Math H185 Lecture Notes

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1 Preliminaries

Here is an important property of complex numbers:

$$|\mathrm{Re}(z)| \leq |z|, |\mathrm{Im}(z)| \leq |z|, |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$$

Lemma 1.1. *Let (a_n) be a sequence in $\mathbb{C}$, and $L \in \mathbb{C}$. Then $\lim_{n\to\infty} a_n = L$ if and only if*

$$\lim_{n\to\infty} Re(a_n) = Re(L), \ \ \lim_{n\to\infty} Im(a_n) = Im(L)$$

Lemma 1.2. *Let $F \subset \mathbb{C}$ be a set. Then the following are equivalent*

(i) F is closed

(ii) for every sequence $(z_n) \in F$, and $z \in \mathbb{C}$, with $\lim_{n\to\infty} z_n = z$, it follows that $z \in F$

Proof: This is the definition of a closed set.

Definition: A **Cauchy sequence** z_n is a sequence in which for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every pair $m, n \geq N$, we have that $d(z_n, z_m) < \epsilon$.

Definition: A set S is **complete** if every Cauchy sequence in S converges to some value in S.

Theorem 1.3. $\mathbb{C}$ *is complete.*

Proof: We will use the property that $\mathbb{R}$ is complete. Suppose we have a Cauchy sequence (z_n) in $\mathbb{C}$. Thus given $\epsilon > 0$ we have N as defined above. Thus, $\forall m, n \geq N$, $|z_m - z_n| < \epsilon$. Then we have that

$$|\mathrm{Re}(z_m - z_n)| \leq |z_m - z_n| < \epsilon, \ \ |\mathrm{Im}(z_m - z_n)| \leq |z_m - z_n| < \epsilon$$

Note that $\mathrm{Re}(z_m - z_n) = \mathrm{Re}(z_m) - \mathrm{Re}(z_n)$ and the same for the imaginary component. Thus the real components and imaginary components of z_n are Cauchy and thus convergent, so (z_n) is convergent.

Definition 1.4. A set $K \in \mathbb{C}$ is called **sequentially compact** if every sequence in K has a convergent subsequence which converges to a point in K.

Proposition 1.5. *If $K \in \mathbb{C}$, then K is sequentially compact if and only if K is closed and bounded.*

Proof: This proof is made trivial if you use the fact that sequential compactness is equivalent to covering compactness. Suppose K is sequentially compact. Then let (x_n) be a sequence in K that converges to some L in $\mathbb{C}$. By compactness, there exists a subsequence of (x_{n_k}) that is convergent to some value in K. Since in a convergent sequence every subsequence converges to the same value, $L \in K$ and so K is closed.

To verify boundedness, suppose K is not bounded. This implies that $\forall x \in K$, and $r \in \mathbb{R}$, there exists $y \in K$ such that $|x - y| > r$ (otherwise K would be bounded). Let $r \in \mathbb{R}^+$ and construct the sequence inductively as follows. Let $x_1 \in K$. Assuming $x_1, ..., x_n$ is defined, pick x_{n+1} such that $d(x_{n+1}, x_i) > r$. This must exist, otherwise $K \subset \cup_{i=1}^{n} B_r(x_i)$ where $B_r(x_i)$ is the ball centered around x_i with radius r implying that K is bounded. Then in our sequence (x_n) we have that distance between every pair of points is at least r, which is a property that carries over to every subsequence. Therefore every subsequence does not converge and so K is not sequentially compact, proving the contrapositive.

The other way is a bit more complicated and I don't really want to type it up, so it is left as an exercise to the reader.

Proposition 1.6. *Let K_i be a sequence of nonempty sequentially compact subsets of $\mathbb{C}$, and suppose we have that $K_{i+1} \subset K_i$ for every i. Then $\cap_{n=1}^{\infty} K_n$ is nonempty.*

Proof: For each K_i, pick element $x_i \in K_i$. We have that the sequence $(x_n) \in K_1$, so it contains a convergent subsequence, (x_{n_k}) in K_1. Let x be the value of which it converges to. x is clearly in K_i for every i, and thus $x \in \cap_{n=1}^{\infty} K_n$.

Proposition 1.7. *Let $D \in \mathbb{C}$, and $f : D \to \mathbb{C}$ a function with $z_0 \in D$. Then the following are equivalent:*

(i) For every sequence $(d_n) \in D$ with $\lim_{n\to\infty} d_n = z_0$, it follows that $\lim_{n\to\infty} f(d_n) = f(z_0)$

(ii) $\forall \epsilon > 0\ \exists \delta > 0$ such that $\forall d \in D$ with $|d - z_0| < \delta$ it follows that $|f(d) - f(z_0)| < \epsilon$

(iii) For every open set $V \in \mathbb{C}$ with $f(z_0) \in V$, there exists open $U \in \mathbb{C}$ with $z_0 \in U$ and $U \cap D \subset f^{-1}(V)$

Proof: Suppose that (ii) is false, that $\exists \epsilon > 0$ such that $\forall \delta$, $\exists d \in D$ with $|d - z_0| > \delta$ such that $|f(d) - f(z_0)| > \epsilon$. Then construct sequence (d_n) such that $|d_n - z_0| > \frac{1}{n}$ and $|f(d_n) - f(z_0)| > \epsilon$. We have that $\lim_{n\to\infty} d_n = z_0$, but since for all n, $|f(d_n) - f(z_0)| > \epsilon$, $f(d_n)$ does not converge to z_0. Thus (i) implies (ii) by contraposition, as desired.

Now suppose that (ii) is true. By definition of open, $\exists \epsilon > 0$ such that $|f(z_0) - v| < \epsilon$ implies that $v \in V$. By (ii), $\exists \delta > 0$ such that $d \in B_\delta(z_0) \cap D$ (which implies that $d \in D$ and $|d - z_0| < \delta$) implies $|f(d) - f(z_0)| < \epsilon$, and so $d \in f^{-1}(V)$ and we conclude $B_\delta(z_0) \cap D \subset f^{-1}(V)$

Finally, suppose (iii) is true. Consider a sequence $(d_n) \to z_0$. Take $\epsilon > 0$ and consider $B_\epsilon(f(z_0))$. It is open so there exists open D with $z_0 \in D$ and $U \cap D \subset f^{-1}(B_\epsilon(f(z_0)))$. Thus there exists $\delta > 0$ such that $B_\delta(z_0) \in D$, and by definition of convergence $\exists N$ such that $\forall n \geq N$ we have $|d_n - z_0| < \delta$. Then we have that since $d_n \in U \cap D$, we have that $f(d_n) \in B_\epsilon(f(z_0))$ so $|f(d_n) - f(z_0)| < \epsilon$ for all $n \geq N$, and so

$$\lim_{n\to\infty} f(d_n) = f(z_0)$$

Definition 1.8. A function f is **continuous** if it satisfies one of the three conditions stipulated above.

Remark 1. If $f : D \to \mathbb{C}$ and $E \subset D$, define $g = f_{|E}$ (ie f restricted to E), then f being continuous implies that g is continuous.

Lemma 1.9. *Let $K \in \mathbb{C}$ be sequentially compact and $f : K \to \mathbb{C}$ continuous, then $f(K)$ is sequentially compact.*

Proof: Let (a_n) be a sequence in $f(K)$. Define $d_n = f^{-1}(a_n)$ and if there are multiple, arbitrarily pick one. We have by sequential compactness of K, some subsequence d_{n_k} converges to a $d \in K$. Thus by continuity, $f(d_{n_k}) \to f(d)$ as $k \to \infty$, and thus (a_{n_k}) converges.

1.1 Differentiability

Definition 1.10. Let $D \subset \mathbb{C}$, and $z_0 \in \mathbb{C}$, then z_0 is called a **cluster point** in D if there exists a sequence $(d_n) \in D - \{z_0\}$ such that $\lim_{n\to\infty} d_n = z_0$.

Definition 1.11. Let $D \subset \mathbb{C}$, $f : D \to \mathbb{C}$, and $z_0 \in \mathbb{C}$, and z_0 is a cluster point of D. Let $L \in \mathbb{C}$, then we say f has a limit L at z_0 if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall d \in D$ such that $0 < |d - z_0| < \delta$, it follows that $|f(d) - L| < \epsilon$.

Lemma 1.12. *$D \subset \mathbb{C}$, z_0 is a cluster of D, $f : D \to \mathbb{C}$ and $L \in \mathbb{C}$. Define $g : D \cup \{z_0\} \to \mathbb{C}$ as*

$$g(z) = \begin{cases} f(z) & \text{if } z \in D - \{z_0\} \\ L & \text{if } z = z_0 \end{cases}$$

Then f has a limit L at z_0 if and only if g is continuous at z_0.

Proof: This holds from the definition of continuity and Definition 1.11.

Definition 1.13. Let $D \subset \mathbb{C}$, $f : D \to \mathbb{C}$, z_0 a cluster of D. Then f is differentiable at z_0 if $\exists L \in \mathbb{C}$ such that

$$z \mapsto \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0 \\ L & \text{if } z = z_0 \end{cases}$$

is continuous at z_0. We then write $f'(z_0) = L$.

Remark 2. f is differentiable at z if and only if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Proposition 1.14. *Let $D \subset \mathbb{C}$, $f : D \to \mathbb{C}$, z a cluster of D, $L \in \mathbb{C}$, then the following are equivalent:*

(i) f is differentiable at z_0 with $f'(z_0) = L$

(ii) There exists function $\varphi : D \to \mathbb{C}$ continuous at z_0 such that $\varphi(z_0) = L$ and

$$f(z) = f(z_0) + (z - z_0)\varphi(z) \ \ \forall z \in D$$

Proof: We will show that (i) implies (ii), and the remaining are trivial. Define

$$\varphi(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0 \\ L & \text{if } z = z_0 \end{cases}$$

Then we have that

$$f(z) = f(z_0) + (z - z_0)\varphi(z)$$

and that $\varphi(z)$ is continuous at z_0.

Proposition 1.15. *Differentiability implies continuity.*

Proof: Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon$$

Thus

$$|f(z) - f(z_0)| < \epsilon|z - z_0| < \epsilon\delta < \epsilon$$

for sufficiently small δ.

Proposition 1.16. *Let $D \subset \mathbb{C}$, z_0 cluster of D, $f, g : D \to \mathbb{C}$ be differentiable at z_0. Then $f + g$, λf, and fg are differentiable. If $f(z_0) \neq 0$, then $\frac{1}{f}$ is differentiable.*

Proof: Follows from the definition of differentiability. Only the proof for fg is a little more involved.

Theorem 1.17. *Let $D_1, D_2 \subset \mathbb{C}$, z_i a cluster for D_i, $f : D_1 \to \mathbb{C}$ differentiable at z_1, and $g : D_2 \to \mathbb{C}$ differentiable at z_2 with $f(z_1) = z_2$. Suppose $f(D_1) \subset D_2$. Then $g \circ f$ is differentiable at z_1 and*

$$(g \circ f)'(z_1) = g'(f(z_1))f'(z_1)$$

Proof: Proposition 1.14 gives functions φ_1, φ_2 continuous at z_1, z_2 respectively, such that

$$\varphi_1(z_1) = f'(z_1), \;\; \varphi_2(z_2) = g'(z_2)$$

and we have that

$$(g \circ f)(z) = g(z_2) + (f(z) - z_2)\varphi_2(f(z))$$

$$= (g \circ f)(z_1) + (z - z_1)\varphi_1(z)\varphi_2(f(z)) = (g \circ f)(z_1) + (z - z_1)\varphi(z)$$

with $\varphi : D_1 \to \mathbb{C}$ continuous at z_1 and defined as $\varphi(z) = \varphi_1(z)\varphi_2(f(z))$ proving the desired.

Definition 1.18. A **holomorphic** function is a differentiable function $f : U \to \mathbb{C}$ hwere U is an open subset of $\mathbb{C}$.

Define $\Phi : \mathbb{R}^2 \to \mathbb{C}$ as $(x, y) \mapsto x + iy$ and $\Psi : \mathbb{C} \to \mathbb{R}^2$ as the inverse. Let $U \subset \mathbb{C}$, and define $\tilde{U} = \Psi(U)$. If $f : U \to \mathbb{C}$, we can define $\tilde{f} = \Psi \circ f \circ \Phi$. Note $\tilde{f} : \tilde{U} \to \mathbb{R}^2$.

Let $z_0 = z_0 + iy_0$. By definition, $\tilde{f}$ is differentiable at (x_0, y_0) if and only if there exists 2×2 matrix M and a function $r : \tilde{U} \to \mathbb{R}^2$ such that for every $(x, y) \in \tilde{U}$ we have

$$\tilde{f}(x, y) = \tilde{f}(x_0, y_0) + M \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix}^\top + r(x, y)$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{r(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

We say that M is the derivative of $\tilde{f}$ at (x_0, y_0).

If $a, b \in \mathbb{R}$ with $L = a + ib$, then we have that

$$\Psi((z - z_0)L) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Suppose f is differentiable at z_0, and $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Then we have that $\tilde{f} = \Psi \circ f \circ \Phi$ is differentiable at (x_0, y_0) with derivative M and

$$r(x, y) = \Psi\left((x + iy) - z_0) * \psi(x + iy)\right)$$

where the ψ comes from Proposition 1.14(III).

Suppose $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) with derivative $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$. Let

$$\psi(x + iy) = \begin{cases} \frac{\Phi(r(x,y))}{x+iy-z} & x + iy \neq z_0 \\ 0 & x + iy = z_0 \end{cases}$$

These satisfies Proposition 1.14(III) with $L = a + bi$, so f is complex differentiable.

Take $f : U \to \mathbb{C}$ continuous. Define $u, v : \tilde{U} \to \mathbb{R}$ via

$$\tilde{f}(x, y) = (u(x, y), v(x, y))$$

Proposition 1.19. *The following are equivalent*

(i) f is differentiable at z_0

(ii) the functions u, v are differentiable at (x_0, y_0) and

$$(D_1u)(x_0, y_0) = (D_2v)(x_0, y_0)$$

$$(D_2u)(x_0, y_0) = -(D_1v)(x_0, y_0)$$

where $D_1u = \delta_x u$, $D_2u = \delta_y u$.

Definition 1.20. The equations are called the **Cauchy-Riemann equations**.

1.2 Series in $\mathbb{C}$

Let $a_0, a_1, ... \in \mathbb{C}$, $\forall n \in N_0$. Define $s_n = \sum_{k=0}^{n} a_n \in \mathbb{C}$. We call s_n the n^{th} partial sum. We say that $\sum_{a_k}$ is convergent if and only if (s_n) converges, and we write

$$\sum_{k=0}^{\infty} a_k = \lim_{n\to\infty} s_n$$

Remark 3. $\sum a_n$ converges if and only if $\sum \mathrm{Re}(a_n)$ and $\sum \mathrm{Im}(a_n)$ converges.

We say that the series $\sum a_k$ is absolutely convergent if $\sum |a_n|$ is convergent in $\mathbb{R}$.

Theorem 1.21. *Every absolutely convergent sequence is convergent in $\mathbb{C}$.*

Proof:

$$\sum_{n=0}^{k} a_n \leq \left|\sum_{n=0}^{k} a_n\right| \leq \sum_{n=0}^{k} |a_n|$$

and then by the sequence comparison test we have $\sum a_n$ converges (I think).

Definition 1.22. Let $D \subset \mathbb{C}$ and $f, f_0, f_1, ... : D \to \mathbb{C}$ be a sequence of functions. We say $\lim_{n\to\infty} f_n = f$ uniformly if $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall x \in D$, $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

Remark 4. Consider $\sum f_k$ where $f_k : D \to \mathbb{C}$. Define $S_k = \sum_{i=0}^{k} f_i$. The series $\sum f_k$ is uniformly convergent on D if the sequence of partial sums S_k is uniformly convergent.

The following result is called the Weierstrass M-test.

Proposition 1.23. *Let $D \subset \mathbb{C}$ and $f_0, f_1, ... : D \to \mathbb{C}$ be a sequence of functions. Let $a_0, a_1, ... \in \mathbb{R}^+$. Suppose $|f_k(z)| \leq a_k$ for all $z \in D$ and all $k \in \mathbb{N}_0$. Moreover suppose that the series $\sum a_n$ is convergent. Then the series $\sum f_n$ is uniformly convergent on D.*

Proof: $\forall z \in D$, the series $\sum_n f_n(z)$ is absolutely convergent clearly (and so therefore convergent). Define $S : D \to \mathbb{C}$ by $S(z) = \sum_{n=0}^{\infty} f(z)$. Let $z \in D$ and if $N \geq n$, then

$$\left|\sum_{k=0}^{N} f_k(z) - \sum_{k=0}^{n} f_k(z)\right| = \left|\sum_{k=n+1}^{N} f_k(z)\right| \leq \sum_{k=n+1}^{N} |f_k(z)| \leq \sum_{k=n+1}^{N} a_k \leq \sum_{k=n+1}^{\infty} a_k$$

Now take the limit as $N \to \infty$, we get that

$$\left|S(z) - \sum_{k=0}^{n} f_k(z)\right| \leq \sum_{k=n+1}^{\infty} a_k$$

Thus since $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_k = 0$, we have the desired result.

1.3 Integration

Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{C}$. This is called Riemann integrable over $[a, b]$ if both Ref and Imf are Riemann integrable on $[a, b] \to \mathbb{R}$, and

$$\int_a^b f(t)dt = \int_a^b \mathrm{Re} f(t)dt + i \int_a^b \mathrm{Im} f(t)dt$$

Lemma 1.24. *The integral is linear.*

Lemma 1.25. *Let $a, b \in \mathbb{R}$, with $a < b$, and $f : [a, b] \to \mathbb{R}$ continuous. Then $|f|$ is Riemann integrable over $[a, b]$ and*

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$$

Proof: Since $|f|$ is continuous, it is Riemann integrable. Let $r \in [0, \infty)$ and $\theta \in \mathbb{R}$ such that $\int_a^b f(t)dt = r(\cos\theta + i\sin\theta)$, and define $\zeta = \cos\theta - i\sin\theta$. Then

$$\left|\int_a^b f(t)dt\right| = \zeta \int_a^b f(t)dt = \mathrm{Re}(\zeta \int_a^b f(t)dt) = \int_b^a \mathrm{Re}(\zeta f(t))dt$$

$$\leq \int_a^b \left|\mathrm{Re}(\zeta f(t))\right| dt \leq \int_a^b |\zeta f(t)| dt = \int_a^b |f(t)| dt$$

2 Analytic Functions

2.1 Power Series

Let $a \in \mathbb{C}$. Then a power series about a is a series of the form

$$\sum \alpha_n (z - \alpha)^n$$

where $\alpha_k \in \mathbb{C}$, $\forall k \in \mathbb{N}_0$. For each $z \in \mathbb{C}$, the series might or might not converge.

If $U \in \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is infinitely differentiable (smooth) then

$$\sum \frac{f^{(n)}(a)}{n!}(z - a)^n$$

is called the power series of f about $a \in U$.

Define $R \in [0, \infty]$ the supremum of all $r \in [0, \infty)$ such that the series

$$\sum |\alpha_n| r^n$$

is convergent. Call R the radius of convergence of the power series.

Lemma 2.1. *Let R be the radius of convergence of $\sum \alpha_k (z - a)^k$. Then the following are equivalent*

(I) If $z \in \mathbb{C}$ with $|z - a| < R$ then the series converges

(II) If $z \in \mathbb{C}$ with $z \notin \overline{B_R(a)}$, then the series diverges

(III) If $0 < r < R$ then $\sum \alpha_n (z - a)^n$ and $\sum |\alpha_n (z - a)^n|$ are convergent uniformly

Proof: WLOG assume $a = 0$. We will prove (II) and (III), and (I) will follow from (III).

We will prove (II) via contraposition. Let $z \in \mathbb{C} - \{0\}$ and suppose that $\sum \alpha_n z^n$ is convergent. Thus $\{\alpha_n z^n : n \in \mathbb{N}\}$ is bounded. Let $M > 0$ be such that $|\alpha_n z^n| \leq M$ for all $n \in \mathbb{N}_0$. Let $r \in (0, |z|)$, then

$$|\alpha_n| r^n \leq M \left(\frac{r}{|z|}\right)^n, \forall n \in \mathbb{N}_0$$

Since $\frac{r}{|z|} < 1$, the series $\sum |\alpha_n| r^n$ converges, by definition of radius of convergence, $r < R$, so $|z| \leq R$.

To prove (III), let $r \in (0, R)$, so $\sum |\alpha_n| r^n$ is convergent. If $z \in \overline{B_r(a)}$ then $|\alpha_n z^n| \leq |\alpha_n| r^n$ for all $n \in \mathbb{N}$. Thus by Weierstrass, we get that $\sum \alpha_n z^n$ and $\sum |\alpha_n z^n|$ are uniformly convergent for all $z \in \overline{B_r(0)}$.

Example 2.2. The radius of convergence of $\sum z^n$ is 1, so it is uniformly convergent on $\overline{B_r(0)}$, $\forall r \in (0, 1)$

Definition 2.3. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ and $a \subset U$. Then f is analytic at a if $\exists R > 0$ and a power series $\sum \alpha_n (z - a)^n$ about a such that $B_R(a) \subset U$, the power series has positive radius of convergence at least R, and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - a)^n, \quad \forall z \in B_R(a)$$

We say that f is analytic if it is analytic $\forall a \in D$.

Note that every polynomial is analytic.

Lemma 2.4. *The power series $\sum \alpha_n (z - a)^n$ and $\sum n\alpha_n (z - a)^{n-1}$ have the same radius of convergence.*

Proof: WLOG assume $a = 0$. Let $R, \hat{R}$ be the respective radii of convergence. Let $r \in [0, R)$. Then there exists $\rho \in \mathbb{R}$ with $r < \rho < R$ such that $\sum |\alpha_n| \rho^n$ is convergent and $\lim_{n\to\infty} n \left(\frac{r}{\rho}\right)^n = 0$. Then $\exists M \in \mathbb{R}$ such that $n \left(\frac{r}{\rho}\right)^n \leq M$ for all $n \in \mathbb{N}_0$. Then

$$n|\alpha_n| r^n = n \left(\frac{r}{\rho}\right)^n |\alpha_n| \rho^n \leq M |\alpha_n| \rho^n, \quad \forall n \in \mathbb{N}$$

Thus since $\sum |\alpha_n| \rho^n$ converges, so too does $\sum n|\alpha_n| r^n$ (and thus $\sum n|\alpha_n| r^{n-1}$). Thus $\hat{R} \geq r$ and so $\hat{R} \geq R$. The other direction is similar.

Theorem 2.5. *Let $R \in (0,\infty)$ and suppose the radius of convergence of the power series $\sum \alpha_n(z-a)^n$ is also at least R. Define $f : B_R(a) \to \mathbb{C}$ as $f(z) = \sum_{n=0}^{\infty} \alpha_n(z-a)^n$. Then f is holomorphic (differentiable). Moreover, $f'(z) = \sum_{n=0}^{\infty} n\alpha_n(z-a)^n$, $\forall z \in B_R(a)$. Hence f is infinitely differentiable and $\alpha_n = \frac{f^{(n)}(a)}{n!}$ for all $n \in \mathbb{N}_0$.*

Proof: WLOG let $a = 0$. Fix $z_0 \in B_R(0)$, and let $\epsilon > 0$. Fix an $r \in (|z_0|, R)$. By lemma 2.4, $\exists n \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} n|\alpha_n|r^{n-1} \leq \frac{\epsilon}{4}$$

Then $\forall z \in B_r(0) - \{z_0\}$, we have

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=1}^{\infty} n\alpha_n z_0^{n-1}\right| = \left|\sum_{n=0}^{\infty} \alpha_n \left(\frac{z^n - z_0^n}{z-z_0} - nz_0^{n-1}\right)\right|$$

$$= \left|\sum_{n=1}^{\infty} \alpha_n \left((\sum_{k=0}^{n-1} z^k z_0^{n-1-k}) - nz_0^{n-1}\right)\right|$$

$$\leq \left|\sum_{n=1}^{N} \alpha_n \sum_{k=0}^{n-1} (z^k z_0^{n-1-k} - z_0^{n-1})\right| + \sum_{n=N+1}^{\infty} 2n|\alpha_n|r^{n-1}$$

The right sum is bounded by $\frac{\epsilon}{2}$. Note that $z \mapsto z^k z_0^{n-1-k} - z_0^{n-1}$ is continuous at z_0, with value 0 for all $n \in \{1, ..., N\}$ and for all $k \in \{0, ..., n-1\}$, so we can bound it and get from the above expression

$$\lim_{z \to z_0} \left(\frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=1}^{\infty} \alpha_n z_0^{n-1}\right) = 0$$

Thus f is differentiable at z_0 with $f'(z_0) = \sum_{n=1}^{\infty} \alpha_n n z_0^{n-1}$. The proof that $\alpha_n = \frac{f(n)(a)}{n!}$ follows from repeated differentiation.

Corollary 2.6. *Every analytic function is differentiable.*

Corollary 2.7. *Let $\sum \alpha_n(z-a)^n$ and $\sum \beta_n(z-a)^n$ be two power series with radii of convergence R_1, R_2 respectively. Let $\epsilon \leq \min(R_1, R_2)$. Suppose that $\sum_{n=1}^{\infty} \alpha_n(z-a)^n = \sum_{n=1}^{\infty} \beta_n(z-a)^n$ for all $z \in B_\epsilon(a)$. Then $\alpha_n = \beta_n$ for all $n \in \mathbb{N}$.*

Proof: Consider $\sum(\alpha_n - \beta_n)(z-a)^n$. It is identically 0, so the n^{th} order derivatives are 0, so $\alpha_n = \beta_n$.

Define

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

These have infinite radius of convergence, and so we take these as a definition for functions $\mathbb{C} \to \mathbb{C}$. Note that we can show

$$e^{iz} = \sum i^n \frac{z^n}{n!} = ... = \cos z + i \sin z$$

We also have the formulas

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2}$$

Let $w, z \in \mathbb{C}$ and $f : \mathbb{R} \to \mathbb{C}$ such that $f(t) = e^{-tz}e^{w+tz}$. We can verify by the product rule that $f' = 0$, and in particular $f(1) = f(0)$. Thus $e^{-z}e^{w+z} = e^w$. Let $a, b \in \mathbb{C}$, and choose $z = -a$, $w = a + b$. Then $e^a e^b = e^{a+b}$.

Let $x, y, u, v \in \mathbb{R}$. Suppose $e^{x+iy} = e^{u+iv}$. Thus $e^x = e^u$, so $x = u$. This implies $e^{iy} = e^{iv}$, so $y - v \in 2\pi\mathbb{Z}$. Thus if $z, w \in \mathbb{C}$, then $e^z = e^w$ if and only if $z - w \in 2\pi i\mathbb{Z}$.

2.2 Curves

Definition 2.8. A curve is a continuous map $\gamma : [a, b] \to \mathbb{C}$ where $a < b \in \mathbb{R}$. Let $\gamma(a)$ be the initial point and $\gamma(b)$ be the final point. Let $\gamma*$ denote the image of γ in $\mathbb{C}$.

If $A \subset \mathbb{C}$, we say γ is in A if $\gamma* \subset A$. Lets say γ is closed if $\gamma(a) = \gamma(b)$. γ is called smooth if it is $\mathbb{C}^1$ continuously differentiable. There is also a notion of being piecewise smooth (it is smooth every except at a finite number of points).

If $\gamma : [a,b] \to \mathbb{C}$ and $\mu : [c,d] \to \mathbb{C}$ with $\gamma(b) = \mu(c)$ then we define the combined curve $\gamma \oplus \mu : [a, b+d-c] \to \mathbb{C}$ as

$$(\gamma \oplus \mu)(t) = \begin{cases} \gamma(t) & t \in [a,b] \\ \mu(c+t-b) & t \in (b, b+d-c] \end{cases}$$

Example 2.9. Define $\gamma : [-1,1] \to \mathbb{C}$ by

$$\gamma(t) = \begin{cases} t^2(1+i) & t \in [0,1] \\ t^2(-1+i) & t \in [-1,0) \end{cases}$$

is smooth. But

$$\gamma(t) = \begin{cases} t(1+i) & t \in [0,1] \\ 2t(1+i) & t \in [-1,0) \end{cases}$$

is not smooth.

Since γ is continuous and $[a,b]$ is sequentially compact, it follows that $\gamma*$ is sequentially compact (and in particular it is closed).

Definition 2.10. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve, $D \subset \mathbb{C}$, and if $f : D \to \mathbb{C}$ a continuous function. Suppose $\gamma* \subset D$. If γ is smooth, we define the contour integral of f along γ to be

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

If γ is only piecewise smooth, with $a = s_0 < s_1 < ... < s_n = b$ and $\gamma|_{[s_{k-1}, s_k]}$ is smooth, then

$$\int_\gamma f(z)dz = \sum_{i=1}^n \int_{\gamma_{[s_{i-1}, s_i]}} f(z)dz$$

The lengths of curve γ is given by

$$l(y) = \int_a^b |\gamma'(t)|dt$$

For piecewise γ,

$$l(\gamma) = \sum_{i=1}^n l(\gamma_i)$$

Example 2.11. Define $\gamma : [0.2\pi] \to \mathbb{C}$, as $t \mapsto e^{it}$. Moreover define $f : \mathbb{C} - \{0\} \to \mathbb{C}$ to be $f(z) = \frac{1}{z}$.

$$\int_\gamma f(z)dz = \int_0^{2\pi} \frac{1}{e^{it}} * ie^{it} dt = i \int_0^{2\pi} dt = 2\pi i$$

Lemma 2.12. *Let $\gamma : [a,b] \to \mathbb{C}$ be piecewise smooth, and $f, \tilde{f} : \gamma* \to \mathbb{C}$. Then we have*

(I) $\int_\gamma (f + \tilde{f})(z)dz = \int_\gamma f(z)dz + \int_\gamma \tilde{f}(z)dz$

(II) $\int_\gamma \lambda f(z)dz = \lambda \int_\gamma f(z)dz$

(III) $|\int_\gamma f(z)dz| \leq Ml(\gamma)$ *where* $M = \sup\{|f(z)| : z \in \gamma*\}$

(IV) Let $c, d \in \mathbb{R}$, $c < d$ and $\varphi : [c,d] \to [a,b]$ be a continuously differentiable function such that $\varphi(c) = a$, and $\varphi(d) = b$. Suppose γ is smooth or φ is strictly increasing. Then $\gamma \circ \varphi$ is piecewise smooth and $\int_\gamma f(z)dz = \int_{\gamma\circ\varphi} f(z)dz$

(V) $f_k : \gamma* \to \mathbb{C}$ be continuous functions with $\lim f_k = f$ uniformly on $\gamma*$. Then

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$$

(VI) If γ_1, γ_2 are curves with $\gamma = \gamma_1 \oplus \gamma_2$, then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

Proof of (IV): Suppose γ is smooth, then

$$\int_{\gamma\circ\varphi} f(z)dz = \int_c^d f(\gamma \circ \varphi(t))(\gamma \circ \varphi)'(t)dt = \int_c^d f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)dt$$

$$= \int_a^b f(\gamma(\tilde{t}))\gamma'(\tilde{t})d\tilde{t} = \int_\gamma f(z)dz$$

Proposition 2.13. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ continuously differentiable, $\gamma : [a,b] \to \mathbb{C}$ be piecewise smooth with $\gamma^* \subset U$. Then*

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$$

Proof: Suppose γ is smooth (proof is almost the same). Then we have that

$$\int_\gamma f'(z)dz = \int_a^b f'(\gamma(t))\gamma'(t)dt = \int_a^b (f\circ\gamma)'(t)dt = f(\gamma(b)) - f(\gamma(a))$$

where the last equality follows from FTC.

Corollary 2.14. *Let $U \subset \mathbb{C}$ be open with $f : U \to \mathbb{C}$ be holomorphic and $f' = 0$. Suppose $\forall z_1, z_2 \in U$, there exists $\gamma : [0,1] \to U$ with $\gamma(0) = z_1$ and $\gamma(1) = z_2$ where γ is smooth. Then f is constant.*

Remark 5. The above requirement is for f to be constant when $f' = 0$ also includes that U must be path connected. In general $f' = 0$ does not imply f is constant.

Theorem 2.15. *Suppose $\gamma : [a,b] \to \mathbb{C}$ be piecewise smooth and $g : \gamma^* \to \mathbb{C}$ a continuous function. Let $U = \mathbb{C} - \gamma*$ (it is open) and define $f : U \to \mathbb{C}$ as*

$$f(z) = \int_\gamma \frac{g(w)}{w-z}dw$$

Then f is analytic. More specifically, let $z_0 \in U$, and

$$R = \inf\{|w - z_0| : w \in \gamma*\}$$

Then $R > 0$ and $\forall n \in N$ let $\alpha_n = \int_\gamma \frac{g(w)}{(w-z_0)^{n+1}}dw$. Then the power series $\sum \alpha_n (z-z_0)^n$ has a radius of convergence $> R$, and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z-z_0)^n$$

Proof: Note $\gamma*$ is closed since γ is continuous, so U is open. Let z_0 be given and R defined as above. Let $z \in B_R(z_0)$. For $w \in \gamma*$,

$$\left|\frac{z-z_0}{w-z_0}\right| \le \frac{|z-z_0|}{R} < 1$$

Therefore,

$$\frac{1}{w-z} = \frac{1}{(w-z_0)-(z-z_0)} = \frac{1}{w-z_0} * \frac{1}{1-\frac{z-z_0}{w-z_0}}$$

$$= \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$$

since the series is absolutely convergent for $z \in B_R(z_0)$, and thus is convergent with the given formula. It is also true $\forall w in \gamma^*$. Now define $h, h_0, h_1, ... : \gamma* \to \mathbb{C}$ as

$$h(w) = \frac{g(w)}{w - z}, \quad h_n(w) = \frac{g(w)(z - z_0)^n}{(w - z_0)^{n+1}}$$

then on γ^*, $\lim_{n\to\infty} \sum_i^n h_i = h$. By Lemma 2.12 (VI) we have that

$$f(z) = \int_\gamma h(w)dw = \sum_{n=0}^{\infty} \int_\gamma h_n(w)dw = \sum_{n=0}^{\infty}(z - z_0)^n \int_\gamma \frac{g(w)}{(w - z_0)^{n-1}}dw$$

$$= \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

In particular, the series $\sum \alpha_n (z - z_0)^n$ converges $\forall z \in B_R(z_0)$. By Lemma 2.1, the power series has radius of convergence at least R.

Definition 2.16. Let γ be a closed piecewise smooth curve. Define on $\mathbb{C} - \gamma^*$

$$\mathrm{Ind}_\gamma : \mathbb{C} - \gamma^* \to \mathbb{C}$$
$$z \mapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} dw$$

This is called the index function of γ with respect to z.

Note that Ind_γ is analytic.

Proposition 2.17. *Let γ be a piecewise smooth closed curve. Then $Ind_\gamma(z) \in \mathbb{Z}$ for all $z \in \mathbb{C} - \gamma^*$. Moreover there exists $R > 0$ such that $Ind_\gamma(z) = 0$ for all $z \in \mathbb{C} - B_R(0)$*

Proof: Let $\gamma : [a, b] \to \mathbb{C}$ be smooth. Define $f : [a, b] \to \mathbb{C}$ as $t \mapsto \int_a^t \frac{\gamma'(s)}{\gamma(s)-z} ds$. Then f is differentiable and $f'(t) = \frac{\gamma'(t)}{\gamma(t)-z}$. Define $g : [a, b] \to \mathbb{C}$ by $g(t) = e^{-f(t)}(\gamma(t) - z)$. Then $g'(t) = 0$ (easily verifiable) and $[a, b]$ is path connected, so g is constant. Therefore

$$\frac{e^{f(t)}}{e^{f(a)}} = \frac{\gamma(t) - z}{\gamma(a) - z}$$

Since $f(a) = 0$ and $\gamma(b) = \gamma(a)$, it follows that $e^{f(b)} = 1$ so $f(b) \in 2\pi i\mathbb{Z}$, but $f(b) = 2\pi i \mathrm{Ind}_\gamma(z)$, so we have $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$.

Since γ^* is bounded, $\exists r > 0$ such that $\gamma^* \subset B_r(0)$ and we have that

$$|\mathrm{Ind}_\gamma(z)| \leq \frac{Kl(\gamma)}{R-r}$$

where $R > r$ and $z \in \mathbb{C} - B_R(0)$ and K is a constant in $\mathbb{R}$. Since $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$, for large enough R we have $\mathrm{Ind}_\gamma(z) = 0$. Since $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$, we conclude for sufficiently large R, we conclude $\mathrm{Ind}_\gamma(z) = 0$.

Corollary 2.18. *Let γ be a closed piecewise smooth curve, and $\varphi : [0,1] \to \mathbb{C} - \gamma^*$ be continuous. Then $Ind_\gamma(\varphi(0)) = Ind_\gamma(\varphi(1))$.*

Definition 2.19. For all $z_0 \in \mathbb{C}$, $r > 0$ define $\gamma : [0, 2\pi] \to \mathbb{C}$, as

$$t \mapsto z_0 + re^{it}$$

Then denote this curve by $\Gamma_r(z_0) = \gamma$.

Example 2.20. Let $z_0 \in \mathbb{C}$, $r > 0$. Then we have $\mathrm{Ind}_{\Gamma_r(z_0)}(z_0) = 1$. moreover, if $|z - z_0| > 3r$, then estimates show

$$|\mathrm{Ind}_{\Gamma_r(z_0)}(z)| \leq \frac{1}{2}$$

and by Corollary 2.18 we have that

$$\mathrm{Ind}_{\Gamma_r(z_0)}(z) = \begin{cases} 1 & |z - z_0| < r \\ 0 & |z - z_0| > r \end{cases}$$

2.3 Cauchy's Theorem for a triangle and a convex set

Definition 2.21. Let $z_1, z_2 \in \mathbb{C}$. Denote by $[z_1, z_2]$ the curve $\gamma : [0,1] \to \mathbb{C}$ which maps $t \mapsto z_1 + t(z_2 - z_1)$. Let $z_3 \in C$. Then define

$$\Delta(z_1, z_2, z_3) = \{t_1 z_1 + t_2 z_2 + t_3 z_3 | t_i \geq 0, \sum_i t_i = 1\}$$

Denote also by $\partial\Delta(z_1, z_2, z_3)$ the piecewise smooth curve $\gamma : [0,3] \to \mathbb{C}$ by

$$\gamma = [z_1, z_2] \oplus [z_2, z_3] \oplus [z_3, z_1]$$

Remark 6.

$$\int_{\partial\Delta} = \int_{[z_1,z_2]} + \int_{[z_2,z_3]} + \int_{[z_3,z_1]}$$

If z_1, z_2, z_3 are collinear, then $\int_{\partial\Delta} = 0$.

Theorem 2.22. *Cauchy-Goursat Theorem: Let $U \subset \mathbb{C}$ open, $p \in U$, $\Delta \subset U$, $f : U \to \mathbb{C}$ be continuous, and suppose $f : U - \{p\}$ be holomorphic. Then $\int_{\partial\Delta} f(z)dz = 0$.*

Proof: There are 3 steps depending on where p is relative to $\partial\Delta$.

Step 1: Suppose $p \notin \Delta$. Then denote the midpoint of the sides of the triangle by z_1', z_2', z_3' where $z_1' = \frac{z_2+z_3}{2}$, and the others are defined similarly. Connect the z_1', z_2', z_3' in the original triangle, and so we get 4 triangles. Label them $\Delta^(i)$ arbitrarily. Note that we have

$$\int_{\partial\Delta} f(z)dz = \sum_{k=1}^{4} \int_{\partial\Delta^(k)} f(z)dz$$

so

$$\int_{\partial\Delta} \leq \left| \int_{\partial\Delta} f(z)dz \right| \leq \sum_{k=1}^{4} \left| \int_{\partial\Delta^{(k)}} f(z)dz \right| \leq 4 \left| \int_{\partial\Delta^{(i)}} f(z)dz \right|$$

for some $i = 1, 2, 3, 4$. Let $\Delta_1 = \Delta^{(i)}$, and note that $l(\partial\Delta_1) = \frac{1}{2} l(\partial\Delta)$. By induction we can continue and make a sequence of closed triangles $\Delta_2, \Delta_3, ...$ with $\Delta_{k+1} \subset \Delta_k$ and

$$\left| \int_{\partial\Delta_k} f(z)dz \right| \leq 4 \left| \int_{\partial\Delta_{k+1}} f(z)dz \right|$$

$$l(\Delta_{k+1}) = \frac{1}{2} l(\Delta_k)$$

Note that for all $z \in \partial\Delta_n$, $|z - z_0| < l(\partial\Delta_n)$.

Thus

$$\left| \int_{\partial\Delta} f(z)dz \right| \leq 4^n \left| \int_{\partial\Delta_n} f(z)dz \right|$$

$$l(\Delta_n) = \frac{1}{2^n} l(\Delta)$$

By compactness, there exists $z_0 \in \cap_{n=1}^{\infty} \Delta_n$ and $z_0 \neq p$. So f is differentiable at z_0 and so

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

with $\psi : U \to \mathbb{C}$ continuous at z_0, and $\psi(z_0) = 0$. The map

$$z \mapsto f(z_0) + f'(z_0)(z - z_0)$$

comes from the derivative of the map

$$z \mapsto f(z_0)z + \frac{1}{2}f'(z_0)(z - z_0)^2$$

Thus for all n,

$$\int_{\partial\Delta_n} f(z_0) + f'(z_0)(z - z_0)dz = 0$$

Therefore we get that

$$\left|\int_{\partial\Delta_n} f(z)dz\right| = \left|\int_{\partial\Delta_n} f(z) - (f(z_0) + f'(z_0)(z - z_0))dz\right| = \left|\int_{\partial\Delta_n} (z - z_0)\psi(z)dz\right|$$

$$\leq l(\partial\Delta_n)\sup\{|(z - z_0)\psi(z)| : z \in \partial\Delta_n\}$$

$$\leq (l(\partial\Delta_n))^2 \sup\{|\psi(z)| : z \in \partial\Delta_n\}$$

Since ψ is continuous at z_0 with $\psi(z_0) = 0$, for all $\epsilon > 0$ we have $\exists\delta > 0$ such that $|\psi(z)| < \epsilon$ for $z \in B_\delta(z_0) \cap U$. Since $\lim_{n\to\infty} l(\partial\Delta_n) = 0$ (then the diameter also goes to 0) we have that $\exists n \in \mathbb{N}$ such that $\bar{\Delta_n} \subset B_\delta(z_0)$. Then $\sup\{|\psi(z)| : z \in \partial\Delta_n\} \leq \epsilon$ and

$$\left|\int_{\partial\Delta_n} f(z)dz\right| \leq \epsilon(l(\partial\Delta))^2 = \frac{\epsilon}{4^n}$$

Hence

$$\left|\int_{\partial\Delta} f(z)dz\right| \leq 4^n\frac{\epsilon}{4^n} = \epsilon$$

and we conclude that

$$\int_{\partial\Delta} f(z)dz = 0$$

In the next step, suppose p is a vertex. Then WLOG let $\Delta = \Delta(p, z_2, z_3)$. Let $\epsilon \in (0, 1)$ and set $p_2 = \epsilon z_2 + (1 - \epsilon)p$ and $p_3 = \epsilon z_3 + (1 - \epsilon)p$. Then we have that

$$\int_{\partial\Delta} = \int_{\partial\Delta(p,p_2,p_3)} + \int_{\partial\Delta(p_2,z_2,z_3)} + \int_{\partial\Delta(p_3,p_2,z_3)}$$

We have that the 2^{nd} and 3^{rd} integrals are zero by the earlier case, so we only need to consider the first integral. Denoting $\partial\Delta(p_1, p_2, p_3)$ by $\partial\Delta p$ we have that

$$\left|\int_{\partial\Delta p} f(z)dz\right| \leq l(\partial\Delta_p)\sup\{|f(z)| : z \in \partial\Delta_p\}$$

There exists $r > 0$ such that $B_{\bar{r}}(p) \subset U$ (by openness of U), and since f is continuous $\exists M > 0$ such that $|f(z)| < M$ for all $z \in B_{\bar{r}}(p)$. If ϵ is small enough, then $\partial\Delta_p \subset B_r(p)$ and since $\lim_{\epsilon\to 0} l(\partial\Delta_p) = 0$, we get that

$$l(\partial\Delta_p)\sup\{|f(z)| : z \in \partial\Delta_p\} \leq \epsilon'$$

proving the desired.

For the case that p is not a vertex but $p \in \Delta$, consider the integral

$$\int_{\partial\Delta(z_1,z_2,z_3)} = \int_{\partial\Delta(z_1,z_2,p)} + \int_{\partial\Delta(p,z_2,z_3)} + \int_{\partial\Delta(z_3,z_1,p)}$$

All three of the integrals are 0 by case 2, and we are done.

We define a set $A \subset \mathbb{C}$ to be convex if

$$tz_1 + (1-t)z_2 \in A$$

for all $z_1, z_2 \in A$, $t \in [0, 1]$.

Proposition 2.23. *Let $U \subset \mathbb{C}$ be open and convex, $f : U \to \mathbb{C}$ continuous such that*

$$\int_{\partial\Delta} f(z)dz = 0$$

for all $\Delta \subset U$. Then $\exists F : U \to C$ holomorphic such that $F' = f$.

Proof: Fix $a \in U$ and define $F : U \to \mathbb{C}$ by $F(z) = \int_{[a,z]} f(w)dw$. Fix $z_0 \in U$ and let $z \in U$. Then we have that

$$0 = \int_{\partial\Delta(a,z,z_0)} f(w)dw = \int_{[a,z]} f(w)dw + \int_{[z,z_0]} f(w)dw + \int_{[z_0,a]} f(w)dw$$

$$= F(z) + \int_{[z,z_0]} f(w)dw - F(z_0)$$

Now let $\epsilon > 0$, since f is continuous at z_0 we have $\exists\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$, for all $|z - z_0| < \delta$. We can write

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{[z_0,z]} (f(w) - f(z_0))dz$$

and so for all $z \in B_r(z_0)$,

$$\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| \leq \frac{1}{|z - z_0|}\int_{[z_0,z]} |f(w) - f(z_0)|dw \leq \frac{1}{|z - z_0|} l([z_0, z])\epsilon \leq \epsilon$$

for all $zinU$ δ close to z_0. Thus $F'(z_0) = f(z_0)$.

Corollary 2.24. *Let U be open and convex with $p \in U$, $f : U \to \mathbb{C}$ continuous who is holomorphic off p. Then $f = F'$ for some holomorphic function $F : U \to \mathbb{C}$. Explicitly, $\forall a \in U$, we have that $F(z) = \int_{[a,z]} f(w)dw$.*

Proof: Because f is holomorphic off p, by Cauchy-Goursat, $\int_\Delta f(z)dz = 0$ for all $\Delta \in U$, and thus by Theorem 2.23 we have F as defined in the proof of the theorem.

Theorem 2.25. *Cauchy's theorem for a convex set. Let U be convex and open, and $p \in U$. Suppose $f : U \to \mathbb{C}$ continuous on U and holomorphic on $U - \{p\}$. Then $\int_\gamma f(z)dz = 0$ where γ is a closed piecewise smooth curve inside of U.*

Proof: $\exists F$ such that

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = 0$$

2.4 Holomorphicity implies Analycity

Theorem 2.26. *Let U be open and convex., γ be piecewise smooth closed, and $\gamma* \subset U$. $f : U \to \mathbb{C}$ holomorphic. Then $\forall z \in U - \gamma^*$, we have*

$$f(z)Ind_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$$

Proof: Let $z \in U - \gamma^*$, and define

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & w \in U - \{z\} \\ f'(z) & w = z \end{cases}$$

Note that g is continuous and holomorphic on $U - \{z\}$. By Theorem 2.25,

$$0 = \frac{1}{2\pi i}\int_\gamma g(w)dw = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{w-z}dw$$

as desired.

Theorem 2.27. *Every holomorphic function is analytic. Stronger, U is open, $f : U \to \mathbb{C}$ is holomorphic. Let $z_0 \in U$, $r > 0$ such that $B_r(z_0) \subset U$. Then there exists power series $\sum \alpha_n(z-z_0)^n$ which has radius of convergence at least r, and $f(z) = \sum_{n=0}^\infty \alpha_n(z-z_0)^n$ for all $z \in B_r(z_0)$.*

Proof: Consider $\rho \in (0, r)$. Then $\Gamma_\rho(z_0) \subset U$ and $\forall z \in B_\rho(z_0)$ we have $\mathrm{Ind}_{\Gamma_\rho(z_0)}(z) = 1$. Then restrict f to $B_{\frac{r+\rho}{2}}(z_0)$ and we get that

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho(z_0)} \frac{f(w)}{w-z} dw$$

for all $z \in B_\rho(z_0)$. Consider the function

$$z \mapsto_{\Gamma_\rho(z_0)} \frac{f(w)}{w-z} dw$$

Theorem 2.15 says this function is analytic. Thus it is infinitely differentiable on $B_\rho(z_0)$ and the power series $\sum \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$ has radius of convergence of at least ρ. Thus we get that it has radius of convergence $\geq r$ and

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Corollary 2.28. *f holomorphic implies that f' is holomorphic.*

Theorem 2.29. *Morera's Theorem: Let U be open, $f : U \to \mathbb{C}$ continuous. Then f is holomorphic if and only if $\forall \Delta \subset U$ we have*

$$\int_{\partial \Delta} f(z) dz = 0$$

Proof: The forward is obvious from Cauchy-Goursat. For the reverse, it is sufficient to prove it for an open ball. Suppose U is convex, then by Theorem 2.23, $f = F'$ on the convex set. Then applying the Fundamental Theorem of Calculus and using the fact that $\partial\Delta$ is closed, we get the desired result.

Lemma 2.30. *Let $a \in U$ which is open, and $f : U \to \mathbb{C}$ continuous on U and holomorphic off a. Then f is holomorphic.*

2.5 Estimates and Consequences

Lemma 2.31. *Let $U \subset \mathbb{C}$ be open, $a \in U$, and $r > 0$. Then $\overline{B_r(a)} \subset U$ if and only if $\exists R \in (r, \infty)$ such that $B_R(a) \subset U$.*

Proposition 2.32. *Let U be open, $f : U \to \mathbb{C}$ holomorphic, $a \in U$, $r > 0$, such that $\overline{B_r(a)} \subset U$. Then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma_r(a)} \frac{f(z)}{(w-z)^{n+1}} dw$$

Proof: Cauchy's Formula (Theorem 2.26) gives

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_r(a)} \frac{f(w)}{w - z} dw$$

for all $z \in B_r(a)$. Then by Theorem 2.15 the result is true.

Corollary 2.33. *Cauchy's Inequality: Let U, $f : U \to \mathbb{C}$, a, r defined as above. Then we have that*

$$|f^{(n)}(z)| \leq n! \frac{r}{(r - |z-a|)^{n+1}} \max_{w \in \partial B_r(a)} |f(w)|$$

Moreover if $z \in B_{\frac{r}{2}}(a)$, then

$$|f^{(n)}(z)| \leq \frac{n! 2^{n+1}}{r^n} \max_{w \in \partial B_r(a)} |f(w)|$$

Proof: By Theorem 2.22, the fact that for all $w \in \partial B_r(a)$,

$$|w - z| \geq r - |z - a|$$

and lemma 2.12(III) this is true.

Definition 2.34. A holomorphic function whose domain in $\mathbb{C}$ is called an "entire" function.

Theorem 2.35. *Liouville: Every bounded holomorphic function f is constant.*

Proof: Let f be entire, bounded by M. It follows from Corollary 2.23 that

$$f'(a) \leq \frac{M}{r}$$

for all $a \in C$, $r > 0$. Therefore $f' = 0$ and vanishes on the connected set $\mathbb{C}$ and thus is constant.

Lemma 2.36. *Let f be nonzero polynomial of degree $n \in \mathbb{N}_0$, then there exists $\mu > 0$ and $R > 1$ such that*

$$|f(z)| \geq \mu |z|^n$$

for all $z \in \mathbb{C}$ with $|z| > R$. In particular, $|f(z)| > \mu$.

Corollary 2.37. *Fundamental Theorem of Algebra*

Proof: Let f be a polynomial without roots. By Lemma 2.36, there exists $R > 0$ such that $\frac{1}{f}$ is bounded on $\mathbb{C} - B_R(0)$. Obviously $\frac{1}{f}$ is bounded on $\overline{B_R(0)}$, and so by Liouville's Theorem, $\frac{1}{f}$ is constant.

Proposition 2.38. *Let $r > 0$, $\sum \alpha_n z^n$, $\sum \beta_n z^n$ be a power series both with radius of convergence at least r. Then for all $n \in \mathbb{N}_0$ define*

$$\gamma_n = \alpha_n\beta_0 + \alpha_{n-1}\beta_1 +,,, + \alpha_0\beta_n$$

Then the power series $\sum \gamma_n z^n$ has radius of convergence at least r.

Proof: Let $f(z) = \sum \alpha_n z^n$ and $g(z) = \sum \beta_n z^n$ and $f, g : B_r(0) \to \mathbb{C}$ be holomorphic on its domain. Then fg is also holomorphic on $B_r(0)$ and thus is analytic by Theorem 2.27; the power series of fg has radius at least r. If $z \in B_r(0)$, then

$$(fg)(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(0)}{n!} z^n$$

Then we can verify that

$$\frac{(f * g)^{(n)}(x)}{n!} = \frac{1}{n!} \sum_{k=0}^{\infty} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) = ... = \gamma_n$$

Proposition 2.39. *Let $U \subset \mathbb{C}$ be open, $R > 0$ such that $\overline{B_R(0)} \subset U$, and $f : U \to \mathbb{C}$ be holomorphic. Then $\forall z \in B_R(0)$ we have that*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} d\theta$$

Proof: Let $z_0 \in B_R(0)$, then

$$z \mapsto \frac{f(z)}{R^2 - \bar{z}_0 z}$$

is holomorphic on $B_{R^2/|z_0|^2} \cap U$. Then we have that $\forall z \in B_R(0)$,

$$\frac{f(z)}{R^2 - \bar{z}_0 z} = \frac{1}{2\pi i} \int_{\Gamma_R(0)} \frac{f(w)}{R^2 - \bar{z}_0 w} \frac{1}{w - z} dw$$

Now replace z_0 by z, and R^2 by $w\bar{w}$ to get

$$\frac{f(z)}{R^2 - |z|^2} = \frac{1}{2\pi i} \int_{\Gamma_R(0)} \frac{f(w)}{w\bar{w} - \bar{z}w} \frac{1}{w - z} dw = \frac{1}{2\pi i} \int_{\Gamma_R(0)} \frac{f(w)}{|w - z|^2} \frac{1}{w} dw$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\theta})}{|Re^{i\theta} - z|^2} dw$$

as desired.

2.6 Locally Uniform Convergence

Definition 2.40. Let $U \subset \mathbb{C}$ be open, $f, f_1, f_2, ... : U \to \mathbb{C}$ be continuous. We say $\lim_{n\to\infty} f_n = f$ locally uniform on U if $\forall a \in U$, there exists $r > 0$ such that $B_r(a) \subset U$ and $\lim_{n\to\infty} f_n|_{B_r(a)} = f|_{B_r(a)}$ uniformly.

Example 2.41. Place-holder

Theorem 2.42. *Weierstrass: Let $U \subset \mathbb{C}$ be open, $f_1, f_2, f_3, ... : U \to \mathbb{C}$ with f_k holomorphic, and $\lim_{n\to\infty} f_n = f$ locally uniformly. Then f is holomorphic and $\lim_{n\to\infty} f'_n = f'$ locally uniformly.*

Proof: Let $a \in U$. So on $B_R(a)$ the convergence is uniform. Then f is continuous and by Theorem 2.29,

$$\int_{\partial\Delta} f_k(z)dz = 0, \quad \forall k, \quad \forall \Delta \subset B_R(a)$$

Thus this implies

$$\int_{\partial\Delta} f(z)dz = 0, \quad \forall k, \quad \forall \Delta \subset B_R(a)$$

and so by Morera's Theorem, f is holomorphic on $B_R(a)$. Now let $a \in U$, $r > 0$ and $B_{2r}(a) \subset U$ such that $\lim f_n = f$ luniformly on $\overline{B_{2r}(a)}$. Thus we have that $\lim f_n = f$ uniformly on $\partial B_r(a)$. We have that for all $z \in B_{\frac{r}{2}}(a)$,

$$|f^{(n)}(z)| \leq \frac{n!2^{n+1}}{r^n} \max_{w\in\partial B_r(a)} |f(w)|$$

Thus we have that

$$|f'(z) - f'_n(z)| \leq \frac{4}{r} \max_{w\in B_r(a)} |f(w) - f_n(w)|$$

Thus $\lim f'_n = f'$ uniformly on $B_{\frac{r}{2}}(a)$.

Corollary 2.43. *Let $r > 0$, $f, f_1, f_2, ... : B_r(0) \to \mathbb{C}$ holomorphic, and let $\sum \alpha_n^{(k)} z^n$ be the power series of f_k. Then if $\lim f_k = f$ locally uniformly, then $\alpha_n = \lim_{k\to\infty} \alpha_n^{(k)}$ for all $n \in \mathbb{N}$ where $\sum \alpha_n z^n$ is the power series of f.*

Proof: By induction on the previous theorem, $\lim_{k\to\infty} f_k^{(n)} = f^{(n)}$. In particular, the result follows at $z = 0$,

$$f_k^{(n)}(0) = n!\alpha_n^{(k)}$$

3 Basic Theory

3.1 Introduction

Proposition 3.1. *Let U be open, nonempty. Then the following are equivalent*

(I) For any open U_1, U_2, with $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, then $U = U_1$ or $U = U_2$.

(II) $\forall p, q \in U$, there exists piecewise smooth path $\gamma^* \subset U$ such that $\gamma(a) = p$, $\gamma(b) = q$

(III) Every continuous function $f : U \to \{0, 1\}$ is constant.

Proof: Suppose (I) is true. Let $p \in U$ and define $U_1 = \{q \in U | \exists \gamma, \gamma(a) = p, \gamma(b) = q\}$, ie the set of points which are path connected to p. U is open, so given $q \in U_1$, there exists $r > 0$ such that $B_r(q) \subset U$. We have that for all $q' \in B_r(q)$, there is a path γ' from q to q' (a ball is convex). Thus $B_r(q) \subset U_1$, and so U_1 is open. Define $U_2 = U - U_1$. Note that U_2 must be open: let $u \in U_2$ and suppose there exists $r > 0$ such that $B_r(u) \subset U$. If it has nontrivial intersection with U_1, then since $B_r(u) \subset U$ and there is a path between the intersection and u, we have that $u \in U_1$, a contradiction. Thus $B_r(u) \subset U_2$, and su U_2 is open. Thus by (I), $U = U_1$, so (II) is true.

Suppose (II) is true. Let $\varphi : U \to \{0, 1\}$ be a continuous function such that it is non-constant, there exists p, q such that $\varphi(p) = 0$, $\varphi(q) = 1$. By (II) there exists a piecewise smooth curve γ inside U, such that $\gamma(a) = p$, $\gamma(b) = q$. We have that $\varphi \circ \gamma : [a, b] \to \{0, 1\}$ is continuous, and so the intermediate value theorem gives us c in $[a, b]$ such that $\varphi(\gamma(c)) = \frac{1}{2}$, a contradiction.

Suppose (III) is true. Let U_1, U_2 be open with $U_1 \cup U_2 = U$ and $U_1 \cap U_2 = \emptyset$. Define $\varphi : U \to \{0, 1\}$ by

$$\varphi(p) = \begin{cases} 0 & p \in U_1 \\ 1 & p \in U_2 \end{cases}$$

We can verify that φ is continuous because its inverse image for every open set in $\{0, 1\}$ is open, and so it is constant. Thus $U = U_1$ or $U = U_2$.

Definition 3.2. The above conditions define connectedness of an open set.

Definition 3.3. We can define a domain or region as an open non-empty connected subset of $\mathbb{C}$.

Corollary 3.4. *If U is open and connected, $f : U \to \mathbb{C}$ holomorphic with $f' = 0$, then f is constant.*

3.2 Zeros of Holmorphic Functions

Definition 3.5. Let U be open, $a \in U$ with $f : U \to \mathbb{C}$ and $f(a) = 0$. Then we say that a is an **isolated singularity** for f if there exists $r > 0$ such that $B_r(a) \subset U$ and $\forall z \in B_r(a) - \{a\}$, we have $f(z) \neq 0$.

We say a is a zero of infinite order if $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$.

We say a is a zero of finite order (m) if there exists $m \in \mathbb{N}$ such that $f^{(m)}(a) \neq 0$, and $f^{(k)}(a) = 0$ for all $k < m$.

Proposition 3.6. *Let U be open, $a \in U$, and $f : U \to \mathbb{C}$ be holomorphic, and that $f(a) = 0$. Then there exists $r > 0$ such that $B_r(a) \subset U$ and either*

(I) $f(z) = 0$ for all $z \in B_r(a)$

(II) $f(z) \neq 0$ for all $z \in B_r(a) - \{a\}$

Case (I) occurs if and only if a is a zero of infinite order. If a is a zero of finite order, say $N \in \mathbb{N}_0$ then there exists unique $g : B_r(a) \to \mathbb{C}$ holomorphic, $g(a) = 0$ and $f(z) = (z-a)^n g(z)$ for all $z \in B_r(a)$.

Proof: Take $R > 0$ such that $B_R(a) \subset U$. BY Theorem 2.27, f has a power series, $\sum \alpha_n (z-a)^n$ which is convergent uniformly on $B_R(a)$. Recall $\alpha_n = \frac{f^{(n)}(a)}{n!}$. If $\alpha_n = 0$ for all n then $f(z) = 0$ for all $z \in B_R(a)$ and Case (I) is satisfied with $r = R$.

Suppose it is not a zero of order ∞. Take $N \in \mathbb{N}$ minimal such that $f^{(N)}(a) \neq 0$, so $\alpha_0 = \alpha_1 = ...\alpha_{n-1} = 0$. Then

$$f(z) = \sum_{n=N}^{\infty} \alpha_n (z-a)^n = (z-a)^N \sum_{k=0}^{\infty} \alpha_{N+k} (z-a)^k$$

for all $z \in B_R(a)$. Define $g : B_R(0) \to \mathbb{C}$ by $g(z) = \sum_{n=0}^{\infty} \beta_n (z-a)^n$ where $\beta_n = \alpha_{N+n}$. Restrict the domain of g such that the image of g does not have 0 (possible by continuity). Then g is holomorphic (because its analytic?), and satisfies the conditions of Case (II).

Lemma 3.7. *Let U be open, $u \in U$, and f holomorphic. Let $N \in \mathbb{N}$, then f has a zero at a of order N if and only if*

$$\lim_{z \to a} \frac{f(z)}{(z-a)^N}$$

exists and is nonzero.

Proof: Forward direction comes from the previous proposition. Now consider

$$\lim_{z \to a} \frac{f(z)}{(z-a)^N}$$

where $f(a) = 0$. We know by the previous proposition that a is not a zero of infinite order. Let M be the order of a. Then there exists $g : U \to \mathbb{C}$ non-zero at a and $f(z) = (z-a)^M g(z)$. Therefore

$$\lim_{z \to a} \frac{(z-a)^M g(z)}{(z-a)^N}$$

exists and is nonzero. Thus

$$\lim_{z \to a} (z-a)^{M-N} g(z) \neq 0$$

and so $M = N$.

Theorem 3.8. *Let U be open and connected with $a \in U$, $r > 0$. Suppose $f : U \to \mathbb{C}$ be holomorphic and $f|_{B_r(a)} = -$. Then $f = 0$ on U.*

Proof: Let

$$U_1 = \{p \in U | \exists s > 0, f|_{B_s(p)} = 0\}$$
$$U_2 = \{p \in U | \exists s > 0, f(z) \neq 0, z \in B_s(p) - \{p\}\}$$

Clearly both U_1 and U_2 are open, and we have that $U = U_1 \cup U_2$ by Proposition 3.6. Thus since U is connected, $U = U_1$ or $U = U_2$. Since $a \in U_1$, we have $U = U_1$, and we are done.

Corollary 3.9. *Let U be open, connected, $V \subset U$ be open and non-empty. Then $f, g : U \to \mathbb{C}$ holomorphic and $f|_V = g|_V$ implies $f = g$.*

Corollary 3.10. *Let U be open and connected, $f : U \to \mathbb{C}$. Suppose there exists a sequence of different zeroes of f which converge to a point in U. Then $f = 0$.*

Proof: By Proposition 3.6, f is zero on some open ball centered at the limit point. Thus f is 0 on U by Theorem 3.8.

3.3 Isolated Singularities

Definition 3.11. Let U be open, $a \in U$, and $f : U - \{a\}$ be holomorphic, We say that a is an **isolated singularity** of f. We say that a is a **removable singularity** of f is there exists holomorphic function $g : U \to \mathbb{C}$ such that $g|_{U-\{a\}} = f$.

Theorem 3.12. *Riemann: Let U be open, $a \in U$, and $f : U - \{a\} \to \mathbb{C}$ be holomorphic. Suppose $\exists r > 0$ with $B_r(a) \subset U$ and $f|_{B_r(a)-\{a\}}$ is bounded, then a is a removable singularity.*

Proof: Define the function $h : U \to \mathbb{C}$ by

$$h(z) = \begin{cases} (z-a)f(z) & z \in U - \{a\} \\ 0 & z = a \end{cases}$$

h is holomorphic on $U - \{a\}$. Since f is bounded, $\lim_{z \to a} h(z) = 0$, so h is continuous on U. Then h is holomorphic on U by Lemma 2.30 (Cauchy-Goursat's Theorem and Morera's Theorem). Proposition 1.14(II) gives continuous $g : U \to \mathbb{C}$ such that

$$h(z) = h(a) + (z-a)g(z)$$

Note that g is holomorphic on $g : U - \{a\}$ but continuous at a, and so by Lemma 2.30 again g is holomorphic. Since $h(a) = 0$, g is defined as desired.

Theorem 3.13. *Casorati-Weierstrass: Let U be open, $a \in U$, $f : U - \{a\} \to \mathbb{C}$ be holomorphic. Then one of the following 3 occurs:*

(I) point a is removable

(II) There exists $m \in \mathbb{N}$ and $c_1, ..., c_m \in \mathbb{C}$ such that $c_m \neq 0$ and the function

$$z \mapsto f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

gas a removable singularity at a.

(III) For all $r > 0$ such that $B_r(a) \subset U$, the set $f(B_r(a) - \{a\})$ is dense in $\mathbb{C}$

Proof: Clearly only one at a time is possible. Suppose III is not the case. That means there exists $w \in \mathbb{C}$, $r > 0$, and $\mu > 0$ such that $B_r(a) \subset U$, and we have that $|f(z) - w| \geq \mu$ for all $z \in B_r(a) - \{a\}$. The idea is that $\mathbb{C}$ contains a ball centered at w that doesn't contain an element of the image of $f|_{B_r(a)-\{a\}}$. Define $g : B_r(a) - \{a\} \to \mathbb{C}$ as $z \mapsto \frac{1}{f(z)-w}$. g is holomorphic on its domain and bounded, so Riemann's Theorem implies that a is a removable singularity of g. Introduce $h : B_r(a) \to \mathbb{C}$ such that h is equal to g on $B_r(a) - \{a\}$ and that h is holomorphic. We have two cases: the first is that $h(a) \neq 0$, then f is bounded in the neighborhood of a

and so f has removable singularity at a which is case I. Suppose $h(a) = 0$. Then h has a zero at a of finite order, so there exists $m \in \mathbb{N}$ and function $k : B_r(a) \to \mathbb{C}$ holomorphic (and nonzero at a) such that

$$h(z) = (z-a)^m k(z)$$

We can assume $k \neq 0$ on $B_r(a)$ (just restrict it so that it happens). Thus we have that

$$f(z) = w + \frac{1}{(z-a)^m} \frac{1}{k(z)}$$

for all $z \in B_r(a) - \{a\}$. Since $\frac{1}{k(z)}$ is holomorphic, we can take the power series representation, $\sum_n \alpha_n (z-a)^n$ where $\alpha_0 \neq 0$, and we get that

$$f(z) = w + \frac{1}{(z-a)^m} \sum_n \alpha_n (z-a)^n = w + \sum_{n=0}^{\infty} \alpha_n (z-a)^{n-m}$$

as desired in case II, with $c_n = \alpha_{m-n}$.

Definition 3.14. In case III of the previous theorem, a is said to be an essential singularity.

In case II, f is said to have a pole of order m at a. If $m = 1$, then we say that the pole is simple.

Corollary 3.15. *Let U be open, $a \in U$, with $f : U - \{a\} \to \mathbb{C}$ holomorphic. Let $m \in \mathbb{N}$. Then f has a pole of order m at a if and only if*

$$\lim_{z \to a} (z-a)^m f(z)$$

exists and is non-zero.

3.4 The Homotopy Theorem

Definition 3.16. Let $U \subset \mathbb{C}$ be open, $\gamma_0 : [a_0, b_0] \to \mathbb{C}$, $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ be two closed curves. Then γ_0 is U-homotopic to γ_1 if there exists continuous map $\Phi : [0,1] \times [0,1] \to U$ such that

$$\Phi(t, 0) = \gamma_0(a_0 + t(b_0 - a_0))$$

$$\Phi(t, 1) = \gamma_1(a_1 + t(b_1 - a_1))$$

$$\Phi(0, s) = \Phi(1, s), \forall s \in [0,1]$$

We say that γ_0 is null homotopic if it is U-homotopic to a constant curve. Note that U-homotopy is an equivalence relation on curves in U.

Lemma 3.17. *Let $K \subset \mathbb{C}$ be sequentially compact.*

(I) Let $U \subset \mathbb{C}$ be open and $K \subset U$, then $\exists \epsilon > 0$ such that $\forall z \in K$ and $B_\epsilon(z) \subset U$

(II) Let $f : K \to \mathbb{C}$ be continuous and $\epsilon > 0$. Then there exists $\delta > 0$ such that $\forall z, w \in K$ with $|z - w| < \delta$ it follows that $|f(z) - f(w)| < \epsilon$

Proof: (I) Suppose not. Then for all $n \in \mathbb{N}$ there exists $z_n \in K$ such that $B_{\frac{1}{n}}(z_n)$ is not a subset of U. K is compact implies that z_n has a convergent subsequence inside K, $z_{n_k} \to z \in K$. Because $z \in K \subset U$, there exists $N \in \mathbb{N}$ such that $B_{\frac{2}{N}}(z) \subset U$. There also exists $k \geq N$ such that

$$|z_{n_k} - z| < \frac{1}{N}$$

Then for all $w \in B_{\frac{1}{n_k}}(z_{n_k})$ one has

$$|w - z| \leq |w - z_{n_k}| + |z_{n_k} - z| < \frac{1}{n_k} + \frac{1}{N} \leq \frac{2}{N}$$

Thus $B_{\frac{1}{n_k}}(z_{n_k}) \subset U$, a contradiction.