Some Problems in the Modelling and Optimal Control of Multi-Agent Systems

Benoît Bonnet

Laboratoire d'Informatique et Systèmes, Aix-Marseille Université

Main goal

Write a first-order optimality conditions for general MFOCPs

$$\begin{cases} \min\limits_{\boldsymbol{u}_N(\cdot)} \left[\int_0^T L(t, \boldsymbol{x}_N(t), \boldsymbol{u}_N(t)) \mathrm{d}t + \boldsymbol{\varphi}(\boldsymbol{x}_N(T)) \right] \\ \text{s.t.} \begin{cases} \dot{x}_i(t) = \boldsymbol{v}_N[\boldsymbol{x}_N(t)](t, x_i(t)) + u_i(t), \\ x_i(0) = x_i^0, \end{cases} \end{cases}$$

which are ***scalable*** (independent from N).

Global Approach

1) Find a unique equivalent infinite-dimensional formulation for the family of discrete problems
2) Derive 1st-order optimality conditions on the limit problem
3) Design functional-based algorithms that can be applied to discrete problems

Multi-agents systems

ODE-based models

Finite-dimensional system with N agents

$$\dot{x}_i(t) = \boldsymbol{v}_N[\boldsymbol{x}_N(t)](t, x_i(t))$$

where $\boldsymbol{v}_N[\boldsymbol{x}](\cdot,\cdot)$ is invariant under permutation.

Reformulation as a PDE

The systems of ODEs can be reformulated as

$$\boxed{\partial_t \mu_N(t) + \nabla \cdot (v[\mu_N(t)](t,\cdot)\mu_N(t)) = 0}$$

with

- $\mu_N(t) = \frac{1}{N}\sum_{i=1}^N \delta_{x_i(t)}$
- $v[\mu_N](\cdot,\cdot) = \boldsymbol{v}_N[\boldsymbol{x}_N](\cdot,\cdot)$

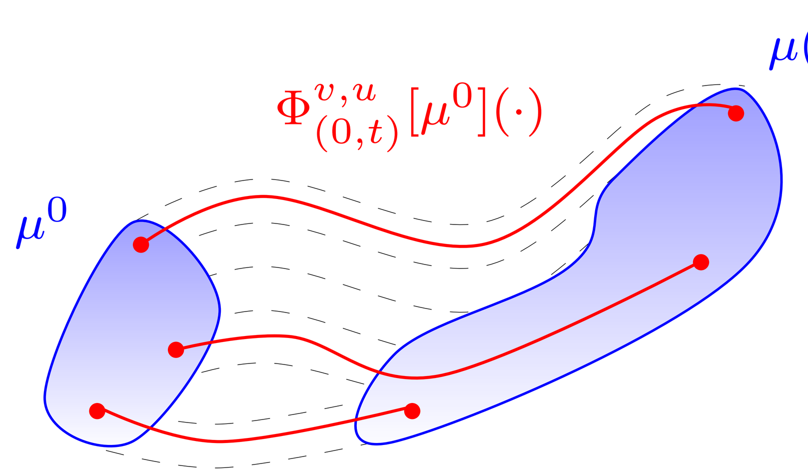
↪ *Mean-field approximation !*

Controlled continuity equations

Non-local continuity equations

$$\partial_t \mu(t) + \nabla \cdot ((v[\mu(t)](t,\cdot) + u(t,\cdot))\mu(t)) = 0,$$

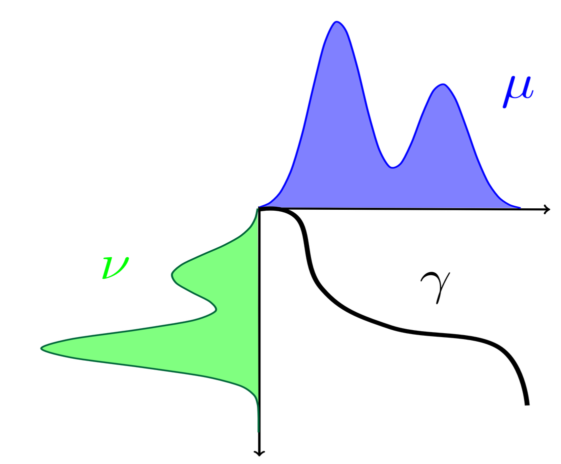
are studied in *Wasserstein spaces* (see right). Classical well-posedness is ensured under **Cauchy-Lipschitz regularity of** $v[\cdot](\cdot,\cdot)$ **and** $u(\cdot,\cdot)$**!**



Optimal Transport

Given $\mu, \nu \in \mathcal{P}_2(\mathbb{R}^d)$, the *Wasserstein distance* between the measures is defined by

$$W_2^2(\mu,\nu) = \min_{\gamma \in \Gamma(\mu,\nu)} \int_{\mathbb{R}^{2d}} |x-y|^2 \mathrm{d}\gamma(x,y).$$



The space $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ can be endowed with a **weak Riemannian structure**

↪ Nice for control !

Pontryagin Maximum Principle in Wasserstein Spaces (B. Bonnet)

Let $(u^*(\cdot,\cdot), \mu^*(\cdot)) \in \mathcal{U} \times \mathrm{Lip}([0,T], \mathcal{P}_c(\mathbb{R}^d))$ be an optimal pair for

$$\min_{u \in \mathcal{U}} \left[\int_0^T L(t, \mu(t), u(t)) \mathrm{d}t + \varphi(\mu(T)) \right]$$

$$\text{s.t.} \begin{cases} \partial_t \mu(t) + \nabla \cdot \big((v[\mu(t)](t,\cdot) + u(t,\cdot))\mu(t) \big) = 0, \\ \mu(0) = \mu^0, \end{cases}$$

$$\text{and} \begin{cases} \Psi_E(\mu(T)) = 0, \quad \Psi_I(\mu(T)) \leq 0, \\ \Lambda(t, \mu(t)) \leq 0 \quad \text{for all } t \in [0,T]. \end{cases}$$

Then, there exists $\nu^*(\cdot) \in \mathrm{Lip}([0,T], \mathcal{P}_c(\mathbb{R}^{2d}))$ solution of

$$\begin{cases} \partial_t \nu^*(t) + \nabla \cdot (\mathbb{J}_{2d} \nabla_\nu \mathcal{H}(t, \nu^*(t), u^*(t)) \nu^*(t)) = 0 \\ \pi^1_{\#}\nu^*(0) = \mu^0, \\ \pi^2_{\#}\nu^*(T) = (-\nabla_\mu \mathcal{S}(\mu^*(T)))_{\#}\mu^*(T) \end{cases}$$

and such that

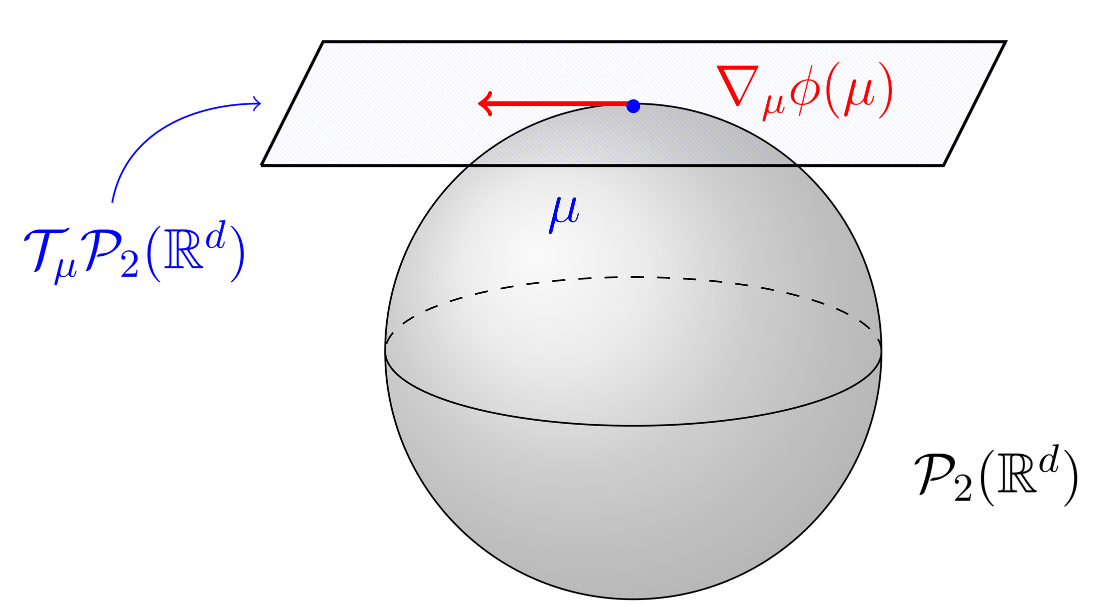
$$\mathcal{H}(t, \nu^*(t), u^*(t)) = \max_{\omega \in U} \mathcal{H}(t, \nu^*(t), \omega)$$

for $\mathcal{L}^1$-almost every $t \in [0,T]$.

Wasserstein differentiability

If $\phi : \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ is "nice enough", there exists $\nabla_\mu \phi(\mu)(\cdot) \in \mathcal{T}_\mu \mathcal{P}_2(\mathbb{R}^d)$ such that

$$\frac{d}{d\epsilon}\left[\phi((I_d + \epsilon \mathcal{F})_{\#}\mu)\right]_{|\epsilon=0} = \langle \nabla_\mu \phi(\mu), \mathcal{F} \rangle_{L^2(\mu)}.$$



Gamkrelidze PMP

- $\nu^*(\cdot)$ is a measure on state-costate pairs.
- Modified Hamiltonian

$$\mathcal{H}(t,\nu,\omega) = \int_{\mathbb{R}^{2d}} \langle r, v[\mu] + \omega \rangle \mathrm{d}\nu(x,r) - L(t,\mu,\omega)$$
$$+ \text{"derivative state constraints"}$$

↪ Costates are Lip instead of BV !

But requires extra regularity of $\Lambda(t,\mu)$.

- Usual non-triviality and transversality conditions.

Main steps of the proof

1) ***N*-needle-variation of the control**

Given $e \in [0, \bar{\epsilon}_N]^N$ and $\{(\omega_k, \tau_k\}_{k=1}^N \subset U \times [0,T]$

$$\tilde{u}_e : t \in [0,T] \mapsto \begin{cases} \omega_k & \text{if } t \in [\tau_k - e_k, \tau_k], \\ u^*(t) & \text{otherwise.} \end{cases}$$

↪ Compute the 1st-order expansion of the total cost in e via the semigroup and differential structure.

2) Non-smooth Lagrange multiplier rule

$e = 0$ is optimal in $[0, \bar{\epsilon}_N]^N$ and write a general non-smooth multiplier rule for the family of finite-dimensional problems.

3) Building the Hamiltonian flow

Given $x \in \mathrm{supp}(\mu^*(T))$, define the curves

$$\sigma^*_{x,N} : t \in [0,T] \mapsto \Psi^{x,N}_{(T,t)}(\cdot)_{\#}\delta_{-\nabla_\mu \mathcal{S}(\mu^*(T))(x)}$$

concentrated on the characteristics generated by $(-\nabla_\mu \mathcal{H})$. Then, define

$$\nu^*_N(t) = \int_{\mathbb{R}^d} \sigma^*_{\Phi^{v,u^*}_{(T,t)}(x),N}(t) \mathrm{d}\mu^*(T)$$

and show that it is s.t. Hamiltonian flow ⊕ relaxed maximization condition hold.

4) Limiting argument

Perform a limiting argument as $N \to +\infty$ to recover the full result.

Future perspectives

1) Apply this result to solve analytically particular problems (e.g. spin synchronization problems, etc...).
2) Implement general shooting algorithms based on our maximum principle, study sufficient optimality condition.
3) Investigate connections with MFG systems.

[1] A Pontryagin Maximum Principle for Constrained Optimal Control Problems in Wasserstein Spaces, B.Bonnet, Accepted in ESAIM COCV (2019)

[2] The Pontryagin Maximum Principle in the Wasserstein Space, B.Bonnet and F.Rossi, Calc.Var.PDEs (2019)