Math H185 Lecture Notes

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1 Preliminaries

Here is an important property of complex numbers:

$$|\mathrm{Re}(z)| \leq |z|, |\mathrm{Im}(z)| \leq |z|, |z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)|$$

Lemma 1.1. *Let (a_n) be a sequence in $\mathbb{C}$, and $L \in \mathbb{C}$. Then $\lim_{n\to\infty} a_n = L$ if and only if*

$$\lim_{n\to\infty} Re(a_n) = Re(L), \quad \lim_{n\to\infty} Im(a_n) = Im(L)$$

Lemma 1.2. *Let $F \subset \mathbb{C}$ be a set. Then the following are equivalent*

(i) F is closed

(ii) for every sequence $(z_n) \in F$, and $z \in \mathbb{C}$, with $\lim_{n\to\infty} z_n = z$, it follows that $z \in F$

Proof: This is the definition of a closed set.

Definition: A **Cauchy sequence** z_n is a sequence in which for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for every pair $m, n \geq N$, we have that $d(z_n, z_m) < \epsilon$.

Definition: A set S is **complete** if every Cauchy sequence in S converges to some value in S.

Theorem 1.3. $\mathbb{C}$ *is complete.*

Proof: We will use the property that $\mathbb{R}$ is complete. Suppose we have a Cauchy sequence (z_n) in $\mathbb{C}$. Thus given $\epsilon > 0$ we have N as defined above. Thus, $\forall m, n \geq N$, $|z_m - z_n| < \epsilon$. Then we have that

$$|\mathrm{Re}(z_m - z_n)| \leq |z_m - z_n| < \epsilon, \quad |\mathrm{Im}(z_m - z_n)| \leq |z_m - z_n| < \epsilon$$

Note that $\text{Re}(z_m - z_n) = \text{Re}(z_m) - \text{Re}(z_n)$ and the same for the imaginary component. Thus the real components and imaginary components of z_n are Cauchy and thus convergent, so (z_n) is convergent.

Definition 1.4. A set $K \in \mathbb{C}$ is called **sequentially compact** if every sequence in K has a convergent subsequence which converges to a point in K.

Proposition 1.5. *If $K \in \mathbb{C}$, then K is sequentially compact if and only if K is closed and bounded.*

Proof: This proof is made trivial if you use the fact that sequential compactness is equivalent to covering compactness. Suppose K is sequentially compact. Then let (x_n) be a sequence in K that converges to some L in $\mathbb{C}$. By compactness, there exists a subsequence of (x_{n_k}) that is convergent to some value in K. Since in a convergent sequence every subsequence converges to the same value, $L \in K$ and so K is closed.

To verify boundedness, suppose K is not bounded. This implies that $\forall x \in K$, and $r \in \mathbb{R}$, there exists $y \in K$ such that $|x - y| > r$ (otherwise K would be bounded). Let $r \in \mathbb{R}^+$ and construct the sequence inductively as follows. Let $x_1 \in K$. Assuming $x_1, ..., x_n$ is defined, pick x_{n+1} such that $d(x_{n+1}, x_i) > r$. This must exist, otherwise $K \subset \cup_{i=1}^n B_r(x_i)$ where $B_r(x_i)$ is the ball centered around x_i with radius r implying that K is bounded. Then in our sequence (x_n) we have that distance between every pair of points is at least r, which is a property that carries over to every subsequence. Therefore every subsequence does not converge and so K is not sequentially compact, proving the contrapositive.

The other way is a bit more complicated and I don't really want to type it up, so it is left as an exercise to the reader.

Proposition 1.6. *Let K_i be a sequence of nonempty sequentially compact subsets of $\mathbb{C}$, and suppose we have that $K_{i+1} \subset K_i$ for every i. Then $\cap_{n=1}^{\infty} K_n$ is nonempty.*

Proof: For each K_i, pick element $x_i \in K_i$. We have that the sequence $(x_n) \in K_1$, so it contains a convergent subsequence, (x_{n_k}) in K_1. Let x be the value of which it converges to. x is clearly in K_i for every i, and thus $x \in \cap_{n=1}^{\infty} K_n$.

Proposition 1.7. *Let $D \in \mathbb{C}$, and $f : D \to \mathbb{C}$ a function with $z_0 \in D$. Then the following are equivalent:*

(i) For every sequence $(d_n) \in D$ with $\lim_{n\to\infty} d_n = z_0$, it follows that $\lim_{n\to\infty} f(d_n) = f(z_0)$

(ii) $\forall \epsilon > 0 \; \exists \delta > 0$ such that $\forall d \in D$ with $|d - z_0| < \delta$ it follows that $|f(d) - f(z_0)| < \epsilon$

(iii) For every open set $V \in \mathbb{C}$ with $f(z_0) \in V$, there exists open $U \in \mathbb{C}$ with $z_0 \in U$ and $U \cap D \subset f^{-1}(V)$

Proof: Suppose that (ii) is false, that $\exists \epsilon > 0$ such that $\forall \delta$, $\exists d \in D$ with $|d - z_0| > \delta$ such that $|f(d) - f(z_0)| > \epsilon$. Then construct sequence (d_n) such that $|d_n - z_0| > \frac{1}{n}$ and $|f(d_n) - f(z_0)| > \epsilon$. We have that $\lim_{n\to\infty} d_n = z_0$, but since for all n, $|f(d_n) - f(z_0)| > \epsilon$, $f(d_n)$ does not converge to z_0. Thus (i) implies (ii) by contraposition, as desired.

Now suppose that (ii) is true. By definition of open, $\exists \epsilon > 0$ such that $|f(z_0) - v| < \epsilon$ implies that $v \in V$. By (ii), $\exists \delta > 0$ such that $d \in B_\delta(z_0) \cap D$ (which implies that $d \in D$ and $|d - z_0| < \delta$) implies $|f(d) - f(z_0)| < \epsilon$, and so $d \in f^{-1}(V)$ and we conclude $B_\delta(z_0) \cap D \subset f^{-1}(V)$

Finally, suppose (iii) is true. Consider a sequence $(d_n) \to z_0$. Take $\epsilon > 0$ and consider $B_\epsilon(f(z_0))$. It is open so there exists open D with $z_0 \in D$ and $U \cap D \subset f^{-1}(B_\epsilon(f(z_0)))$. Thus there exists $\delta > 0$ such that $B_\delta(z_0) \in D$, and by definition of convergence $\exists N$ such that $\forall n \geq N$ we have $|d_n - z_0| < \delta$. Then we have that since $d_n \in U \cap D$, we have that $f(d_n) \in B_\epsilon(f(z_0))$ so $|f(d_n) - f(z_0)| < \epsilon$ for all $n \geq N$, and so

$$\lim_{n\to\infty} f(d_n) = f(z_0)$$

Definition 1.8. A function f is **continuous** if it satisfies one of the three conditions stipulated above.

Remark 1. If $f : D \to \mathbb{C}$ and $E \subset D$, define $g = f_{|E}$ (ie f restricted to E), then f being continuous implies that g is continuous.

Lemma 1.9. *Let $K \in \mathbb{C}$ be sequentially compact and $f : K \to \mathbb{C}$ continuous, then $f(K)$ is sequentially compact.*

Proof: Let (a_n) be a sequence in $f(K)$. Define $d_n = f^{-1}(a_n)$ and if there are multiple, arbitrarily pick one. We have by sequential compactness of K, some subsequence d_{n_k} converges to a $d \in K$. Thus by continuity, $f(d_{n_k}) \to f(d)$ as $k \to \infty$, and thus (a_{n_k}) converges.

1.1 Differentiability

Definition 1.10. Let $D \subset \mathbb{C}$, and $z_0 \in \mathbb{C}$, then z_0 is called a **cluster point** in D if there exists a sequence $(d_n) \in D - \{z_0\}$ such that $\lim_{n\to\infty} d_n = z_0$.

Definition 1.11. Let $D \subset \mathbb{C}$, $f : D \to \mathbb{C}$, and $z_0 \in \mathbb{C}$, and z_0 is a cluster point of D. Let $L \in \mathbb{C}$, then we say f has a limit L at z_0 if $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\forall d \in D$ such that $0 < |d - z_0| < \delta$, it follows that $|f(d) - L| < \epsilon$.

Lemma 1.12. *$D \subset \mathbb{C}$, z_0 is a cluster of D, $f : D \to \mathbb{C}$ and $L \in \mathbb{C}$. Define $g : D \cup \{z_0\} \to \mathbb{C}$ as*

$$g(z) = \begin{cases} f(z) & \text{if } z \in D - \{z_0\} \\ L & \text{if } z = z_0 \end{cases}$$

Then f has a limit L at z_0 if and only if g is continuous at z_0.

Proof: This holds from the definition of continuity and Definition 1.11.

Definition 1.13. Let $D \subset \mathbb{C}$, $f : D \to \mathbb{C}$, z_0 a cluster of D. Then f is differentiable at z_0 if $\exists L \in \mathbb{C}$ such that

$$z \mapsto \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0 \\ L & \text{if } z = z_0 \end{cases}$$

is continuous at z_0. We then write $f'(z_0) = L$.

Remark 2. f is differentiable at z if and only if

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists.

Proposition 1.14. *Let $D \subset \mathbb{C}$, $f : D \to \mathbb{C}$, z a cluster of D, $L \in \mathbb{C}$, then the following are equivalent:*

(i) f is differentiable at z_0 with $f'(z_0) = L$

(ii) There exists function $\varphi : D \to \mathbb{C}$ continuous at z_0 such that $\varphi(z_0) = L$ and

$$f(z) = f(z_0) + (z - z_0)\varphi(z) \ \ \forall z \in D$$

Proof: We will show that (i) implies (ii), and the remaining are trivial. Define

$$\varphi(z) = \begin{cases} \frac{f(z)-f(z_0)}{z-z_0} & \text{if } z \neq z_0 \\ L & \text{if } z = z_0 \end{cases}$$

Then we have that

$$f(z) = f(z_0) + (z - z_0)\varphi(z)$$

and that $\varphi(z)$ is continuous at z_0.

Proposition 1.15. *Differentiability implies continuity.*

Proof: Given $\epsilon > 0$, there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \implies \left| \frac{f(z) - f(z_0)}{z - z_0} \right| < \epsilon$$

Thus

$$|f(z) - f(z_0)| < \epsilon |z - z_0| < \epsilon\delta < \epsilon$$

for sufficiently small δ.

Proposition 1.16. *Let $D \subset \mathbb{C}$, z_0 cluster of D, $f, g : D \to \mathbb{C}$ be differentiable at z_0. Then $f + g$, λf, and fg are differentiable. If $f(z_0) \neq 0$, then $\frac{1}{f}$ is differentiable.*

Proof: Follows from the definition of differentiability. Only the proof for fg is a little more involved.

Theorem 1.17. *Let $D_1, D_2 \subset \mathbb{C}$, z_i a cluster for D_i, $f : D_1 \to \mathbb{C}$ differentiable at z_1, and $g : D_2 \to \mathbb{C}$ differentiable at z_2 with $f(z_1) = z_2$. Suppose $f(D_1) \subset D_2$. Then $g \circ f$ is differentiable at z_1 and*

$$(g \circ f)'(z_1) = g'(f(z_1))f'(z_1)$$

Proof: Proposition 1.14 gives functions φ_1, φ_2 continuous at z_1, z_2 respectively, such that

$$\varphi_1(z_1) = f'(z_1), \;\; \varphi_2(z_2) = g'(z_2)$$

and we have that

$$(g \circ f)(z) = g(z_2) + (f(z) - z_2)\varphi_2(f(z))$$

$$= (g \circ f)(z_1) + (z - z_1)\varphi_1(z)\varphi_2(f(z)) = (g \circ f)(z_1) + (z - z_1)\varphi(z)$$

with $\varphi : D_1 \to \mathbb{C}$ continuous at z_1 and defined as $\varphi(z) = \varphi_1(z)\varphi_2(f(z))$ proving the desired.

Definition 1.18. A **holomorphic** function is a differentiable function $f : U \to \mathbb{C}$ hwere U is an open subset of $\mathbb{C}$.

Define $\Phi : \mathbb{R}^2 \to \mathbb{C}$ as $(x, y) \mapsto x + iy$ and $\Psi : \mathbb{C} \to \mathbb{R}^2$ as the inverse. Let $U \subset \mathbb{C}$, and define $\tilde{U} = \Psi(U)$. If $f : U \to \mathbb{C}$, we can define $\tilde{f} = \Psi \circ f \circ \Phi$. Note $\tilde{f} : \tilde{U} \to \mathbb{R}^2$.

Let $z_0 = z_0 + iy_0$. By definition, $\tilde{f}$ is differentiable at (x_0, y_0) if and only if there exists 2×2 matrix M and a function $r : \tilde{U} \to \mathbb{R}^2$ such that for every $(x, y) \in \tilde{U}$ we have

$$\tilde{f}(x, y) = \tilde{f}(x_0, y_0) + M \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix}^\top + r(x, y)$$

where

$$\lim_{(x,y)\to(x_0,y_0)} \frac{r(x, y)}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0$$

We say that M is the derivative of $\tilde{f}$ at (x_0, y_0).

If $a, b \in \mathbb{R}$ with $L = a + ib$, then we have that

$$\Psi((z - z_0)L) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$$

Suppose f is differentiable at z_0, and $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Then we have that $\tilde{f} = \Psi \circ f \circ \Phi$ is differentiable at (x_0, y_0) with derivative M and

$$r(x, y) = \Psi\left((x + iy) - z_0) * \psi(x + iy)\right)$$

where the ψ comes from Proposition 1.14(III).

Suppose $\tilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at (x_0, y_0) with derivative $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a, b \in \mathbb{R}$. Let

$$\psi(x + iy) = \begin{cases} \frac{\Phi(r(x,y))}{x+iy-z} & x + iy \neq z_0 \\ 0 & x + iy = z_0 \end{cases}$$

These satisfies Proposition 1.14(III) with $L = a + bi$, so f is complex differentiable.

Take $f : U \to \mathbb{C}$ continuous. Define $u, v : \tilde{U} \to \mathbb{R}$ via

$$\tilde{f}(x, y) = (u(x, y), v(x, y))$$

Proposition 1.19. *The following are equivalent*

(i) f is differentiable at z_0

(ii) the functions u, v are differentiable at (x_0, y_0) and

$$(D_1 u)(x_0, y_0) = (D_2 v)(x_0, y_0)$$

$$(D_2 u)(x_0, y_0) = -(D_1 v)(x_0, y_0)$$

where $D_1 u = \delta_x u$, $D_2 u = \delta_y u$.

Definition 1.20. The equations are called the **Cauchy-Riemann equations**.

1.2 Series in $\mathbb{C}$

Let $a_0, a_1, ... \in \mathbb{C}$, $\forall n \in N_0$. Define $s_n = \sum_{k=0}^{n} a_n \in \mathbb{C}$. We call s_n the n^{th} partial sum. We say that $\sum_{a_k}$ is convergent if and only if (s_n) converges, and we write

$$\sum_{k=0}^{\infty} a_k = \lim_{n \to \infty} s_n$$

Remark 3. $\sum a_n$ converges if and only if $\sum \mathrm{Re}(a_n)$ and $\sum \mathrm{Im}(a_n)$ converges.

We say that the series $\sum a_k$ is absolutely convergent if $\sum |a_n|$ is convergent in $\mathbb{R}$.

Theorem 1.21. *Every absolutely convergent sequence is convergent in $\mathbb{C}$.*

Proof:

$$\sum_{n=0}^{k} a_n \leq \left| \sum_{n=0}^{k} a_n \right| \leq \sum_{n=0}^{k} |a_n|$$

and then by the sequence comparison test we have $\sum a_n$ converges (I think).

Definition 1.22. Let $D \subset \mathbb{C}$ and $f, f_0, f_1, ... : D \to \mathbb{C}$ be a sequence of functions. We say $\lim_{n \to \infty} f_n = f$ uniformly if $\forall \epsilon > 0$ there exists $N \in \mathbb{N}$ such that $\forall x \in D$, $\forall n \geq N$ we have $|f_n(x) - f(x)| < \epsilon$

Remark 4. Consider $\sum f_k$ where $f_k : D \to \mathbb{C}$. Define $S_k = \sum_{i=0}^{k} f_i$. The series $\sum f_k$ is uniformly convergent on D if the sequence of partial sums S_k is uniformly convergent.

The following result is called the Weierstrass M-test.

Proposition 1.23. *Let $D \subset \mathbb{C}$ and $f_0, f_1, ... : D \to \mathbb{C}$ be a sequence of functions. Let $a_0, a_1, ... \in \mathbb{R}^+$. Suppose $|f_k(z)| \leq a_k$ for all $z \in D$ and all $k \in \mathbb{N}_0$. Moreover suppose that the series $\sum a_n$ is convergent. Then the series $\sum f_n$ is uniformly convergent on D.*

Proof: $\forall z \in D$, the series $\sum_n f_n(z)$ is absolutely convergent clearly (and so therefore convergent). Define $S : D \to \mathbb{C}$ by $S(z) = \sum_{n=0}^{\infty} f(z)$. Let $z \in D$ and if $N \geq n$, then

$$\left|\sum_{k=0}^{N} f_k(z) - \sum_{k=0}^{n} f_k(z)\right| = \left|\sum_{k=n+1}^{N} f_k(z)\right| \leq \sum_{k=n+1}^{N} |f_k(z)| \leq \sum_{k=n+1}^{N} a_k \leq \sum_{k=n+1}^{\infty} a_k$$

Now take the limit as $N \to \infty$, we get that

$$\left|S(z) - \sum_{k=0}^{n} f_k(z)\right| \leq \sum_{k=n+1}^{\infty} a_k$$

Thus since $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} a_k = 0$, we have the desired result.

1.3 Integration

Let $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{C}$. This is called Riemann integrable over $[a, b]$ if both Ref and Imf are Riemann integrable on $[a, b] \to \mathbb{R}$, and

$$\int_a^b f(t)dt = \int_a^b \mathrm{Re}f(t)dt + i\int_a^b \mathrm{Im}f(t)dt$$

Lemma 1.24. *The integral is linear.*

Lemma 1.25. *Let $a, b \in \mathbb{R}$, with $a < b$, and $f : [a, b] \to \mathbb{R}$ continuous. Then $|f|$ is Riemann integrable over $[a, b]$ and*

$$\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$$

Proof: Since $|f|$ is continuous, it is Riemann integrable. Let $r \in [0, \infty)$ and $\theta \in \mathbb{R}$ such that $\int_a^b f(t)dt = r(\cos\theta + i\sin\theta)$, and define $\zeta = \cos\theta - i\sin\theta$. Then

$$\left|\int_a^b f(t)dt\right| = \zeta \int_a^b f(t)dt = \mathrm{Re}(\zeta \int_a^b f(t)dt) = \int_b^a \mathrm{Re}(\zeta f(t))dt$$

$$\leq \int_a^b \left|\mathrm{Re}(\zeta f(t))\right| dt \leq \int_a^b |\zeta f(t)| dt = \int_a^b |f(t)|dt$$

2 Analytic Functions

2.1 Power Series

Let $a \in \mathbb{C}$. Then a power series about a is a series of the form

$$\sum \alpha_n (z - \alpha)^n$$

where $\alpha_k \in \mathbb{C}$, $\forall k \in \mathbb{N}_0$. For each $z \in \mathbb{C}$, the series might or might not converge.

If $U \in \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is infinitely differentiable (smooth) then

$$\sum \frac{f^{(n)}(a)}{n!}(z - a)^n$$

is called the power series of f about $a \in U$.

Define $R \in [0, \infty]$ the supremum of all $r \in [0, \infty)$ such that the series

$$\sum |\alpha_n| r^n$$

is convergent. Call R the radius of convergence of the power series.

Lemma 2.1. *Let R be the radius of convergence of $\sum \alpha_k (z - a)^k$. Then the following are equivalent*

(I) If $z \in \mathbb{C}$ with $|z - a| < R$ then the series converges

(II) If $z \in \mathbb{C}$ with $z \notin \overline{B_R(a)}$, then the series diverges

(III) If $0 < r < R$ then $\sum \alpha_n (z - a)^n$ and $\sum |\alpha_n (z - a)^n|$ are convergent uniformly

Proof: WLOG assume $a = 0$. We will prove (II) and (III), and (I) will follow from (III).

We will prove (II) via contraposition. Let $z \in \mathbb{C} - \{0\}$ and suppose that $\sum \alpha_n z^n$ is convergent. Thus $\{\alpha_n z^n : n \in \mathbb{N}\}$ is bounded. Let $M > 0$ be such that $|\alpha_n z^n| \leq M$ for all $n \in \mathbb{N}_0$. Let $r \in (0, |z|)$, then

$$|\alpha_n| r^n \leq M \left(\frac{r}{|z|}\right)^n, \forall n \in \mathbb{N}_0$$

Since $\frac{r}{|z|} < 1$, the series $\sum |\alpha_n| r^n$ converges, by definition of radius of convergence, $r < R$, so $|z| \leq R$.

To prove (III), let $r \in (0, R)$, so $\sum |\alpha_n| r^n$ is convergent. If $z \in \overline{B_r(a)}$ then $|\alpha_n z^n| \leq |\alpha_n| r^n$ for all $n \in \mathbb{N}$. Thus by Weierstrass, we get that $\sum \alpha_n z^n$ and $\sum |\alpha_n z^n|$ are uniformly convergent for all $z \in \overline{B_r(0)}$.

Example 2.2. The radius of convergence of $\sum z^n$ is 1, so it is uniformly convergent on $\overline{B_r(0)}$, $\forall r \in (0, 1)$

Definition 2.3. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ and $a \subset U$. Then f is analytic at a if $\exists R > 0$ and a power series $\sum \alpha_n (z - a)^n$ about a such that $B_R(a) \subset U$, the power series has positive radius of convergence at least R, and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - a)^n, \quad \forall z \in B_R(a)$$

We say that f is analytic if it is analytic $\forall a \in D$.

Note that every polynomial is analytic.

Lemma 2.4. *The power series $\sum \alpha_n (z - a)^n$ and $\sum n\alpha_n (z - a)^{n-1}$ have the same radius of convergence.*

Proof: WLOG assume $a = 0$. Let $R, \hat{R}$ be the respective radii of convergence. Let $r \in [0, R)$. Then there exists $\rho \in \mathbb{R}$ with $r < \rho < R$ such that $\sum |\alpha_n| \rho^n$ is convergent and $\lim_{n\to\infty} n \left(\frac{r}{\rho}\right)^n = 0$. Then $\exists M \in \mathbb{R}$ such that $n \left(\frac{r}{\rho}\right)^n \leq M$ for all $n \in \mathbb{N}_0$. Then

$$n|\alpha_n| r^n = n \left(\frac{r}{\rho}\right)^n |\alpha_n| \rho^n \leq M |\alpha_n| \rho^n, \quad \forall n \in \mathbb{N}$$

Thus since $\sum |\alpha_n| \rho^n$ converges, so too does $\sum n|\alpha_n| r^n$ (and thus $\sum n|\alpha_n| r^{n-1}$). Thus $\hat{R} \geq r$ and so $\hat{R} \geq R$. The other direction is similar.

Theorem 2.5. *Let $R \in (0,\infty)$ and suppose the radius of convergence of the power series $\sum \alpha_n(z-a)^n$ is also at least R. Define $f : B_R(a) \to \mathbb{C}$ as $f(z) = \sum_{n=0}^{\infty} \alpha_n(z-a)^n$. Then f is holomorphic (differentiable). Moreover, $f'(z) = \sum_{n=0}^{\infty} n\alpha_n(z-a)^n$, $\forall z \in B_R(a)$. Hence f is infinitely differentiable and $\alpha_n = \frac{f^{(n)}(a)}{n!}$ for all $n \in \mathbb{N}_0$.*

Proof: WLOG let $a = 0$. Fix $z_0 \in B_R(0)$, and let $\epsilon > 0$. Fix an $r \in (|z_0|, R)$. By lemma 2.4, $\exists n \in \mathbb{N}$ such that

$$\sum_{n=N+1}^{\infty} n|\alpha_n|r^{n-1} \leq \frac{\epsilon}{4}$$

Then $\forall z \in B_r(0) - \{z_0\}$, we have

$$\begin{aligned}\left|\frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=1}^{\infty} n\alpha_n z_0^{n-1}\right| &= \left|\sum_{n=0}^{\infty} \alpha_n \left(\frac{z^n - z_0^n}{z-z_0} - nz_0^{n-1}\right)\right| \\ &= \left|\sum_{n=1}^{\infty} \alpha_n \left((\sum_{k=0}^{n-1} z^k z_0^{n-1-k}) - nz_0^{n-1}\right)\right| \\ &\leq \left|\sum_{n=1}^{N} \alpha_n \sum_{k=0}^{n-1} (z^k z_0^{n-1-k} - z_0^{n-1})\right| + \sum_{n=N+1}^{\infty} 2n|\alpha_n|r^{n-1}\end{aligned}$$

The right sum is bounded by $\frac{\epsilon}{2}$. Note that $z \mapsto z^k z_0^{n-1-k} - z_0^{n-1}$ is continuous at z_0, with value 0 for all $n \in \{1, ..., N\}$ and for all $k \in \{0, ..., n-1\}$, so we can bound it and get from the above expression

$$\lim_{z\to z_0} \left(\frac{f(z)-f(z_0)}{z-z_0} - \sum_{n=1}^{\infty} \alpha_n z_0^{n-1}\right) = 0$$

Thus f is differentiable at z_0 with $f'(z_0) = \sum_{n=1}^{\infty} \alpha_n n z_0^{n-1}$. The proof that $\alpha_n = \frac{f(n)(a)}{n!}$ follows from repeated differentiation.

Corollary 2.6. *Every analytic function is differentiable.*

Corollary 2.7. *Let $\sum \alpha_n(z-a)^n$ and $\sum \beta_n(z-a)^n$ be two power series with radii of convergence R_1, R_2 respectively. Let $\epsilon \leq \min(R_1, R_2)$. Suppose that $\sum_{n=1}^{\infty} \alpha_n(z-a)^n = \sum_{n=1}^{\infty} \beta_n(z-a)^n$ for all $z \in B_\epsilon(a)$. Then $\alpha_n = \beta_n$ for all $n \in \mathbb{N}$.*

Proof: Consider $\sum(\alpha_n - \beta_n)(z-a)^n$. It is identically 0, so the n^{th} order derivatives are 0, so $\alpha_n = \beta_n$.

Define

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

These have infinite radius of convergence, and so we take these as a definition for functions $\mathbb{C} \to \mathbb{C}$. Note that we can show

$$e^{iz} = \sum i^n \frac{z^n}{n!} = ... = \cos z + i \sin z$$

We also have the formulas

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin(z) = \frac{e^{iz} - e^{-iz}}{2}$$

Let $w, z \in \mathbb{C}$ and $f : \mathbb{R} \to \mathbb{C}$ such that $f(t) = e^{-tz}e^{w+tz}$. We can verify by the product rule that $f' = 0$, and in particular $f(1) = f(0)$. Thus $e^{-z}e^{w+z} = e^w$. Let $a, b \in \mathbb{C}$, and choose $z = -a$, $w = a + b$. Then $e^a e^b = e^{a+b}$.

Let $x, y, u, v \in \mathbb{R}$. Suppose $e^{x+iy} = e^{u+iv}$. Thus $e^x = e^u$, so $x = u$. This implies $e^{iy} = e^{iv}$, so $y - v \in 2\pi\mathbb{Z}$. Thus if $z, w \in \mathbb{C}$, then $e^z = e^w$ if and only if $z - w \in 2\pi i\mathbb{Z}$.

2.2 Curves

Definition 2.8. A curve is a continuous map $\gamma : [a, b] \to \mathbb{C}$ where $a < b \in \mathbb{R}$. Let $\gamma(a)$ be the initial point and $\gamma(b)$ be the final point. Let $\gamma*$ denote the image of γ in $\mathbb{C}$.

If $A \subset \mathbb{C}$, we say γ is in A if $\gamma* \subset A$. Lets say γ is closed if $\gamma(a) = \gamma(b)$.
γ is called smooth if it is $\mathbb{C}^1$ continuously differentiable. There is also a notion of being piecewise smooth (it is smooth every except at a finite number of points).

If $\gamma : [a,b] \to \mathbb{C}$ and $\mu : [c,d] \to \mathbb{C}$ with $\gamma(b) = \mu(c)$ then we define the combined curve $\gamma \oplus \mu : [a, b+d-c] \to \mathbb{C}$ as

$$(\gamma \oplus \mu)(t) = \begin{cases} \gamma(t) & t \in [a,b] \\ \mu(c+t-b) & t \in (b, b+d-c] \end{cases}$$

Example 2.9. Define $\gamma : [-1,1] \to \mathbb{C}$ by

$$\gamma(t) = \begin{cases} t^2(1+i) & t \in [0,1] \\ t^2(-1+i) & t \in [-1,0) \end{cases}$$

is smooth. But

$$\gamma(t) = \begin{cases} t(1+i) & t \in [0,1] \\ 2t(1+i) & t \in [-1,0) \end{cases}$$

is not smooth.

Since γ is continuous and $[a,b]$ is sequentially compact, it follows that $\gamma*$ is sequentially compact (and in particular it is closed).

Definition 2.10. Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise smooth curve, $D \subset \mathbb{C}$, and if $f : D \to \mathbb{C}$ a continuous function. Suppose $\gamma* \subset D$. If γ is smooth, we define the contour integral of f along γ to be

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

If γ is only piecewise smooth, with $a = s_0 < s_1 < ... < s_n = b$ and $\gamma|_{[s_{k-1}, s_k]}$ is smooth, then

$$\int_\gamma f(z)dz = \sum_{i=1}^{n} \int_{\gamma_{[s_{i-1}, s_i]}} f(z)dz$$

The lengths of curve γ is given by

$$l(y) = \int_a^b |\gamma'(t)|dt$$

For piecewise γ,

$$l(\gamma) = \sum_{i=1}^{n} l(\gamma_i)$$

Example 2.11. Define $\gamma : [0.2\pi] \to \mathbb{C}$, as $t \mapsto e^{it}$. Moreover define $f : \mathbb{C} - \{0\} \to \mathbb{C}$ to be $f(z) = \frac{1}{z}$.

$$\int_\gamma f(z)dz = \int_0^{2\pi} \frac{1}{e^{it}} * ie^{it} dt = i \int_0^{2\pi} dt = 2\pi i$$

Lemma 2.12. *Let $\gamma : [a, b] \to \mathbb{C}$ be piecewise smooth, and $f, \tilde{f} : \gamma* \to \mathbb{C}$. Then we have*

(I) $\int_\gamma (f + \tilde{f})(z)dz = \int_\gamma f(z)dz + \int_\gamma \tilde{f}(z)dz$

(II) $\int_\gamma \lambda f(z)dz = \lambda \int_\gamma f(z)dz$

(III) $|\int_\gamma f(z)dz| \leq Ml(\gamma)$ *where* $M = \sup\{|f(z)| : z \in \gamma*\}$

(IV) Let $c, d \in \mathbb{R}$, $c < d$ and $\varphi : [c, d] \to [a, b]$ be a continuously differentiable function such that $\varphi(c) = a$, and $\varphi(d) = b$. Suppose γ is smooth or φ is strictly increasing. Then $\gamma \circ \varphi$ is piecewise smooth and $\int_\gamma f(z)dz = \int_{\gamma \circ \varphi} f(z)dz$

(V) $f_k : \gamma* \to \mathbb{C}$ be continuous functions with $\lim f_k = f$ uniformly on $\gamma*$. Then

$$\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$$

(VI) If γ_1, γ_2 are curves with $\gamma = \gamma_1 \oplus \gamma_2$, then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$$

Proof of (IV): Suppose γ is smooth, then

$$\int_{\gamma\circ\varphi} f(z)dz = \int_c^d f(\gamma \circ \varphi(t))(\gamma \circ \varphi)'(t)dt = \int_c^d f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)dt$$

$$= \int_a^b f(\gamma(\tilde{t}))\gamma'(\tilde{t})d\tilde{t} = \int_\gamma f(z)dz$$

Proposition 2.13. *Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ continuously differentiable, $\gamma : [a, b] \to \mathbb{C}$ be piecewise smooth with $\gamma^* \subset U$. Then*

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$$

Proof: Suppose γ is smooth (proof is almost the same). Then we have that

$$\int_\gamma f'(z)dz = \int_a^b f'(\gamma(t))\gamma'(t)dt = \int_a^b (f \circ \gamma)'(t)dt = f(\gamma(b)) - f(\gamma(a))$$

where the last equality follows from FTC.

Corollary 2.14. *Let $U \subset \mathbb{C}$ be open with $f : U \to \mathbb{C}$ be holomorphic and $f' = 0$. Suppose $\forall z_1, z_2 \in U$, there exists $\gamma : [0,1] \to U$ with $\gamma(0) = z_1$ and $\gamma(1) = z_2$ where γ is smooth. Then f is constant.*

Remark 5. The above requirement is for f to be constant when $f' = 0$ also includes that U must be path connected. In general $f' = 0$ does not imply f is constant.

Theorem 2.15. *Suppose $\gamma : [a,b] \to \mathbb{C}$ be piecewise smooth and $g : \gamma^* \to \mathbb{C}$ a continuous function. Let $U = \mathbb{C} - \gamma*$ (it is open) and define $f : U \to \mathbb{C}$ as*

$$f(z) = \int_\gamma \frac{g(w)}{w - z} dw$$

Then f is analytic. More specifically, let $z_0 \in U$, and

$$R = \inf\{|w - z_0| : w \in \gamma*\}$$

Then $R > 0$ and $\forall n \in N$ let $\alpha_n = \int_\gamma \frac{g(w)}{(w-z_0)^{n+1}} dw$. Then the power series $\sum \alpha_n (z - z_0)^n$ has a radius of convergence $> R$, and

$$f(z) = \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

Proof: Note $\gamma*$ is closed since γ is continuous, so U is open. Let z_0 be given and R defined as above. Let $z \in B_R(z_0)$. For $w \in \gamma*$,

$$\left| \frac{z - z_0}{w - z_0} \right| \leq \frac{|z - z_0|}{R} < 1$$

Therefore,

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} * \frac{1}{1 - \frac{z - z_0}{w - z_0}}$$

$$= \frac{1}{w - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0} \right)^n$$

since the series is absolutely convergent for $z \in B_R(z_0)$, and thus is convergent with the given formula. It is also true $\forall w in \gamma^*$. Now define $h, h_0, h_1, ... : \gamma* \to \mathbb{C}$ as

$$h(w) = \frac{g(w)}{w - z}, \quad h_n(w) = \frac{g(w)(z - z_0)^n}{(w - z_0)^{n+1}}$$

then on γ^*, $\lim_{n\to\infty} \sum_i^n h_i = h$. By Lemma 2.12 (VI) we have that

$$f(z) = \int_\gamma h(w)dw = \sum_{n=0}^{\infty} \int_\gamma h_n(w)dw = \sum_{n=0}^{\infty} (z - z_0)^n \int_\gamma \frac{g(w)}{(w - z_0)^{n-1}} dw$$

$$= \sum_{n=0}^{\infty} \alpha_n (z - z_0)^n$$

In particular, the series $\sum \alpha_n (z - z_0)^n$ converges $\forall z \in B_R(z_0)$. By Lemma 2.1, the power series has radius of convergence at least R.

Definition 2.16. Let γ be a closed piecewise smooth curve. Define on $\mathbb{C} - \gamma^*$

$$\mathrm{Ind}_\gamma : \mathbb{C} - \gamma^* \to \mathbb{C}$$

$$z \mapsto \frac{1}{2\pi i} \int_\gamma \frac{1}{w - z} dw$$

This is called the index function of γ with respect to z.

Note that Ind_γ is analytic.

Proposition 2.17. *Let γ be a piecewise smooth closed curve. Then $Ind_\gamma(z) \in \mathbb{Z}$ for all $z \in \mathbb{C} - \gamma^*$. Moreover there exists $R > 0$ such that $Ind_\gamma(z) = 0$ for all $z \in \mathbb{C} - B_R(0)$*

Proof: Let $\gamma : [a, b] \to \mathbb{C}$ be smooth. Define $f : [a, b] \to \mathbb{C}$ as $t \mapsto \int_a^t \frac{\gamma'(s)}{\gamma(s) - z} ds$. Then f is differentiable and $f'(t) = \frac{\gamma'(t)}{\gamma(t) - z}$. Define $g : [a, b] \to \mathbb{C}$ by $g(t) = e^{-f(t)}(\gamma(t) - z)$. Then $g'(t) = 0$ (easily verifiable) and $[a, b]$ is path connected, so g is constant. Therefore

$$\frac{e^{f(t)}}{e^{f(a)}} = \frac{\gamma(t) - z}{\gamma(a) - z}$$

Since $f(a) = 0$ and $\gamma(b) = \gamma(a)$, it follows that $e^{f(b)} = 1$ so $f(b) \in 2\pi i\mathbb{Z}$, but $f(b) = 2\pi i \mathrm{Ind}_\gamma(z)$, so we have $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$.

Since γ^* is bounded, $\exists r > 0$ such that $\gamma^* \subset B_r(0)$ and we have that

$$|\mathrm{Ind}_\gamma(z)| \leq \frac{Kl(\gamma)}{R-r}$$

where $R > r$ and $z \in \mathbb{C} - B_R(0)$ and K is a constant in $\mathbb{R}$. Since $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$, for large enough R we have $\mathrm{Ind}_\gamma(z) = 0$. Since $\mathrm{Ind}_\gamma(z) \in \mathbb{Z}$, we conclude for sufficiently large R, we conclude $\mathrm{Ind}_\gamma(z) = 0$.

Corollary 2.18. *Let γ be a closed piecewise smooth curve, and $\varphi : [0,1] \to \mathbb{C} - \gamma^*$ be continuous. Then $Ind_\gamma(\varphi(0)) = Ind_\gamma(\varphi(1))$.*

Definition 2.19. For all $z_0 \in \mathbb{C}$, $r > 0$ define $\gamma : [0, 2\pi] \to \mathbb{C}$, as

$$t \mapsto z_0 + re^{it}$$

Then denote this curve by $\Gamma_r(z_0) = \gamma$.

Example 2.20. Let $z_0 \in \mathbb{C}$, $r > 0$. Then we have $\mathrm{Ind}_{\Gamma_r(z_0)}(z_0) = 1$. moreover, if $|z - z_0| > 3r$, then estimates show

$$|\mathrm{Ind}_{\Gamma_r(z_0)}(z)| \leq \frac{1}{2}$$

and by Corollary 2.18 we have that

$$\mathrm{Ind}_{\Gamma_r(z_0)}(z) = \begin{cases} 1 & |z - z_0| < r \\ 0 & |z - z_0| > r \end{cases}$$

2.3 Cauchy's Theorem for a triangle and a convex set

Definition 2.21. Let $z_1, z_2 \in \mathbb{C}$. Denote by $[z_1, z_2]$ the curve $\gamma : [0,1] \to \mathbb{C}$ which maps $t \mapsto z_1 + t(z_2 - z_1)$. Let $z_3 \in C$. Then define

$$\Delta(z_1, z_2, z_3) = \{t_1 z_1 + t_2 z_2 + t_3 z_3 | t_i \geq 0, \sum_i t_i = 1\}$$

Denote also by $\partial\Delta(z_1, z_2, z_3)$ the piecewise smooth curve $\gamma : [0,3] \to \mathbb{C}$ by

$$\gamma = [z_1, z_2] \oplus [z_2, z_3] \oplus [z_3, z_1]$$

Remark 6.

$$\int_{\partial\Delta} = \int_{[z_1,z_2]} + \int_{[z_2,z_3]} + \int_{[z_3,z_1]}$$

If z_1, z_2, z_3 are collinear, then $\int_{\partial\Delta} = 0$.

Theorem 2.22. *Cauchy-Goursat Theorem: Let $U \subset \mathbb{C}$ open, $p \in U$, $\Delta \subset U$, $f : U \to \mathbb{C}$ be continuous, and suppose $f : U - \{p\}$ be holomorphic. Then $\int_{\partial\Delta} f(z)dz = 0$.*

Proof: There are 3 steps depending on where p is relative to $\partial\Delta$.

Step 1: Suppose $p \notin \Delta$. Then denote the midpoint of the sides of the triangle by z_1', z_2', z_3' where $z_1' = \frac{z_2+z_3}{2}$, and the others are defined similarly. Connect the z_1', z_2', z_3' in the original triangle, and so we get 4 triangles. Label them $\Delta^(i)$ arbitrarily. Note that we have

$$\int_{\partial\Delta} f(z)dz = \sum_{k=1}^{4} \int_{\partial\Delta^(k)} f(z)dz$$

so

$$\int_{\partial\Delta} \leq \left| \int_{\partial\Delta} f(z)dz \right| \leq \sum_{k=1}^{4} \left| \int_{\partial\Delta^{(k)}} f(z)dz \right| \leq 4 \left| \int_{\partial\Delta^{(i)}} f(z)dz \right|$$

for some $i = 1, 2, 3, 4$. Let $\Delta_1 = \Delta^{(i)}$, and note that $l(\partial\Delta_1) = \frac{1}{2} l(\partial\Delta)$. By induction we can continue and make a sequence of closed triangles $\Delta_2, \Delta_3, ...$ with $\Delta_{k+1} \subset \Delta_k$ and

$$\left| \int_{\partial\Delta_k} f(z)dz \right| \leq 4 \left| \int_{\partial\Delta_{k+1}} f(z)dz \right|$$

$$l(\Delta_{k+1}) = \frac{1}{2} l(\Delta_k)$$

Note that for all $z \in \partial\Delta_n$, $|z - z_0| < l(\partial\Delta_n)$.

Thus

$$\left| \int_{\partial\Delta} f(z)dz \right| \leq 4^n \left| \int_{\partial\Delta_n} f(z)dz \right|$$

$$l(\Delta_n) = \frac{1}{2^n} l(\Delta)$$

By compactness, there exists $z_0 \in \cap_{n=1}^{\infty} \Delta_n$ and $z_0 \neq p$. So f is differentiable at z_0 and so

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

with $\psi : U \to \mathbb{C}$ continuous at z_0, and $\psi(z_0) = 0$. The map

$$z \mapsto f(z_0) + f'(z_0)(z - z_0)$$

comes from the derivative of the map

$$z \mapsto f(z_0)z + \frac{1}{2}f'(z_0)(z-z_0)^2$$

Thus for all n,

$$\int_{\partial\Delta_n} f(z_0) + f'(z_0)(z-z_0)dz = 0$$

Therefore we get that

$$\left|\int_{\partial\Delta_n} f(z)dz\right| = \left|\int_{\partial\Delta_n} f(z) - (f(z_0) + f'(z_0)(z-z_0))dz\right| = \left|\int_{\partial\Delta_n} (z-z_0)\psi(z)dz\right|$$

$$\leq l(\partial\Delta_n)\sup\{|(z-z_0)\psi(z)| : z \in \partial\Delta_n\}$$

$$\leq (l(\partial\Delta_n))^2 \sup\{|\psi(z)| : z \in \partial\Delta_n\}$$

Since ψ is continuous at z_0 with $\psi(z_0) = 0$, for all $\epsilon > 0$ we have $\exists\delta > 0$ such that $|\psi(z)| < \epsilon$ for $z \in B_\delta(z_0) \cap U$. Since $\lim_{n\to\infty} l(\partial\Delta_n) = 0$ (then the diameter also goes to 0) we have that $\exists n \in \mathbb{N}$ such that $\bar{\Delta_n} \subset B_\delta(z_0)$. Then $\sup\{|\psi(z)| : z \in \partial\Delta_n\} \leq \epsilon$ and

$$\left|\int_{\partial\Delta_n} f(z)dz\right| \leq \epsilon(l(\partial\Delta))^2 = \frac{\epsilon}{4^n}$$

Hence

$$\left|\int_{\partial\Delta} f(z)dz\right| \leq 4^n\frac{\epsilon}{4^n} = \epsilon$$

and we conclude that

$$\int_{\partial\Delta} f(z)dz = 0$$

In the next step, suppose p is a vertex. Then WLOG let $\Delta = \Delta(p, z_2, z_3)$. Let $\epsilon \in (0,1)$ and set $p_2 = \epsilon z_2 + (1-\epsilon)p$ and $p_3 = \epsilon z_3 + (1-\epsilon)p$. Then we have that

$$\int_{\partial\Delta} = \int_{\partial\Delta(p,p_2,p_3)} + \int_{\partial\Delta(p_2,z_2,z_3)} + \int_{\partial\Delta(p_3,p_2,z_3)}$$

We have that the 2^{nd} and 3^{rd} integrals are zero by the earlier case, so we only need to consider the first integral. Denoting $\partial\Delta(p_1,p_2,p_3)$ by $\partial\Delta p$ we have that

$$\left|\int_{\partial\Delta p} f(z)dz\right| \leq l(\partial\Delta_p)\sup\{|f(z)| : z \in \partial\Delta_p\}$$

There exists $r > 0$ such that $B_r\bar{(}p) \subset U$ (by openness of U), and since f is continuous $\exists M > 0$ such that $|f(z)| < M$ for all $z \in B_r\bar{(}p)$. If ϵ is small enough, then $\partial\Delta_p \subset B_r(p)$ and since $\lim_{\epsilon\to 0} l(\partial\Delta_p) = 0$, we get that

$$l(\partial\Delta_p)\sup\{|f(z)| : z \in \partial\Delta_p\} \le \epsilon'$$

proving the desired.

For the case that p is not a vertex but $p \in \Delta$, consider the integral

$$\int_{\partial\Delta(z_1,z_2,z_3)} = \int_{\partial\Delta(z_1,z_2,p)} + \int_{\partial\Delta(p,z_2,z_3)} + \int_{\partial\Delta(z_3,z_1,p)}$$

All three of the integrals are 0 by case 2, and we are done.

We define a set $A \subset \mathbb{C}$ to be convex if

$$tz_1 + (1-t)z_2 \in A$$

for all $z_1, z_2 \in A$, $t \in [0,1]$.

Proposition 2.23. *Let $U \subset \mathbb{C}$ be open and convex, $f : U \to \mathbb{C}$ continuous such that*

$$\int_{\partial\Delta} f(z)dz = 0$$

for all $\Delta \subset U$. Then $\exists F : U \to C$ holomorphic such that $F' = f$.

Proof: Fix $a \in U$ and define $F : U \to \mathbb{C}$ by $F(z) = \int_{[a,z]} f(w)dw$. Fix $z_0 \in U$ and let $z \in U$. Then we have that

$$0 = \int_{\partial\Delta(a,z,z_0)} f(w)dw = \int_{[a,z]} f(w)dw + \int_{[z,z_0]} f(w)dw + \int_{[z_0,a]} f(w)dw$$

$$= F(z) + \int_{[z,z_0]} f(w)dw - F(z_0)$$

Now let $\epsilon > 0$, since f is continuous at z_0 we have $\exists\delta > 0$ such that $|f(z) - f(z_0)| < \epsilon$, for all $|z - z_0| < \delta$. We can write

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0}\int_{[z_0,z]} (f(w) - f(z_0))dz$$

and so for all $z \in B_r(z_0)$,

$$\left|\frac{F(z) - F(z_0)}{z - z_0} - f(z_0)\right| \le \frac{1}{|z - z_0|}\int_{[z_0,z]} |f(w) - f(z_0)|dw \le \frac{1}{|z - z_0|} l([z_0, z])\epsilon \le \epsilon$$

for all $zinU$ δ close to z_0. Thus $F'(z_0) = f(z_0)$.

Corollary 2.24. *Let U be open and convex with $p \in U$, $f : U \to \mathbb{C}$ continuous who is holomorphic off p. Then $f = F'$ for some holomorphic function $F : U \to \mathbb{C}$. Explicitly, $\forall a \in U$, we have that $F(z) = \int_{[a,z]} f(w)dw$.*

Proof: Because f is holomorphic off p, by Cauchy-Goursat, $\int_\Delta f(z)dz = 0$ for all $\Delta \in U$, and thus by Theorem 2.23 we have F as defined in the proof of the theorem.

Theorem 2.25. *Cauchy's theorem for a convex set. Let U be convex and open, and $p \in U$. Suppose $f : U \to \mathbb{C}$ continuous on U and holomorphic on $U - \{p\}$. Then $\int_\gamma f(z)dz = 0$ where γ is a closed piecewise smooth curve inside of U.*

Proof: $\exists F$ such that

$$\int_\gamma f(z)dz = \int_\gamma F'(z)dz = 0$$

2.4 Holomorphicity implies Analycity

Theorem 2.26. *Let U be open and convex., γ be piecewise smooth closed, and $\gamma* \subset U$. $f : U \to \mathbb{C}$ holomorphic. Then $\forall z \in U - \gamma^*$, we have*

$$f(z)Ind_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$$

Proof: Let $z \in U - \gamma^*$, and define

$$g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & w \in U - \{z\} \\ f'(z) & w = z \end{cases}$$

Note that g is continuous and holomorphic on $U - \{z\}$. By Theorem 2.25,

$$0 = \frac{1}{2\pi i}\int_\gamma g(w)dw = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw - \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{w-z}dw$$

as desired.

Theorem 2.27. *Every holomorphic function is analytic. Stronger, U is open, $f : U \to \mathbb{C}$ is holomorphic. Let $z_0 \in U$, $r > 0$ such that $B_r(z_0) \subset U$. Then there exists power series $\sum \alpha_n(z-z_0)^n$ which has radius of convergence at least r, and $f(z) = \sum_{n=0}^\infty \alpha_n(z-z_0)^n$ for all $z \in B_r(z_0)$.*

Proof: Consider $\rho \in (0, r)$. Then $\Gamma_\rho(z_0) \subset U$ and $\forall z \in B_\rho(z_0)$ we have $\text{Ind}_{\Gamma_\rho(z_0)}(z) = 1$. Then restrict f to $B_{\frac{r+\rho}{2}}(z_0)$ and we get that

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_\rho(z_0)} \frac{f(w)}{w - z} dw$$

for all $z \in B_\rho(z_0)$. Consider the function

$$z \mapsto_{\Gamma_\rho(z_0)} \frac{f(w)}{w - z} dw$$

Theorem 2.15 says this function is analytic. Thus it is infinitely differentiable on $B_\rho(z_0)$ and the power series $\sum \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ has radius of convergence of at least ρ. Thus we get that it has radius of convergence $\geq r$ and

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Corollary 2.28. *f holomorphic implies that f' is holomorphic.*

Theorem 2.29. *Morera's Theorem: Let U be open, $f : U \to \mathbb{C}$ continuous. Then f is holomorphic if and only if $\forall \Delta \subset U$ we have*

$$\int_{\partial\Delta} f(z) dz = 0$$

Proof: The forward is obvious from Cauchy-Goursat. For the reverse, it is sufficient to prove it for an open ball. Suppose U is convex, then by Theorem 2.23, $f = F'$ on the convex set. Then applying the Fundamental Theorem of Calculus and using the fact that $\partial\Delta$ is closed, we get the desired result.

Lemma 2.30. *Let $a \in U$ which is open, and $f : U \to \mathbb{C}$ continuous on U and holomorphic off a. Then f is holomorphic.*

2.5 Estimates and Consequences

Lemma 2.31. *Let $U \subset \mathbb{C}$ be open, $a \in U$, and $r > 0$. Then $\overline{B_r(a)} \subset U$ if and only if $\exists R \in (r, \infty)$ such that $B_R(a) \subset U$.*

Proposition 2.32. *Let U be open, $f : U \to \mathbb{C}$ holomorphic, $a \in U$, $r > 0$, such that $\overline{B_r(a)} \subset U$. Then*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma_r(a)} \frac{f(z)}{(w - z)^{n+1}} dw$$

Proof: Cauchy's Formula (Theorem 2.26) gives

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma_r(a)} \frac{f(w)}{w-z} dw$$

for all $z \in B_r(a)$. Then by Theorem 2.15 the result is true.

Corollary 2.33. *Cauchy's Inequality: Let U, $f : U \to \mathbb{C}$, a, r defined as above. Then we have that*

$$|f^{(n)}(z)| \leq n! \frac{r}{(r - |z-a|)^{n+1}} \max_{w \in \partial B_r(a)} |f(w)|$$

Moreover if $z \in B_{\frac{r}{2}}(a)$, then

$$|f^{(n)}(z)| \leq \frac{n! 2^{n+1}}{r^n} \max_{w \in \partial B_r(a)} |f(w)|$$

Proof: By Theorem 2.22, the fact that for all $w \in \partial B_r(a)$,

$$|w - z| \geq r - |z - a|$$

and lemma 2.12(III) this is true.

Definition 2.34. A holomorphic function whose domain in $\mathbb{C}$ is called an "entire" function.

Theorem 2.35. *Liouville: Every bounded holomorphic function f is constant.*

Proof: Let f be entire, bounded by M. It follows from Corollary 2.23 that

$$f'(a) \leq \frac{M}{r}$$

for all $a \in C$, $r > 0$. Therefore $f' = 0$ and vanishes on the connected set $\mathbb{C}$ and thus is constant.

Lemma 2.36. *Let f be nonzero polynomial of degree $n \in \mathbb{N}_0$, then there exists $\mu > 0$ and $R > 1$ such that*

$$|f(z)| \geq \mu |z|^n$$

for all $z \in \mathbb{C}$ with $|z| > R$. In particular, $|f(z)| > \mu$.

Corollary 2.37. *Fundamental Theorem of Algebra*

Proof: Let f be a polynomial without roots. By Lemma 2.36, there exists $R > 0$ such that $\frac{1}{f}$ is bounded on $\mathbb{C} - B_R(0)$. Obviously $\frac{1}{f}$ is bounded on $\overline{B_R(0)}$, and so by Liouville's Theorem, $\frac{1}{f}$ is constant.

Proposition 2.38. *Let $r > 0$, $\sum \alpha_n z^n$, $\sum \beta_n z^n$ be a power series both with radius of convergence at least r. Then for all $n \in \mathbb{N}_0$ define*

$$\gamma_n = \alpha_n \beta_0 + \alpha_{n-1}\beta_1 +,,, + \alpha_0 \beta_n$$

Then the power series $\sum \gamma_n z^n$ has radius of convergence at least r.

Proof: Let $f(z) = \sum \alpha_n z^n$ and $g(z) = \sum \beta_n z^n$ and $f, g : B_r(0) \to \mathbb{C}$ be holomorphic on its domain. Then fg is also holomorphic on $B_r(0)$ and thus is analytic by Theorem 2.27; the power series of fg has radius at least r. If $z \in B_r(0)$, then

$$(fg)(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(0)}{n!} z^n$$

Then we can verify that

$$\frac{(f * g)^{(n)}(x)}{n!} = \frac{1}{n!} \sum_{k=0}^{\infty} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x) = ... = \gamma_n$$

Proposition 2.39. *Let $U \subset \mathbb{C}$ be open, $R > 0$ such that $\overline{B_R(0)} \subset U$, and $f : U \to \mathbb{C}$ be holomorphic. Then $\forall z \in B_R(0)$ we have that*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(Re^{i\theta}) \frac{R^2 - |z|^2}{|Re^{i\theta} - z|^2} d\theta$$

Proof: Let $z_0 \in B_R(0)$, then

$$z \mapsto \frac{f(z)}{R^2 - \bar{z}_0 z}$$

is holomorphic on $B_{R^2/|z_0|^2} \cap U$. Then we have that $\forall z \in B_R(0)$,

$$\frac{f(z)}{R^2 - \bar{z}_0 z} = \frac{1}{2\pi i} \int_{\Gamma_R(0)} \frac{f(w)}{R^2 - \bar{z}_0 w} \frac{1}{w - z} dw$$

Now replace z_0 by z, and R^2 by $w\bar{w}$ to get

$$\frac{f(z)}{R^2 - |z|^2} = \frac{1}{2\pi i} \int_{\Gamma_R(0)} \frac{f(w)}{w\bar{w} - \bar{z}w} \frac{1}{w - z} dw = \frac{1}{2\pi i} \int_{\Gamma_R(0)} \frac{f(w)}{|w - z|^2} \frac{1}{w} dw$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(Re^{i\theta})}{|Re^{i\theta} - z|^2} dw$$

as desired.

2.6 Locally Uniform Convergence

Definition 2.40. Let $U \subset \mathbb{C}$ be open, $f, f_1, f_2, ... : U \to \mathbb{C}$ be continuous. We say $\lim_{n\to\infty} f_n = f$ locally uniform on U if $\forall a \in U$, there exists $r > 0$ such that $B_r(a) \subset U$ and $\lim_{n\to\infty} f_n|_{B_r(a)} = f|_{B_r(a)}$ uniformly.

Example 2.41. Place-holder

Theorem 2.42. *Weierstrass: Let $U \subset \mathbb{C}$ be open, $f_1, f_2, f_3, ... : U \to \mathbb{C}$ with f_k holomorphic, and $\lim_{n\to\infty} f_n = f$ locally uniformly. Then f is holomorphic and $\lim_{n\to\infty} f_n' = f'$ locally uniformly.*

Proof: Let $a \in U$. So on $B_R(a)$ the convergence is uniform. Then f is continuous and by Theorem 2.29,

$$\int_{\partial\Delta} f_k(z)dz = 0, \quad \forall k, \quad \forall \Delta \subset B_R(a)$$

Thus this implies

$$\int_{\partial\Delta} f(z)dz = 0, \quad \forall k, \quad \forall \Delta \subset B_R(a)$$

and so by Morera's Theorem, f is holomorphic on $B_R(a)$. Now let $a \in U$, $r > 0$ and $B_{2r}(a) \subset U$ such that $\lim f_n = f$ luniformly on $\overline{B_{2r}(a)}$. Thus we have that $\lim f_n = f$ uniformly on $\partial B_r(a)$. We have that for all $z \in B_{\frac{r}{2}}(a)$,

$$|f^{(n)}(z)| \leq \frac{n!2^{n+1}}{r^n} \max_{w\in\partial B_r(a)} |f(w)|$$

Thus we have that

$$|f'(z) - f_n'(z)| \leq \frac{4}{r} \max_{w\in B_r(a)} |f(w) - f_n(w)|$$

Thus $\lim f_n' = f'$ uniformly on $B_{\frac{r}{2}}(a)$.

Corollary 2.43. *Let $r > 0$, $f, f_1, f_2, ... : B_r(0) \to \mathbb{C}$ holomorphic, and let $\sum \alpha_n^{(k)} z^n$ be the power series of f_k. Then if $\lim f_k = f$ locally uniformly, then $\alpha_n = \lim_{k\to\infty} \alpha_n^{(k)}$ for all $n \in \mathbb{N}$ where $\sum \alpha_n z^n$ is the power series of f.*

Proof: By induction on the previous theorem, $\lim_{k\to\infty} f_k^{(n)} = f^{(n)}$. In particular, the result follows at $z = 0$,

$$f_k^{(n)}(0) = n!\alpha_n^{(k)}$$

3 Basic Theory

3.1 Introduction

Proposition 3.1. *Let U be open, nonempty. Then the following are equivalent*

(I) For any open U_1, U_2, with $U = U_1 \cup U_2$, $U_1 \cap U_2 = \emptyset$, then $U = U_1$ or $U = U_2$.

(II) $\forall p, q \in U$, there exists piecewise smooth path $\gamma^* \subset U$ such that $\gamma(a) = p$, $\gamma(b) = q$

(III) Every continuous function $f : U \to \{0, 1\}$ is constant.

Proof: Suppose (I) is true. Let $p \in U$ and define $U_1 = \{q \in U | \exists\gamma, \gamma(a) = p, \gamma(b) = q\}$, ie the set of points which are path connected to p. U is open, so given $q \in U_1$, there exists $r > 0$ such that $B_r(q) \subset U$. We have that for all $q' \in B_r(q)$, there is a path γ' from q to q' (a ball is convex). Thus $B_r(q) \subset U_1$, and so U_1 is open. Define $U_2 = U - U_1$. Note that U_2 must be open: let $u \in U_2$ and suppose there exists $r > 0$ such that $B_r(u) \subset U$. If it has nontrivial intersection with U_1, then since $B_r(u) \subset U$ and there is a path between the intersection and u, we have that $u \in U_1$, a contradiction. Thus $B_r(u) \subset U_2$, and su U_2 is open. Thus by (I), $U = U_1$, so (II) is true.

Suppose (II) is true. Let $\varphi : U \to \{0, 1\}$ be a continuous function such that it is non-constant, there exists p, q such that $\varphi(p) = 0$, $\varphi(q) = 1$. By (II) there exists a piecewise smooth curve γ inside U, such that $\gamma(a) = p$, $\gamma(b) = q$. We have that $\varphi \circ \gamma : [a, b] \to \{0, 1\}$ is continuous, and so the intermediate value theorem gives us c in $[a, b]$ such that $\varphi(\gamma(c)) = \frac{1}{2}$, a contradiction.

Suppose (III) is true. Let U_1, U_2 be open with $U_1 \cup U_2 = U$ and $U_1 \cap U_2 = \emptyset$. Define $\varphi : U \to \{0, 1\}$ by

$$\varphi(p) = \begin{cases} 0 & p \in U_1 \\ 1 & p \in U_2 \end{cases}$$

We can verify that φ is continuous because its inverse image for every open set in $\{0, 1\}$ is open, and so it is constant. Thus $U = U_1$ or $U = U_2$.

Definition 3.2. The above conditions define connectedness of an open set.

Definition 3.3. We can define a domain or region as an open non-empty connected subset of $\mathbb{C}$.

Corollary 3.4. *If U is open and connected, $f : U \to \mathbb{C}$ holomorphic with $f' = 0$, then f is constant.*

3.2 Zeros of Holmorphic Functions

Definition 3.5. Let U be open, $a \in U$ with $f : U \to \mathbb{C}$ and $f(a) = 0$. Then we say that a is an **isolated singularity** for f if there exists $r > 0$ such that $B_r(a) \subset U$ and $\forall z \in B_r(a) - \{a\}$, we have $f(z) \neq 0$.

We say a is a zero of infinite order if $f^{(n)}(a) = 0$ for all $n \in \mathbb{N}$.

We say a is a zero of finite order (m) if there exists $m \in \mathbb{N}$ such that $f^{(m)}(a) \neq 0$, and $f^{(k)}(a) = 0$ for all $k < m$.

Proposition 3.6. *Let U be open, $a \in U$, and $f : U \to \mathbb{C}$ be holomorphic, and that $f(a) = 0$. Then there exists $r > 0$ such that $B_r(a) \subset U$ and either*

(I) $f(z) = 0$ for all $z \in B_r(a)$

(II) $f(z) \neq 0$ for all $z \in B_r(a) - \{a\}$

Case (I) occurs if and only if a is a zero of infinite order. If a is a zero of finite order, say $N \in \mathbb{N}_0$ then there exists unique $g : B_r(a) \to \mathbb{C}$ holomorphic, $g(a) = 0$ and $f(z) = (z - a)^n g(z)$ for all $z \in B_r(a)$.

Proof: Take $R > 0$ such that $B_R(a) \subset U$. BY Theorem 2.27, f has a power series, $\sum \alpha_n (z - a)^n$ which is convergent uniformly on $B_R(a)$. Recall $\alpha_n = \frac{f^{(n)}(a)}{n!}$. If $\alpha_n = 0$ for all n then $f(z) = 0$ for all $z \in B_R(a)$ and Case (I) is satisfied with $r = R$.

Suppose it is not a zero of order ∞. Take $N \in \mathbb{N}$ minimal such that $f^{(N)}(a) \neq 0$, so $\alpha_0 = \alpha_1 = ...\alpha_{n-1} = 0$. Then

$$f(z) = \sum_{n=N}^{\infty} \alpha_n (z - a)^n = (z - a)^N \sum_{k=0}^{\infty} \alpha_{N+k} (z - a)^k$$

for all $z \in B_R(a)$. Define $g : B_R(0) \to \mathbb{C}$ by $g(z) = \sum_{n=0}^{\infty} \beta_n (z - a)^n$ where $\beta_n = \alpha_{N+n}$. Restrict the domain of g such that the image of g does not have 0 (possible by continuity). Then g is holomorphic (because its analytic?), and satisfies the conditions of Case (II).

Lemma 3.7. *Let U be open, $u \in U$, and f holomorphic. Let $N \in \mathbb{N}$, then f has a zero at a of order N if and only if*

$$\lim_{z \to a} \frac{f(z)}{(z - a)^N}$$

exists and is nonzero.

Proof: Forward direction comes from the previous proposition. Now consider

$$\lim_{z\to a}\frac{f(z)}{(z-a)^N}$$

where $f(a)=0$. We know by the previous proposition that a is not a zero of infinite order. Let M be the order of a. Then there exists $g:U\to\mathbb{C}$ non-zero at a and $f(z)=(z-a)^M g(z)$. Therefore

$$\lim_{z\to a}\frac{(z-a)^M g(z)}{(z-a)^N}$$

exists and is nonzero. Thus

$$\lim_{z\to a}(z-a)^{M-N}g(z)\neq 0$$

and so $M=N$.

Theorem 3.8. *Let U be open and connected with $a\in U$, $r>0$. Suppose $f:U\to\mathbb{C}$ be holomorphic and $f|_{B_r(a)}=-$. Then $f=0$ on U.*

Proof: Let

$$U_1=\{p\in U|\exists s>0, f|_{B_s(p)}=0\}$$
$$U_2=\{p\in U|\exists s>0, f(z)\neq 0, z\in B_s(p)-\{p\}\}$$

Clearly both U_1 and U_2 are open, and we have that $U=U_1\cup U_2$ by Proposition 3.6. Thus since U is connected, $U=U_1$ or $U=U_2$. Since $a\in U_1$, we have $U=U_1$, and we are done.

Corollary 3.9. *Let U be open, connected, $V\subset U$ be open and non-empty. Then $f,g:U\to\mathbb{C}$ holomorphic and $f|_V=g|_V$ implies $f=g$.*

Corollary 3.10. *Let U be open and connected, $f:U\to\mathbb{C}$. Suppose there exists a sequence of different zeroes of f which converge to a point in U. Then $f=0$.*

Proof: By Proposition 3.6, f is zero on some open ball centered at the limit point. Thus f is 0 on U by Theorem 3.8.

3.3 Isolated Singularities

Definition 3.11. Let U be open, $a\in U$, and $f:U-\{a\}$ be holomorphic, We say that a is an **isolated singularity** of f. We say that a is a **removable singularity** of f is there exists holomorphic function $g:U\to\mathbb{C}$ such that $g|_{U-\{a\}}=f$.

Theorem 3.12. *Riemann: Let U be open, $a \in U$, and $f : U - \{a\} \to \mathbb{C}$ be holomorphic. Suppose $\exists r > 0$ with $B_r(a) \subset U$ and $f|_{B_r(a)-\{a\}}$ is bounded, then a is a removable singularity.*

Proof: Define the function $h : U \to \mathbb{C}$ by

$$h(z) = \begin{cases} (z-a)f(z) & z \in U - \{a\} \\ 0 & z = a \end{cases}$$

h is holomorphic on $U - \{a\}$. Since f is bounded, $\lim_{z \to a} h(z) = 0$, so h is continuous on U. Then h is holomorphic on U by Lemma 2.30 (Cauchy-Goursat's Theorem and Morera's Theorem). Proposition 1.14(II) gives continuous $g : U \to \mathbb{C}$ such that

$$h(z) = h(a) + (z-a)g(z)$$

Note that g is holomorphic on $g : U - \{a\}$ but continuous at a, and so by Lemma 2.30 again g is holomorphic. Since $h(a) = 0$, g is defined as desired.

Theorem 3.13. *Casorati-Weierstrass: Let U be open, $a \in U$, $f : U - \{a\} \to \mathbb{C}$ be holomorphic. Then one of the following 3 occurs:*

(I) point a is removable

(II) There exists $m \in \mathbb{N}$ and $c_1, ..., c_m \in \mathbb{C}$ such that $c_m \neq 0$ and the function

$$z \mapsto f(z) - \sum_{k=1}^{m} \frac{c_k}{(z-a)^k}$$

gas a removable singularity at a.

(III) For all $r > 0$ such that $B_r(a) \subset U$, the set $f(B_r(a) - \{a\})$ is dense in $\mathbb{C}$

Proof: Clearly only one at a time is possible. Suppose III is not the case. That means there exists $w \in \mathbb{C}$, $r > 0$, and $\mu > 0$ such that $B_r(a) \subset U$, and we have that $|f(z) - w| \geq \mu$ for all $z \in B_r(a) - \{a\}$. The idea is that $\mathbb{C}$ contains a ball centered at w that doesn't contain an element of the image of $f|_{B_r(a)-\{a\}}$. Define $g : B_r(a) - \{a\} \to \mathbb{C}$ as $z \mapsto \frac{1}{f(z)-w}$. g is holomorphic on its domain and bounded, so Riemann's Theorem implies that a is a removable singularity of g. Introduce $h : B_r(a) \to \mathbb{C}$ such that h is equal to g on $B_r(a) - \{a\}$ and that h is holomorphic. We have two cases: the first is that $h(a) \neq 0$, then f is bounded in the neighborhood of a

and so f has removable singularity at a which is case I. Suppose $h(a) = 0$. Then h has a zero at a of finite order, so there exists $m \in \mathbb{N}$ and function $k : B_r(a) \to \mathbb{C}$ holomorphic (and nonzero at a) such that

$$h(z) = (z-a)^m k(z)$$

We can assume $k \neq 0$ on $B_r(a)$ (just restrict it so that it happens). Thus we have that

$$f(z) = w + \frac{1}{(z-a)^m} \frac{1}{k(z)}$$

for all $z \in B_r(a) - \{a\}$. Since $\frac{1}{k(z)}$ is holomorphic, we can take the power series representation, $\sum_n \alpha_n (z-a)^n$ where $\alpha_0 \neq 0$, and we get that

$$f(z) = w + \frac{1}{(z-a)^m} \sum_n \alpha_n (z-a)^n = w + \sum_{n=0}^{\infty} \alpha_n (z-a)^{n-m}$$

as desired in case II, with $c_n = \alpha_{m-n}$.

Definition 3.14. In case III of the previous theorem, a is said to be an essential singularity.

In case II, f is said to have a pole of order m at a. If $m = 1$, then we say that the pole is simple.

Corollary 3.15. *Let U be open, $a \in U$, with $f : U - \{a\} \to \mathbb{C}$ holomorphic. Let $m \in \mathbb{N}$. Then f has a pole of order m at a if and only if*

$$\lim_{z \to a} (z-a)^m f(z)$$

exists and is non-zero.

3.4 The Homotopy Theorem

Definition 3.16. Let $U \subset \mathbb{C}$ be open, $\gamma_0 : [a_0, b_0] \to \mathbb{C}$, $\gamma_1 : [a_1, b_1] \to \mathbb{C}$ be two closed curves. Then γ_0 is U-homotopic to γ_1 if there exists continuous map $\Phi : [0,1] \times [0,1] \to U$ such that

$$\Phi(t, 0) = \gamma_0(a_0 + t(b_0 - a_0))$$

$$\Phi(t, 1) = \gamma_1(a_1 + t(b_1 - a_1))$$

$$\Phi(0, s) = \Phi(1, s), \forall s \in [0, 1]$$

We say that γ_0 is null homotopic if it is U-homotopic to a constant curve. Note that U-homotopy is an equivalence relation on curves in U.

Lemma 3.17. *Let $K \subset \mathbb{C}$ be sequentially compact.*

(I) Let $U \subset \mathbb{C}$ be open and $K \subset U$, then $\exists \epsilon > 0$ such that $\forall z \in K$ and $B_\epsilon(z) \subset U$

(II) Let $f : K \to \mathbb{C}$ be continuous and $\epsilon > 0$. Then there exists $\delta > 0$ such that $\forall z, w \in K$ with $|z - w| < \delta$ it follows that $|f(z) - f(w)| < \epsilon$

Proof: (I) Suppose not. Then for all $n \in \mathbb{N}$ there exists $z_n \in K$ such that $B_{\frac{1}{n}}(z_n)$ is not a subset of U. K is compact implies that z_n has a convergent subsequence inside K, $z_{n_k} \to z \in K$. Because $z \in K \subset U$, there exists $N \in \mathbb{N}$ such that $B_{\frac{2}{N}}(z) \subset U$. There also exists $k \geq N$ such that

$$|z_{n_k} - z| < \frac{1}{N}$$

Then for all $w \in B_{\frac{1}{n_k}}(z_{n_k})$ one has

$$|w - z| \leq |w - z_{n_k}| + |z_{n_k} - z| < \frac{1}{n_k} + \frac{1}{N} \leq \frac{2}{N}$$

Thus $B_{\frac{1}{n_k}}(z_{n_k}) \subset U$, a contradiction.

Theorem 3.18. *(Homotopy Theorem): Let U be open, $f : U \to \mathbb{C}$ holomorphic. Let γ_0, γ_1 be two closed curves in U which are U-homotopic. Then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

If γ_0 is null-homotopic, then $\int_{\gamma_0} f(z)dz = 0$.

Proof: Without loss of generality let $\gamma_0, \gamma_1 : [0,1] \to \mathbb{C}$. Let Φ be a homotopy of γ_0 and γ_1. By Theorem 3.17, there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that

$$B_\epsilon(\Phi(t,s)) \subset U, \quad \forall (t,s) \in [0,1] \times [0,1]$$

(because continuous image of compact is compact) and we have that

$$|\Phi(t_1, s_1) - \Phi(t_2, s_2)| < \epsilon \quad \forall (t_i, s_i) \in [0,1]^2$$

with $|(t_1, s_1) - (t_2, s_2)| \leq \frac{2}{N}$. For all $m, n \in \mathbb{Z}_N$, and let $z_{n,m} = \Phi(\frac{n}{N}, \frac{m}{N})$. Then we have that

$$z_{n-1,m-1}, z_{n,m-1}, z_{n-1,m}, z_{n,m} \in B_\epsilon(z_{n,m})$$

Note the counterclockwise curve formed by the 4 points listed above is within the convex set of the ball, so by Cauchy's theorem the integral around it is 0. Thus starting from the bottom left corner, the integral right and up is integral to the path integral up and right. (and some more stuff like adding up things)

Definition 3.19. An open set $U \subset \mathbb{C}$ is simply connected if every closed curve in U is null-homotopic in U.

Remark: The ball is simply connected. Any open convex set is simply connected. $B_1(0) \cup B_1(37)$ is simply connected.

Theorem 3.20. *Let U be open, simply connected, $f : U \to \mathbb{C}$ holomorphic. If γ is closed, then $\int_\gamma f(z)dz = 0$*

Corollary 3.21. *Let U be open, connected and simply connected, with $f : U \to \mathbb{C}$ holomorphic. Then there exists $F : U \to \mathbb{C}$ holomorphic such that $F' = f$. If $p \in U$ fixed, then $F(z) = \int_\gamma f(w)dw$ where $\gamma(0) = p$, $\gamma(1) = z$.*

Proof: Fix p, for z choose γ piecewise smooth from p to z (by connectedness). Then $F(z) = \int_\gamma f(z)dz$ is independent of the choice of γ by Theorem 3.18. If $a \in U$, $r > 0$ such that $B_r(a) \subset U$. Then

$$F(z) = F(a) + \int_{[a,z]} f(z)dz$$

Then Corollary 2.24 gives F holomorphic on $B_r(a)$.

Corollary 3.22. *Let U be open, connected, and simply connected, with $f : U \to \mathbb{C}$ holomorphic. Suppose $f(z) \neq 0$ for all $z \in U$. Then there exists holomorphic function $g : U \to \mathbb{C}$ such that $e^g = f$*

Proof: Consider $\frac{f'}{f}$ which is holomorphic. Then there exists $F : U \to \mathbb{C}$ holomorphic such that $F' = \frac{f'}{f}$. Then we have that

$$(fe^{-F})' = f'e^{-F} - fF'e^{-F} = 0$$

so fe^{-F} is constant since U is connected. Then fix $a \in U$ and then there exists $w \in \mathbb{C}$ such that $e^w = f(a)$. Establish for $z \in U$

$$f(z) = f(z)e^{-F(z)}e^{F(z)} = f(a)e^{-F(a)}e^{F(z)} = e^{w-F(a)+F(z)}$$

as desired.

Corollary 3.23. *Let U be open, $f : U \to \mathbb{C}$ holomorphic, $\gamma_0, \gamma_1 : [0,1] \to U$ be piecewise smooth curves such that $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$. If we have Φ continuous such that $\Phi(0,s) = \gamma_0(s)$ for all s and $\Phi(1,s) = \gamma_1(s)$ for all s, then $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$.*

Proof: Modify Φ to create $\tilde{\Phi}$ in the setting of Theorem 3.18.

Remark 7. There does not exist function $f : \mathbb{C} - \{0\} \to \mathbb{C}$ holomorphic such that $f' = \frac{1}{z}$.

Proof: Let $\gamma = \Gamma_1(0)$. Then we have that $\int_\gamma \frac{1}{z}dz = 2\pi i$, but if the statement was true, by Cauchy's Theorem we have $\int_\gamma \frac{1}{z}dz = 0$.

3.5 The Complex Logarithm Function

Definition 3.24. For all $x \in (0,\infty)$ we have that $\log x = \int_1^x \frac{1}{t}dt$. Thus we define

$$\log : \mathbb{C} - (-\infty, 0] \to \mathbb{C}, \qquad z \mapsto \int_{[1,z]} \frac{1}{w}dw$$

Proposition 3.25. *1.* $\log$ *is holomorphic,* $\log'(z) = \frac{1}{z}$

2. $e^{\log z} = z$

3. We have that

$$\log z = \log|z| + 2i arctan\frac{\Im(z)}{\Re(z) + |z|}$$

In addition, $\Im(\log z) \in (-\pi, \pi)$.

4. $\log$ *is a bijection onto* $\{w \in \mathbb{C} : |\Im(w)| < \pi\}$

Lemma 3.26. *Let $a \in \mathbb{C} - (-\infty, 0]$ and $n \in \mathbb{N}$. Then $a^n = e^{n \log a}$ and $a^{-n} = e^{-n\log a}$.*

Definition 3.27. Let $a \in \mathbb{C} - (-\infty, 0], z \in \mathbb{C}$ and define $a^z = e^{z \log a}$.

Note that $(ab)^z \neq a^z b^z$ with this definition.

Example 3.28. Any example that messes around with the branch cut illustrates the above.

Example 3.29. Another example that demonstrates $(a^b)^c \neq a^{bc}$.

3.6 Laurent Series

Definition 3.30. If $0 \leq R_1 < R_2 \leq \infty$, and $a \in \mathbb{C}$. Then define the **annulus** to be

$$A(a, R_1, R_2) = \{z \in \mathbb{C} | R_1 < |z - a| < R_2\}$$

Definition 3.31. Let $z_k \in \mathbb{C}$, $k \in \mathbb{Z}$. We say that the series $\sum z_k$ is convergent if both the sequences $N \mapsto \sum_{k=1}^{\infty} z_k$ and $N \mapsto \sum_{k=1}^{\infty} z_{-k}$ are convergent. In that case we write

$$\sum_{k=-\infty}^{\infty} z_k = \sum_{k=0}^{\infty} z_k + \sum_{k=1}^{\infty} z_{-k}$$

If $a \in \mathbb{C}$, then a Laurent series about a is a series of the form $\sum \alpha_k (z-a)^k$ with $\alpha_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$. The α_k are called the coefficients of the Laurent series.

Theorem 3.32. *Let $0 \leq R_1 < R_2 \leq \infty$, and $a \in \mathbb{C}$. Suppose $f : A(a, R_1, R_2) \to \mathbb{C}$ holomorphic, and $r \in (R_1, R_2)$ for all $k \in \mathbb{Z}$ define*

$$\alpha_k = \frac{1}{2\pi i} \int_{\Gamma_r(a)} \frac{f(w)}{(w-a)^{k+1}} dw$$

Then α_k is independent of r, the series $\sum \alpha_k (z-a)^k$ is convergent and

$$f(z) = \sum_{k=-\infty}^{\infty} \alpha_k (z-a)^k$$

Proof: WLOG let $a = 0$. The independence of α_k is direct from the homotopy theorem. Let $z \in A(0, R_1, R_2)$ and choose r_1, r_2 such that $R_1 < r_1 < |z| < r_2 < R_2$. Define $g : A(0, R_1, R_2) \to \mathbb{C}$ by

$$g(w) = \begin{cases} \frac{f(w) - f(z)}{w - z} & w \neq z \\ f'(z) & w = z \end{cases}$$

By Riemann's Theorem (Theorem 3.12) we have that g is bounded near z, and we get that g is holomorphic. By the homotopy theorem,

$$\int_{\Gamma_{r_1}(0)} g(w) dw = \int_{\Gamma_{r_2}(0)} g(w) dw$$

Note that since z does not lie on both of the curves, we can expand the integrand without fear to get the expression

$$\int_{\Gamma_{r_2}(0)} \frac{f(w)}{w-z}dw - \int_{\Gamma_{r_1}(0)} \frac{f(w)}{w-z}dw = f(z)\left(\int_{\Gamma_{r_2}(0)} \frac{1}{w-z}dw - \int_{\Gamma_{r_2}(0)} \frac{1}{w-z}dw\right)$$

$$= f(z) * 2\pi i$$

We have $r_1 < |z|$ so for $w \in \partial B_{r_1}(0)$ we define

$$h_n(w) = \left(\frac{w}{z}\right)^n, \quad \forall n \in \mathbb{N}$$

By Weierstrass's M-test, we have $\sum h_n$ is uniformly convergent. We also have that

$$-\frac{1}{2\pi i}\int_{\Gamma_{r_1}(0)} \frac{f(w)}{w-z}dw = \frac{1}{2\pi i * z}\int_{\Gamma_{r_1}(0)} \frac{f(w)}{1-\frac{w}{z}}dw$$

$$= \frac{1}{2\pi i}\frac{1}{z}\int_{\Gamma_{r_1}(0)} f(w) * \sum_{n=0}^{\infty}\left(\frac{w}{z}\right)^n dw = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\frac{1}{z}\int_{\Gamma_{r_1}(0)} f(w)\left(\frac{w}{z}\right)^n dw$$

$$= \sum_{k=1}^{\infty}\alpha_{-k}z^{-k}$$

By Theorem 2.15,

$$\frac{1}{2\pi i}\int_{\Gamma_{r_2}(0)} \frac{f(w)}{w-z}dw = \sum_{n=0}^{\infty}\alpha_n z^n$$

proving the desired.

Lemma 3.33. *For all $k \in \mathbb{Z}$, $\alpha_k \in \mathbb{C}$, and let $0 \le R_1 < R_2 \le \infty$. Suppose $\forall z \in A(a, R_1, R_2)$ the series $\sum \alpha_k(z-a)^k$ converges. Define $f : A(a, R_1, R_2) \to \mathbb{C}$ by $f(z) = \sum_{k=-\infty}^{\infty}\alpha_k(z-a)^k$. Then f is holomorphic and $\forall n \in \mathbb{Z}$, $r \in (R_1, R_2)$ we have*

$$\alpha_n = \frac{1}{2\pi i}\int_{\Gamma_r(a)} \frac{f(w)}{(w-z)^{n+1}}dw$$

Definition 3.34. From above $\sum \alpha_k(z-a)^k$ is the Laurent series of f. Define $h : A(a, R_1, R_2)$ by $h(z) = \sum_{k=-\infty}^{-1}\alpha_k(z-a)^k$ called the principal part of f about a. In the case that $R_1 = 0$ we say that $alpha_{-1}$ is the residue of f at a. We write

$$\alpha_{-1} = \mathrm{Res}_\alpha(f)$$

Proposition 3.35. *Let $f : A(a, 0, R_2) \to \mathbb{C}$, and $f(z) = \sum_{k=-\infty}^{\infty} \alpha_k (z-a)^k$. Let $m \in \mathbb{N}$. Then one has the following:*

(a) a is a removable singularity if and only if $\alpha_k = 0$ for all negative k.

(b) a is an isolated pole $\alpha_k = 0$ if and only if for all $k \in \mathbb{Z}$ with $k < -m$ we have that $\alpha_{-m} \neq 0$.

(c) a is an essential singularity of f if and only if the set $\{\alpha_k | k \in \mathbb{N}\}$ is of infinite size.

(d) If h is the principal part, then $f - h$ has a removable singularity at a.

Lemma 3.36. *Let $a \in \mathbb{C}$, $r > 0$, $f : B_r(a) - \{a\} \to \mathbb{C}$ holomorphic. Suppose f has pole of order m at a. Then*

$$Res_a(f) = \frac{1}{(m-1)!} \lim_{z \to a} g^{(m-1)}(z)$$

where $g(z) = (z-a)^m f(z)$

3.7 The Residue Theorem

Let $U \subset \mathbb{C}$ be open, $a \in U$ and $f : U - \{a\} \to \mathbb{C}$ holomorphic. Let $r > 0$ such that $B_r(a) \subset U$. Then f has a Laurent series $\sum \alpha_k (z-a)^k$ about a on the annulus $B_r(a) - \{a\}$. Let h be the principal part of f about a. Let γ be a piecewise smooth closed curve in $U - \{a\}$ which is null-homotopic in U. Then the function $z \mapsto \sum_{k=2}^{\infty} \alpha_{-k}(z-a)^{-k}$ is the derivative of the holomorphic function $z \mapsto \sum_{k=2}^{\infty} \frac{\alpha_{-k}}{1-k}(z-a)^{-k+1}$ on $\mathbb{C} - \{a\}$. So

$$\int_\gamma \sum_{k=2}^{\infty} \alpha_{-k}(z-a)^{-k} dz = 0$$

by Proposition 2.13. Hence

$$\frac{1}{2\pi i} \int_\gamma h(z) dz = \frac{1}{2\pi i} \frac{\alpha_{-1}}{z-a} dz = (\text{Res}_a f) * \text{Ind}_\gamma(a)$$

Theorem 3.37. *Let $U \subset \mathbb{C}$ be open, and $A \subset U$. Suppose $U - A$ is open. Let $f : U \backslash A \to \mathbb{C}$ be a holomorphic function. Suppose for all $a \in A$ the function f has an isolated singularity at a. Let γ be a closed piecewise smooth curve in $U \backslash A$ which is null-homotopic in U. Then the set*

$$\{a \in A : Ind_\gamma(a) \neq 0\}$$

is finite. Write

$$\{a \in A : Ind_\gamma(a) \neq 0\} = \{a_1, ..., a_N\}$$

with $N \in \mathbb{N}$ chosen minimal. Then

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{k=1}^{N}(Res_{a_k} f) Ind_\gamma(a_k)$$

Proof: Let Φ be a homotopy from γ to a constant curve in U. If $a \in A\backslash\Phi([0,1]\times[0,1])$ then Φ is a homotopy from γ to a constant curve in $\mathbb{C}\backslash\{a\}$. Hence $\mathrm{Ind}_\gamma(a) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z-a}dz = 0$ by the homotopy theorem. So

$$\{a \in A : \mathrm{Ind}_\gamma(a) \neq 0\} \subset A \cap \Phi([0,1] \times [0,1])$$

Note that $\Phi([0,1] \times [0,1])$ is sequentially compact. Therefiore if the set $A \cap \Phi([0,1] \times [0,1])$ is infinite, there exists a convergent sequene of different elements of A. This sequence converges to a point $p \in \Phi([0,1] \times [0,1]) \subset U$. But there is an $r > 0$ such that f is holomorphic on $B_r(p)\backslash\{p\}$, a contradiction. This establishes the first part of the theorem.

Let $V = (U\backslash A)\cup\{a_1, ..., a_N\}$. Then V is open and γ is null-homotopic in V via Φ. For all $k \in \{1, ..., N\}$ let h_k be the principal part of f at b_k. Then $f - \sum_{k=1}^{N} h_k$ is holomorphic on $V\backslash\{a_1, ..., a_N\}$ with removable singularities at $a_1, .., a_N$ by Proposition 3.35(IV). Therefore

$$\int_\gamma (f - \sum_{k=1}^{N} h_k(z)dz = 0$$

by the homotopy theorem. Hence

$$\frac{1}{2\pi i}\int_\gamma f(z)dz = \sum_{k=1}^{N}\frac{1}{2\pi i}\int_\gamma h_k(z)dz = \sum_{k=1}^{N}(\mathrm{Res}_{a_k} f)\mathrm{Ind}_\gamma(a_k)$$

Lemma 3.38. *Joran's Lemma: Let $f : \{z \in \mathbb{C} : \Im z \geq 0\} \to \mathbb{C}$ be a continuous function and suppose that*

$$\lim_{R\to\infty} \sup\{|f(Re^{it})| : t \in [0, \pi]\} = 0$$

For all $R > 0$ define $\gamma_R : [0, \pi] \to \mathbb{C}$ by $\gamma_R(t) = Re^{it}$. Fix $m > 0$. Then

$$\lim_{R\to\infty}\int_{\gamma_R} e^{imz} f(z)dz = 0$$

Example 3.39. Note that using the residue theorem we can calculate $\int_0^\infty \frac{\sin x}{x} dx$ by considering the integral of $f(z) = \frac{e^{iz}}{z}$ around the closed curve of the half circle with radius R and taking the limit as $R \to \infty$.

Example 3.40. We can also use the residue theorem to verify that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

3.8 The Maximum Principle

Lemma 3.41. *Let U be open, $f : U \to \mathbb{C}$ holomorphic, γ piecewise smooth closed curve in U, $b \in U \setminus \gamma*$. Then*

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - b} dz = Ind_{f \circ \gamma}(b)$$

Proof: We have that

$$\text{Ind}_{f \circ \gamma}(b) = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{1}{z - b} dz = \frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t)) - b} dt$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z) - b} dz$$

Definition 3.42. A simple curve γ is a closed curve such that

$$\text{Ind}_\gamma(a) \in \{0, 1\}$$

for all $a \in \mathbb{C} \setminus \gamma^*$

Theorem 3.43. *Argument Principle: Let U be open, connected, γ simple curve and null-homotopic in U. Define*

$$U_1 = \{a \in \mathbb{C} \setminus \gamma^* | Ind_\gamma(a) = 1\} \subset U$$

Let $A \subset \mathbb{C}$ be a finite set, $f : U \setminus A \to \mathbb{C}$ be holormophic. Suppose f has a pole at all $a \in A$. Suppose f has no pole/zero in γ^*. Then f has finitely many zeroes in U_1. Let N_f, P_f be the number of zeroes, poles respectively in U_1 counting multiplicty. Then

$$N_f - P_f = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = Ind_{f \circ \gamma}(0)$$

Proof: Note that the poles are isolated since A is a finite set. We have that U is connected, and so $U \setminus A$ is connected (isolated implies ball that doesn't contain other elements of A, we can then reroute the path if it goes through an element of A around the center). f also has finitely many zeros, otherwise we can create a sequence of unique points that has a convergent subsequence (compactify U_1 since it is bounded) and so we have that f is identically zero. Then we have that $V = U \setminus (A \cup \{z \in U | f(z) = 0\})$ is open (because zeros of finite multiplicities are isolated). Thus define $g : V \to \mathbb{C}$ by $g(z) = \frac{f'(z)}{f(z)}$. g is holomorphic. First consider zeroes. Let $a \in U$ and $m \in \mathbb{N}$, and suppose f has a zero at a of order m. So there exists $h : U \to \mathbb{C}$ such that $h(a) \neq 0$ and $f(z) = (z-a)^m h(z)$. Locally,

$$g(z) = \frac{f'(z)}{f(z)} = \frac{m(z-a)^{m-1}h(z) + (z-a)^m h'(z)}{(z-a)^m h(z)} = \frac{m}{z-a} + \frac{h'(z)}{h(z)}$$

We have that the right function is holomorphic near a, so $\mathrm{Res}_a(g) = m$. If we do a similar thing for poles, where a is one of order n, we get that

$$f(z) = \frac{h(z)}{(z-a)^n}$$

and so we have that $\mathrm{Res}_a(g) = 0$. Thus we get that answer after applying the residue theorem.

Definition 3.44. $A \subset \mathbb{C}$ is **discrete** if $\exists r > 0$ such that for all $z \in A$ we have that $B_r(z) \cap A$ is finite. Let U be open, A discrete in U. We say f is meromorphic on U if $\mathrm{Dom}(f) = U \setminus A$ and f is holomorphic with a pole at every $a \in A$. We then say f is meromorphic on U with finite singularities.

Corollary 3.45. *Let U be open and connected, γ simple and null-homotopic in U. Suppose $f : U \to \mathbb{C}$ holomorphic and take $b \in \mathbb{C} \setminus \gamma^*$. Let $U_1 = \{a \in U | Ind_\gamma(a) = 1\} \subset U$. Then $f - b$ has finitely many zeros in U_1, and with multiplicity equal to $\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)-b} dz$.*

Theorem 3.46. *Rouche: Let U be open, connected, γ is simple and null-homotopic. Set $U_1 = \{a \in \mathbb{C} \setminus \gamma^* | Ind_\gamma(a) = 1\} \subset U$. Take $f, g : U \to \mathbb{C}$ holomorphic and suppose on γ^* that $|g| < |f|$. Then let N_h be the number of zeros counting multiplicity of a function h on U_1. Then we have that*

$$N_f = N_{f+g}$$

Proof: For $s \in [0,1]$, define $f_s = f + sg$. We have that f_s is holomorphic. Suppose for the sake of contradiction, $f_s(z) = 0$ for all $z \in \gamma^*$. Then we have

$$0 = f(z) + sg(z) \implies f(z) = -sg(z) \implies |f(z)| \leq |g(z)|$$

a contradiction. Thus f_s is always non-vanishing on γ^*. Define $\varphi : [0,1] \to \mathbb{Z}, s \mapsto \frac{1}{2\pi i} \int_\gamma \frac{f_s'(z)}{f_s(z)} dz$. We have that φ is continuous, and since it maps to the integers we have that it is constant. Thus $N_f = \varphi(0) = \varphi(1) = N_{f+g}$, as desired.

Theorem 3.47. *Let U be open, connected and $f_1, f_2, \ldots : U \to \mathbb{C}$ holomorphic. Let $f : U \to \mathbb{C}$ be such that $\lim_{k\to\infty} f_k = f$ locally uniform. Suppose $f \neq 0$ and let $a \in U$, $m \in \mathbb{N}$. Then the following are equivalent*

(I) f has a zero at a with multiplicity m

(II) $\exists r > 0$ with $B_r(a) \subset U$ such that $\forall s \in (0,r)$ there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ the function f_n has m zeros in $B_s(a)$ counted with multiplicities

Proof: Find $r > 0$ such that $B_r(a) \subset U$, local uniform convergence is uniform convergence on $B_r(a)$, and $f(z) \neq 0$ for all $z \in B_r(a) \setminus \{a\}$. Let $s \in (0,r)$ and $\varepsilon = \min_{z \in \partial B_s(a)} |f(z)| > 0$. There also exists $N \in \mathbb{N}$ such that $\forall n \geq N$, $|f_n(z) - f(z)| < \epsilon$ for all $z \in B_r(a)$. Thus we have on $\Gamma_s(a)^*$, $|f(z)| > |f_n(z) - f(z)|$, and so Rouche's Theorem gives us the forward direction.

Theorem 3.48. *Hurwitz: Let U be open, connected, $f_1, f_2, \ldots : U \to \mathbb{C}$ holomorphic and converge locally uniformly to f. If f_k's don't have any zeroes, then either f has no zeros, or $f = 0$.*

Corollary 3.49. *Let $f_1, f_2, \ldots : U \to \mathbb{C}$ be holomorphic, injective and converge locally uniformly to f. Then f is constant or f is injective.*

Proof: Let $a \in U$. Then we have that $V = .U \setminus \{a\}$ is open and connected. Set $g_n(z) = f_n(z) - f_n(a)$. Then g_n converges locally uniformly to $f(z) - f(a)$ which we define to be $g(z)$. Since g_n has no zero, then g doesn't have zero (or is constant). Thus $f(z) \neq f(a)$ for all $z \in U \setminus \{a\}$.

Theorem 3.50. *Let U be open, $f : U \to \mathbb{C}$ holomorphic, and nonconstnat. Then $f(U)$ is open.*

Precisely, let $a \in U$, $b = f(a)$. Suppose $f - b$ has a zero at a of order N. Then there exists open U_0, V_0 such that $a \in U_0$, $b \in V_0$ with $f(U_0) = V_0$.

Additionally, for all $w \in V_0 \setminus \{b\}$ we have that $f(z) - w$ has precisely N zeros on U_0 with multiplicity one.

Proof: f is not constant so $f' \neq 0$ implying that we can't have a sequence of zeros converging in U (otherwise f would have a zero of infinite order). There exists $r > 0$ such that $B_{2r}(a) \subset U$ and for $z \neq a$, $z \in B_r(a)$ we have that $f(z) - b \neq 0$ and $f'(z) \neq 0$. By the argument principle with $\gamma = \Gamma_r(a)$ and $f - b$ gives that

$$\mathrm{Ind}_{f \circ \gamma}(b) = N$$

Let $V_0 = \{w \in \mathbb{C} \setminus (f \circ \gamma) | \mathrm{Ind}_{f \circ \gamma}(w) = N\}$. It is open because $\mathrm{Ind}_{f \circ \gamma}$ is continuous and takes integer values. Let $U_0 = B_r(a) \cap f^{-1}(V_0)$ which is open by continuity of f. We have that $f(U_0) \subset V_0$. Let $winV_0 \setminus \{b\}$, so $\mathrm{Ind}_{f \circ \gamma}(w) = N$. Argument principle implies that $f - w$ has N zeros counted with multiplicities in $B_r(a)$. Let z be one such zero. So $f(z) = w$, which implies $V_0 \subset f(U_0)$. Then $f'(z_0) \neq 0$ so $(f - w)'(z) \neq 0$ meaning that $f - w$ has a zero of order one at z.

Theorem 3.51. *Maximum Modulus Principle: If U is open, connected, $f : U \to \mathbb{C}$ holomorphic, then if $|f|$ obtains a maximum, it is constant.*

Theorem 3.52. *If $|f|$ obtains its minimum, its constant or $|f| = 0$.*

Theorem 3.53. *If Ref or Imf obtains a maximum then f is constant.*

Corollary 3.54. *Let U be open, connected, and bounded. Suppose $f : \overline{U} \to \mathbb{C}$ be continuous with $f|_U$ holomorphic. Then*

$$\sup_{z \in U} |f(z)| = \max_{z \in \overline{U}} |f(z)| = \max_{z \in \partial U} |f(z)|$$

Proof: The first equality follows form compactness. For the second, $\geq$ is obviously true. $\leq$ is true by the maximum modulus principle on U.

Corollary 3.55. *Let U be open, $f : U \to \mathbb{C}$ holomorphic, $a \in U, r > 0$ and $\overline{B_r(a)} \subset U$. Let*

$$M = \max_{z \in \overline{B_r(a)}} |f(z)|$$

If f is not constant, then $|f|_{B_r(a)}| < M$.

Example 3.56. Some example that really didn't make sense in lecture

Example 3.57. Placeholder

Theorem 3.58. *(Schwarz's Lemma) Let $f : B_1(0) \to B_1(0)$ be holomorphic such that $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in B_1(0)$ and $|f'(0)| \leq 1$. Suppose for some $a \in B_1(0) \setminus \{0\}$ we have $|f(z)| = |z|$. Then $f(z) = \lambda z$ for some $|\lambda| = 1$.*

Proof: Define $g : B_1(0) \to \mathbb{C}$ as

$$g(z) = \begin{cases} \frac{f(z)}{z} & z \neq 0 \\ f'(z) & z = 0 \end{cases}$$

g is holomorphic on $B_1(0) \setminus \{0\}$ and continuous at 0, so g is holomorphic. We have that for all $z \in \partial B_r(0)$, where $r \in (0, 1)$ we have that $|g(z)| = \frac{|f(z)|}{|z|} \leq \frac{1}{r}$. Thus we get $|g(z)| \leq 1$ on $\partial B_1(0)$, and so by Corollary 3.55, $|g(z)| \leq 1$ on $B_1(0)$. Thus $|f(z)| \leq |z|$ on $B_1(0)$. If $|g(z)| = 1$ somewhere in $B_1(0)$, then $|g(z)|$ is constant, so $f = \lambda$ (ie $f(z). = \lambda z$).

Corollary 3.59. *Let $f : B_1(0) \to B_1(0)$ be a bijection with $f(0) = 0$. Suppose f, f^{-1} are holomorphic. Then there exists $|\lambda| = 1$ such that $f = \lambda$.*

Proof: By Schwarz's Lemma, we have that $|f^{-1}(z)| \leq |z|$ so that implies $|f(z)| \geq |z|$ for all $z \in B_1(0)$. Schwarz's Lemma also implies that $|f(z)| \leq |z|$, so $|f(z)| = |z|$ giving us the desired.

Theorem 3.60. *Let U be open, connected, $f : U \to \mathbb{C}$ injective holomorphic. Let $V = f(U)$. Then V is open. Let $g : V \to U$ be the inverse of f. Then g is holomorphic, and $f'(a) \neq 0$ for all $a \in U$.*

Proof: f is not constant U so V is open by open mapping theorem. Let $a \in U$, $b = f(a)$. Then for all $r > 0$ we have that $f(U \cap B_r(a))$ is open, so g is continuous at b. Suppose $f'(a) = 0$, then $f(a) - b$ has a zero of order 2 and so by the open mapping theorem there exists $r > 0$ such that $\forall w \in B_r(b) \setminus \{b\}$ the function $f - w$ has 2 different zeroes in U. Since f is injective this is a contradiction so $f'(a) \neq 0$. Define

$$H(z) = \begin{cases} \frac{z-a}{f(z)-b} & z \neq a \\ \frac{1}{f'(z)} & z = a \end{cases}$$

We have that H is continuous at a and $H \circ g$ is continuous at b. Thus

$$\lim_{u \to b} \frac{g(u) - g(b)}{u - b} = \lim_{u \to b} H \circ g(u) = \frac{1}{f'(a)}$$

so g is differentiable at b and so it is holomorphic.