PSL UNIVERSITÉ PARIS

IMJ-PRG
Institut de Mathématiques de Jussieu-Paris Rive Gauche

Mean-Field Control and Continuity Inclusions

Benoît Bonnet
(*in collaboration with H. Frankowska*)

59th Conference on Decision and Control

From my kitchen to Jeju Island

November 5, 2020

Outline of the talk

Introduction to mean-field control

From differential inclusions to continuity inclusions

Main contributions

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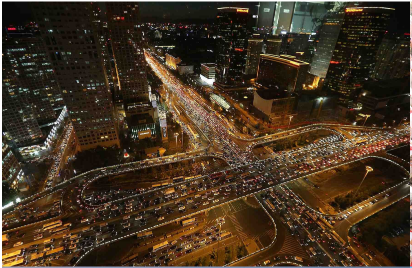
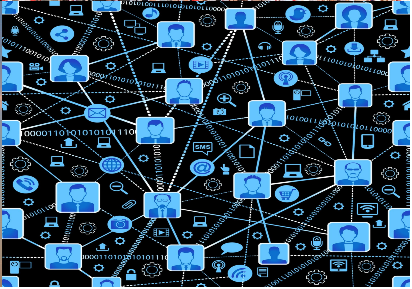
Introduction – *Multi-agent systems*

A multi-agent system is a large ensemble of interacting things



Introduction – *Mean-field limits of particle systems*

First observation (dimensionality-related problems)

The number N of agents is usually very large $\rightsquigarrow$ **numerical issues**

Idea: Approximation procedure using **mean-field limits!**

System of ODEs on N agents $(x_1,\dots,x_N)\in(\mathbb{R}^d)^N$

$$\mu_N=\frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$$

Single PDE on the density of agents $\mu:\mathbb{R}^d\to\mathbb{R}$.

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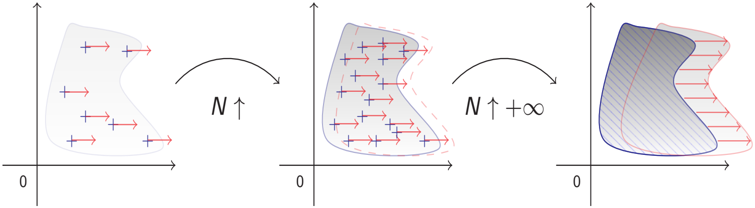

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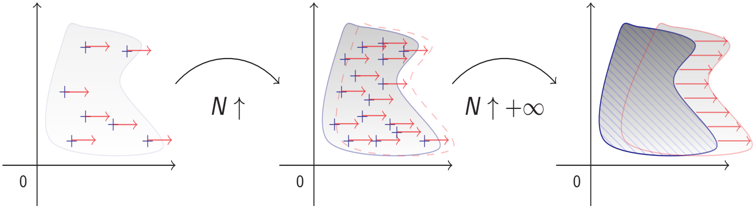

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Introduction – *Continuity equations*

Multi-agent dynamics are modelled via **continuity equations**

$$\partial_t \mu(t) + \mathrm{div}\big(\boldsymbol{v}(t, \mu(t), \cdot)\mu(t)\big) = 0,$$

where $\mu(t) \in \mathcal{P}_c(\mathbb{R}^d)$ and $\boldsymbol{v} : [0, T] \times \mathcal{P}_c(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$.

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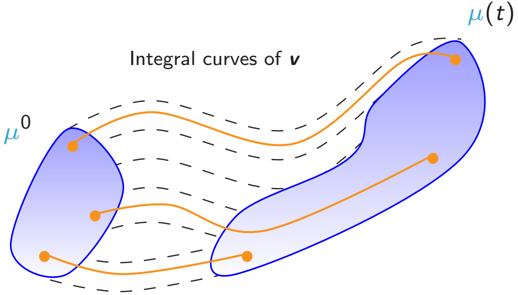

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Mean-field control $\rightsquigarrow$ control problems on **continuity equations**

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- **Optimal feedback synthesis** $\rightsquigarrow$ Open for now

Natural question : General framework to study these problems ?
$\longrightarrow$ Yes! Reformulate controlled dynamics as differential inclusions

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From differential inclusions to continuity inclusions

Main contributions

Differential inclusions – *The classical case*

Intuition: Differential inclusions $\iff$ ODEs with **set-valued** r.h.s

Definition (differential inclusion)

Let $\mathcal{M}$ be a smooth manifold and $F : \mathcal{M} \rightrightarrows T\mathcal{M}$. Then

$$\dot{x}(t) \in F(x(t)) \iff \begin{cases} \dot{x}(t) = f(t), \\ f(t) \in F(x(t)), \end{cases} \quad \mathcal{L}^1\text{-a.e. in } [0, T]$$

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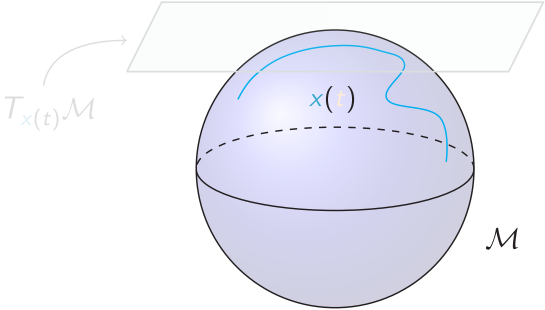

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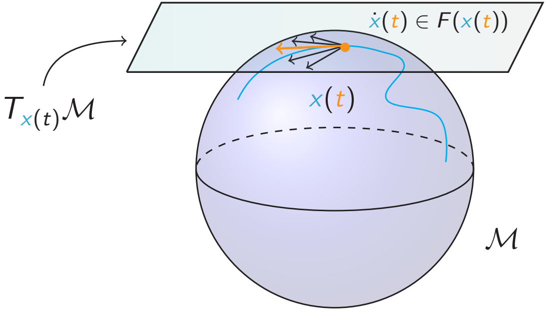

Differential inclusions – *Link with control system*

Key point: Control systems are **equivalent** to diff. inclusions

Equivalence (heuristic statement)

If $f : [0, T] \times \mathcal{M} \times U \to T\mathcal{M}$ is **Carathéodory**, then

$$\dot{x}(t) = f(t, x(t), u(t)) \iff \dot{x}(t) \in F(t, x(t)),$$

where

$$F(t, x) := \Big\{ f(t, x, u) \text{ s.t. } u \in U \Big\}.$$

Recap': Building differential inclusions

1) Check that the ambient space has a manifold-like structure
2) Identify an ODE structure $\leadsto$ make the velocities set-valued

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Differential inclusions – *What about continuity equations ?*

Idea: Endow $\mathcal{P}_2(\mathbb{R}^d)$ with the **optimal transport** metric W_2

$$W_2(\mu,\nu) := \inf_{\gamma\in\Gamma(\mu,\nu)} \left(\int_{\mathbb{R}^{2d}} |x-y|^2 \mathrm{d}\gamma(x,y) \right)^{1/2}$$

Wasserstein geometry & ODEs [Ambrosio, Gigli, McCann, Otto, Savaré]

- $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ is a **"manifold"** with $T_\mu \mathcal{P}_2(\mathbb{R}^d) \subset L^2(\mu)$
- Continuity equations

$$\partial_t \mu(t) + \mathrm{div}\big(\boldsymbol{v}(t,\mu(t),\cdot)\mu(t)\big) = 0,$$

are **ODEs** over $\mathcal{P}_2(\mathbb{R}^d)$, with velocities $\boldsymbol{v}(t,\mu(t),\cdot) \in L^2(\mu(t))$

Conclusion: Make $(t,\mu) \mapsto \boldsymbol{v}(t,\mu,\cdot) \in L^2(\mu)$ **set-valued** !

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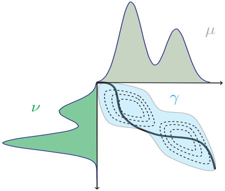

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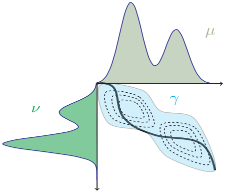

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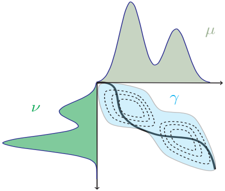

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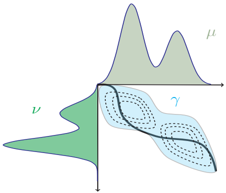

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Conclusion: Make $(t,\mu) \mapsto \boldsymbol{v}(t,\mu,\cdot) \in L^2(\mu)$ **set-valued** !

Outline of the talk

Introduction to mean-field control

From differential inclusions to continuity inclusions

Main contributions

Continuity inclusions – *Definition* & *controlled systems*

Definition (Continuity inclusions) [B., Frankowska]

Let $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \rightrightarrows V(t,\mu) \subset L^2(\mu)$. Then, $\mu(\cdot)$ solves

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Certificate: Is this notion suited for control systems ? $\rightsquigarrow$ **Yes!**

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Take $v : [0,T] \times \mathcal{P}_c(\mathbb{R}^d) \times U \times \mathbb{R}^d \to \mathbb{R}^d$ is **Cauchy-Lip** and set

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Continuity inclusions – *Compactness* & *Relaxation*

Theorem (Compactness of solution sets)[B., Frankowska]

Suppose that $V : [0,T] \times \mathcal{P}_c(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d)$ is **Cauchy-Lip**, with **closed** **convex** images. Then for any μ^0, the solution set

$$\mathcal{S}(\mu^0) := \Big\{ \mu(\cdot) \ \text{ s.t. } \ \partial_t \mu(t) \in -\mathrm{div}\big(V(t,\mu(t))\mu(t)\big), \ \mu(0) = \mu^0 \Big\},$$

is **compact** in the $C^0 W_2$-topology $\rightsquigarrow$ **Existence of minimisers!**

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Thank you for your attention !