Optimal Control Problems in Wasserstein Spaces
Pontryagin Maximum Principle

Benoît Bonnet

LIS, Aix-Marseille Université, FRANCE

LabEx Archimède

22/01/2018

12th International Young Researcher Workshop on Mechanics, Geometry and Control

Outline of the talk

1. Introduction and motivations
2. Optimal transport theory
3. Optimal control problems & Pontryagin Maximum Principle

Outline of the talk

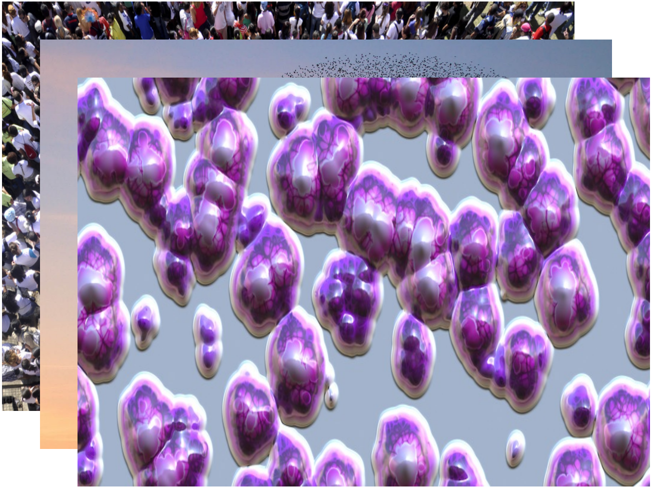
Introduction and motivation - Pedestrian modelling

Motivations coming from crowds, animal swarms, cells & opinion dynamics on large networks...

Introduction and motivation - Pedestrian modelling

Motivations coming from crowds, animal swarms, cells & opinion dynamics on large networks...

Motivations coming from crowds, animal swarms, cells & opinion dynamics on large networks...



Motivations coming from crowds, animal swarms, cells & opinion dynamics on large networks...

Example (1-dimensional voting intentions)

Many voting systems are subject to the **spoiler effect**
$\hookrightarrow$ *Opinions tend to cluster locally around strong poles*

Winning strategy for a centrist candidate ?

Goal : Enforce a centrist majority at minimal cost
$\hookrightarrow$ Bounded actions (daily allowances) and Lipschitz constraints (coherent impact on the opinion)

[0] YouTube channel : CGP Grey

Example (Finite dim. Drift + Convolution)

Given a system of N agents $(x_1, ..., x_N) \in (\mathbb{R}^d)^N$ evolving according to

$$\dot{x}_i(t) = v_d(x_i) + \frac{1}{N}\sum_{j=1}^{N} \phi(x_i - x_j) + u_i(t), \qquad (S_N)$$

Objective : *design a set of control laws $t \mapsto (u_1(t), ..., u_N(t))$ achieving a certain goal, e.g. forming a consensus, etc...*

Issues related to the formulation (S_N)

- No a priori knowledge of the crowd (number of agents, exact positions, etc...)
- Not relevant consider individual controls for large models
- Extremely demanding computationally speaking

Example (Finite dim. Drift + Convolution)

Given a system of N agents $(x_1, ..., x_N) \in (\mathbb{R}^d)^N$ *evolving according to*

$$\dot{x}_i(t) = v_d(x_i) + \frac{1}{N}\sum_{j=1}^{N} \phi(x_i - x_j) + u_i(t), \qquad (S_N)$$

Objective : *design a set of control laws* $t \mapsto (u_1(t), ..., u_N(t))$ *achieving a certain goal, e.g. forming a consensus, etc...*

Issues related to the formulation (S_N)

- No a priori knowledge of the crowd (number of agents, exact positions, etc...)
- Not relevant consider individual controls for large models
- Extremely demanding computationally speaking

Example (Finite dim. Drift + Convolution)

Given a system of N agents $(x_1, ..., x_N) \in (\mathbb{R}^d)^N$ evolving according to

$$\dot{x}_i(t) = v_d(x_i) + \frac{1}{N}\sum_{j=1}^{N} \phi(x_i - x_j) + u_i(t), \qquad (S_N)$$

Objective : *design a set of control laws $t \mapsto (u_1(t), ..., u_N(t))$ achieving a certain goal, e.g. forming a consensus, etc...*

Issues related to the formulation (S_N)

- ◇ No a priori knowledge of the crowd (number of agents, exact positions, etc...)
- ◇ Not relevant consider individual controls for large models
- ◇ Extremely demanding computationally speaking

Example (Finite dim. Drift + Convolution)

Given a system of N agents $(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$ *evolving according to*

$$\dot{x}_i(t) = v_d(x_i) + \frac{1}{N}\sum_{j=1}^{N} \phi(x_i - x_j) + u_i(t), \qquad (S_N)$$

Objective : *design a set of control laws* $t \mapsto (u_1(t), \ldots, u_N(t))$ *achieving a certain goal, e.g. forming a consensus, etc...*

Issues related to the formulation (S_N)

- No a priori knowledge of the crowd (number of agents, exact positions, etc...)
- Not relevant consider individual controls for large models
- Extremely demanding computationally speaking

Example (Finite dim. Drift + Convolution)

Given a system of N agents $(x_1, ..., x_N) \in (\mathbb{R}^d)^N$ *evolving according to*

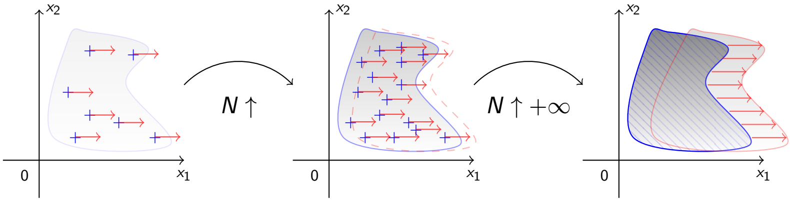
$$\dot{x}_i(t) = v_d(x_i) + \frac{1}{N}\sum_{j=1}^{N} \phi(x_i - x_j) + u_i(t), \qquad (S_N)$$

Objective : *design a set of control laws* $t \mapsto (u_1(t), ..., u_N(t))$ *achieving a certain goal, e.g. forming a consensus, etc...*

Issues related to the formulation (S_N)

- No a priori knowledge of the crowd (number of agents, exact positions, etc...)
- Not relevant consider individual controls for large models
- Extremely demanding computationally speaking

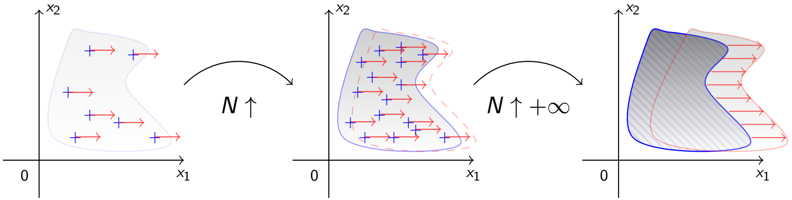
Idea : Approximate the large crowd by a single PDE through a *mean-field process*



Interesting for pedestrian modelling and crowd control

$\hookrightarrow$ Deal with intractable control / optimal control problems on large crowds through the PDE model

Idea : Approximate the large crowd by a single PDE through a *mean-field process*



Interesting for pedestrian modelling and crowd control

$\hookrightarrow$ Deal with intractable control / optimal control problems on large crowds through the PDE model

In the **mean-field limit**, crowds are described by *transport equations with non-local velocities*

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](\cdot)\mu(t)) = 0. \qquad (S_\infty)$$

Example (Drift + Convolution model)

$$v[\mu](x) = v_d(x) + \int_{\mathbb{R}^d} \phi(x-y)d\mu(y)$$

If $\mu(t) \equiv \mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$ *sum of Dirac masses, then*

$$(S_\infty) \iff \dot{x}_i = v_d(x_i) + \frac{1}{N}\sum_{j=1}^N \phi(x_i - x_j),$$

Hint : *There is a natural correspondence between ODEs and Transport PDEs through* $(x_1, ..., x_N) \in (\mathbb{R}^d)^N \mapsto \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$

Introduction and motivations - *Transport PDE*

In the **mean-field limit**, crowds are described by *transport equations with non-local velocities*

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](\cdot)\mu(t)) = 0. \qquad (S_\infty)$$

Example (Drift + Convolution model)

$$v[\mu](x) = v_d(x) + \int_{\mathbb{R}^d} \phi(x-y)d\mu(y)$$

If $\mu(t) \equiv \mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$ *sum of Dirac masses, then*

$$(S_\infty) \iff \dot{x}_i = v_d(x_i) + \frac{1}{N}\sum_{j=1}^N \phi(x_i - x_j),$$

Hint : *There is a natural correspondence between ODEs and Transport PDEs through* $(x_1, ..., x_N) \in (\mathbb{R}^d)^N \mapsto \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$

In the **mean-field limit**, crowds are described by *transport equations with non-local velocities*

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](\cdot)\mu(t)) = 0. \qquad (S_\infty)$$

Example (Drift + Convolution model)

$$v[\mu](x) = v_d(x) + \int_{\mathbb{R}^d} \phi(x-y) d\mu(y)$$

If $\mu(t) \equiv \mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$ *sum of Dirac masses, then*

$$(S_\infty) \iff \dot{x}_i = v_d(x_i) + \frac{1}{N}\sum_{j=1}^N \phi(x_i - x_j),$$

Hint : *There is a natural correspondence between ODEs and Transport PDEs through* $(x_1, ..., x_N) \in (\mathbb{R}^d)^N \mapsto \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$

In the **mean-field limit**, crowds are described by *transport equations with non-local velocities*

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](\cdot)\mu(t)) = 0. \qquad (S_\infty)$$

Example (Drift + Convolution model)

$$v[\mu](x) = v_d(x) + \int_{\mathbb{R}^d} \phi(x-y)d\mu(y)$$

If $\mu(t) \equiv \mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$ *sum of Dirac masses, then*

$$(S_\infty) \iff \dot{x}_i = v_d(x_i) + \frac{1}{N}\sum_{j=1}^N \phi(x_i - x_j),$$

Hint : *There is a natural correspondence between ODEs and Transport PDEs through* $(x_1, ..., x_N) \in (\mathbb{R}^d)^N \mapsto \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$

In the **mean-field limit**, crowds are described by *transport equations with non-local velocities*

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](\cdot)\mu(t)) = 0. \qquad (S_\infty)$$

Example (Drift + Convolution model)

$$v[\mu](x) = v_d(x) + \int_{\mathbb{R}^d} \phi(x-y)d\mu(y)$$

If $\mu(t) \equiv \mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$ *sum of Dirac masses, then*

$$(S_\infty) \iff \dot{x}_i = v_d(x_i) + \frac{1}{N}\sum_{j=1}^N \phi(x_i - x_j),$$

Hint : *There is a natural correspondence between ODEs and Transport PDEs through* $(x_1, ..., x_N) \in (\mathbb{R}^d)^N \mapsto \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$

The controlled version of (S_∞) writes

$$\partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](\cdot) + u(t,\cdot))\mu(t)) = 0,$$

where the controls depend on time and space.

Question : Choice of the state space

Dirac masses $\Longrightarrow$ distributional spaces (measures, distributions...)

Problem : Distributional topologies are not very 'explicit'

$\hookrightarrow$ Space of probability measures endowed with an optimal transportation metric !

The controlled version of (S_∞) writes

$$\partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](\cdot) + u(t,\cdot))\mu(t)) = 0,$$

where the controls depend on time and space.

Question : Choice of the state space

Dirac masses $\Longrightarrow$ distributional spaces (measures, distributions...)

Problem : Distributional topologies are not very 'explicit'

$\hookrightarrow$ Space of probability measures endowed with an optimal transportation metric !

The controlled version of (S_∞) writes

$$\partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](\cdot) + u(t,\cdot))\mu(t)) = 0,$$

where the controls depend on time and space.

Question : Choice of the state space

Dirac masses $\Longrightarrow$ distributional spaces (measures, distributions...)

Problem : Distributional topologies are not very 'explicit'

$\hookrightarrow$ Space of probability measures endowed with an optimal transportation metric !

Outline of the talk

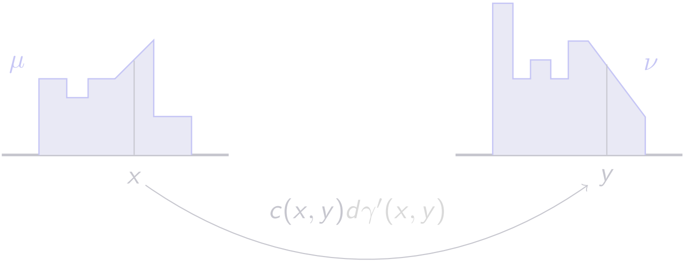
1. Introduction and motivations
2. Optimal transport theory
3. Optimal control problems & Pontryagin Maximum Principle

Kantorovich problem (1942) - (OT_K)

Given two probability measures $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$ and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, **find a measure** $\gamma \in \mathcal{P}_c(\mathbb{R}^{2d})$ such that

$$(i) \quad \gamma \in \Gamma(\mu, \nu) = \Big\{ \gamma \in \mathcal{P}_c(\mathbb{R}^{2d}) \text{ s.t. } \pi^1_{\#}\gamma = \mu \; , \; \pi^2_{\#}\gamma = \nu \Big\},$$

$$(ii) \quad \int_{\mathbb{R}^{2d}} c(x,y) \mathrm{d}\gamma(x,y) = \min_{\gamma' \in \Gamma(\mu,\nu)} \left[\int_{\mathbb{R}^{2d}} c(x,y) \mathrm{d}\gamma'(x,y) \right].$$

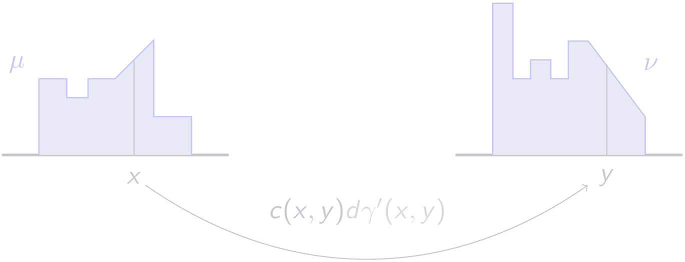


Kantorovich problem (1942) - (OT_K)

Given two probability measures $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$ and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, **find a measure** $\gamma \in \mathcal{P}_c(\mathbb{R}^{2d})$ such that

$$(i) \quad \gamma \in \Gamma(\mu,\nu) = \Big\{ \gamma \in \mathcal{P}_c(\mathbb{R}^{2d}) \text{ s.t. } \pi^1_{\#}\gamma = \mu \ , \ \pi^2_{\#}\gamma = \nu \Big\},$$

$$(ii) \quad \int_{\mathbb{R}^{2d}} c(x,y) \mathrm{d}\gamma(x,y) = \min_{\gamma' \in \Gamma(\mu,\nu)} \left[\int_{\mathbb{R}^{2d}} c(x,y) \mathrm{d}\gamma'(x,y) \right].$$



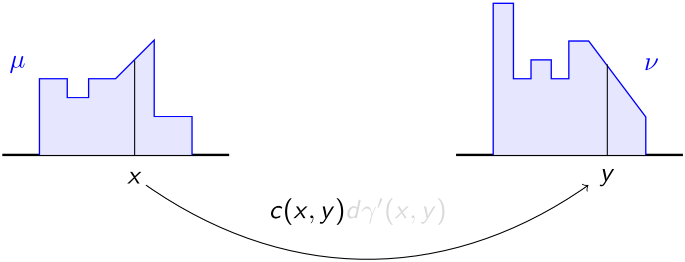
Optimal transport theory - *Kantorovich problem*

Kantorovich problem (1942) - (OT_K)

Given two probability measures $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$ and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, **find a measure** $\gamma \in \mathcal{P}_c(\mathbb{R}^{2d})$ such that

$$(i) \quad \gamma \in \Gamma(\mu,\nu) = \left\{ \gamma \in \mathcal{P}_c(\mathbb{R}^{2d}) \text{ s.t. } \pi^1_{\#}\gamma = \mu \ , \ \pi^2_{\#}\gamma = \nu \right\},$$

$$(ii) \quad \int_{\mathbb{R}^{2d}} c(x,y)\mathrm{d}\gamma(x,y) = \min_{\gamma' \in \Gamma(\mu,\nu)} \left[\int_{\mathbb{R}^{2d}} c(x,y)\mathrm{d}\gamma'(x,y) \right].$$

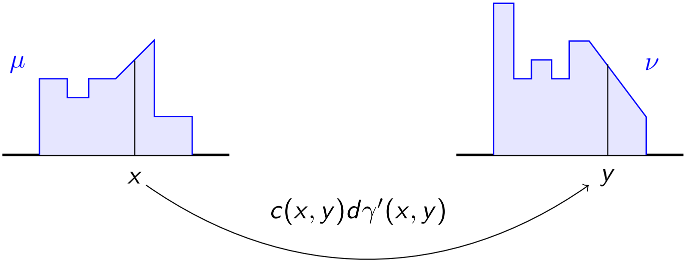


Kantorovich problem (1942) - (OT_K)

Given two probability measures $\mu, \nu \in \mathcal{P}_c(\mathbb{R}^d)$ and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, **find a measure** $\gamma \in \mathcal{P}_c(\mathbb{R}^{2d})$ such that

$$(i) \quad \gamma \in \Gamma(\mu,\nu) = \left\{ \gamma \in \mathcal{P}_c(\mathbb{R}^{2d}) \text{ s.t. } \pi^1_{\#}\gamma = \mu \ , \ \pi^2_{\#}\gamma = \nu \right\},$$

$$(ii) \quad \int_{\mathbb{R}^{2d}} c(x,y)\mathrm{d}\gamma(x,y) = \min_{\gamma' \in \Gamma(\mu,\nu)} \left[\int_{\mathbb{R}^{2d}} c(x,y)\mathrm{d}\gamma'(x,y) \right].$$



Definition (Wasserstein distance and spaces)

$\hookrightarrow$ *When* $c(x,y) = |x-y|^p$, (OT_K) *defines a distance over* $\mathcal{P}_c(\mathbb{R}^d)$ *called the Wasserstein distance of order* p*.*

$$W_p(\mu,\nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2d}} |x-y|^p d\gamma(x,y) \right)^{1/p} \right\}$$

$\hookrightarrow$ *It metrizes the weak-* topology of measures.*

$\hookrightarrow$ *The space* $(\mathcal{P}_c(\mathbb{R}^d), W_p)$ *is a complete and separable metric space called Wasserstein space of order* p*.*

Additional structure when $p = 2$ (Ambrosio,Gigli,Gangbo,...)

$\hookrightarrow$ The space $(\mathcal{P}_c(\mathbb{R}^d), W_2)$ has the structure of a weak Riemannian manifold : geodesic space, tangent bundle, 1^{st} & 2^{nd} order differentiation theory... $\rightarrow$ **Perfect for control theory!**

Definition (Wasserstein distance and spaces)

$\hookrightarrow$ *When* $c(x,y) = |x-y|^p$, (OT_K) *defines a distance over* $\mathcal{P}_c(\mathbb{R}^d)$ *called the Wasserstein distance of order* p.

$$W_p(\mu,\nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2d}} |x-y|^p d\gamma(x,y) \right)^{1/p} \right\}$$

$\hookrightarrow$ *It metrizes the weak-* topology of measures.*

$\hookrightarrow$ *The space* $(\mathcal{P}_c(\mathbb{R}^d), W_p)$ *is a complete and separable metric space called Wasserstein space of order* p.

Additional structure when $p = 2$ (Ambrosio,Gigli,Gangbo,...)

$\hookrightarrow$ The space $(\mathcal{P}_c(\mathbb{R}^d), W_2)$ has the structure of a weak Riemannian manifold : geodesic space, tangent bundle, 1st & 2nd order differentiation theory... $\rightarrow$ **Perfect for control theory!**

Optimal transport theory - *Wasserstein spaces*

Definition (Wasserstein distance and spaces)

$\hookrightarrow$ *When* $c(x,y) = |x-y|^p$, (OT_K) *defines a distance over* $\mathcal{P}_c(\mathbb{R}^d)$ *called the Wasserstein distance of order* p.

$$W_p(\mu,\nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2d}} |x-y|^p d\gamma(x,y) \right)^{1/p} \right\}$$

$\hookrightarrow$ *It metrizes the weak-* topology of measures.*

$\hookrightarrow$ *The space* $(\mathcal{P}_c(\mathbb{R}^d), W_p)$ *is a complete and separable metric space called Wasserstein space of order* p.

Additional structure when $p = 2$ (Ambrosio, Gigli, Gangbo,...)

$\hookrightarrow$ The space $(\mathcal{P}_c(\mathbb{R}^d), W_2)$ has the structure of a weak Riemannian manifold : geodesic space, tangent bundle, 1st & 2nd order differentiation theory... $\rightarrow$ **Perfect for control theory!**

Optimal transport theory - *Wasserstein spaces*

Definition (Wasserstein distance and spaces)

$\hookrightarrow$ *When* $c(x,y) = |x-y|^p$, (OT_K) *defines a distance over* $\mathcal{P}_c(\mathbb{R}^d)$ *called the Wasserstein distance of order* p.

$$W_p(\mu,\nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2d}} |x-y|^p d\gamma(x,y) \right)^{1/p} \right\}$$

$\hookrightarrow$ *It metrizes the weak-* topology of measures.*

$\hookrightarrow$ *The space* $(\mathcal{P}_c(\mathbb{R}^d), W_p)$ *is a complete and separable metric space called Wasserstein space of order* p.

Additional structure when $p=2$ (Ambrosio,Gigli,Gangbo,...)

$\hookrightarrow$ The space $(\mathcal{P}_c(\mathbb{R}^d), W_2)$ has the structure of a weak Riemannian manifold : geodesic space, tangent bundle, 1st & 2nd order differentiation theory... $\rightarrow$ **Perfect for control theory!**

Definition (Wasserstein distance and spaces)

$\hookrightarrow$ *When* $c(x,y) = |x-y|^p$, (OT_K) *defines a distance over* $\mathcal{P}_c(\mathbb{R}^d)$ *called the Wasserstein distance of order* p.

$$W_p(\mu,\nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2d}} |x-y|^p d\gamma(x,y) \right)^{1/p} \right\}$$

$\hookrightarrow$ *It metrizes the weak-* topology of measures.*

$\hookrightarrow$ *The space* $(\mathcal{P}_c(\mathbb{R}^d), W_p)$ *is a complete and separable metric space called Wasserstein space of order* p.

Additional structure when $p = 2$ (Ambrosio,Gigli,Gangbo,...)

$\hookrightarrow$ The space $(\mathcal{P}_c(\mathbb{R}^d), W_2)$ has the structure of a weak Riemannian manifold : geodesic space, tangent bundle, 1st & 2nd order differentiation theory... $\rightarrow$ **Perfect for control theory!**

Optimal transport theory - *Wasserstein spaces*

Definition (Wasserstein distance and spaces)

$\hookrightarrow$ *When* $c(x,y) = |x-y|^p$, (OT_K) *defines a distance over* $\mathcal{P}_c(\mathbb{R}^d)$ *called the Wasserstein distance of order* p.

$$W_p(\mu,\nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \left\{ \left(\int_{\mathbb{R}^{2d}} |x-y|^p d\gamma(x,y) \right)^{1/p} \right\}$$

$\hookrightarrow$ *It metrizes the weak-* topology of measures.*

$\hookrightarrow$ *The space* $(\mathcal{P}_c(\mathbb{R}^d), W_p)$ *is a complete and separable metric space called Wasserstein space of order* p.

Additional structure when $p = 2$ (Ambrosio,Gigli,Gangbo,...)

$\hookrightarrow$ The space $(\mathcal{P}_c(\mathbb{R}^d), W_2)$ has the structure of a weak Riemannian manifold : geodesic space, tangent bundle, 1^{st} & 2^{nd} order differentiation theory... $\rightarrow$ **Perfect for control theory!**

Theorem (Ambrosio,Gangbo '07 - Piccoli,Rossi '13)

Under Cauchy-Lipschitz type assumptions, the Cauchy problem

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](t,\cdot)\mu(t)) = 0 \ , \ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$$

has a **unique Lipschitz solution** *and is* **continuous w.r.t. its initial datum**.
The solution $t \mapsto \mu(t)$ *is given explicitly by*

$$\mu(t) = \Phi^v_{(0,t)}(\cdot)_{\#}\mu^0$$

where $(\Phi^v_{(0,t)}(\cdot))_t$ *is the* **flow generated by** $v[\cdot]$, *i.e.*

$$\partial_t \Phi^v_{(0,t)}(x) = v[\mu(t)]\left(t, \Phi^v_{(0,t)}(x)\right) \ , \ \Phi^v_{(0,0)}(x) = x, \ \forall x \in \mathbb{R}^d.$$

$\hookrightarrow$ **Transport PDEs are Wasserstein ODEs**

Theorem (Ambrosio,Gangbo '07 - Piccoli,Rossi '13)

Under Cauchy-Lipschitz type assumptions, the Cauchy problem

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](t,\cdot)\mu(t)) = 0 \ , \ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$$

has a **unique Lipschitz solution** *and is* **continuous w.r.t. its initial datum**.

The solution $t \mapsto \mu(t)$ is given explicitly by

$$\mu(t) = \Phi^v_{(0,t)}(\cdot)_{\#}\mu^0$$

where $(\Phi^v_{(0,t)}(\cdot))_t$ is the **flow generated by** *$v[\cdot]$, i.e.*

$$\partial_t \Phi^v_{(0,t)}(x) = v[\mu(t)]\left(t, \Phi^v_{(0,t)}(x)\right) \ , \ \Phi^v_{(0,0)}(x) = x, \ \forall x \in \mathbb{R}^d.$$

$\hookrightarrow$ **Transport PDEs are Wasserstein ODEs**

Theorem (Ambrosio,Gangbo '07 - Piccoli,Rossi '13)

Under Cauchy-Lipschitz type assumptions, the Cauchy problem

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](t,\cdot)\mu(t)) = 0 \ , \ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$$

has a **unique Lipschitz solution** *and is* **continuous w.r.t. its initial datum**.

The solution $t \mapsto \mu(t)$ is given explicitly by

$$\mu(t) = \Phi^v_{(0,t)}(\cdot)_{\#}\mu^0$$

where $(\Phi^v_{(0,t)}(\cdot))_t$ is the **flow generated by** $v[\cdot]$, *i.e.*

$$\partial_t \Phi^v_{(0,t)}(x) = v[\mu(t)]\left(t, \Phi^v_{(0,t)}(x)\right) \ , \ \Phi^v_{(0,0)}(x) = x, \ \forall x \in \mathbb{R}^d.$$

$\hookrightarrow$ **Transport PDEs are Wasserstein ODEs**

Theorem (Ambrosio,Gangbo '07 - Piccoli,Rossi '13)

Under Cauchy-Lipschitz type assumptions, the Cauchy problem

$$\partial_t \mu(t) + \nabla \cdot (v[\mu(t)](t,\cdot)\mu(t)) = 0 \;,\; \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d)$$

has a **unique Lipschitz solution** *and is* **continuous w.r.t. its initial datum**.
The solution $t \mapsto \mu(t)$ is given explicitly by

$$\mu(t) = \Phi^v_{(0,t)}(\cdot)_{\#}\mu^0$$

where $(\Phi^v_{(0,t)}(\cdot))_t$ is the **flow generated by** $v[\cdot]$, *i.e.*

$$\partial_t \Phi^v_{(0,t)}(x) = v[\mu(t)]\left(t, \Phi^v_{(0,t)}(x)\right) \;,\; \Phi^v_{(0,0)}(x) = x,\; \forall x \in \mathbb{R}^d.$$

$\hookrightarrow$ **Transport PDEs are Wasserstein ODEs**

Outline of the talk

General optimal control problem in the space $(\mathcal{P}_c(\mathbb{R}^d), W_1)$

$$(\mathcal{P}_{\text{OC}}) \begin{cases} \min\limits_{u \in \mathcal{U}} \left[\int_0^T L(\mu(t), u(t)) \mathrm{d}t + \varphi(\mu(T)) \right] \\ \text{s.t.} \begin{cases} \partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](t,\cdot) + u(t,\cdot))\mu(t)) = 0, \\ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d), \end{cases} \end{cases}$$

Classical questions: Existence of minimizers, optimality conditions, efficient algorithms, etc...

Wasserstein OCP - General problem

General optimal control problem in the space $(\mathcal{P}_c(\mathbb{R}^d), W_1)$

$$(\mathcal{P}_{\mathrm{OC}}) \begin{cases} \min\limits_{u \in \mathcal{U}} \left[\int_0^T L(\mu(t), u(t)) \mathrm{d}t + \varphi(\mu(T)) \right] \\ \text{s.t. } \begin{cases} \partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](t, \cdot) + u(t, \cdot)) \mu(t)) = 0, \\ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d), \end{cases} \end{cases}$$

Classical questions: Existence of minimizers, optimality conditions, efficient algorithms, etc...

Wasserstein OCP - *State of the art*

- **Existence results :** Several results involving mean field and Γ-convergence approaches (*Bongini, Fornasier, Piccoli, Rossi, Solombrino*) or classical PDE methods (*Achdou, Laurière*).
- **Numerical methods :** Good methods in the transport + diffusion case (*Albi, Pareschi, Toscani, Zanella,...*). More difficult for pure transport.
- **Optimality conditions :** Hamilton-Jacobi Optimality conditions (*Cavagnari, Marigonda*)

$\hookrightarrow$ Recent results on Pontryagin-type conditions (one paper, *Bongini,Fornasier,Rossi,Solombrino*).

- **Existence results :** Several results involving mean field and Γ-convergence approaches (*Bongini, Fornasier, Piccoli, Rossi, Solombrino*) or classical PDE methods (*Achdou, Laurière*).
- **Numerical methods :** Good methods in the transport + diffusion case (*Albi, Pareschi, Toscani, Zanella,...*). More difficult for pure transport.
- **Optimality conditions :** Hamilton-Jacobi Optimality conditions (*Cavagnari, Marigonda*)

$\hookrightarrow$ Recent results on Pontryagin-type conditions (one paper, *Bongini,Fornasier,Rossi,Solombrino*).

Wasserstein OCP - *State of the art*

- ◇ **Existence results :** Several results involving mean field and Γ-convergence approaches (*Bongini, Fornasier, Piccoli, Rossi, Solombrino*) or classical PDE methods (*Achdou, Laurière*).
- ◇ **Numerical methods :** Good methods in the transport + diffusion case (*Albi, Pareschi, Toscani, Zanella,...*). More difficult for pure transport.
- ◇ **Optimality conditions :** Hamilton-Jacobi Optimality conditions (*Cavagnari, Marigonda*)

↪ Recent results on Pontryagin-type conditions (one paper, *Bongini,Fornasier,Rossi,Solombrino*).

- ◇ **Existence results :** Several results involving mean field and Γ-convergence approaches (*Bongini, Fornasier, Piccoli, Rossi, Solombrino*) or classical PDE methods (*Achdou, Laurière*).
- ◇ **Numerical methods :** Good methods in the transport + diffusion case (*Albi, Pareschi, Toscani, Zanella,...*). More difficult for pure transport.
- ◇ **Optimality conditions :** Hamilton-Jacobi Optimality conditions (*Cavagnari, Marigonda*)

↪ Recent results on Pontryagin-type conditions (one paper, *Bongini,Fornasier,Rossi,Solombrino*).

PMP - *Problem formulation*

Goal : PMP for the general problem

$$(\mathcal{P}_{OC}) \begin{cases} \min\limits_{u \in \mathcal{U}} \left[\int_0^T L(\mu(t), u(t)) \mathrm{d}t + \varphi(\mu(T)) \right] \\ \text{s.t.} \begin{cases} \partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](t,\cdot) + u(t,\cdot))\mu(t)) = 0, \\ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d), \end{cases} \end{cases}$$

where the control set $\mathcal{U}$ is defined by

$$\mathcal{U} = \left\{ u \in L^1([0,T], C^1(\mathbb{R}^d, \mathbb{R}^d)) \text{ s.t. } \|u(t)\|_{C^1(\mathbb{R}^d,\mathbb{R}^d)} \leq M \right\}$$

Question : Choice of the control set $\mathcal{U}$

$\hookrightarrow$ Cauchy-Lipschitz requires $u(t) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ + sublinear.

$\hookrightarrow$ PMP requires $C^1(\mathbb{R}^d, \mathbb{R}^d)$ + uniformly Lipschitz

Goal : PMP for the general problem

$$(\mathcal{P}_{\text{OC}}) \begin{cases} \min\limits_{u \in \mathcal{U}} \left[\int_0^T L(\mu(t), u(t)) \mathrm{d}t + \varphi(\mu(T)) \right] \\ \text{s.t.} \begin{cases} \partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](t, \cdot) + u(t, \cdot))\mu(t)) = 0, \\ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d), \end{cases} \end{cases}$$

where the control set $\mathcal{U}$ is defined by

$$\mathcal{U} = \left\{ u \in L^1([0,T], C^1(\mathbb{R}^d, \mathbb{R}^d)) \text{ s.t. } \|u(t)\|_{C^1(\mathbb{R}^d,\mathbb{R}^d)} \leq M \right\}$$

Question : Choice of the control set $\mathcal{U}$

$\hookrightarrow$ Cauchy-Lipschitz requires $u(t) \in \mathrm{Lip}_{\text{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ + sublinear.

$\hookrightarrow$ PMP requires $C^1(\mathbb{R}^d, \mathbb{R}^d)$ + uniformly Lipschitz

Goal : PMP for the general problem

$$(\mathcal{P}_{\rm OC}) \begin{cases} \min\limits_{u \in \mathcal{U}} \left[\int_0^T L(\mu(t), u(t)) \mathrm{d}t + \varphi(\mu(T)) \right] \\ \text{s.t.} \begin{cases} \partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](t, \cdot) + u(t, \cdot))\mu(t)) = 0, \\ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d), \end{cases} \end{cases}$$

where the control set $\mathcal{U}$ is defined by

$$\mathcal{U} = \left\{ u \in L^1([0, T], C^1(\mathbb{R}^d, \mathbb{R}^d)) \text{ s.t. } \|u(t)\|_{C^1(\mathbb{R}^d, \mathbb{R}^d)} \leq M \right\}$$

Question : Choice of the control set $\mathcal{U}$

$\hookrightarrow$ Cauchy-Lipschitz requires $u(t) \in \mathrm{Lip}_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ + sublinear.

$\hookrightarrow$ PMP requires $C^1(\mathbb{R}^d, \mathbb{R}^d)$ + uniformly Lipschitz

PMP - *Problem formulation*

Goal : PMP for the general problem

$$
(\mathcal{P}_{\mathrm{OC}}) \quad \begin{cases} \displaystyle\min_{u \in \mathcal{U}} \left[\int_0^T L(\mu(t), u(t)) \mathrm{d}t + \varphi(\mu(T)) \right] \\ \text{s.t.} \quad \begin{cases} \partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](t, \cdot) + u(t, \cdot))\mu(t)) = 0, \\ \mu(0) = \mu^0 \in \mathcal{P}_c(\mathbb{R}^d), \end{cases} \end{cases}
$$

where the control set $\mathcal{U}$ is defined by

$$
\mathcal{U} = \left\{ u \in L^1([0, T], C^1(\mathbb{R}^d, \mathbb{R}^d)) \text{ s.t. } \|u(t)\|_{C^1(\mathbb{R}^d, \mathbb{R}^d)} \leq M \right\}
$$

Question : Choice of the control set $\mathcal{U}$

$\hookrightarrow$ Cauchy-Lipschitz requires $u(t) \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^d, \mathbb{R}^d)$ + sublinear.
$\hookrightarrow$ PMP requires $C^1(\mathbb{R}^d, \mathbb{R}^d)$ + uniformly Lipschitz

PMP - *Unconstrained finite dimensional case*

Theorem (Unconstrained and smooth PMP on $\mathbb{R}^d$)

Let $(u^*(\cdot), x^*(\cdot))$ *be an optimal pair control-trajectory for the problem*

$$\begin{cases} \min_{u \in \mathcal{U}} \left[\int_0^T L(x(t), u(t)) dt + \varphi(x(T)) \right] \\ \dot{x}(t) = f(t, x(t), u(t)) \ , \ x(0) = x^0. \end{cases}$$

Then, there exists a curve $p^*(\cdot)$ *called costate such that*

$$\begin{cases} \dot{p}^*(t) = -\nabla_x \mathcal{H}(t, x^*(t), p^*(t)) \ , \ p^*(T) = -\nabla \varphi(x^*(T)), \\ \dot{x}^*(t) = \nabla_p \mathcal{H}(t, x^*(t), p^*(t)) \quad , \ x^*(0) = x^0, \end{cases}$$

where

$$\mathcal{H}(t, x^*(t), p^*(t)) = \max_{\omega \in \mathcal{U}} \left[\langle p^*(t), f(t, x^*(t), \omega) \rangle - L(t, x^*(t), \omega) \right].$$

Theorem (Unconstrained and smooth PMP on $\mathbb{R}^d$)

Let $(u^*(\cdot), x^*(\cdot))$ be an optimal pair control-trajectory for the problem

$$\begin{cases} \min\limits_{u \in \mathcal{U}} \left[\int_0^T L(x(t), u(t)) dt + \varphi(x(T)) \right] \\ \dot{x}(t) = f(t, x(t), u(t)) \ , \ x(0) = x^0. \end{cases}$$

Then, there exists a curve $p^*(\cdot)$ called costate such that

$$\begin{cases} \dot{p}^*(t) = -\nabla_x \mathcal{H}(t, x^*(t), p^*(t)) \ , \ p^*(T) = -\nabla \varphi(x^*(T)), \\ \dot{x}^*(t) = \nabla_p \mathcal{H}(t, x^*(t), p^*(t)) \quad , \ x^*(0) = x^0, \end{cases}$$

where

$$\mathcal{H}(t, x^*(t), p^*(t)) = \max_{\omega \in \mathcal{U}} \left[\langle p^*(t), f(t, x^*(t), \omega) \rangle - L(t, x^*(t), \omega) \right].$$

Transcription to the Wasserstein setting

◇ Couple state-costate $(x^*(\cdot), p^*(\cdot)) \in \mathrm{Lip}([0,T], \mathbb{R}^{2d})$

↪ *measures* $\nu^*(\cdot) \in \mathrm{Lip}([0,T], \mathcal{P}_c(\mathbb{R}^{2d}))$ *on the product space.*

◇ Euclidean scalar product $\langle p^*(t), f(t, x^*(t), u^*(t))\rangle$ and maximized Hamiltonian $\mathcal{H}(t, x^*(t), p^*(t))$

↪ *Hilbertian* $L^2(\nu^*(t))$*-scalar product and infinite-dimensional maximized Hamiltonian*

$$\mathbb{H}(t, \nu^*(t)) = \max_{\omega \in \mathcal{U}} \left[\int_{\mathbb{R}^{2d}} \langle r, v[\pi^1_{\#}\nu^*(t)](t,x) + \omega(x)\rangle \mathrm{d}\nu^*(t)(x,r) - L(\pi^1_{\#}\nu^*(t), \omega) \right]$$

Transcription to the Wasserstein setting

- Couple state-costate $(x^*(\cdot), p^*(\cdot)) \in \mathrm{Lip}([0,T], \mathbb{R}^{2d})$

$\hookrightarrow$ *measures* $\nu^*(\cdot) \in \mathrm{Lip}([0,T], \mathcal{P}_c(\mathbb{R}^{2d}))$ *on the product space.*

- Euclidean scalar product $\langle p^*(t), f(t, x^*(t), u^*(t))\rangle$ and maximized Hamiltonian $\mathcal{H}(t, x^*(t), p^*(t))$

$\hookrightarrow$ *Hilbertian* $L^2(\nu^*(t))$*-scalar product and infinite-dimensional maximized Hamiltonian*

$$\mathbb{H}(t, \nu^*(t)) = \max_{\omega \in \mathcal{U}} \left[\int_{\mathbb{R}^{2d}} \langle r, v[\pi^1_{\#}\nu^*(t)](t,x) + \omega(x)\rangle \mathrm{d}\nu^*(t)(x,r) - L(\pi^1_{\#}\nu^*(t), \omega) \right]$$

Transcription to the Wasserstein setting

- Couple state-costate $(x^*(\cdot), p^*(\cdot)) \in \mathrm{Lip}([0,T], \mathbb{R}^{2d})$

$\hookrightarrow$ *measures* $\nu^*(\cdot) \in \mathrm{Lip}([0,T], \mathcal{P}_c(\mathbb{R}^{2d}))$ *on the product space.*

- Euclidean scalar product $\langle p^*(t), f(t, x^*(t), u^*(t))\rangle$ and maximized Hamiltonian $\mathcal{H}(t, x^*(t), p^*(t))$

$\hookrightarrow$ *Hilbertian $L^2(\nu^*(t))$-scalar product and infinite-dimensional maximized Hamiltonian*

$$\mathbb{H}(t, \nu^*(t)) = \max_{\omega \in \mathcal{U}} \left[\int_{\mathbb{R}^{2d}} \langle r, v[\pi^1_{\#}\nu^*(t)](t,x) + \omega(x)\rangle \mathrm{d}\nu^*(t)(x,r) - L(\pi^1_{\#}\nu^*(t), \omega) \right]$$

Transcription to the Wasserstein setting

◇ Couple state-costate $(x^*(\cdot), p^*(\cdot)) \in \mathrm{Lip}([0,T], \mathbb{R}^{2d})$

$\hookrightarrow$ *measures* $\nu^*(\cdot) \in \mathrm{Lip}([0,T], \mathcal{P}_c(\mathbb{R}^{2d}))$ *on the product space.*

◇ Euclidean scalar product $\langle p^*(t), f(t, x^*(t), u^*(t)) \rangle$ and maximized Hamiltonian $\mathcal{H}(t, x^*(t), p^*(t))$

$\hookrightarrow$ *Hilbertian* $L^2(\nu^*(t))$*-scalar product and infinite-dimensional maximized Hamiltonian*

$$\mathbb{H}(t, \nu^*(t)) = \max_{\omega \in \mathcal{U}} \left[\int_{\mathbb{R}^{2d}} \langle r, v[\pi^1_{\#}\nu^*(t)](t,x) + \omega(x) \rangle \mathrm{d}\nu^*(t)(x,r) \right. \\ \left. - L(\pi^1_{\#}\nu^*(t), \omega) \right]$$

PMP - *Generalizations 2*

Question : *How do we compute derivatives in $\mathcal{P}_c(\mathbb{R}^d)$?*

Definition (Subdifferential in $(\mathcal{P}_c(\mathbb{R}^d), W_2)$)

The **Wasserstein subdifferential** $\partial\phi(\mu)$ *of* $\phi : \mathcal{P}_c(\mathbb{R}^d) \mapsto \mathbb{R}$ *at* $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ *is the set of maps* $\xi \in L^2(\mu)$ *satisfying*

$$\phi(\nu) - \phi(\mu) \geq \inf_{\gamma \in \Gamma_o(\mu,\nu)} \int_{\mathbb{R}^{2d}} \langle \xi(x), y - x \rangle d\gamma(x,y) + o(W_2(\mu,\nu))$$

for all $\nu \in \mathcal{P}_c(\mathbb{R}^d)$.

The element of minimal $L^2(\mu)$*-norm in* $\partial\phi(\mu)$ *is the* **Wasserstein gradient** *of the functional*

$$\nabla_\mu \phi(\mu) = \operatorname{argmin} \left\{ \|\xi\|_{L^2(\mu)} \text{ s.t. } \xi \in \partial\phi(\mu) \right\}$$

Question : *How do we compute derivatives in $\mathcal{P}_c(\mathbb{R}^d)$?*

Definition (Subdifferential in $(\mathcal{P}_c(\mathbb{R}^d), W_2)$)

The **Wasserstein subdifferential** $\partial\phi(\mu)$ *of* $\phi : \mathcal{P}_c(\mathbb{R}^d) \mapsto \mathbb{R}$ *at* $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ *is the set of maps* $\xi \in L^2(\mu)$ *satisfying*

$$\phi(\nu) - \phi(\mu) \geq \inf_{\gamma \in \Gamma_o(\mu,\nu)} \int_{\mathbb{R}^{2d}} \langle \xi(x), y - x \rangle d\gamma(x,y) + o(W_2(\mu,\nu))$$

for all $\nu \in \mathcal{P}_c(\mathbb{R}^d)$.

The element of minimal $L^2(\mu)$*-norm in* $\partial\phi(\mu)$ *is the* **Wasserstein gradient** *of the functional*

$$\nabla_\mu \phi(\mu) = \text{argmin} \left\{ \|\xi\|_{L^2(\mu)} \text{ s.t. } \xi \in \partial\phi(\mu) \right\}$$

Question : *How do we compute derivatives in $\mathcal{P}_c(\mathbb{R}^d)$?*

Definition (Subdifferential in $(\mathcal{P}_c(\mathbb{R}^d), W_2)$)

The **Wasserstein subdifferential** $\partial\phi(\mu)$ *of* $\phi : \mathcal{P}_c(\mathbb{R}^d) \mapsto \mathbb{R}$ *at* $\mu \in \mathcal{P}_c(\mathbb{R}^d)$ *is the set of maps* $\xi \in L^2(\mu)$ *satisfying*

$$\phi(\nu) - \phi(\mu) \geq \inf_{\gamma \in \Gamma_o(\mu,\nu)} \int_{\mathbb{R}^{2d}} \langle \xi(x), y - x \rangle d\gamma(x,y) + o(W_2(\mu,\nu))$$

for all $\nu \in \mathcal{P}_c(\mathbb{R}^d)$.

The element of minimal $L^2(\mu)$*-norm in* $\partial\phi(\mu)$ *is the* **Wasserstein gradient** *of the functional*

$$\nabla_\mu \phi(\mu) = \text{argmin} \left\{ \|\xi\|_{L^2(\mu)} \text{ s.t. } \xi \in \partial\phi(\mu) \right\}$$

Theorem (B., Rossi '17)

If $(u^*(\cdot), \mu^*(\cdot))$ *is an optimal pair for* $(\mathcal{P}_{OC})$*, there exists a curve* $\nu^*(\cdot) \in \text{Lip}([0, T], \mathcal{P}_c(\mathbb{R}^{2d}))$*, Lipschitz in the* W_1 *metric, and solution of*

$$\begin{cases} \partial_t \nu^*(t) + \nabla \cdot (\mathbb{J}_{2d} \nabla_\nu \mathbb{H}(t, \nu^*(t))(\cdot,\cdot)\nu^*(t)) = 0, \\ \pi^1_{\#}\nu^*(0) = \mu^0, \\ \pi^2_{\#}\nu^*(T) = (-\nabla_\mu \varphi(\mu^*(T))_{\#}\mu^*(T), \end{cases}$$

where $(x, r) \mapsto \nabla_\nu \mathbb{H}(t, \nu^*(t))(x, r)$ *is the Wasserstein gradient of the maximized Hamiltonian and* $\mathbb{J}_{2d}$ *is the symplectic matrix.*

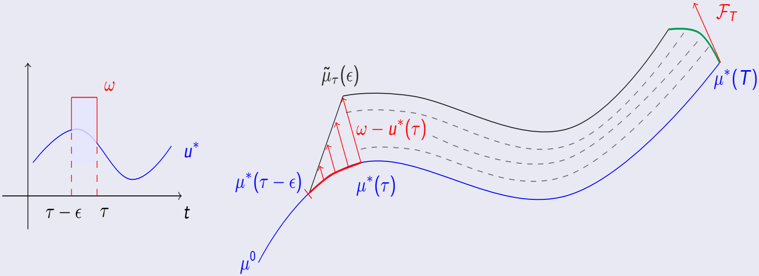
Theorem (B., Rossi '17)

If $(u^*(\cdot), \mu^*(\cdot))$ *is an optimal pair for* $(\mathcal{P}_{OC})$*, there exists a curve* $\nu^*(\cdot) \in \mathrm{Lip}([0, T], \mathcal{P}_c(\mathbb{R}^{2d}))$*, Lipschitz in the* W_1 *metric, and solution of*

$$\begin{cases} \partial_t \nu^*(t) + \nabla \cdot (\mathbb{J}_{2d} \nabla_\nu \mathbb{H}(t, \nu^*(t))(\cdot,\cdot) \nu^*(t)) = 0, \\ \pi^1_{\#} \nu^*(0) = \mu^0, \\ \pi^2_{\#} \nu^*(T) = (-\nabla_\mu \varphi(\mu^*(T))_{\#} \mu^*(T), \end{cases}$$

where $(x, r) \mapsto \nabla_\nu \mathbb{H}(t, \nu^*(t))(x, r)$ *is the Wasserstein gradient of the maximized Hamiltonian and* $\mathbb{J}_{2d}$ *is the symplectic matrix.*

1) Define a *needle-variation* $\tilde{u}_\epsilon^{\omega,\tau}(\cdot)$ of $u^*(\cdot)$



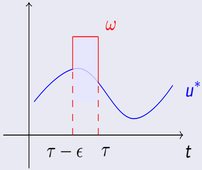
2) First-order expansion at time T using the subdifferential calculus of Wasserstein spaces.

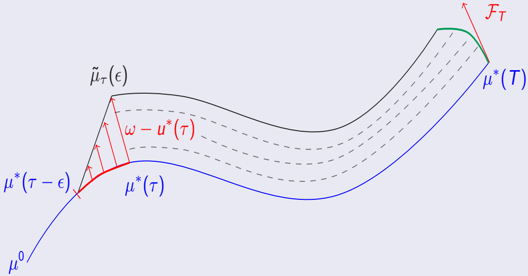
3) Propagate from T to τ using the Hamiltonian dynamics.

4) Show that this implies the maximality condition and conclude.

PMP - *Scheme of proof*

1) Define a *needle-variation* $\tilde{u}_{\epsilon}^{\omega,\tau}(\cdot)$ of $u^*(\cdot)$



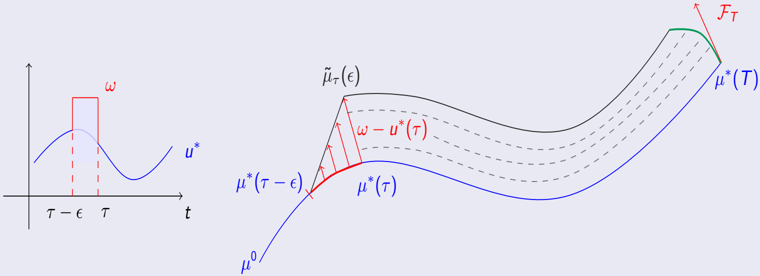


2) First-order expansion at time T using the subdifferential calculus of Wasserstein spaces.

3) Propagate from T to τ using the Hamiltonian dynamics.

4) Show that this implies the maximality condition and conclude.

1) Define a *needle-variation* $\tilde{u}_\epsilon^{\omega,\tau}(\cdot)$ of $u^*(\cdot)$

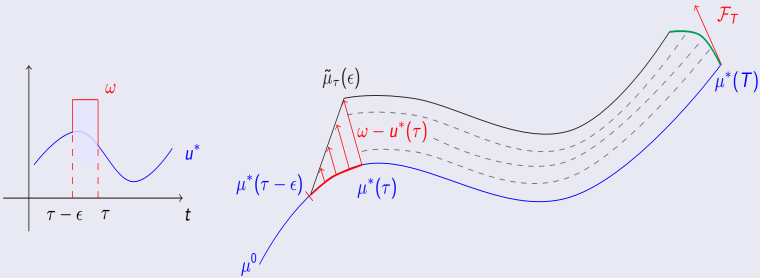


2) First-order expansion at time T using the subdifferential calculus of Wasserstein spaces.

3) Propagate from T to τ using the Hamiltonian dynamics.

4) Show that this implies the maximality condition and conclude.

1) Define a *needle-variation* $\tilde{u}_\epsilon^{\omega,\tau}(\cdot)$ of $u^*(\cdot)$



2) First-order expansion at time T using the subdifferential calculus of Wasserstein spaces.

3) Propagate from T to τ using the Hamiltonian dynamics.

4) Show that this implies the maximality condition and conclude.

Developments with (and without) PMP flavours

- Numerical simulations and indirect methods using the PMP.
- Geometric PMP formulated in terms of the symplectic structure in Wasserstein spaces.
- Applications to optimal covering problems.

Thank you for your attention !