Math H185 Homework 1

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(1) Let $f : \{z \in \mathbb{C} : \text{Re}(z) \geq 0\} \to \mathbb{C}$ be a continuous function and suppose that

$$\lim_{R\to\infty} \sup\{|f(Re^{it})| : t \in [0, \pi]\} = 0$$

For all $R > 0$ and define $\gamma_R : [0, \pi] \to C$ by $\gamma_R(t) = Re^{it}$. Fix $m > 0$. Prove that

$$\lim_{R\to\infty} \int_{\gamma_R} e^{imz} f(z)dz = 0$$

Solution: Note we have that for every $\epsilon > 0$ there exists R_ϵ such that for all $R > R_\epsilon$ we have that $|f(Re^{it})| < \epsilon$. Then we have that for sufficiently large R. We also have that

$$\int_{\gamma_R} e^{imz} f(z)dz = \int_0^\pi e^{imRe^{it}} iRe^{it} f(Re^{it})dt$$

Taking an absolute value we get

$$|\int_{\gamma_R} e^{imz} f(z)dz| \leq \int_{\gamma_R} |e^{imz} f(z)|dz \leq \int_0^\pi |e^{imRe^{it}}||iRe^{it}||f(Re^{it})|dt$$

$$\leq \epsilon R \int_0^\pi |e^{imR(\cos t + i\sin t)}|dt = \epsilon R \int_0^\pi |e^{-Rm\sin t}||e^{iRm\cos t}|dt$$

Note we have that $|e^{iRm\cos t}| \leq 1$ since $Rm\cos t \in \mathbb{R}$. We get the integral is less than

$$\epsilon R \int_0^\pi e^{-Rm\sin t}dt$$

Note that we have

$$\int_0^\pi e^{-Rm\sin t}dt = \pi e^{-Rm\sin x}$$

where $x \in [0, \pi]$ by the Mean-Value Theorem. Note that $\sin x \neq 0$ because we have $\sin t > 0$ for our interval and

$$\int_0^{\frac{\pi}{4}} (e^{-Rm \sin t} - 1)dt + \int_{\frac{\pi}{4}}^{\pi} (e^{-Rm \sin t} - 1)dt < 0$$

Since the integrand of the left integral is < 0 on the entire region of integration and the integrand of the right integral is ≤ 0. Thus

$$\pi e^{-Rm \sin x} = \int_0^{\pi} e^{-Rm \sin t} < \pi$$

and so $\sin x$ must be positive, otherwise the above inequality fails. Thus we have that

$$\epsilon R \int_0^{\pi} e^{-Rm \sin t} dt = \epsilon R \pi e^{-cR} = \epsilon \pi \frac{R}{e^{cR}}$$

where $c > 0$. And the limit as $R \to \infty$ (starting from $R > R_\epsilon$) of this expression is 0 (since exponential terms dominates polynomials) proving the desired.

(2) Let $U \subset \mathbb{C}$ be an open set with $0 \in U$, and $f : U \to \mathbb{C}$ a continuous function. Prove that

$$\lim_{r \to 0^+} \int_0^{\pi} f(re^{it})dt = \pi f(0)$$

Solution: We have that f is continuous at 0, so for all $\epsilon > 0$ there exists $\delta > 0$ such that $|z| < \delta$ implies that $|f(z) - f(0)| < \frac{\epsilon}{\pi}$. Thus if $0 < r < \delta$, then $|f(re^{it}) - f(0)| < \frac{\epsilon}{\pi}$. Thus we get that

$$\pi(f(0) - \frac{\epsilon}{\pi}) < \int_0^{\pi} f(re^{it})dt < \pi(f(0) + \frac{\epsilon}{\pi})$$

and so we get that

$$|\int_0^{\pi} f(re^{it})dt - \pi f(0)| < \epsilon$$

as desired.

(3) (a) Let $a \in \mathbb{C}$, $f \in (0, \infty)$ and $f : B_r(a) \to C$ be a holomorphic function. Suppose that $f(B_r(a)) \subset \mathbb{R}$. Prove that f is constant.

Solution: Pick $z \in B_r(a)$ and a ball $B_\delta(z)$. Let $z = a + bi$, and consider the derivative of f approaching from the real side, ie $z' = a' + bi$. We have that

$$f'(z) = \lim_{a' \to a} \frac{f(z) - f(z')}{z - z'} = \lim_{b \to b'} \frac{c}{a - a'}$$

Note that both the numerator and denominator are real so $f'(z)$ is real. Also now let z' approach from the imaginary side, ie $z' = a + b'i$. We also have that

$$f'(z) = \lim_{b' \to b} \frac{f(z) - f(z')}{z - z'} = \lim_{b' \to b} \frac{c}{(b - b')i}$$

which is an imaginary number whose real component is zero. Thus the only number which is simultaneously real and imaginary with no real component is 0, and so $f'(z) = 0$ for all z. Note that since $B_r(a)$ is path-connected (draw a line between two points and let that be the path), we have that by Corollary 2.14 the function is constant on the domain.

(b) Give an example of an open $U \subset \mathbb{C}$ and a holomorphic function $f : U \to \mathbb{C}$ such that $f(U) \subset \mathbb{R}$ and f is not constant.
Solution: Let $U = B_1(-2) \cup B_1(2)$ and

$$f(B_1(-2)) = \{-1\}, f(B_1(2)) = \{1\}$$

U is open since it is the union of open sets, and f is clearly holomorphic with derivative 0 everywhere, mapping to values only on the real line, but it is not connected.

(4) Let $a, b \in R$ with $a < b$. Let $f : [a, b] \to \mathbb{C}$ be a continuous function. Define $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = \int_a^b e^{-tz} f(t) dt$$

Prove that F is holomorphic and

$$F'(z) = -\int_a^b e^{-tz} t f(t) dt$$

for all $z \in \mathbb{C}$.

Solution: We can expand using the power series.

$$F(z) = \int_a^b e^{-tz} f(t)dt = \int_a^b (1 - tz + \frac{t^2z^2}{2!} - \frac{t^3z^3}{3!} + ...)f(t)dt$$

Because of uniform convergence extends to the derivative of the power series, we can just differentiate with respect to z (or for those who want more rigor, we break the integral of sums to a sum of integrals, then factor the z out and differentiate, then put it back together) and we get that

$$F'(z) = \int_a^b (-t + t^2z - \frac{t^3z^2}{2!} + \frac{t^4z^3}{3!} - ...)f(t)dt$$

$$= \int_a^b \left(1 - tz + \frac{t^2z^2}{2!} - \frac{t^3z^3}{3!} + ...\right)(-t)f(t)dt = -\int_a^b e^{-tz}tf(t)dt$$

as desired.

(5) Let $(a_n)_{n\in\mathbb{N}}$ be the Fibonacci sequence. Consider the power series $\sum a_n z^n$.

(a) Prove that the radius of convergence of the power series is at least $\frac{\sqrt{5}-1}{2}$.

Solution: Note clearly that $|a_0| \leq 1$. We will then show by induction that $|a_n| \leq \left(\frac{\sqrt{5}+1}{2}\right)^n$. Suppose that it is true for n and $n-1$. Then we have that

$$|a_{n+1}| = |a_n + a_{n-1}| \leq |a_n| + |a_{n-1}| = \left(\frac{\sqrt{5}+1}{2}\right)^n + \left(\frac{\sqrt{5}+1}{2}\right)^{n-1}$$

$$= \left(\frac{\sqrt{5}+1}{2}\right)^{n-1}\left(\left(\frac{\sqrt{5}+1}{2}\right) + 1\right) = \left(\frac{\sqrt{5}+1}{2}\right)^{n-1}\left(\frac{\sqrt{5}+3}{2}\right) = \left(\frac{\sqrt{5}+1}{2}\right)^{n+1}$$

using the fact that $\left(\frac{\sqrt{5}+1}{2}\right)^2 = \left(\frac{\sqrt{5}+3}{2}\right)$, getting the desired result. Using the root test on $\left(\frac{\sqrt{5}+1}{2}\right)^n$ (and ignoring the limit supremum since its the same for all n) we have

$$\lim_{n\to\infty} \sup a_n \leq \lim_{n\to\infty} \sqrt[n]{\left(\frac{\sqrt{5}+1}{2}\right)^n} = \lim_{n\to\infty}\left(\frac{\sqrt{5}+1}{2}\right) = \left(\frac{\sqrt{5}+1}{2}\right)$$

and so the radius of convergence $R \geq \frac{1}{\frac{\sqrt{5}+1}{2}} = \frac{\sqrt{5}-1}{2}$

(b) Define $f : B_{\frac{\sqrt{5}-1}{2}}(0) \to \mathbb{C}$ by

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

Prove that $(1 - z - z^2)f(z) = z$ for all $z \in B_{\frac{\sqrt{5}-1}{2}}(0)$.

Solution: We have that

$$(1 - z - z^2)f(z) = \sum_{n=1}^{\infty} a_n z^n - \sum_{n=1}^{\infty} a_n z^{n+1} - \sum_{n=1} z^{n+2}$$

$$= a_1 z + a_2 z^2 - a_1 z^2 + \sum_{n=3} a_n z^n - \sum_{n=3} a_{n-1} z^n - \sum_{n=3}^{\infty} a_{n-2} z^n$$

$$= z + \sum_{n=3} (a_n - a_{n-1} - a_{n-2}) z^n = z + \sum_{n=3}^{\infty} 0(z^n) = z$$

(6) (a) Show that the radius of convergence of the power series $\sum z^{2^n}$ is equal to 1.

Solution: Note that the series $\sum z^n$ is absolutely convergent with radius of convergence 1. We then have if $|z|^n < 1$, $|z|^{2^n} < 1$, so that implies $\sum z^{2^n}$ is absolutely convergent (and thus convergent) for $z \in B_1(0)$. Now if $z = 1$, we have that $\sum 1^{2^n}$ does not converge, so the radius of convergence is not greater than 1, and thus $R = 1$.

Define the function $f : B_1(0) \to \mathbb{C}$ by

$$f(z) = \sum_{n=0}^{\infty} z^{2^n}$$

(b) Let $\delta > 0$. Prove that f is not bounded on $B_\delta(1) \cap B_1(0)$.

Solution: Pick an arbitrary $k \in N$; by continuity we have that there exists $\delta' \in \mathbb{R}$ such that if $|z - 1| < \delta'$, $z^{2^k} > 1 - \frac{1}{k}$. Note that if $m < k$, then $z^{2^m} > z^{2^k} > 1 - \frac{1}{k}$. Take $z \in B_\delta(1) \cap B_{\delta'}(1) \cap B_1(0)$, then

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = \sum_{n=0}^{k} z^{2^n} + \sum_{n=k+1}^{\infty} z^{2^n} > \sum_{n=0}^{k} (1 - \frac{1}{k}) = k - \frac{1}{k}$$

Since the choice of k was arbitrary, we have that f is unbounded.

(c) Prove that $f(z) = z + f(z^2)$ for all $z \in B_1(0)$

Solution: Note that since $z^{2^n} = (z^2)^{2^{n-1}}$ and if $z \in B_1(0)$, $z^2 \in B_1(0)$, we have that

$$f(z) = z + \sum_{n=1}^{\infty} z^{2^n} = z + \sum_{n=1} (z^2)^{2^{n-1}} = z + \sum_{n=0}^{\infty} (z^2)^{2^n} = z + f(z^2)$$

(d) Let $a \in \mathbb{C}$ with $|a| = 1$. Suppose that there exists a $\delta > 0$ such that f is bounded on the set $B_\delta(a) \cap B_1(0)$. Prove that there exists an $\epsilon > 0$ such that f is bounded on $B_\epsilon(a^2) \cap B_1(0)$

Solution: By the previous part, it is sufficient to show that if for all z in the ball of radius ϵ around a^2, we can find a square root that lies within $B_\delta(a)$ we are done. WLOG suppose δ is small enough such that $B_\delta(a)$ is one quadrant (if not, just make δ smaller). Suppose this region is contained in the sector of the region swept out by the radius with angles $(\theta - \alpha, \theta + \alpha)$ where $e^{i\theta} = a$ (ie this is the region between the two radii at the angles). Note that this sector is also within the same quadrant. Then we have that the equivalent region around a^2 is contained within $(2\theta - \alpha, 2\theta + \alpha)$ ($a^2 = e^{i2\theta}$, can be verified by basic plane geometry). If $z \in B_\delta(a^2)$, consider its square root z' "near" $B_\delta(a)$. If it has angle $2\theta + \beta$ with $\beta < \alpha$, then z' will have angle $\theta + \beta/2$ and so z' will belong in the sector. Thus we have that

$$|z' + a| = |z'|^2 + |a|^2 + 2\text{Re}(a\bar{z})$$

Since a and z' are in the same quadrant, we have that $\text{Re}(a\bar{z'}) = 2(\text{Re}(z')\text{Re}(a) + \text{Im}(z')\text{Im}(a)) > 0$, and so we have that $|z' + a| > 1$. Thus we get that

$$\delta > |z - a^2| = |z' - a||z' + a| > |z' - a|$$

and so $z' \in B_\delta(a)$, as desired, and $f(z') = z' + f(z)$, giving that $f(z)$ is bounded.

(e) Let $a \in \mathbb{C}$, $k \in \mathbb{N}_0$ and $\delta > 0$. Suppose that $a^{2^k} = 1$. Prove that f is not bounded on $B_\delta(a) \cap B_1(0)$.

Solution: Note that

$$|f(z)| = |\sum_{n=0}^{\infty} z^{2^n}| > |\sum_{n=k}^{\infty} z^{2^n}| - |\sum_{n=0}^{k-1} z^{2^n}| > |\sum_{n=k}^{\infty} z^{2^n}| - \sum_{n=0}^{k-1} |z^{2^n}|$$

$$> |\sum_{n=k}^{\infty} z^{2^n}| - k$$

Thus if we show that the left expression is unbounded, we are done. Now suppose $a = e^{i\theta}$. Then we have that $a^{2^i} = 1$ for all $i > k$, since $a^{2^{i+1}} = (a^{2^i})^2$. Pick $z = re^{i\theta}$ and we have that $z^{2^i} = r^{2^i}$ for all $i > k$. Arbitrarily pick $m \in \mathbb{N}$ and by continuity there exists δ' such that $1 - \delta' < r < 1$ implies $r^{2^{k+m}} > 1 - \frac{1}{m}$ (and note that $|a - re^{i\theta}| = 1 - r < \delta'$). We also have that $r^{2^{k+j}} > r^{2^{k+m}} > 1 - \frac{1}{m}$ for $j < m$. Thus pick $r \in (\max(1-\delta, 1-\delta'), 1)$ and we have that

$$|\sum_{n=k}^{\infty} z^{2^n}| - k = \sum_{n=k}^{\infty} r^{2^n} - k > \sum_{n=k}^{k+m} (1 - \frac{1}{m}) - k = m - \frac{1}{m} - k$$

and since the choice of m was arbitrary, we have that f is unbounded on $B_\delta(a)$.

(f) Let $a \in \mathbb{C}$ and $\delta > 0$. Suppose that $|a| = 1$. Prove that f is not bounded on $B_\delta(a) \cap B_1(0)$

Solution: Note that the set of numbers of the form $\pi * \frac{z}{2^k}$ where z, k are integers are dense in the real numbers. Suppose we have a real r and we want to find a number within ϵ of it. Then pick k large enough such that $\frac{2^k}{\pi} > 1$. Thus we have that there exists an integer $z \in (\frac{2^k}{\pi}(r - \epsilon), \frac{2^k}{\pi}(r + \epsilon))$, and so we have our number $\pi * \frac{z}{2^k}$.

We have that

$$|e^{i\theta} - e^{i\theta'}| \le |\cos\theta - \cos\theta'| + |\sin\theta - \sin\theta'|$$

By continuity of cos and sin, pick ϵ such that $|\theta - \theta'| < \epsilon$ implies that $|\cos\theta - \cos\theta'|, |\sin\theta - \sin\theta'| < \frac{\delta}{4}$, so $|e^{i\theta} - e^{i\theta'}| < \frac{\delta}{2}$.

Now note that if $b = e^{i\theta}$ where $\theta = 2\pi\frac{z}{2^k}$, then $b^{2^j} = 1$ for all $j \ge k$, and since $2^k > k$, $z^{2^k} = 1$. Let $a = e^{i\theta}$. By denseness, we have that there exists m such that $\theta' = 2\pi\frac{m}{2^k}$ is within ϵ of θ. Define $w = e^{i\theta'}$, and by the earlier parts we have that $|w - a| < \frac{\delta}{2}$, and $w^{2^k} = 1$. Therefore by part (d)

$$\sum_{n=0}^{\infty} z^{2^n}$$

is unbounded for $z \in B_{\frac{\delta}{2}}(w) \cap B_1(0)$. We can also verify that $B_{\frac{\delta}{2}}(w) \subset B_\delta(a)$: let $z \in B_{\frac{\delta}{2}}(w)$, then we have that $|z - w| < \frac{\delta}{2}$,

$|w - a| < \frac{\delta}{2}$, so we get that $|z - a| < \delta$. Thus we have that f is unbounded on $B_{\delta(a)} \cap B_1(0)$, proving the desired.