Exponential Flocking under Communication Failures for some Cucker-Smale models

Benoît Bonnet

(*in collaboration with É. Flayac*)

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LJLL, Paris-Sorbonne University

March 8, 2021

Outline of the talk

Multi-agent systems and pattern formation

A quick overview of the Cucker-Smale flocking

Cucker-Smale flocking under communication failures

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Multi-agent systems – *Some illustrations*

A multi-agent system is a large ensemble of interacting things

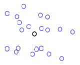
Introduction – *Pattern formation*

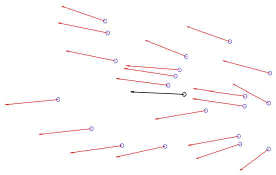
Central observation (pattern formation)

Interacting multi-agent systems may form **interesting global structures** starting from **elementary** interaction rules.

Examples of classical patterns

- **Consensus** (everybody tends to agree on something) :
  ↪ *aggregation models in biology, opinion models, etc...*
- **Flocking** (everybody goes in the same direction) :
  ↪ *flocks of birds, herds analysis, opinion formation, etc...*





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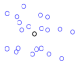

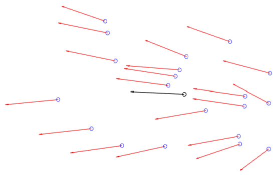

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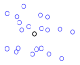

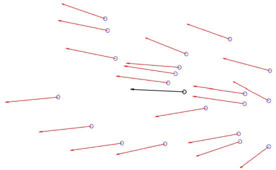

Multi-agent systems – *Weighted Cucker-Smale model*

Let $N \geq 1$ and $(\boldsymbol{x}, \boldsymbol{v}) := (x_1, \dots, x_N, v_1, \dots, v_N) \in (\mathbb{R}^{2d})^N$ be s.t.

$$\text{(CS')} \quad \begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} \xi_{ij}(t)\phi(|x_i(t) - x_j(t)|)(v_j(t) - v_i(t)), \end{cases}$$

where

- $\phi(\cdot)$ is **radial** $\rightsquigarrow$ interactions depend on **relative distance**,
- $\xi_{ij}(\cdot) \in L^\infty(\mathbb{R}_+, [0,1])$ are **symmetric communication rates**

$\rightsquigarrow$ **Intuitive idea** : each agent tries to **align** its velocity with that of the other agents, with a **weight** depending on their relative distance.

Main question for today

Under which hypotheses will (CS') converge towards alignment ?

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Alignement models (Cucker & Smale '07)

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where $\phi \in \text{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$ is a **positive** and **non-increasing** kernel.

Definition (Asymptotic flocking for (CS))

A solution $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ of (CS) **converges to flocking** if

$$\sup_{t \geq 0} |x_i(t) - \bar{\boldsymbol{x}}(t)| < +\infty \qquad \text{and} \qquad \lim_{t \to +\infty} |v_i(t) - \bar{\boldsymbol{v}}| = 0.$$

where $\bar{\boldsymbol{x}}(\cdot)$ and $\bar{\boldsymbol{v}}$ are the position-velocity barycenters.

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Cucker-Smale systems – *Convergence analysis framework*

First question

How do we characterise flocking formation ?

$\rightsquigarrow$ **Good idea** : find a suitable **dissipative Lyapunov** structure

Definition (Variance and standard deviation)

Consider the **variance bilinear form**

$$B : (\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mapsto \frac{1}{N} \sum_{i=1}^{N} \langle x_i, y_i \rangle - \langle \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}} \rangle.$$

Characterisation of flocking

A solution $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ of (CS) converges to flocking **if and only if**

$$\sup_{t \geq 0} X(t) < +\infty \qquad \text{and} \qquad \lim_{t \to +\infty} V(t) = 0,$$

where $X(t) := \sqrt{B(\boldsymbol{x}(t), \boldsymbol{x}(t))}$ and $V(t) := \sqrt{B(\boldsymbol{v}(t), \boldsymbol{v}(t))}$.

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Generalised flocking – *Underlying dissipative structure*

Definition (Graph Laplacians)

Consider the operator $\boldsymbol{L} : (\mathbb{R}^d)^N \to \mathcal{L}((\mathbb{R}^d)^N)$ defined by

$$\big(\boldsymbol{L}(\boldsymbol{x})\boldsymbol{v}\big)_i = \frac{1}{N}\sum_{j=1}^{N} \phi\big(|x_i - x_j|\big)(v_i - v_j),$$

for all $i \in \{1, \ldots, N\}$.

Proposition (Interesting things about graph Laplacians)

- Allows to rewrite (CS) as the **semilinear** system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t), \qquad \dot{\boldsymbol{v}}(t) = -\boldsymbol{L}(\boldsymbol{x}(t))\boldsymbol{v}(t).$$

- It holds that $B(\boldsymbol{L}(\boldsymbol{x})\boldsymbol{v}, \boldsymbol{v}) \geq 0$ for all $\boldsymbol{x}, \boldsymbol{v} \in (\mathbb{R}^d)^N$.
- The strength of the interactions in the system is quantified by the **eigenvalues** of $\boldsymbol{L}(\boldsymbol{x}(t))$.

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Cucker-Smale systems – *Known convergence results*

Observation : $\dot{X}(t) \leq V(t)$ and $\dot{V}(t) \leq -\phi(2\sqrt{N}X(t))V(t)$

Theorem (Flocking formation) [Cucker & Smale '07, Ha & Liu '09]

Suppose that $\phi(\cdot) \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$ is **positive** and **non-increasing**.

- If $\phi(\cdot)$ is **lower-bounded**, then flocking always occurs
  $\rightsquigarrow$ $V(\cdot)$ uniformly exponentially converges towards 0,
- If $\phi(\cdot)$ **vanishes at infinity** but $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+)$, then flocking always occurs as well!
  $\rightsquigarrow$ one has $\phi \notin L^1 \Longrightarrow X(t) \leq X_M$ for some $X_M > 0$,
- If $\phi(\cdot)$ **vanishes at infinity** and $\phi \in L^1(\mathbb{R}_+, \mathbb{R}_+)$, then flocking may fail to occur... $\rightsquigarrow$ One needs small $(X(0), V(0))$.

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Suppose that $\phi(\cdot) \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R}_+^*)$ is **positive** and **non-increasing**.

- If $\phi(\cdot)$ is **lower-bounded**, then flocking always occurs
  $\rightsquigarrow$ $V(\cdot)$ uniformly exponentially converges towards 0,
- If $\phi(\cdot)$ **vanishes at infinity** but $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+)$, then flocking always occurs as well!
  $\rightsquigarrow$ one has $\phi \notin L^1 \implies X(t) \leq X_M$ for some $X_M > 0$,
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Outline of the talk

Multi-agent systems and pattern formation

A quick overview of the Cucker-Smale flocking

Cucker-Smale flocking under communication failures

Generalised flocking – *Modelling communication failures*

Cucker-Smale system (Back to communication failures)

The weighted Cucker-Smale model can be rewritten as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t), \qquad \dot{\boldsymbol{v}}(t) = -\boldsymbol{L}(t, \boldsymbol{x}(t))\boldsymbol{v}(t),$$

where we introduce the **time-dependent** graph-Laplacian

$$\big(\boldsymbol{L}(t, \boldsymbol{x})\boldsymbol{v}\big)_i = \frac{1}{N}\sum_{j=1}^{N} \xi_{ij}(t)\phi\big(|x_i - x_j|\big)(v_i - v_j)$$

Problem (Communication weights)

⇝ The $\xi_{ij}(\cdot)$ may vanish or be small on possibly long time-intervals.

Main question

Under what kind of assumption on $\xi_{ij}(\cdot)$ do we recover flocking ?

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Generalised flocking – *What structure for the weights ?*

First idea : Agents i and j directly communicate if $\xi_{ij}(t) > 0$, but they can still **indirectly communicate** when $\xi_{ij}(t) = 0$.

Figure: *In both situation* $\xi_{34}(t) = 0$ *but agents* 3 *and* 4 *communicate*

Second idea : All the agents do not **need** to interact at all times,
$\hookrightarrow$ we only need a lower-bound on the average interactions.

$\rightsquigarrow$ **Persistence conditions** on the $\xi_{ij}(\cdot)$ using a graph-Laplacian

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Generalised flocking – *Persistence and graph Laplacian*

Definition (Graph Laplacian of the weights)

Consider the operator $\boldsymbol{L}_\xi : \mathbb{R}_+ \to \mathcal{L}((\mathbb{R}^d)^N)$ defined by

$$\big(\boldsymbol{L}_\xi(t)\boldsymbol{v}\big)_i = \frac{1}{N}\sum_{j=1}^{N} \xi_{ij}(t)(v_i - v_j).$$

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The weights $\xi_{ij}(\cdot)$ satisfy the **persistence condition** (PE) if

$$B\Big(\big(\tfrac{1}{\tau}\textstyle\int_t^{t+\tau} \boldsymbol{L}_\xi(s)\mathrm{d}s\big)\boldsymbol{v}, \boldsymbol{v}\Big) \geq \mu B(\boldsymbol{v}, \boldsymbol{v}), \qquad (\mathrm{PE}_{\tau,\mu})$$

for all $(t, \boldsymbol{v}) \in \mathbb{R}_+ \times (\mathbb{R}^d)^N$, where $(\tau, \mu) \in \mathbb{R}_+^* \times (0, 1]$.

Heuristic meaning

On every time window of length τ, the total average of interactions (both **direct** and **indirect**) between pairs of agents is **at least** μ.

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Generalised flocking – *Main convergence result*

Theorem (Flocking formation)[B. & Flayac '21]

Let $(\boldsymbol{x}^0, \boldsymbol{v}^0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and assume the following holds.

(i) The weights $\xi_{ij}(\cdot)$ satisfy (PE) for some $(\tau, \mu) \in \mathbb{R}_+ \times (0, 1]$.

(ii) There exists $\sigma, K > 0$ and $\beta \in (0, \frac{1}{2})$ such that

$$\phi(r) \geq \frac{K}{(\sigma + r)^\beta}.$$

Then, the solution $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ of (CS') **converges to flocking**.

Remarks (sharpness of the assumptions)

- Hypothesis (ii) is probably not sharp... we expect $\beta \in (0, 1)$
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$\longrightarrow$ Average connectedness is necessary, even for consensus

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Generalised flocking – *Elements of proof*

Problem : Lack of dissipation for $V(\cdot)$ $\rightsquigarrow$ strict Lyapunov design !

1) Define the **time-varying** family of operators

$$\psi_\tau(t) = (1 + c^2)\tau \mathrm{Id} - \frac{1}{\tau}\int_t^{t+\tau}\int_t^s \boldsymbol{L}(\sigma, \boldsymbol{x}(\sigma))\mathrm{d}\sigma\mathrm{d}s$$

and consider the candidate **Lyapunov functional**

$$\mathcal{V}_\tau(t) = \lambda(t)V(t) + \sqrt{B(\psi_\tau(t)\boldsymbol{v}(t), \boldsymbol{v}(t))}$$

where $\lambda(\cdot)$ is a tuning parameter (e.g. Mazenc & Malisoff '09).

2) After some (heavy) Lyapunov using (PE) and (ii), we have

$$V(T) \lesssim V(0)\exp\left(-C_{\tau,\mu}T_\lambda^{\frac{1-2\beta}{2(1-\beta)}}\right)$$

where T_λ is the **existence horizon** of $\lambda(\cdot)$ $\rightsquigarrow$ $T \to +\infty$!

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$$\psi_\tau(t) = (1+c^2)\tau \mathrm{Id} - \frac{1}{\tau}\int_t^{t+\tau}\int_t^s \boldsymbol{L}(\sigma, \boldsymbol{x}(\sigma))\mathrm{d}\sigma\mathrm{d}s$$

and consider the candidate **Lyapunov functional**

$$\mathcal{V}_\tau(t) = \lambda(t)V(t) + \sqrt{B(\psi_\tau(t)\boldsymbol{v}(t), \boldsymbol{v}(t))}$$

where $\lambda(\cdot)$ is a tuning parameter (e.g. Mazenc & Malisoff '09).

2) After some (heavy) Lyapunov using (PE) and (ii), we have

$$V(T) \lesssim V(0)\exp\left(-C_{\tau,\mu}T_\lambda^{\frac{1-2\beta}{2(1-\beta)}}\right)$$

where T_λ is the **existence horizon** of $\lambda(\cdot)$ $\rightsquigarrow$ $T \to +\infty$!

Generalised flocking – *Elements of proof*

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Generalised flocking – *Conclusion and open perspective*

Conclusive remarks

We have identified a synthetic and fairly minimalist persistence condition for multi-agent flocking via Lyapunov methods $\rightsquigarrow$ **Nice !**

Open questions and coming work

1) Recover the **sharp** exponent range $\beta \in (0, 1)$
   $\hookrightarrow$ Try a **better Lyapunov design**!

2) Carry out the analysis using L^∞-Lyapunov functionals
   $\hookrightarrow$ Necessary to obtain **uniform estimates** w.r.t. N and study **graph-limits** as $N \rightarrow +\infty$!

3) Adapt our methodology to **random communication failures**
   $\hookrightarrow$ Case where $(t, \omega) \mapsto \xi_{ij}(t, \omega)$ are **stochastic processes**

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Thank you for your attention !