Sufficient Conditions for the Lipschitz Regularity of Mean-Field Optimal Controls

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(*in collaboration with F.Rossi*)

Groupe de Travail de Calcul des Variations

March 29, 2021

Outline of the talk

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Multi-agent systems and mean-field optimal control

Lipschitz feedbacks in classical OCPs

Lipschitz feedbacks in mean-field OCPs

Example in 1D – Maximisation of the variance

Conclusion and open problems

Multi-agent systems – *Some illustrations*

Multi-agent system $\rightsquigarrow$ **large** ensemble of **interacting** things

Introduction – *Pattern formation*

Central observation (pattern formation)

Simple **microscopic** rules $\rightsquigarrow$ complex **macroscopic** structures!

- **Consensus** (everybody tends to agree on something) :
  $\hookrightarrow$ *aggregation models in biology, herds analysis, etc...*

- **Flocking** (everybody goes in the same direction) :
  $\hookrightarrow$ *flocks of birds, opinion models, etc..*

- **Shock-waves** & **mills** (periodic motions in the system) :
  $\hookrightarrow$ *Unidimensional traffic networks, schools of fishes, etc...*

Multi-agent control (main themes)

1. First, **model** agent dynamics and **analyse** pattern formations,
2. Then, **stir** systems **towards/away** from relevant patterns.

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Multi-agent control – *Finite dimensional setting*

Controlled multi-agent dynamics $\rightsquigarrow$ **systems of ODEs**

$$\dot{x}_i(t) = \boldsymbol{v}_N(t, \boldsymbol{x}_N(t), x_i(t)) + u_i(t),$$

where

- $\boldsymbol{x}_N(\cdot) = (x_1(\cdot), ..., x_N(\cdot)) \in \mathrm{Lip}([0,T], (\mathbb{R}^d)^N)$ are the **states**,
- $\boldsymbol{v}_N : [0,T] \times (\mathbb{R}^d)^N \times \mathbb{R}^d \to \mathbb{R}^d$ is a **non-local** drift,
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$\hookrightarrow$ Controllability, explicit synthesis, optimal control !

$$\begin{cases} \min\limits_{u(\cdot)} \left[\dfrac{1}{N} \sum\limits_{i=1}^{N} \displaystyle\int_0^T \Big(\boldsymbol{L}_N(t, \boldsymbol{x}_N(t), x_i(t)) + \psi(u_i(t)) \Big) \mathrm{d}t + \boldsymbol{\varphi}_N(\boldsymbol{x}_N(T)) \right] \\ \text{s.t.} \begin{cases} \dot{x}_i(t) = \boldsymbol{v}_N(t, \boldsymbol{x}_N(t), x_i(t)) + u_i(t), \\ x_i(0) = x_i^0. \end{cases} \end{cases}$$

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Multi-agent control – *Dimension and mean-field limits*

First observation (dimensionality-related problems)

The number N of agents is usually **very large** $\rightsquigarrow$ **numerical issues**

Idea: Approximation procedure using **mean-field** limits!

System of ODEs on N agents $(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$

$$\mu_N = \frac{1}{N}\sum_{i=1}^{N}\delta_{x_i}$$

Single PDE on the density of agents $\mu : \mathbb{R}^d \to \mathbb{R}$.

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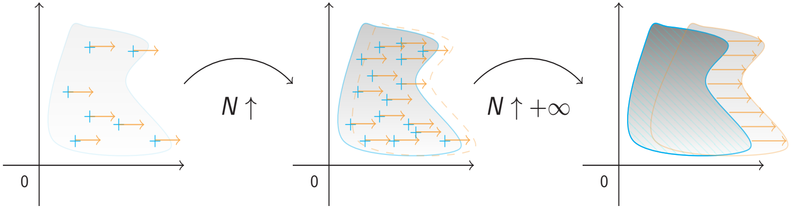

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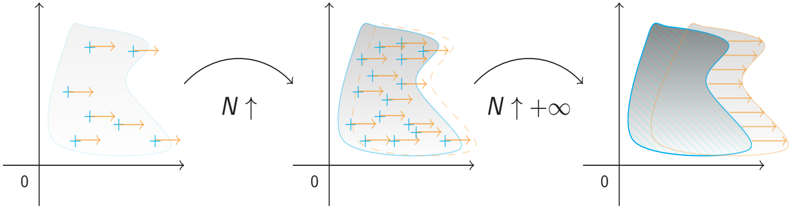

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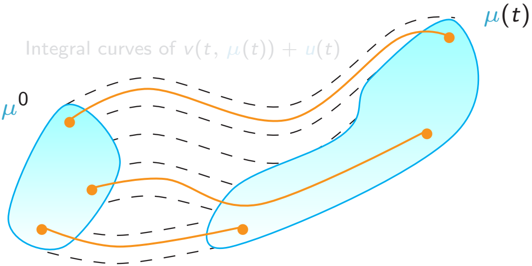
Deterministic mean-field dynamics $\rightsquigarrow$ **continuity equations**

$$\partial_t \mu(t) + \mathrm{div}_x\Big((v(t,\mu(t),\cdot) + u(t,\cdot))\mu(t)\Big) = 0,$$

where

- $\mu(t) \in \mathcal{P}(\mathbb{R}^d)$ is a **probability measure** $\rightsquigarrow$ micro-macro ok!
- $v : [0,T] \times \mathcal{P}_c(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is a **non-local** velocity field,
- the controls $u : [0,T] \times \mathbb{R}^d \to U$ are **vector fields**



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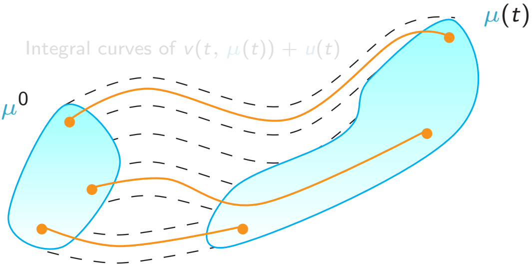

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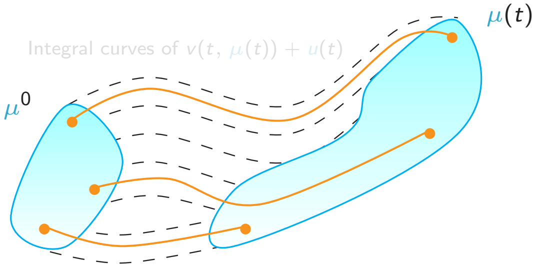

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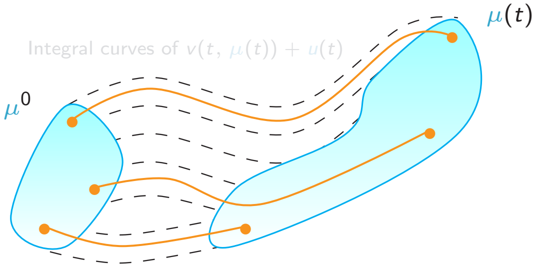

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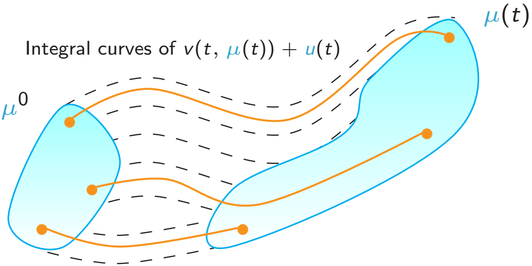

Multi-agent control – *Mean-field optimal control problems*

Idea: From $(\mathbf{OCP}_N)$ $\rightsquigarrow$ **formally associate** an OCP on $\mu(\cdot)$

General mean-field optimal control problem **(MFOCP)**

$$\begin{cases} \min\limits_{u(\cdot,\cdot)} \left[\int_0^T \int_{\mathbb{R}^d} \Big(L(t,\mu(t),x) + \psi(u(t,x)) \Big) \mathrm{d}\mu(t)(x)\mathrm{d}t + \varphi(\mu(T)) \right] \\ \text{s.t.} \begin{cases} \partial_t \mu(t) + \mathrm{div}_x \Big((v(t,\mu(t),\cdot) + u(t,\cdot))\mu(t) \Big) = 0, \\ \mu(0) = \mu^0, \end{cases} \end{cases}$$

where the dynamics and cost functionals satisfy

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Mean-field
limit process

Infinite dimensional (MFOCP)
Single PDE on μ

Lots of mathematics
(PDE analysis, optimal transport...)

Optimal control theory
for (MFOCP)
(optimality conditions, etc...)

Numerical
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Synthesis of control laws
Compute $u^*(\cdot, \cdot)$ *vector field*

Apply and solve

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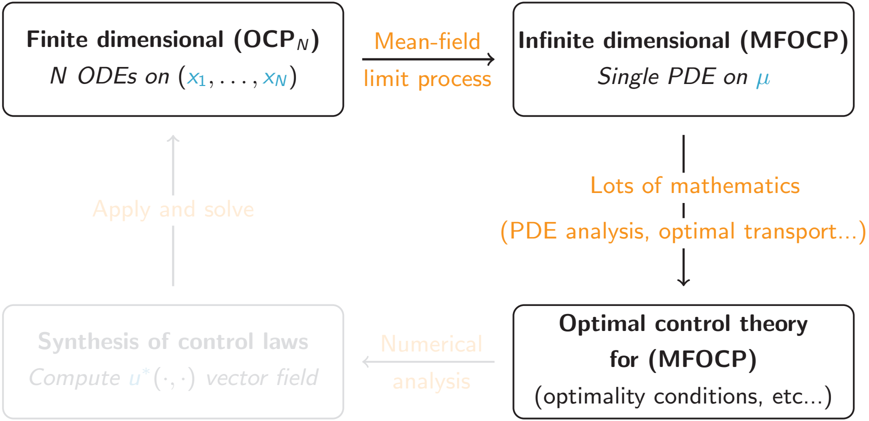
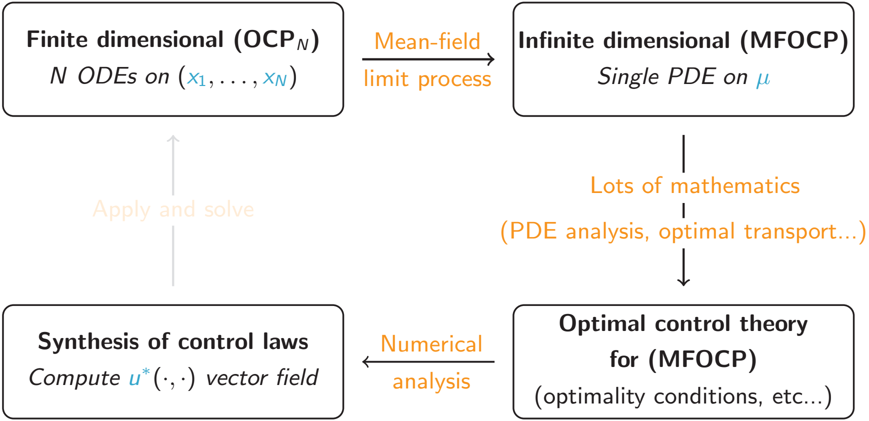


Multi-agent control – *Summing up the whole scheme*



Multi-agent control – *Summing up the whole scheme*

Finite dimensional (OCP$_N$)
N ODEs on $(x_1, \dots, x_N)$

Mean-field limit process

Infinite dimensional (MFOCP)
Single PDE on μ

Lots of mathematics
(PDE analysis, optimal transport...)

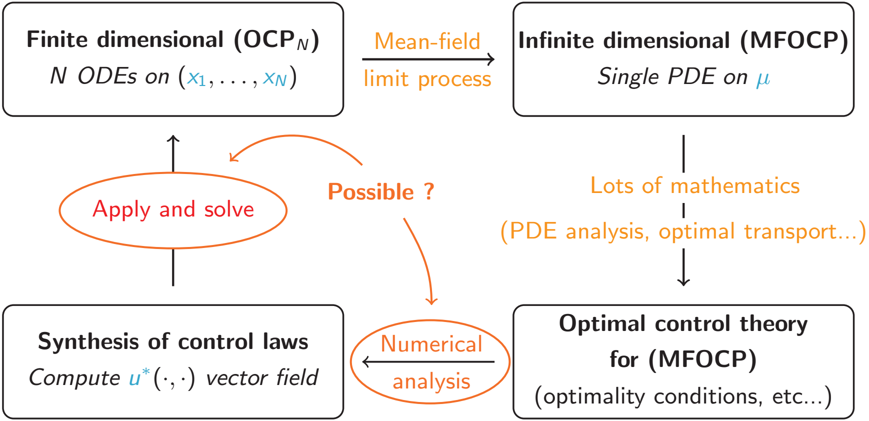
Optimal control theory for (MFOCP)
(optimality conditions, etc...)

Numerical analysis

Synthesis of control laws
Compute $u^*(\cdot,\cdot)$ vector field

Apply and solve

Multi-agent control – *Summing up the whole scheme*



Multi-agent systems – *Micro-macro correspondence*

Mean-field controls (back to discrete problems)

(MFOCP) on measures $\mu(\cdot)$ $\Longrightarrow$ sol. $u^*(\cdot,\cdot)$

Plug in $\mu_N(\cdot) = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i(\cdot)}$

discrete classical trajectories $(x_1(\cdot), ..., x_N(\cdot))$

Stirring $(x_1(\cdot), \dots, x_N(\cdot))$ with the mean-field control $u^*(\cdot,\cdot)$ yields

$$\dot{x}_i(t) = \boldsymbol{v}_N(t, \boldsymbol{x}_N(t), x_i(t)) + u^*(t, x_i(t))$$

May be ill-defined !

Well-posedness of continuity equations (general idea)[*DpL'89* & *A'04*]

Arbitrary $\mu^0 \in \mathcal{P}(\mathbb{R}^d)$ $\leftrightsquigarrow$ **Cauchy-Lipschitz regularity** on $u^*(t,\cdot)$

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Multi-agent systems – *Lipschitz regularity of MFOC*

Theorem 1 (Sufficient Lip conditions for (MFOCP))[B. & Rossi'21]

Suppose that the functionals in **(MFOCP)** are $C^{2,1}_{\text{loc}}$ in (μ, x) and that $\psi : U \to \mathbb{R}$ is **strongly convex** with $\lambda_\psi > 0$ large enough.

Then, there exist a constant $\mathcal{L}_U > 0$ and an optimal control $u^*(\cdot,\cdot)$ for **(MFOCP)** such that $\text{Lip}(u^*(t,\cdot);\mathbb{R}^d) \leq \mathcal{L}_U$ for a.e. $t \in [0,T]$.

Scheme of proof

1. Discretize**(MFOCP)** with $\mu_N^0 = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^0} \rightharpoonup \mu^0 \rightsquigarrow$ **(OCP$_N$)**
2. Build a sequence of **uniformly Lipschitz** $u_N^*(\cdot,\cdot)$ for **(OCP$_N$)**
3. Show that $u^*(\cdot,\cdot) = \lim_{k\to+\infty} u_{N_k}^*(\cdot,\cdot)$ is optimal for **(MFOCP)**

$\hookrightarrow$ Let us look into the case $N = 1$, i.e. classical OCPs !

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Outline of the talk

Classical OCPs – *Formulation of the problem*

Let $(u^*(\cdot), x^*(\cdot))$ be an optimal pair for

$$\textbf{(OCP)} \quad \left\{ \begin{aligned} & \min_{u(\cdot)\in\mathcal{U}} \left[\int_0^T \Big(L(t, x(t)) + \psi(u(t)) \Big) \mathrm{d}t + \varphi(x(T)) \right] \\ & \text{s.t.} \quad \begin{cases} \dot{x}(t) = f(t, x(t)) + u(t), \\ x(0) = x^0. \end{cases} \end{aligned} \right.$$

Main question (Lipschitz feedbacks)

When does there exist a map $\bar{u} : [0, T] \times \mathbb{R}^d \to U$ such that

(i) $u^*(t) = \bar{u}(t, x^*(t))$ for all times $t \in [0, T]$,

(ii) the map $\bar{u}(\cdot, \cdot)$ is locally optimal for (OCP),

(iii) the application $x \in \mathbb{R}^d \mapsto \bar{u}(t, x) \in U$ is Lipschitz

Hamilton-Jacobi approach & $C^{1,1}$ **value function** ?

$\hookrightarrow$ **Pontryagin** approach following [Dontchev et.al 19] !

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Classical OCPs – *Necessary optimality conditions*

Idea: Apply 1$^{\text{st}}$-order necessary optimality conditions $\rightsquigarrow$ **PMP**

Theorem (Pontryagin's maximum principle)

Define the **Hamiltonian function** associated to **(OCP)** by

$$H(t, x, p, u) = \langle p, f(t, x) + u \rangle - L(t, x) - \psi(u).$$

Then, there exists a **costate curve** $p^*(\cdot) \in \mathrm{Lip}([0, T], \mathbb{R}^d)$ such that the **Pontryagin triple** $(x^*(\cdot), p^*(\cdot), u^*(\cdot))$ satisfies

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$$u^*(t) \in \underset{\omega \in U}{\mathrm{argmax}}\, H(t, x^*(t), p^*(t), \omega),$$

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$$\begin{cases} \dot{x}^*(t) = \nabla_p H(t, x^*(t), p^*(t), u^*(t)), & x^*(0) = x^0, \\ \dot{p}^*(t) = -\nabla_x H(t, x^*(t), p^*(t), u^*(t)), & p^*(T) = -\nabla\varphi(x^*(T)), \end{cases}$$

and

$$u^*(t) \in \underset{\omega \in U}{\mathrm{argmax}}\, H(t, x^*(t), p^*(t), \omega),$$

for $\mathcal{L}^1$-almost every $t \in [0, T]$.

Locally optimal Lipschitz feedback – *Statement*

Theorem (Pontryagin approach to Lipschitz feedbacks) [DKV'19]

Suppose that the data of **(OCP)** are $C^{2,1}_{\mathrm{loc}}$, that $\psi(\cdot)$ is **strictly convex**, and that there exists $\rho > 0$ such that any **linearised pairs**

$$\dot{y}(t) = \mathrm{D}_x f(t, x^*(t)) y(t) + w(t), \quad u^*(t) + w(t) \in U$$

satisfies the following **uniform coercivity estimate** holds

$$\begin{aligned} &\langle \nabla^2_x \varphi(x^*(T))\, y(T), y(T) \rangle \\ &- \int_0^T \langle \nabla^2_x H(t, x^*(t), p^*(t), u^*(t)) y(t), y(t) \rangle \mathrm{d}t \\ &- \int_0^T \langle \nabla^2_u H(t, x^*(t), p^*(t), u^*(t)) w(t), w(t) \rangle \mathrm{d}t \geq \rho \int_0^T |w(t)|^2 \mathrm{d}t. \end{aligned}$$

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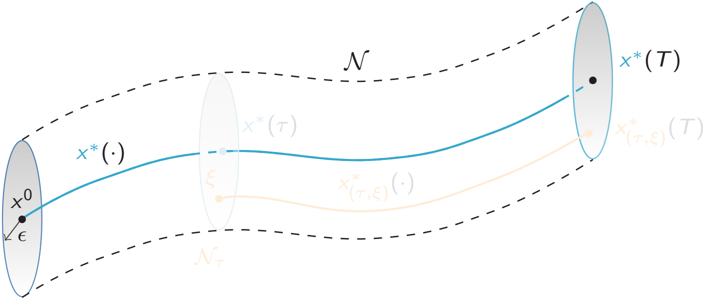

1. Consider the curve $x^*(\cdot)$ and the ϵ-neighbourhood $\mathcal{N}$,
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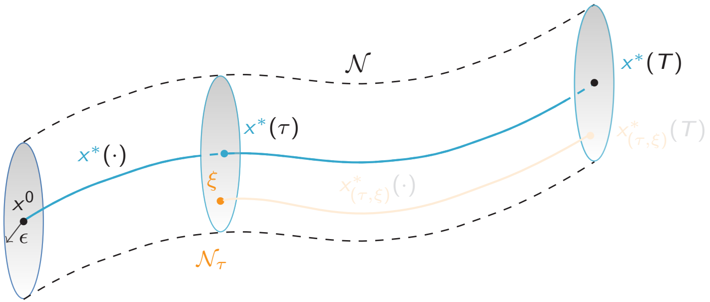

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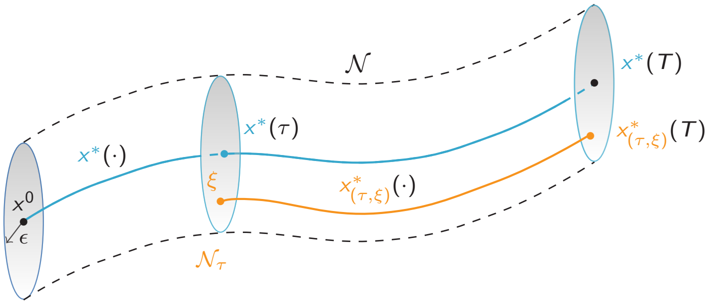

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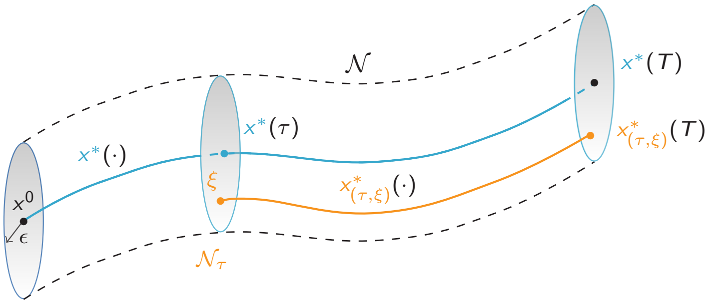

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Locally optimal Lipschitz feedback – *Proof scheme (1)*

Proof (General idea)

Show that the PMP system is **invertible** in a **Lipschitz way** with respect to perturbations of the form $x(\tau) = \xi \in \mathcal{N}_\tau$.

1. Rewrite the maximisation condition as

$$\nabla_u H(t, x^*(t), p^*(t), u^*(t)) \in N_U(u^*(t)).$$

2. Rewrite the PMP as the **dynamical differential inclusion**

$$0 \in F(x^*(\cdot), p^*(\cdot), u^*(\cdot)) + G(x^*(\cdot), p^*(\cdot), u^*(\cdot))$$

where F encodes the **state-costate dynamics** and G the **maximisation condition**.

$\hookrightarrow$ **Quantitative** version of the non-smooth IFT (**Robinson**)

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Locally optimal Lipschitz feedback – *Proof scheme (2)*

3. The **partially linearised** dynamical inclusion

$$0 \in \mathrm{D}F(x^*(\cdot), p^*(\cdot), u^*(\cdot))(y, q, w) + G(y, q, w)$$

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4. **Coercivity estimate:** quantitative **second-order convexity condition** for the LQ problem $\rightsquigarrow$ **Lipschitz inverse!** [HD'93]

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Outline of the talk

Back to multi-agent systems – *Adaptation to* $\mathbf{(CO}_N\mathbf{)}$

Going back to general systems with $N \geq 1$ interacting agents

General multi-agent optimal control problem $\mathbf{(OCP}_N\mathbf{)}$

$$\begin{cases} \min\limits_{\boldsymbol{u}(\cdot)} \left[\dfrac{1}{N} \displaystyle\sum_{i=1}^{N} \int_0^T \Big(\boldsymbol{L}_N(t, \boldsymbol{x}_N(t), x_i(t)) + \psi(u_i(t)) \Big) \mathrm{d}t + \varphi_N(\boldsymbol{x}_N(T)) \right] \\ \text{s.t.} \begin{cases} \dot{x}_i(t) = \boldsymbol{v}_N(t, \boldsymbol{x}_N(t), x_i(t)) + u_i(t), \\ x_i(0) = x_i^0. \end{cases} \end{cases}$$

Idea: See $\mathbf{(OCP}_N\mathbf{)}$ as a classical OCP in $(\mathbb{R}^d)^N$ with trajectory-control pairs $(\boldsymbol{x}(\cdot), \boldsymbol{u}(\cdot)) \in \mathrm{Lip}([0,T], (\mathbb{R}^d)^N) \times L^\infty([0,T], U^N)$.

Question: How do we write a coercivity estimate adapted to the mean-field structure, i.e. **uniform in** N ? $\rightsquigarrow$ **Wasserstein calculus!**

Back to multi-agent systems – *Adaptation to* (CO$_N$)

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Optimal transport toolbox – *The modern foundations*

Optimal transport problem (Kantorovich's formulation)

Given $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$ and a cost function $c : \mathbb{R}^{2d} \to (-\infty, +\infty]$, find a probability measure $\gamma^* \in \mathcal{P}(\mathbb{R}^{2d})$ s.t.

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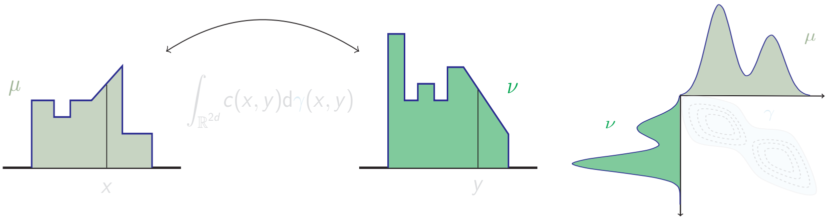

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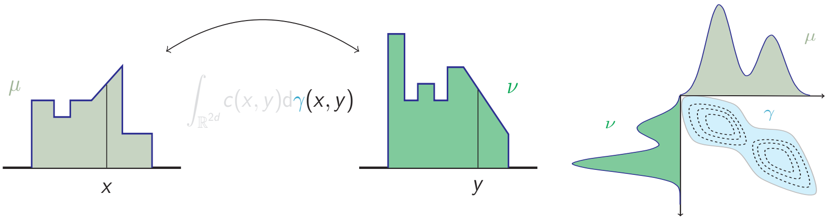

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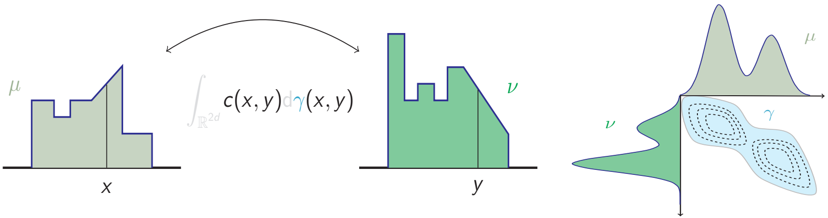

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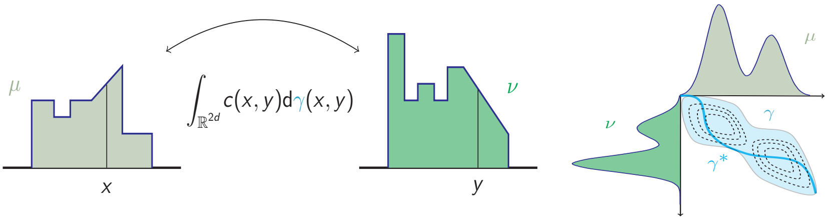

Optimal transport toolbox – *Wasserstein spaces*

Idea: choose $c(x,y) := |x-y|^2 \rightsquigarrow$ **distance** between measures

Definition (Wasserstein distance)

The quantity

$$W_2(\mu,\nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \left(\int_{\mathbb{R}^d} |x-y|^2 \mathrm{d}\gamma(x,y) \right)^{1/2}$$

defines a **distance** over $\mathcal{P}_2(\mathbb{R}^d)$.

Theorem (Structure of Wasserstein spaces)

The metric space $(\mathcal{P}_2(\mathbb{R}^d), W_2)$

(i) gives an **intrinsic meaning** to continuity equations [AGS'08]

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Wasserstein calculus (Heuristic definitions)

If $\varphi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is "nice enough" at μ, there exists

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If $\mu_N := \mu[\boldsymbol{x}] = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$, then

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Idea: Formulate **coercivity estimates** for **(OCP$_N$)** using **Wasserstein derivatives** in the natural Hilbert space $\left((\mathbb{R}^d)^N, \langle \cdot, \cdot \rangle_N \right)$

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Lipschitz regularity in $(\mathbf{OCP}_N)$ – *The coercivity condition*

Define the **mean-field Hamiltonian** associated to $(\mathbf{OCP}_N)$

$$\mathbb{H}_N(t,\boldsymbol{x},\boldsymbol{r},\boldsymbol{u}) = \frac{1}{N}\sum_{i=1}^N \Big(\langle r_i, \boldsymbol{v}_N(t,\boldsymbol{x},x_i)+u_i\rangle - \boldsymbol{L}_N(t,\boldsymbol{x},x_i) - \psi(u_i)\Big)$$

and apply the **PMP** $\rightsquigarrow$ **Pontryagin triple** $(\boldsymbol{x}_N^*(\cdot),\boldsymbol{r}_N^*(\cdot),\boldsymbol{u}_N^*(\cdot))$

Mean-field coercivity estimate (MFCE_N)

(MFCE_N) holds along $(\boldsymbol{x}_N^*(\cdot),\boldsymbol{r}_N^*(\cdot),\boldsymbol{u}_N^*(\cdot))$ iff there exists $\rho > 0$ s.t.

$$\begin{aligned}
&\mathbf{Hess}_{\boldsymbol{x}}\,\varphi_N[\boldsymbol{x}_N^*(T)](\boldsymbol{y}(T),\boldsymbol{y}(T)) \\
&- \int_0^T \mathbf{Hess}_{\boldsymbol{x}}\,\mathbb{H}_N[t,\boldsymbol{x}_N^*(t),\boldsymbol{r}_N^*(t),\boldsymbol{u}_N^*(t)](\boldsymbol{y}(t),\boldsymbol{y}(t))\mathrm{d}t \\
&- \int_0^T \mathbf{Hess}_{\boldsymbol{u}}\,\mathbb{H}_N[t,\boldsymbol{x}_N^*(t),\boldsymbol{r}_N^*(t),\boldsymbol{u}_N^*(t)](\boldsymbol{w}(t),\boldsymbol{w}(t))\mathrm{d}t \geq \rho \int_0^T |\boldsymbol{w}(t)|_N^2\mathrm{d}t,
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for any pair $(\boldsymbol{y}(\cdot),\boldsymbol{w}(\cdot))$ sol. the **mean-field linearised dynamics**.

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Lipschitz regularity in $(\mathbf{OCP}_N)$ – *Convergence of feedbacks*

Theorem 2 (Convergence of Lipschitz feedbacks)[B. & Rossi'21]

Suppose that for all $N \geq 1$, there exists an **optimal Pontryagin triple** $(\boldsymbol{x}_N^*(\cdot), \boldsymbol{r}_N^*(\cdot), \boldsymbol{u}_N^*(\cdot))$ for $(\mathbf{OCP}_N)$ which satisfies (MFCE_N) for a given constant $\rho > 0$ **independent of** $N \geq 1$.

Then, there exists $\mathcal{L}_U > 0$ and $u_N^*(\cdot,\cdot) \in L^\infty([0,T], \mathrm{Lip}(\mathbb{R}^d, U))$ s.t.

(i) $u_N^*(t, x_i^*(t)) = u_i^*(t)$ for $\mathcal{L}^1$-almost every $t \in [0,T]$.

(ii) The maps $x \in \mathbb{R}^d \mapsto u_N^*(t,x) \in U$ are $\mathcal{L}_U$-**Lipschitz**.

(iii) For $p \in (d, +\infty)$, the cluster points of $(u_N^*(\cdot,\cdot))$ in the weak-topology of $L^2([0,T], W^{1,p}_{\mathrm{loc}}(\mathbb{R}^d, U))$ are **optimal** for **(MFOCP)**.

1. By adapting [DKV'19] in $\big((\mathbb{R}^d)^N, \langle \cdot,\cdot \rangle_N\big)$ there exists a **locally optimal feedback** mapping $\bar{\boldsymbol{u}}_N : \mathcal{N} \to U^N$ s.t.

$$|\bar{\boldsymbol{u}}(t,\boldsymbol{y}) - \bar{\boldsymbol{u}}(t,\boldsymbol{x})|_N \leq \mathcal{L}_U |\boldsymbol{y} - \boldsymbol{x}|_N \quad \text{and} \quad \boldsymbol{u}_N^*(t) = \bar{\boldsymbol{u}}(t, \boldsymbol{x}_N^*(t)).$$

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Lipschitz regularity in **(OCP$_N$)** – *Proof of Theorem 2*

2. The **agent-based** feedbacks $\bar{u}_i : [0,T] \times \mathcal{N}_i \to \mathbb{R}^d$ defined by

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4. It holds $u_{N_k}^*(\cdot,\cdot) \rightharpoonup u^*(\cdot,\cdot)$ where $\mathrm{Lip}(u^*(t,\cdot), \mathbb{R}^d) \leq \mathcal{L}_U$ and $u^*(\cdot,\cdot)$ is **optimal** for **(MFOCP)** $\rightsquigarrow$ Γ-conv. [FS'14,FLOS'19] $\square$

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Lipschitz regularity in **(MFOCP)** – *Wrapping up everything*

Question: Precise link with Theorem 1 dealing with **(MFOCP)** ?

Proposition (Sufficient condition for mean-field coercivity)

Suppose that the dynamics and cost functionals in **(MFOCP)** are $C^{2,1}_{\mathrm{loc}}$ in the **measure** and **space variables**.

Then for all $N \geq 1$, the coercivity condition (MFCE_N) holds along **every** optimal Pontryagin triples for **(OCP_N) if and only if** $\lambda_\psi > \lambda_{\mathrm{MF}}$ for some $\lambda_{\mathrm{MF}} \geq 0$. In this case, one has $\rho = \lambda_\psi - \lambda_{\mathrm{MF}} > 0$.

Remarks (Interpretation and link with MFGs)

$\lambda_{\mathrm{MF}} \sim$ global **concavity constant** for **(MFOCP)** over $\mathcal{P}_2(\mathbb{R}^d)$

- If **(MFOCP)** is **sufficiently convex** $\rightsquigarrow$ Lip sol. for any $\lambda_\psi > 0$
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- If **(MFOCP)** is **sufficiently convex** $\rightsquigarrow$ Lip sol. for any $\lambda_\psi > 0$
- Otherwise, one needs $\lambda_\psi > 0$ **large** w.r.t. data of **(MFOCP)**

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Suppose that the dynamics and cost functionals in **(MFOCP)** are $C^{2,1}_{\text{loc}}$ in the **measure** and **space variables**.

Then for all $N \geq 1$, the coercivity condition (MFCE_N) holds along **every** optimal Pontryagin triples for **(OCP$_N$)** **if and only if** $\lambda_\psi > \lambda_{\text{MF}}$ for some $\lambda_{\text{MF}} \geq 0$. In this case, one has $\rho = \lambda_\psi - \lambda_{\text{MF}} > 0$.

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Outline of the talk

Variance maximisation – *Formulation of the problem*

Consider the following **(MFOCP)** and the corresponding **(OCP$_N$)**

Infinite dimensional problem **(MFOCP)**

$$\begin{cases} \min\limits_{u(\cdot,\cdot)} \left[\dfrac{\lambda}{2} \displaystyle\int_0^T \int_{\mathbb{R}} |u(t,x)|^2 \mathrm{d}\mu(t)(x)\mathrm{d}t - \dfrac{1}{2} \int_{\mathbb{R}} |x - \bar{\mu}(T)|^2 \mathrm{d}\mu(T) \right] \\ \text{s.t. } \begin{cases} \partial_t \mu(t) + \mathrm{div}_x(u(t,\cdot)\mu(t)) = 0, \quad \|u^*(t,\cdot)\|_{L^\infty(\mu(t))} \leq 1, \\ \mu(0) = \frac{1}{2} \mathbb{1}_{[-1,1]} \cdot \mathcal{L}^1. \end{cases} \end{cases}$$

Approximating sequence of finite dimensional problems **(OCP$_N$)**

$$\begin{cases} \min\limits_{\boldsymbol{u}(\cdot)} \left[\dfrac{\lambda}{2N} \displaystyle\sum_{i=1}^N \int_0^T |u_i(t)|^2 \mathrm{d}t - \dfrac{1}{2N} \sum_{i=1}^N |x_i(T) - \bar{\boldsymbol{x}}(T)|^2 \right] \\ \text{s.t. } \begin{cases} \dot{x}_i(t) = u_i(t), \quad |u_i(t)| \leq 1 \\ x_i(0) = x_i^0. \end{cases} \end{cases}$$

Variance maximisation – *The role of coercivity*

Idea: there are roughly two ways of increasing the variance

1. **Spreading** the initial indicator function (**smooth** action)
2. **Breaking** the initial indicator function (**non-smooth** action)

$\hookrightarrow$ the **coercivity estimate determines** each scenario!

Proposition (Variance maximisation) [B. & Rossi'21]

The following statements are equivalent

(i) (MFCE_N) holds with constant $\rho > 0$ along every triple.

(ii) It holds $T < \lambda$ and $\rho = \lambda - T > 0$, i.e. $\lambda_{\mathrm{MF}} = T$.

(iii) The finite-dimensional controls $\{u_i^*(\cdot)\}_{i=1}^N$ satisfy

$$|u_i^*(t) - u_j^*(t)| \leq \frac{1}{\rho}|x_i^*(t) - x_j^*(t)|.$$

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We have a *"sharp-ish"* condition for **Lipschitz regularity**... Hence Cauchy-Lipschitz regularity is **very demanding** for **(MFOCP)**

Open problem (more general well-poesdness analysis)

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$\hookrightarrow$ **Regular Lagrangian Flows** [DpL'89,A'04]

Idea: Considering rough vector fields on **absolutely continuous** densities & select solutions that **do not concentrate**.

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Thank you for your attention !