B_1(0)$, we have the coefficients are

$$c_k = \begin{cases} -\frac{1}{2^{k+1}} & k \text{ odd} \\ (-1)^{k/2} + \frac{1}{2^{k+1}} - \frac{1}{i^k} & \text{otherwise} \end{cases}$$

$$c_{-k} = \begin{cases} 0 & k \text{ odd} \\ -i^k & \text{otherwise} \end{cases}$$

For $\mathbb{C}\setminus B_2(0)$ we have the coefficients are

$$c_k = \begin{cases} 0 & k \text{ odd} \\ (-1)^{k/2} - \frac{1}{i^k} & \text{otherwise} \end{cases}$$

$$c_{-k} = \begin{cases} 2^{k-1} & k \text{ odd} \\ -i^k - 2^{k-1} & \text{otherwise} \end{cases}$$

(4) A continuous function f on a compact subset $K \subset \mathbb{R}$ can be uniformly approximated: $p_n \to f$ uniformly on K for some sequence $\{p_n\}$ of polynomials. For $K \subset \mathbb{C}$ this is not true in general.

(a) Show that $K = \overline{A(a, R_1, R_2)} = \{z : R_1 \leq |z-a| \leq R_2\}$ (the closure of the annulus) provides a counterexample.

Solution: Suppose we do have polynomials p_n that converges uniformly to a continuous f. Then this implies that f is holomorphic since the p_n are holomorphic by Theorem 2.42, but this is a contradiction since not every continuous function is holomorphic (take $\text{Re}(z)$)

(b) Let $K = \overline{A(0, 1, 2)}$ and $f(z) = \frac{z}{3} + \frac{1}{z}$. Find an explicit value for $\epsilon > 0$ such that $\sup_{z \in L} |f(z) - p(z)| > \epsilon$ for all polynomials p.

Solution: Note that this is equivalent to finding a bound for

$$\sup_{z \in L} |\frac{1}{z} - p(z)| > \epsilon$$

since $\frac{z}{3} + p(z)$ is a polynomial for polynomials p. We have that f is holomorphic on $K' = A(0, 1 - \delta, 2 + \delta)$ (which is also an open set), and so for every $z \in K'$ we have a ball $\overline{B_r(z)} \subset K'$. Let $\gamma = \partial B_r(z)$. By Cauchy's Integral Formula we have

$$\frac{1}{z} - p(z) = \frac{1}{2\pi i} \int_\gamma \frac{\frac{1}{w} - p(w)}{w - z} dw$$

Note that we have $p(w) = a_0 + a_1 w + ... + a_n w^n$ and it can be rewritten as $p(w) = b_0 + b_1(w - z) + ... + b_n(w - z)^n$ (if we set up the system of linear equations to solve for this, we get a lower/upper triangular matrix depending on how you order the equations where the diagonal entries are all one, meaning the system is invertible, and thus there exists a solution). Thus we get that

$$\int_\gamma \frac{\frac{1}{w} - p(w)}{w - z} dw = \int_\gamma \frac{\frac{1}{w} - b_0}{w - z} + g(w) dw$$

where $g(w) = b_1 + ... + b_n(w-z)^{n-1}$. Note that since g is polynomial, it is holomorphic and thus the integral around the closed curve is 0, so we get that the integral becomes

$$\frac{1}{2\pi i} \int_\gamma \frac{\frac{1}{w} - b_0}{w - z} dw = \frac{1}{z} - b_0$$

using Cauchy's Integral Formula again. Thus we try and find a lower bound for

$$|\frac{1}{z} - b_0|$$

Note that no matter how we choose $z \in K$, then $\frac{1}{z} \in \overline{A(0, \frac{1}{2}, 1)}$, and no matter how we choose b_0, there will always exists a $\frac{1}{z}$ such that $|\frac{1}{z} - b_0| > \frac{1}{2}$. Suppose otherwise, then we have that

$$|-1 - b_0| \leq \frac{1}{2}, \quad |b_0 - 1| \leq \frac{1}{2}$$

Thus we get by the triangle inequality

$$|-2| \leq |-1 - b_0| + |b_0 - 1| \leq 1$$

a contradiction. Thus we have that $\sup_{z \in K} |\frac{1}{z} - b_0| > \frac{1}{2}$, and so we have that

$$\sup_{z \in L} |\frac{1}{z} - p(z)| > |\frac{1}{2\pi i}\frac{1}{2}| = \frac{1}{4\pi}$$

(5) Let $\gamma : [0, 1] \to \mathbb{C}$ be the curve given by $\gamma(t) = \frac{6e^{2\pi it} + 2e^{6\pi it}}{10e^{4\pi it} - 1}$.

(a) Find f holomorphic on a neighbourhood of the unit circle $\{|z| = 1\}$ such that $f \circ \Gamma = \gamma$ (where Γ is the curve that parametrizes the unit circle

Solution: We have that

$$f(z) = \frac{6z + 2z^3}{10z^2 - 1}$$

satisfies the requirements.

(b) Use the argument principle to calculate $\mathrm{Ind}_\gamma(0)$

Solution:

$$\mathrm{Ind}_\gamma(0) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w}dw = \int_0^1 \frac{\gamma'(t)}{\gamma(t)}dt = \int_0^1 \frac{f'(\Gamma(t))\Gamma'(t)}{f(\Gamma(t))}dt = \int_\Gamma \frac{f'(t)}{f(t)}dt$$

Thus we see that $f(z)$ has a root at $z = 0$ insider $B_1(0)$ and has two simple poles at $z = \pm\frac{1}{\sqrt{10}}$. Thus by the argument principle, we have that

$$\mathrm{Ind}_\gamma(0) = -1$$

(c) Calculate $\text{Ind}_\gamma(1+i)$

Solution: Let $g(z) = f(z) - (1+i)$. Then

$$g(\Gamma(t)) = f(\Gamma(t)) - (1+i) = \gamma(t) - (1+i)$$

and

$$\frac{d}{dt}g(\Gamma(t)) = g'(\Gamma(t))\Gamma'(t) = f'(\Gamma(t))\Gamma'(t)$$

since g and f differ by a constant. Thus we have that

$$\text{Ind}_\gamma(1+i) = \frac{1}{2\pi i}\int_\gamma \frac{1}{w-(1+i)}dw = \frac{1}{2\pi i}\int_0^1 \frac{\gamma'(t)}{\gamma(t)-(1+i)}dt$$

$$= \frac{1}{2\pi i}\int_0^1 \frac{g'(\Gamma(t))\Gamma'(t)}{g(\Gamma(t))}dt = \frac{1}{2\pi i}\int_\Gamma \frac{g'(z)}{g(z)}dz$$

Note that the poles for g are the same as the poles for f, so it has two simple poles. Then solving for when g is zero (using Wolfram Alpha), we have that there are two places inside $B_1(0)$ where g is zero. Thus by the argument principle, $\text{Ind}_\gamma(1+i) = 0$