Optimal Control Problems in Wasserstein Spaces

A PDE Approach to Multi-Agent Systems

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Outline of the talk

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1. Introduction and Motivations
2. Optimal Transport Theory & Wasserstein Spaces
3. Optimal Control Problems and Pontryagin Maximum Principle

Idea : Understand and control the global behaviour of large systems of interacting agents

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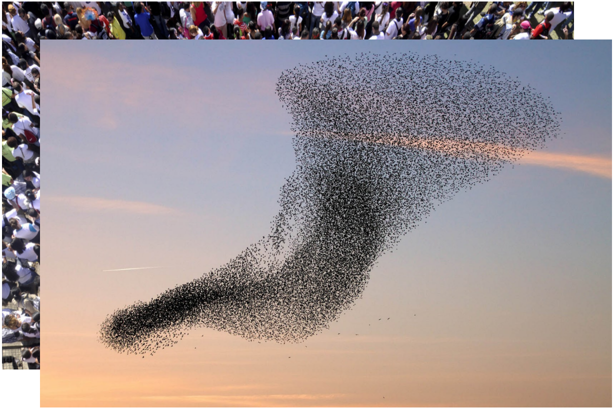

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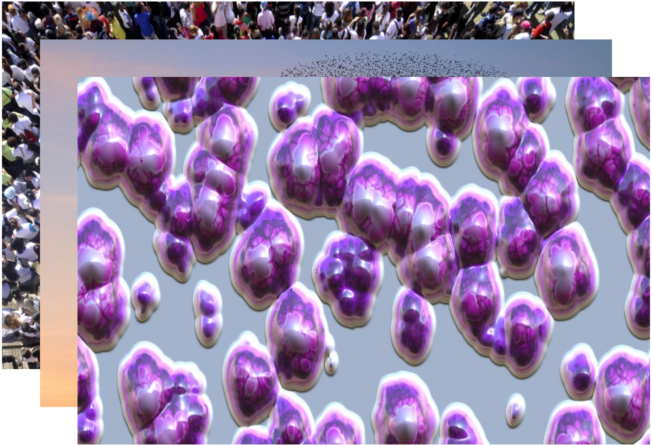

Idea : Understand and control the global behaviour of large systems of interacting agents

Example (Finite dim. Drift + Convolution + control)

Consider N agents $(x_1, ..., x_N) \in (\mathbb{R}^d)^N$ evolving according to

$$\dot{x}_i(t) = v_d(x_i) + \frac{1}{N}\sum_{j=1}^{N} K(x_i - x_j) + u_i(t), \qquad (S_N)$$

Objective : *design a set of control laws $t \mapsto (u_1(t), ..., u_N(t))$ achieving a certain goal, e.g. forming a consensus, etc...*

Issues related to the formulation (S_N)

- No a priori knowledge of the system (number of agents, exact positions, etc...)
- Not so relevant to consider a discrete model for very large systems of interacting agents
- Extremely demanding computationally speaking

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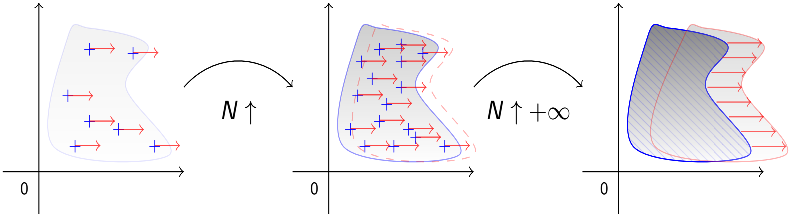

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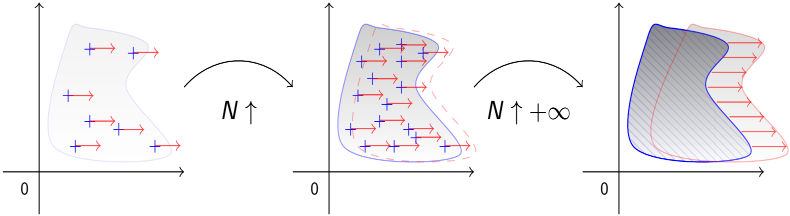

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Mean field controlled system described by *transport equations with non-local velocities*

$$\partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](\cdot) + u(t, \cdot))\mu(t)) = 0. \qquad (S_\infty)$$

where the controls depend on time and space.

Question : Choice of the state space ?

Discrete $\oplus$ continuous objects $\rightsquigarrow$ *'distributional spaces'*

Problem : Distributional topologies are not very nice...

$\hookrightarrow$ Space of probability measures endowed with an optimal transportation metric !

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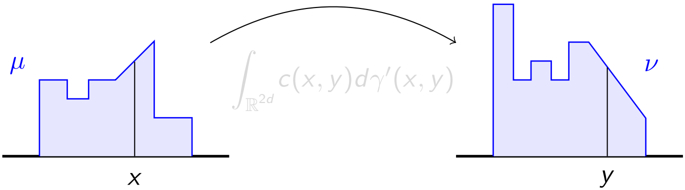
Given two probability measures $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$ and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, **find a measure** $\gamma \in \mathcal{P}_c(\mathbb{R}^{2d})$ such that

$$(i) \quad \gamma \in \Gamma(\mu,\nu) = \left\{ \gamma' \in \mathcal{P}_c(\mathbb{R}^{2d}) \text{ s.t. } \pi^1_{\#}\gamma' = \mu \ , \ \pi^2_{\#}\gamma' = \nu \right\},$$

$$(ii) \quad \int_{\mathbb{R}^{2d}} c(x,y)\mathrm{d}\gamma(x,y) = \min_{\gamma' \in \Gamma(\mu,\nu)} \left[\int_{\mathbb{R}^{2d}} c(x,y)\mathrm{d}\gamma'(x,y) \right].$$



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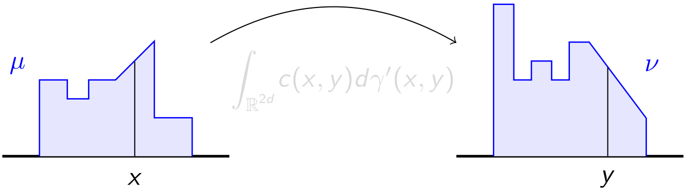

Optimal transport theory - *Kantorovich problem*

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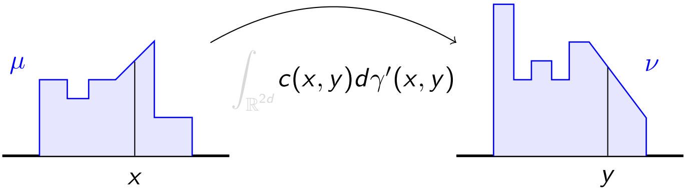

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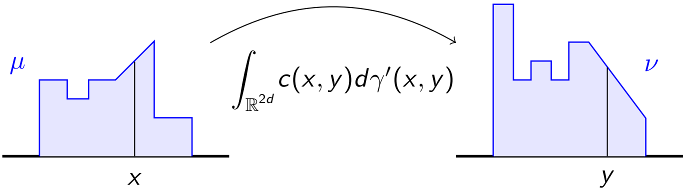

Definition (Wasserstein distance)

Taking $c(x,y) = |x-y|^2$, *the quantity*

$$W_2(\mu,\nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2d}} |x-y|^2 d\gamma(x,y) \right)^{1/2} \right\}$$

defines a distance over $\mathcal{P}_c(\mathbb{R}^d)$.

$\sim$ *Infimum of* L^2*-distances over the couplings* $\gamma \in \Gamma(\mu,\nu)$.

Proposition (Some useful properties)

- W_2 *metrizes the usual weak-* topology of measures.*
- $(\mathcal{P}_c(\mathbb{R}^d), W_2)$ *is a complete and separable metric space.*
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Theorem (Ambrosio,Gangbo '07 - Piccoli,Rossi '13)

Under Cauchy-Lipschitz assumptions on v, the Cauchy problem

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](t,\cdot)\mu(t)) = 0 \ , \ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$$

has a **unique Lipschitz solution** *which is* **continuous w.r.t. its initial datum**.

The solution $t \mapsto \mu(t)$ is given explicitly by

$$\mu(t) = \Phi^v_{(0,t)}(\cdot)_{\#}\mu^0 \qquad \text{(image measure)}$$

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Optimal control problem in $(\mathcal{P}_c(\mathbb{R}^d), W_2)$

$$\begin{cases} \min_{u \in \mathcal{U}} \left[\int_0^T L(\mu(t), u(t)) \mathrm{d}t + \varphi(\mu(T)) \right] \\ \text{s.t.} \begin{cases} \partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](t, \cdot) + u(t, \cdot))\mu(t)) = 0, \\ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d), \end{cases} \end{cases}$$

where $\mathcal{U}$ is defined by

$$\mathcal{U} = \left\{ u \in L^\infty([0, T], C^1(\mathbb{R}^d, \mathbb{R}^d)) \text{ s.t. } \|u(t)\|_{C^1(\mathbb{R}^d, \mathbb{R}^d)} \leq M \right\}$$

Question : Choice of the control set $\mathcal{U}$

$\hookrightarrow$ Cauchy-Lipschitz requires $u(t) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d) \oplus$ sublinear.
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- **Existence results :** Mean-field and Γ-convergence approaches (*Bongini, Fornasier, Piccoli, Rossi, Solombrino*) or classical PDE methods (*Achdou, Laurière*).

- **Numerical methods :** Good methods in the transport ⊕ diffusion case (*Albi, Pareschi, Toscani, Zanella,...*).
  ⇝ More ad hoc and difficult for pure transport.

- **Optimality conditions :** Hamilton-Jacobi Optimality conditions (*Cavagnari, Marigonda*)

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Theorem (Unconstrained and smooth PMP on $\mathbb{R}^d$)

Let $(u^*(\cdot), x^*(\cdot))$ be an optimal pair control-trajectory for the problem

$$\begin{cases} \min\limits_{u \in \mathcal{U}} \left[\int_0^T L(t, x(t), u(t)) dt + \varphi(x(T)) \right] \\ \dot{x}(t) = f(t, x(t), u(t)) \ , \ x(0) = x^0. \end{cases}$$

Then, there exists a curve $p^*(\cdot)$ called costate such that

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$\diamond$ Couple state-costate $(x^*(\cdot), p^*(\cdot)) \in \mathrm{Lip}([0,T], \mathbb{R}^{2d})$

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$\diamond$ Classical Hamiltonian flow of $\mathcal{H}(t,x,p)$ with respect to $|\cdot|_2$.

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Theorem (B., Rossi '18)

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- Design of a shooting method
- Study gradient type methods with the same kind of duality involved in this PMP (measure on the product space)
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Thank you for your attention !