# Continuity Inclusions and Applications in Mean-Field Optimal Control

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(*in collaboration with H. Frankowska*)

**Seminar of Analysis, Stochastic Phenomena and Applications**

*February 2, 2021*

# Outline of the talk

Mean-field control problems

From differential to continuity inclusions

Main structural results in the Cauchy-Lipschitz case

Application : geometric derivation of the PMP

Conclusion

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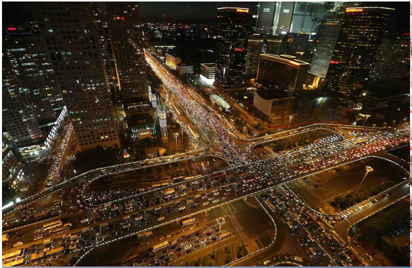


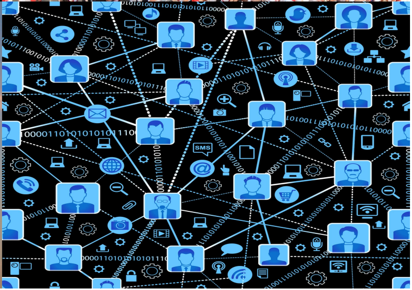

# Introduction – *Mean-field limits of particle systems*

**First observation (dimensionality-related problems)**

The number $N$ of agents is usually **very large** $\rightsquigarrow$ **numerical issues**

**Idea:** Approximation procedure using **mean-field** limits!




System of ODEs on $N$ agents $(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$$

Single PDE on the density of agents $\mu : \mathbb{R}^d \to \mathbb{R}$.

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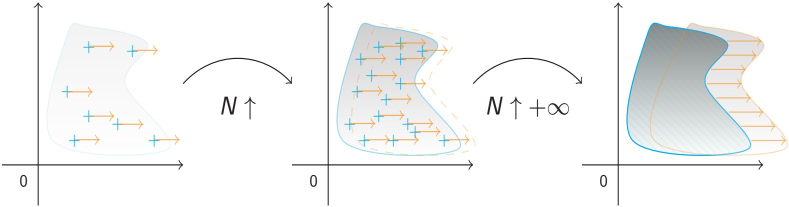


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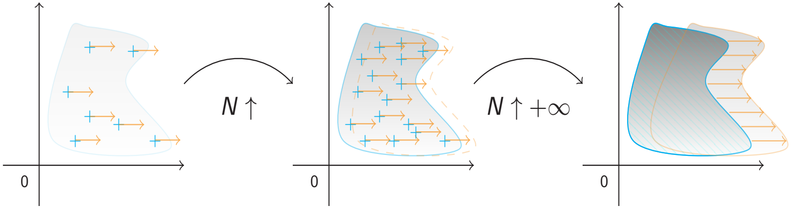


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# Introduction – *Continuity equations*

Deterministic mean-field dynamics $\rightsquigarrow$ **continuity equations**

$$\partial_t \mu(t) + \mathrm{div}\big(v(t, \mu(t), \cdot)\mu(t)\big) = 0,$$

where $\mu(t) \in \mathcal{P}_c(\mathbb{R}^d)$ and $v : [0, T] \times \mathcal{P}_c(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$.




Figure: *Solution of a continuity equation with smooth driving field*

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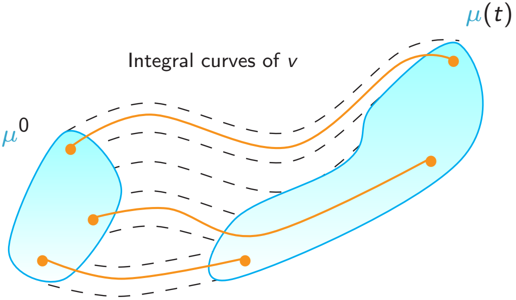


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# Differential inclusions – *The classical case*

**Intuition:** Differential inclusions $\Longleftrightarrow$ ODEs with **set-valued** r.h.s

### Definition (differential inclusion)

Let $\mathcal{M}$ be a smooth manifold and $F : \mathcal{M} \rightrightarrows T\mathcal{M}$. Then

$$\dot{x}(t) \in F(x(t)) \Longleftrightarrow \begin{cases} \dot{x}(t) = f(t), \\ f(t) \in F(x(t)), \end{cases} \quad \mathcal{L}^1\text{-a.e. in } [0,T]$$

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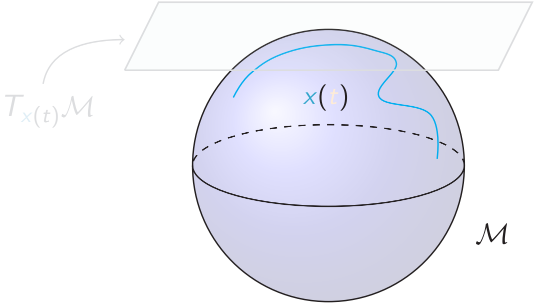

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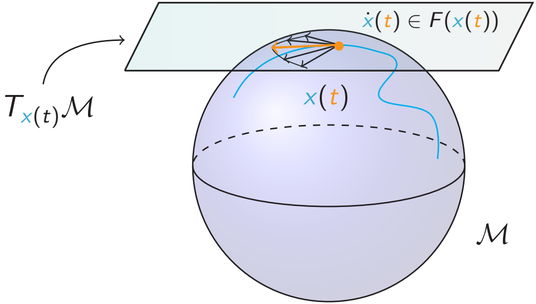

# Differential inclusions – *Link with control system*

**Key point:** Control systems are **equivalent** to diff. inclusions

Equivalence (heuristic statement)

If $f : [0, T] \times \mathcal{M} \times U \to T\mathcal{M}$ is **Carathéodory**, then

$$\dot{x}(t) = f(t, x(t), u(t)) \quad \Longleftrightarrow \quad \dot{x}(t) \in F(t, x(t)),$$

where

$$F(t, x) := \Big\{ f(t, x, u) \text{ s.t. } u \in U \Big\}.$$

**Recap':** Building differential inclusions

1) Check that the ambient space has a **manifold-like** structure,
2) Identify an ODE structure,
3) Make the velocities **set-valued** !

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**Idea:** Endow $\mathcal{P}_c(\mathbb{R}^d)$ with **Wasserstein** metrics $W_p$

$$W_p(\mu,\nu) := \min_{\gamma\in\Gamma(\mu,\nu)} \left( \int_{\mathbb{R}^{2d}} |x-y|^p \mathrm{d}\gamma(x,y) \right)^{1/p}$$




Wasserstein geometry & ODEs [Ambrosio, Gigli, McCann, Otto, Savaré]

- $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ is a **"manifold"** with $T_\mu\mathcal{P}_2(\mathbb{R}^d) \subset L^2(\mu)$
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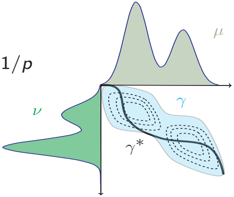


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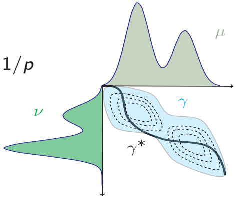


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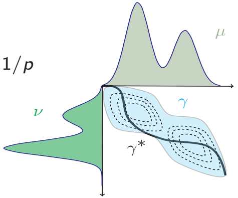


Wasserstein geometry & ODEs [Ambrosio, Gigli, McCann, Otto, Savaré]

- $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ is a **"manifold"** with $T_\mu\mathcal{P}_2(\mathbb{R}^d) \subset L^2(\mu)$
- Continuity equations

$$\partial_t \mu(t) + \mathrm{div}\big(v(t,\mu(t),\cdot)\mu(t)\big) = 0,$$

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**Conclusion:** Make $(t,\mu) \mapsto v(t,\mu,\cdot) \in L^2(\mu)$ **set-valued** !

# Differential inclusions – *What about continuity equations ?*

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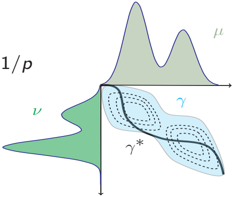


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# Continuity inclusions – *Definition*

**Definition (Continuity inclusions)** [B., Frankowska '21]

Let $(t,\mu) \in [0,T] \times \mathcal{P}_2(\mathbb{R}^d) \rightrightarrows V(t,\mu) \subset L^2(\mu)$. Then, $\mu(\cdot)$ solves

$$\partial_t \mu(t) \in -\mathrm{div}\big(V(t,\mu(t))\mu(t)\big), \tag{CI}$$

if there exists a meas. selection $t \in [0,T] \mapsto \boldsymbol{v}(t) \in V(t,\mu(t))$ s.t.

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**Literature comparison (Gradient and Hamiltonian flows)**

- $V(t,\mu(t)) = -\partial\phi(\mu(t)) \rightsquigarrow$ **Gradient flow** [AGS'08 & many]
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**Certificate:** Is this notion suited to control systems ? $\rightsquigarrow$ **Yes!**

Theorem (Correspondence with control systems) [B., Frankowska '21]

Take $v : [0, T] \times \mathcal{P}_c(\mathbb{R}^d) \times U \times \mathbb{R}^d \to \mathbb{R}^d$ is **Cauchy-Lip** and set

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# Outline of the talk

Mean-field control problems

From differential to continuity inclusions

Main structural results in the Cauchy-Lipschitz case

Application : geometric derivation of the PMP

Conclusion

# Continuity inclusions – *Filippov estimates (1)*

**Theorem (Filippov)**[B., Frankowska '21]

Suppose that $V : [0,T] \times \mathcal{P}_c(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d)$ is **Cauchy-Lip** with **compact** images, let $\nu(\cdot)$ solve

$$\partial_t \nu(t) + \mathrm{div}(\boldsymbol{w}(t)\nu(t)) = 0,$$

with $\boldsymbol{w} : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ Carathéodory, and assume that

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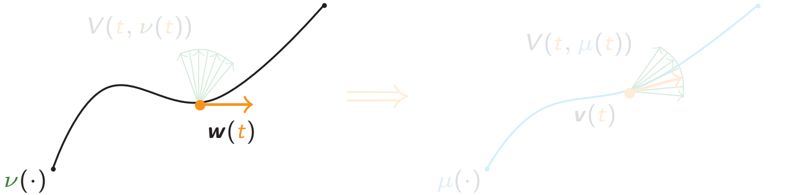


Figure: *Illustration of Filippov's estimate*

### Proof (Main ideas)

1) Build a sequence $(\mu_k(\cdot), \boldsymbol{v}_k(\cdot))$ s.t.

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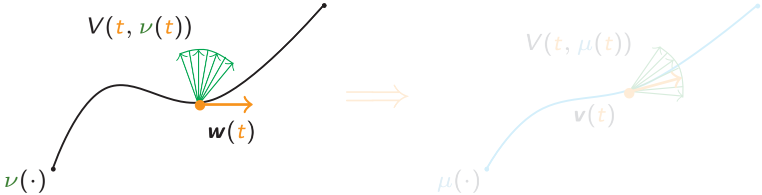


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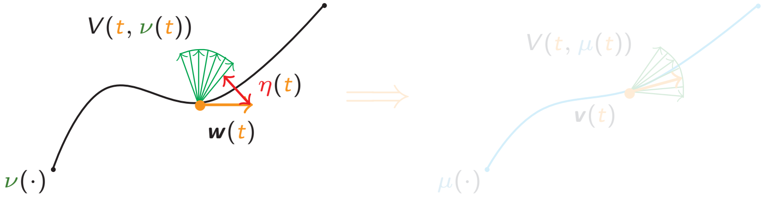


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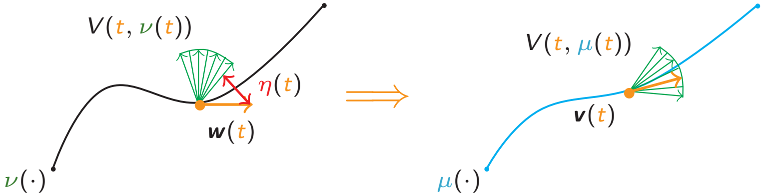


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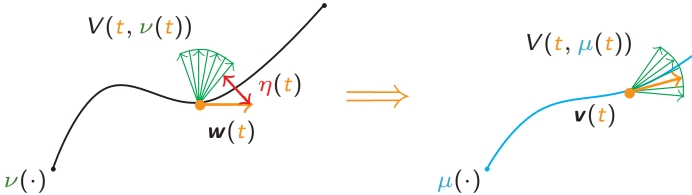


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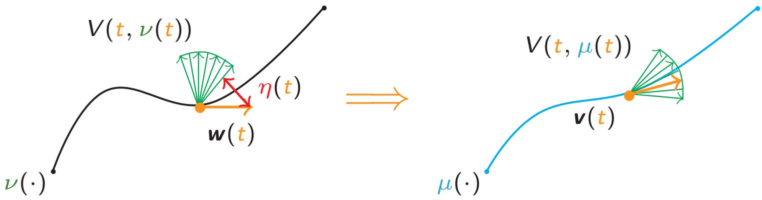


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# Continuity inclusions – *Filippov's estimates (2)*

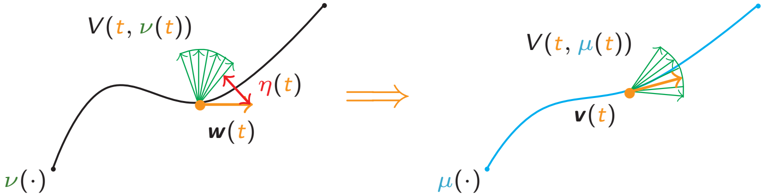


Figure: *Illustration of Filippov's estimate*

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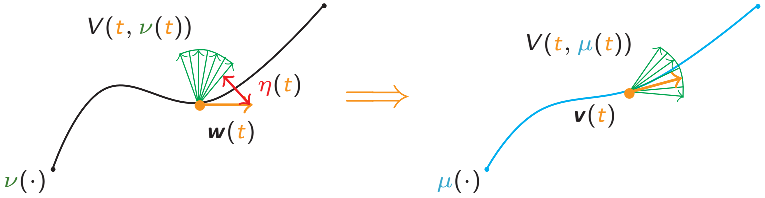


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**Theorem (Compactness of solutions)** [B., Frankowska '21]

Suppose that $V : [0,T] \times \mathcal{P}_c(\mathbb{R}^d) \rightrightarrows C^0(\mathbb{R}^d, \mathbb{R}^d)$ is **Cauchy-Lip**, with **compact** **convex** images. Then for any $\mu^0$, the solution set

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## Proof (Main ideas)

1) Let $\mu(\cdot)$ be a relaxed traj. By **Aumann**, there exists $\nu(\cdot)$ s.t.

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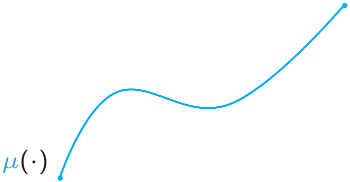

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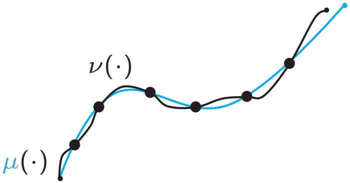

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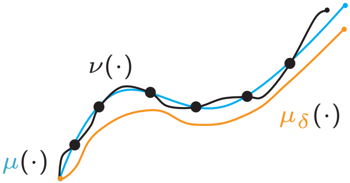

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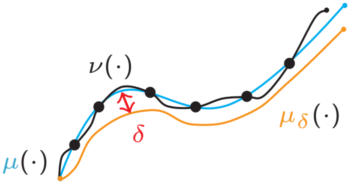

# Outline of the talk

# Mean-field OCP – *Definition and existence*

Consider the Mayer problem

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(\mathbf{OCP}) \quad \begin{cases} \min\limits_{u(\cdot)\in\mathcal{U}} \Big[\varphi(\mu(T))\Big], \\ \text{s.t.} \quad \begin{cases} \partial_t \mu(t) + \operatorname{div}\big(v(t,\mu(t),u(t))\mu(t)\big) = 0, \\ \mu(0) = \mu^0, \\ \Psi_i(\mu(T)) \leq 0 \qquad \text{for } i \in \{1,\dots,m\}, \end{cases} \end{cases}
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where

- $u : [0,T] \to U$ is $\mathcal{L}^1$-meas. with $(U, d_U)$ comp. metric space.
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If $v(t,\mu,U) \subset C^0(\mathbb{R}^d,\mathbb{R}^d)$ is convex, then $(\mathbf{OCP})$ admits a solution.

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If $v(t,\mu,U) \subset C^0(\mathbb{R}^d,\mathbb{R}^d)$ is convex, then **(OCP)** admits a solution.

# Mean-field OCP – $C^1_{\mathrm{loc}}$-*functionals*

**Definition (Local $W_2$-sub and superdifferentials)**[B., Frankowska '21]

Let $\phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R} \cup \{\pm\infty\}$ be s.t. $\mathcal{P}_c(\mathbb{R}^d) \subset D(\phi)$ and $\mu \in \mathcal{P}_c(\mathbb{R}^d)$. Then, $\xi \in \partial^-_{\mathrm{loc}}\phi(\mu)$ if and only if

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## Pontryagin Maximum Principle for **(OCP)** [B., Frankowska '21]

Let $(\mu^*(\cdot), u^*(\cdot))$ be a strong local min. for **(OCP)**. Then, there exist **non-trivial multipliers** $(\lambda_0, \lambda_1, \dots, \lambda_n) \in \{0,1\} \times \mathbb{R}^n_+$ and a **state-costate curve** $\nu^*(\cdot) \in \mathrm{AC}([0,T], \mathcal{P}_c(\mathbb{R}^{2d}))$ s.t.

$$\begin{cases} \partial_t \nu^*(t) + \mathrm{div}\Big( \mathbb{J}_{2d} \nabla_\nu \mathbb{H}(t, \nu^*(t), u^*(t)) \, \nu^*(t) \Big) = 0, \\ \pi^1_{\#} \nu^*(t) = \mu^*(t) \qquad \text{for all times } t \in [0,T], \\ \nu^*(T) = \Big( \mathrm{Id}, -\lambda_0 \nabla \varphi(\mu^*(T)) - \sum_{i=1}^n \lambda_i \nabla \Psi_i(\mu^*(T)) \Big)_{\#} \mu^*(T), \end{cases}$$

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# Mean-field OCP – *Proof of the PMP*

## Proof of the PMP (Main steps)

1) Reformulate the dynamics as (CI) with $V(t,\mu) := v(t,\mu,U)$.

2) For all $\epsilon > 0$ and $t \mapsto \boldsymbol{w}(t) \in T_{\overline{\text{co}}V(t,\mu^*(t))}\big(v(t,\mu^*(t),u^*(t))\big)$, there exists $\tilde{\mu}_\epsilon(\cdot)$ sol. of (CI) s.t.

$$\sup_{t\in[0,T]} W_1\Big(\tilde{\mu}_\epsilon(t), (\text{Id} + \epsilon\mathcal{F}_{\boldsymbol{w}}(t,\cdot))_{\#}\mu^*(t)\Big) = o(\epsilon),$$

where $t \mapsto \mathcal{F}_{\boldsymbol{w}}(t,x) \in \mathbb{R}^d$ "solves" a lin. system [B'19,BR'19].

3) Contradiction $\oplus$ Separation $\implies \exists\,(\lambda_0,\lambda_1,\dots,\lambda_n) \neq 0$ s.t.

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# Outline of the talk

Mean-field control problems

From differential to continuity inclusions

Main structural results in the Cauchy-Lipschitz case

Application : geometric derivation of the PMP

Conclusion

# Conclusion – *Recap' and open perspectives*

## Recap' of the presentation

- **Geometric** notion of **set-valued** dynamics for continuity equations $\rightsquigarrow$ **Continuity inclusions** [B., Frankowska JDE'21]
- **Localised** differentiability and **variational** linearisations $\rightsquigarrow$ **Pontryagin Principle** [B., Frankowska AMO'21 (soon)]

## Work (in progress and to come)

1) **Semiconcavity** and **sensitivity** value function $\rightsquigarrow$ submitted !
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# Thank you for your attention !