Flocking Formation under Communication Failures for a Class of Cucker-Smale Systems

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(*in collaboration with É. Flayac*)

Seminar of the CAGE INRIA Team
LJLL, Paris-Sorbonne University

May 15, 2020

Outline of the talk

Multi-agent systems and pattern formation

A quick overview of the Cucker-Smale flocking

Cucker-Smale flocking under communication failures

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Multi-agent systems – *Some illustrations*

A multi-agent system is a large ensemble of interacting things

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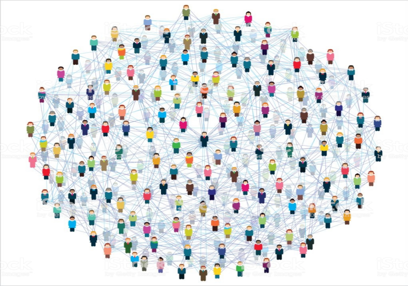
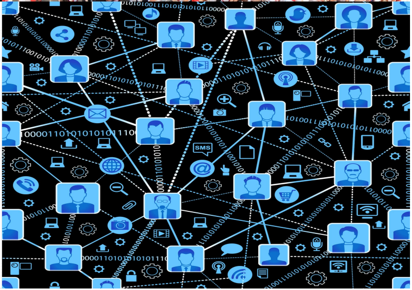



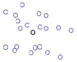
Introduction – *Pattern formation*

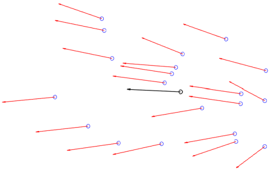
Central observation (pattern formation)

Interacting multi-agent systems may form interesting global structures starting from elementary interaction rules.

Examples of classical patterns

- **Consensus** (everybody tends to agree on something) :
  ↪ *aggregation models in biology, opinion models, etc...*
- **Flocking** (everybody goes in the same direction) :
  ↪ *flocks of birds, herds analysis, opinion formation, etc...*





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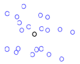

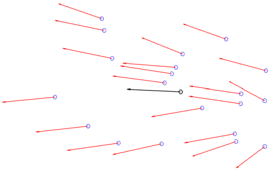

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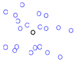

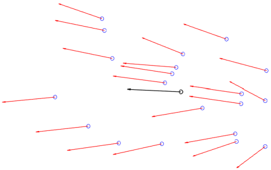

Multi-agent systems – *ODE models*

Multi-agents dynamics are described by **systems of ODEs**

$$\dot{x}_i(t) = \boldsymbol{V}_N[\boldsymbol{x}(t)](t, x_i(t)),$$

where

- $\boldsymbol{x}(\cdot) = (x_1(\cdot), ..., x_N(\cdot)) \in (\mathbb{R}^d)^N$ is the state of the system,
- $\boldsymbol{V}_N : [0, T] \times (\mathbb{R}^d)^N \times \mathbb{R}^d \to \mathbb{R}^d$ is a **non-local** velocity field.

Example (Discrete convolution with a kernel)

$$\boldsymbol{V}_N[\boldsymbol{x}](t, x_i) := \frac{1}{N}\sum_{j=1}^{N} K(t, x_j - x_i).$$

Question : What choice for K ?

$\hookrightarrow$ Depends on the **agent modelling** (pedestrians, animals, opinions...) and the **interactions** (physics-based, etc...).

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Multi-agent systems – *Weighted Cucker-Smale model*

$$(\mathrm{CS'}) \quad \begin{cases} \dot{x}_i(t) = v_i(t), \\ \dot{v}_i(t) = \dfrac{1}{N}\displaystyle\sum_{j=1}^{N} \xi_{ij}(t)\phi\big(|x_i(t) - x_j(t)|\big)(v_j(t) - v_i(t)), \end{cases}$$

where

- $\phi(\cdot)$ is **radial** $\rightsquigarrow$ interactions depend on relative distance,
- $\xi_{ij}(\cdot) \in L^\infty(\mathbb{R}_+, [0,1])$ are **symmetric** communication rates
  $\rightsquigarrow$ agents i and j fully communicate only when $\xi_{ij}(t) = 1$.

$\rightsquigarrow$ Intuitive idea : each agent tries to align its velocity with that of the other agents, with a weight depending on their relative distance.

Main question for today

Under which hypotheses will (CS') converge towards alignment ?

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Alignement models (Cucker & Smale '07)

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Definition (Asymptotic flocking for (CS))

A solution $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ of (CS) **converges to flocking** if

$$\sup_{t \geq 0} |x_i(t) - \bar{\boldsymbol{x}}(t)| < +\infty \qquad \text{and} \qquad \lim_{t \to +\infty} |v_i(t) - \bar{\boldsymbol{v}}| = 0.$$

where $\bar{\boldsymbol{x}}(\cdot)$ and $\bar{\boldsymbol{v}}$ are the position-velocity barycenters.

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Cucker-Smale systems – *Convergence analysis framework*

First question

How do we characterise flocking formation ?

$\rightsquigarrow$ Good idea : find a suitable **dissipative Lyapunov** structure

Definition (Variance bilinear form and standard deviation)

Consider the variance bilinear form

$$B : (\boldsymbol{x}, \boldsymbol{y}) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N \mapsto \frac{1}{N}\sum_{i=1}^{N} \langle x_i, y_i \rangle - \langle \bar{\boldsymbol{x}}, \bar{\boldsymbol{y}} \rangle.$$

Characterisation of flocking

A solution $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ of (CS) converges to flocking **if and only if**

$$\sup_{t \geq 0} X(t) < +\infty \qquad \text{and} \qquad \lim_{t \to +\infty} V(t) = 0,$$

where $X(t) := \sqrt{B(\boldsymbol{x}(t), \boldsymbol{x}(t))}$ and $V(t) := \sqrt{B(\boldsymbol{v}(t), \boldsymbol{v}(t))}$.

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Generalised flocking – *Underlying dissipative structure*

Definition (Graph Laplacians)

Consider the operator $\boldsymbol{L} : (\mathbb{R}^d)^N \to \mathcal{L}((\mathbb{R}^d)^N)$ defined by

$$\big(\boldsymbol{L}(\boldsymbol{x})\boldsymbol{v}\big)_i = \frac{1}{N}\sum_{j=1}^{N} \phi\big(|x_i - x_j|\big)(v_i - v_j),$$

for all $i \in \{1, \dots, N\}$.

Proposition (Interesting things about graph Laplacians)

- Allows to rewrite (CS) as the **semilinear** system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t), \qquad \dot{\boldsymbol{v}}(t) = -\boldsymbol{L}(\boldsymbol{x}(t))\boldsymbol{v}(t).$$

- It holds that $B(\boldsymbol{L}(\boldsymbol{x})\boldsymbol{v}, \boldsymbol{v}) \geq 0$ for all $\boldsymbol{x}, \boldsymbol{v} \in (\mathbb{R}^d)^N$.
- The strength of the interactions in the system is quantified by the **eigenvalues** of $\boldsymbol{L}(\boldsymbol{x}(t))$.

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Cucker-Smale systems – *Known convergence results*

Observation : $\dot{X}(t) \leq V(t)$ and $\dot{V}(t) \leq -\phi(2NX(t))V(t)$

Theorem (Flocking formation) [Cucker & Smale '07, Ha & Liu '09]

Suppose that $\phi(\cdot) \in \mathrm{Lip}(\mathbb{R}_+, \mathbb{R}_+)$ is **positive** and **non-increasing**.

- If $\phi(\cdot)$ is **lower-bounded**, then flocking always occurs
  $\rightsquigarrow$ $V(\cdot)$ exponentially converges towards 0,
- If $\phi(\cdot)$ **vanishes at infinity** but $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+)$, then flocking always occurs
  $\rightsquigarrow$ one has $\phi \notin L^1 \Longrightarrow X(t) \leq X_M$ for some $X_M > 0$,
- If $\phi(\cdot)$ **vanishes at infinity** and $\phi \in L^1(\mathbb{R}_+, \mathbb{R}_+)$, then flocking may fail to occur... $\rightsquigarrow$ One needs small $(X(0), V(0))$.

$\hookrightarrow$ We will henceforth assume that $\phi \notin L^1(\mathbb{R}_+, \mathbb{R}_+)$.

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Cucker-Smale systems – *Known convergence results*

Observation : $\dot{X}(t) \leq V(t)$ and $\dot{V}(t) \leq -\phi(2NX(t))V(t)$

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Outline of the talk

Multi-agent systems and pattern formation

A quick overview of the Cucker-Smale flocking

Cucker-Smale flocking under communication failures

Generalised flocking – *Modelling communication failures*

Back to the weighted model

Consider the Cucker-Smale system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{v}(t), \qquad \dot{\boldsymbol{v}}(t) = -\boldsymbol{L}(t, \boldsymbol{x}(t))\boldsymbol{v}(t)$$

with the time-dependent graph Laplacian

$$\big(\boldsymbol{L}(t, \boldsymbol{x})\boldsymbol{v}\big)_i = \frac{1}{N}\sum_{j=1}^{N} \xi_{ij}(t)\phi\big(|x_i - x_j|\big)(v_i - v_j)$$

Problem (communication weights)

$\rightsquigarrow$ The $\xi_{ij}(\cdot)$ may vanish or be small on possibly long time-intervals.

Main question

Under what kind of assumption on $\xi_{ij}(\cdot)$ do we recover flocking ?

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Second idea : Agents i and j directly communicate if $\xi_{ij}(t) > 0$, but they can still indirectly communicate when $\xi_{ij}(t) = 0$.

Figure: In both situation $\xi_{34}(t) = 0$ but agents 3 and 4 communicate

First idea : All the agents do not **need** to interact at all times, we only need a lower-bound on the average interactions.

$\rightsquigarrow$ Formulate a suitable **persistence conditions** on the weights $\xi_{ij}(\cdot)$

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Definition (Graph Laplacian of the weights)

Consider the operator $\boldsymbol{L}_\xi : \mathbb{R}_+ \to \mathcal{L}((\mathbb{R}^d)^N)$ defined by

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for all $(t, \boldsymbol{v}) \in \mathbb{R}_+ \times (\mathbb{R}^d)^N$, where $(\tau, \mu) \in \mathbb{R}_+ \times (0, 1]$.

Heuristic meaning

On every time window of length τ, the average of interaction magnitudes (both **direct** and **indirect**) between agents is **at least** μ.

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Generalised flocking – *Main convergence result*

Theorem (Flocking formation) [B. & Flayac '20]

Let $(\boldsymbol{x}^0, \boldsymbol{v}^0) \in (\mathbb{R}^d)^N \times (\mathbb{R}^d)^N$ and assume the following holds.

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$$\phi(r) \geq \frac{K}{(\sigma + r)^{\beta}}. \tag{$\mathbf{H}$}$$

Then, the solution $(\boldsymbol{x}(\cdot), \boldsymbol{v}(\cdot))$ of (CS') **converges to flocking**.

Remarks (sharpness of the assumptions)

- Hypothesis ($\mathbf{H}$) is probably not sharp... we expect $\beta \in (0, 1)$
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$\longrightarrow$ Average connectedness is necessary [illegible]

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Generalised flocking – *Elements of proof*

Problem : Lack of dissipation for $V(\cdot)$ $\rightsquigarrow$ strict Lyapunov design !

1) Define the **time-varying** family of operators

$$\psi_\tau(t) = (1+c^2)\tau \mathrm{Id} - \frac{1}{\tau}\int_t^{t+\tau}\int_t^s \boldsymbol{L}(\sigma, \boldsymbol{x}(\sigma))\mathrm{d}\sigma\mathrm{d}s$$

and consider the candidate Lyapunov functional

$$\mathcal{V}_\tau(t) = \lambda(t)V(t) + \sqrt{B(\psi_\tau(t)\boldsymbol{v}(t), \boldsymbol{v}(t))}$$

where $\lambda(\cdot)$ is a tuning parameter (e.g. Mazenc & Malisoff '09).

2) After some (heavy) Lyapunov using (PE) and (**H**), we have

$$V(T) \leq V(0)\exp\left(-C_{\tau,\mu}T^{\frac{1-2\beta}{2(1-\beta)}}\right)$$

where T is the **maximal existence time** of $\lambda(\cdot)$!

Generalised flocking – *Elements of proof*

Problem : Lack of dissipation for $V(\cdot)$ $\rightsquigarrow$ strict Lyapunov design !

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Conclusive remarks

We have identified a synthetic and fairly minimalist persistence condition for multi-agent flocking via Lyapunov methods $\rightsquigarrow$ **Nice !**

Open questions and coming work

1) Recover the **sharp** exponent range $\beta \in (0, 1)$
   $\hookrightarrow$ Try a better Lyapunov design / perform better estimates ?

2) Carry out the analysis using L^∞-Lyapunov functionals
   $\hookrightarrow$ Necessary to obtain **uniform estimates** w.r.t. N and perform mean-field limit approximations as $N \to +\infty$!

3) Adapt our methodology to **random communication failures**
   $\hookrightarrow$ Case where $(t, \omega) \mapsto \xi_{ij}(t, \omega)$ are stochastic processes

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Thank you for your attention !